AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(1): 412–439.
DOI: 10.3934/math.2024023
Received: 14 September 2023
Revised: 04 November 2023
Accepted: 17 November 2023
Published: 29 November 2023

Research article

On single-valued neutrosophic soft uniform spaces

Yaser Saber[1,*], Hanan Alohali[2], Tawfik Elmasry[1] and Florentin Smarandache[3]

[1] Department of Business Administration, College of Science and Human Studies, Hotat Sudair, Majmaah University, Majmaah, 11952, Saudi Arabia

[2] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia

[3] Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA

* **Correspondence:** Email: y.sber@mu.edu.sa; Tel: +966509022945.

Abstract: In this paper, we introduce the notion of single-valued neutrosophic soft uniform spaces as a view point of the entourage approach. We investigate the relationship among single-valued neutrosophic soft uniformities, single-valued neutrosophic soft topologies and single-valued neutrosophic soft interior operators. Also, we study several single-valued neutrosophic soft topologies induced by a single-valued neutrosophic soft uniform space.

Keywords: single-valued neutrosophic soft sets; single-valued neutrosophic soft uniformity; Stratification
Mathematics Subject Classification: 54A10, 54A40

1. Introduction

There are many theories that have been suggested for dealing with uncertainties in an efficient way such as the theory of fuzzy sets [1], the theory of intuitionistic fuzzy sets [2], the theory of rough sets [3], and the theory of neutrosophic sets [4]. However, the idea of fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets are not sufficient to cope with parametrization tools. In 1999, Molodtsov [5] proposed the idea of a soft set that has the ability to deal with this difficulty. The idea of fuzzy soft (FS) sets and neutrosophic soft sets was proposed by Maji et al. [6, 7], and some properties of FS sets were discussed by Ahmad and Kharal [8]. Wang et al. [9] proposed the idea of single-valued neutrosophic sets. Saber et al. [10–13] introduced several concepts including, r-single-valued neutrosophic compact modulo, and r-single-valued neutrosophic connected sets in single-valued neutrosophic topological spaces, single-valued neutrosophic ideal open local function, single-valued neutrosophic θ£-separated. Single-valued neutrosophic fuzzy set and multi-attribute decision-making were introduced by Sasirekha et al. [14]. Masri et al. [15] introduced the idea of a single-valued trapezoidal neutrosophic number.

Šostak's single-valued neutrosophic soft topological spaces and single-valued neutrosophic soft sets were constructed by Saber et al. [16]. The concept of single-valued neutrosophic soft has been thoroughly explored and advanced by numerous researchers, such as (Shahzadi et al. [17], Cano et al. [18], Özkan et al. [19], Al-Hijjawi et al. [20], Jana et al. [21] and Kamal et al. [22]) There are three alternative approaches to uniformity in the fuzzy case: Lowen's [23] entourage approach based on power sets of the form $\zeta^{\mathcal{X}\times\mathcal{X}}$, Kotzé's [24] uniform covering approach, and Hutton's [25] uniform operator approach.

It is well known that the theory of neutrosophic sets has been regarded as a generalization of the theory of fuzzy sets, the theory of intuitionistic fuzzy sets and the theory of rough sets. Furthermore, this is an important mathematical tool to deal with uncertainty. One of the main contributions of this paper is to introduce the concepts of single-valued neutrosophic uniformity in the sense of entourage, which is a generalization of the concepts introduced in Lowen [23], Kotzé [24], Hutton [25] and Abbas et al. [26].

Motivated by the above discussion, the present work deals with the single-valued neutrosophic uniformity in the sense of entourage. We introduce the notions of single-valued neutrosophic soft uniform spaces and single-valued neutrosophic soft uniform bases. The notion of this single-valued neutrosophic soft uniformities to be stratified is ensured. We investigate the relationship among single-valued neutrosophic soft uniformities, single-valued neutrosophic soft topologies and single-valued neutrosophic soft interior operators. We study several single-valued neutrosophic soft topologies induced by a single-valued neutrosophic soft uniform structure. Finally, we introduce the product single-valued neutrosophic soft uniformity of a given family of single-valued neutrosophic soft uniform spaces.

2. Preliminaries

In this section, we give all the basic definitions and results that we need to go through our work. First, we give the definition of a single-valued neutrosophic set (svn-set) and a single-valued neutrosophic soft set (svns-set). For more details about svn-set theory and svns-set theory, we refer to [9, 16]. As usual, $\widehat{(\mathcal{X}, \mathrm{E})}$ denotes the family of all svns-sets on $\mathcal{X}$, and E is the set of all parameters. Additionally, $\mathcal{X}$ indicates an initial universe and $\zeta^{\mathcal{X}}$ are the sets of all svn-sets on $\mathcal{X}$ (where, $\zeta = [0, 1]$ and $\zeta_0 = (0, 1]$).

Definition 1. *[4]. Let $\mathcal{X}$ be a universe set. A neutrosophic set (n-set) Θ on $\mathcal{X}$ defined as*

$$\Theta = \{\langle y, \gamma_{\Theta}(y), \pi_{\Theta}(y), \varsigma_{\Theta}(y) \mid y \in \mathcal{X}, \gamma_{\Theta}(y), \pi_{\Theta}(y), \varsigma_{\Theta}(y) \in]^{-}0, 1^{+}\lfloor\},$$

where $\gamma_{\Theta}(y)$, $\pi_{\Theta}(y)$ and $\varsigma_{\Theta}(y)$ are the truth, the indeterminacy, and the falsity membership functions respectively.

Definition 2. *[9]. Let $\mathcal{X}$ be a non-null set. Then, svn-set Θ on $\mathcal{X}$ is defined as*

$$\Theta = \{\langle y, \gamma_{\Theta}(y), \pi_{\Theta}(y), \varsigma_{\Theta}(y) \mid y \in \mathcal{X}, \gamma_{\Theta}(y), \pi_{\Theta}(y), \varsigma_{\Theta}(y) \in \zeta\},$$

where $\gamma_{\Theta}, \pi_{\Theta}, \varsigma_{\Theta} : \mathcal{X} \to \zeta$ and $0 \leq \gamma_{\Theta}(y) + \pi_{\Theta}(y) + \varsigma_{\Theta}(y) \leq 3$.

Remark 1. *To clarify the relationship between intuitionistic fuzzy sets if-set, neutrosophic sets n-set, and single-valued neutrosophic sets svn-set, let us confirm that both neutrosophic sets and single-valued neutrosophic sets are a generalization of the concept of intuitionistic fuzzy sets, as follows:*

In IFS, paraconsistent, dialtheist and incomplete information cannot be characterized. This most important distinction between if-set and n-set is shown in the below neutrosophic cube A' B' C' D' E' F' G' H' introduced by J. Dezert [27].

Because only the classical interval [0,1]is used as a range for the neutrosophic parameters in technical applications (truth, indeterminacy and falsity), we call the cube ABCDEDGH the technical neutrosophic cube and its extension A' B' C' D' E' F' G' H' the neutrosophic cube or nonstandard neutrosophic cube, used in the fields where we need to differentiate between absolute and relative notions like philosophy.

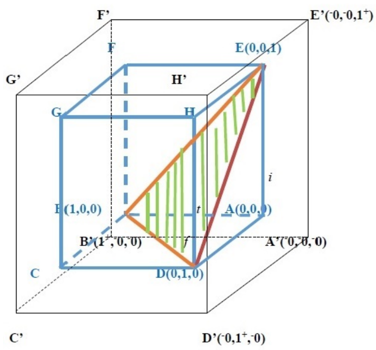

Figure 1. Neutrosophic cube.

Definition 3. *[16]. f_A is an svns-set on $\mathcal{X}$, where $f : E \to \zeta^{\mathcal{X}}$; i.e., $f_e \triangleq f(e)$ is an svn-set on $\mathcal{X}$, for all $e \in A$ and $f(e) = \langle 0, 1, 1 \rangle$, if $e \notin A$.*

The svn-set $f(e)$ is termed as an element of the svns-set f_A. Thus, an svns-set f_E on $\mathcal{X}$ can be defined as:

$$\begin{aligned}(f, E) &= \left\{(e, f(e)) \mid e \in E, f(e) \in \zeta^{\mathcal{X}}\right\} \\ &= \left\{e, \langle \gamma_f(e), \pi_f(e), \varsigma_f(e) \rangle) \mid e \in E, f(e) \in \zeta^{\mathcal{X}}\right\},\end{aligned}$$

where $\gamma_f : E \to \zeta$ (γ_f is termed as a membership function), $\pi_f : E \to \zeta$ (π_f is termed as indeterminacy function) and $\varsigma_f : E \to \zeta$ (ς_f is termed as a nonmembership function) of svns-set.

An svns-set f_E on $\mathcal{X}$ is termed as a null svns-set (for short, $\widehat{\Phi}$), if $\gamma_f(e) = 0, \pi_f(e) = 1$ and $\varsigma_f(e) = 1$, for any $e \in E$.

An svns-set f_E on $\mathcal{X}$ is termed as an absolute svns-set (for short, $\widehat{\widetilde{E}}$), if $\gamma_f(e) = 1, \pi_f(e) = 0$ and $\varsigma_f(e) = 0$, for any $e \in E$.

Definition 4. *[16]. Let $f_A, g_B \in \widehat{(\mathcal{X}, E)}$ be an svns-sets on $\mathcal{X}$. Then,*

(1) Inclusion of two sets (for short, $f_A \preceq g_B$) defined as:

$$\gamma_f(e) \leq \gamma_g(e), \quad \pi_f(e) \geq \pi_g(e), \quad \varsigma_f(e) \geq \varsigma_g(e).$$

(2) The complemented of the set f_A denoted by (for short, f^c_A) defined as:

$$f^c_A = \left\{\left(e, \langle \varsigma_f(e), \tilde{\pi}_{f^c}(e), \gamma_f(e)\rangle\right) \mid e \in E\right\}.$$

Definition 5. *[16]. A mapping $\mathfrak{T}^\gamma, \mathfrak{T}^\pi, \mathfrak{T}^\varsigma : E \to \zeta^{\widehat{(\mathcal{X},E)}}$ is said to be a single-valued neutrosophic soft topology (svnst) on $\mathcal{X}$ if it meets the next criteria, for every $e \in E$:*

$(\mathfrak{T}_1)$ $\mathfrak{T}^\gamma_e(\widehat{\Phi}) = 1$, $\mathfrak{T}^\pi_e(\widehat{\Phi}) = 0$, $\mathfrak{T}^\varsigma_e(\widehat{\Phi}) = 0$ *and* $\mathfrak{T}^\gamma_e(\widehat{E}) = 1$ $\mathfrak{T}^\pi_e(\widehat{E}) = 0$, $\mathfrak{T}^\varsigma_e(\widehat{E}) = 0$,

$(\mathfrak{T}_2)$ $\mathfrak{T}^\gamma_e(f_A \sqcap g_B) \geq \mathfrak{T}^\gamma_e(f_A) \wedge \mathfrak{T}^\gamma_e(g_B)$, $\mathfrak{T}^\pi_e(f_A \sqcap g_B) \leq \mathfrak{T}^\pi_e(f_A) \vee \mathfrak{T}^\pi_e(g_B)$,
$\mathfrak{T}^\varsigma_e(f_A \sqcap g_B) \leq \mathfrak{T}^\varsigma_e(f_A) \vee \mathfrak{T}^\varsigma_e(g_B)$, $\forall\, f_A, g_B \in \widehat{(\mathcal{X},E)}$,

$(\mathfrak{T}_3)$ $\mathfrak{T}^\gamma_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \geq \bigwedge_{j\in\Gamma} \mathfrak{T}^\gamma_e((f_A)_j)$, $\mathfrak{T}^\pi_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \leq \bigvee_{j\in\Gamma} \mathfrak{T}^\pi_e((f_A)_j)$,
$\mathfrak{T}^\varsigma_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \leq \bigvee_{j\in\Gamma} \mathfrak{T}^\varsigma_e((f_A)_j)$, $\forall\, (f_A)_j \in \widehat{(\mathcal{X},E)}$, $j \in \Gamma$.

(Note that $\sqcap$ and $\sqcup$ in the definition are clarified in Molodtsov [5]). The quadruple $(\mathcal{X}, \mathfrak{T}^\gamma, \mathfrak{T}^\pi, \mathfrak{T}^\varsigma)$ is said to be a single-valued neutrosophic soft topological space (svnst-space), where $(\mathfrak{T}^\gamma_e(f_A))$ representing the degree of openness, $(\mathfrak{T}^\pi_e(f_A))$ the degree of indeterminacy and $(\mathfrak{T}^\varsigma_e(f_A))$ the degree of non-openness; of a svns-set with respect to that parameter $e \in E$. Sometimes, we will write $\mathfrak{T}^{\gamma\pi\varsigma}$ for $(\mathfrak{T}^\gamma, \mathfrak{T}^\pi, \mathfrak{T}^\varsigma)$.

Let $(\mathcal{X}, \mathfrak{T}^{\gamma\pi\varsigma})$ and $(\mathcal{G}, \mathfrak{T}^{\star\gamma\pi\varsigma})$ be svnst-space. An svns-mapping $\psi_\varphi : \widehat{(\mathcal{X},E)} \to \widehat{(\mathcal{G},\mathcal{R})}$ is said to be a single-valued neutrosophic soft continuous mapping (svnsc-map) if

$$\mathfrak{T}^\gamma_e(\psi^{-1}_\varphi(g_B)) \geq \mathfrak{T}^{\star\gamma}_{\varphi(e)}(g_B), \qquad \mathfrak{T}^\pi_e(\psi^{-1}_\varphi(g_B)) \leq \mathfrak{T}^{\star\pi}_{\varphi(e)}(g_B),$$

$$\mathfrak{T}^\varsigma_e(\psi^{-1}_\varphi(g_B)) \leq \mathfrak{T}^{\star\varsigma}_{\varphi(e)}(g_B),$$

for all $g_B \in \widehat{(\mathcal{G},\mathcal{R})}$ and $e \in E$ [Saber et al. (2022) [16].

Definition 6. *A map $\mathcal{I} : E \times \widehat{(\mathcal{X},E)} \times \zeta_0 \to \widehat{(\mathcal{X},E)}$ is said to be single-valued neutrosophic soft interior operator (svnsi-operator) on $\mathcal{X}$ if it meets the next criteria, $\forall$, $e \in E$, $f_A, g_B \in \widehat{(\mathcal{X},E)}$ and $r, s \in \zeta$:*

$(\mathcal{I}_1)$ $\mathcal{I}(e, \widehat{E}, r) = \widehat{E}$,
$(\mathcal{I}_2)$ $\mathcal{I}(e, f_A, r) \leq f_A$,
$(\mathcal{I}_3)$ if $f_A \leq g_B$ and $r \leq s$ then $\mathcal{I}(e, f_A, r) \leq \mathcal{I}(e, g_B, s)$,
$(\mathcal{I}_4)$ $\mathcal{I}(e), f_A \sqcap g_B, r \wedge s) \geq \mathcal{I}(e, f_A, r) \sqcap \mathcal{I}(e, g_B, s)$,
$(\mathcal{I}_5)$ $\mathcal{I}(e, \mathcal{I}(e, f_A, r), r) = \mathcal{I}(e, f_A, r)$.

Definition 7. *[16]. A map $\mathcal{C} : E \times \widehat{(\mathcal{X},E)} \times \zeta_0 \to \widehat{(\mathcal{X},E)}$ is said to be single-valued neutrosophic soft closure operator (svnsc-operator) on $\mathcal{X}$ if it meets the next criteria, $\forall$, $e \in E$, $f_A, g_B \in \widehat{(\mathcal{X},E)}$ and $r, s \in \zeta$:*

$(\mathcal{C}_1)$ $\mathcal{C}(e, \widehat{\Phi}, r) = \widehat{\Phi}$,
$(\mathcal{C}_2)$ $\mathcal{C}(e, f_A, r) \geq f_A$,
$(\mathcal{C}_3)$ if $f_A \leq g_B$ and $r \leq s$ then $\mathcal{C}(e, f_A, r) \leq \mathcal{C}(e, g_B, s)$,
$(\mathcal{C}_4)$ $\mathcal{C}(e, f_A \sqcup g_B, r \wedge s) \leq \mathcal{C}(e, f_A, r) \sqcup \mathcal{C}(e, g_B, s)$,
$(\mathcal{C}_5)$ $\mathcal{C}(e, \mathcal{C}(e, f_A, r), r) \leq \mathcal{C}(e, f_A, r)$,
$(\mathcal{C}_6)$ $\mathcal{C}(e, f_A, r) = [\mathcal{I}(e, f^c_A, r)]^c$.

3. Single-valued neutrosophic soft soft uniform spaces

The main objective of this section is to define and discuss the concepts of single-valued neutrosophic soft uniformity (svns-uniformity), single-valued neutrosophic soft uniform base (svns-uniform base) and stratified single-valued neutrosophic soft uniform space (ssvns-uniform space). Several basic properties and theorems related to these concepts are explored.

In this section, we indicate that $(\widehat{\mathcal{X}\times\mathcal{X}}, \mathrm{E})$ is the family of all svns-sets on $\mathcal{X}\times\mathcal{X}$ and $\zeta^{\mathcal{X}\times\mathcal{X}}$ are the sets of all svn-sets on $\mathcal{X}\times\mathcal{X}$. Additionally, for $\varrho\in\zeta$, $\bar{\varrho}(x,y)=\varrho$ for any $(x,y)\in\mathcal{X}\times\mathcal{X}$.

Definition 8. *Let $\mathcal{X}$ be a set. A mappings $\pounds^{\gamma},\pounds^{\pi},\pounds^{\varsigma}: E\to\zeta^{(\widehat{\mathcal{X}\times\mathcal{X}},E)}$ is called an svns-uniformity on $\mathcal{X}$ if it meets the next criteria:*

($\pounds_1$) *for any $e\in E$, there exists $\upsilon_A\in(\widehat{\mathcal{X}\times\mathcal{X}},E)$ such that $\pounds^{\gamma}_e(\upsilon_A)=1$, $\pounds^{\pi}_e(\upsilon_A)=0$, $\pounds^{\varsigma}_e(\upsilon_A)=0$,*
($\pounds_2$) *if $\upsilon_A\preceq\mu_B$, then $\pounds^{\gamma}_e(\upsilon_A)\leq\pounds^{\gamma}_e(\mu_B)$, $\pounds^{\pi}_e(\upsilon_A)\geq\pounds^{\pi}_e(\mu_B)$, $\pounds^{\varsigma}_e(\upsilon_A)\geq\pounds^{\varsigma}_e(\mu_B)$,*
($\pounds_3$) *for every $\upsilon_A,\mu_B\in(\widehat{\mathcal{X}\times\mathcal{X}},E)$, then*

$$\pounds^{\gamma}_e(\upsilon_A\sqcap\mu_B)\geq\pounds^{\gamma}_e(\upsilon_A)\wedge\pounds^{\gamma}_e(\mu_B),\quad \pounds^{\pi}_e(\upsilon_A\sqcap\mu_B)\leq\pounds^{\pi}_e(\upsilon_A)\vee\pounds^{\pi}_e(\mu_B)$$

$$\pounds^{\varsigma}_e(\upsilon_A\sqcap\mu_B)\leq\pounds^{\varsigma}_e(\upsilon_A)\vee\pounds^{\varsigma}_e(\mu_B),$$

($\pounds_4$) $(\top)_C\npreceq\upsilon_A$ *implies that $\pounds^{\gamma}_e(\upsilon_A)=0$, $\pounds^{\pi}_e(\upsilon_A)=1$, $\pounds^{\varsigma}_e(\upsilon_A)=1$, where, $\forall\ e\in E$,*

$$(\top)_e(x,y)=\begin{cases}\langle 1,0,0\rangle, & \text{if } x=y,\\ \langle 0,1,1\rangle, & \text{otherwise},\end{cases}$$

($\pounds_5$) $\pounds^{\gamma}_e(\upsilon_A)\leq\pounds^{\gamma}_e(\upsilon^s_A)$, $\pounds^{\pi}_e(\upsilon_A)\geq\pounds^{\pi}_e(\upsilon^s_A)$, $\pounds^{\varsigma}_e(\upsilon_A)\geq\pounds^{\varsigma}_e(\upsilon^s_A)$, *where $\upsilon^s_e(x,y)=\upsilon_e(y,x)$ for every $e\in E$,*
($\pounds_6$) *for each $\upsilon_A\in(\widehat{\mathcal{X}\times\mathcal{X}},E)$, $e\in E$,*

$$\pounds^{\gamma}_e(\upsilon_A)\leq\bigvee\{\pounds^{\gamma}_e(\mu_B):\ (\mu_B\circ\mu_B)\preceq\upsilon_A\},\qquad \pounds^{\pi}_e(\upsilon_A)\geq\bigwedge\{\pounds^{\pi}_e(\mu_B):\ (\mu_B\circ\mu_B)\preceq\upsilon_A\},$$

$$\pounds^{\varsigma}_e(\upsilon_A)\geq\bigwedge\{\pounds^{\varsigma}_e(\mu_B):\ (\mu_B\circ\mu_B)\preceq\upsilon_A\},$$

where $(\upsilon_B\circ\mu_B)=\bigvee_{z\in\mathcal{X}}\{\upsilon_e(x,z)\wedge\mu_e(z,y)\}$ for each $x,y\in\mathcal{X}$.

A svns-uniformity $\pounds^{\gamma},\pounds^{\pi},\pounds^{\varsigma}: E\to\zeta^{(\widehat{\mathcal{X}\times\mathcal{X}},E)}$ is called stratified if

($\pounds_{st}$) $\pounds^{\gamma}_e(\widehat{E_\varrho})=1$, $\pounds^{\pi}_e(\widehat{E_\varrho})=0$, $\pounds^{\varsigma}_e(\widehat{E_\varrho})=0$, *where $\upsilon_E=\widehat{E_\varrho}$ if $\upsilon_e=\bar{\varrho}\ \forall$, $e\in E$.*

After adding the last condition $(\mathcal{X},\pounds^{\gamma},\pounds^{\pi},\pounds^{\varsigma})$ is called ssvns-uniform space. Sometimes, we will write $\pounds^{\gamma\pi\varsigma}_E$ for $(\pounds^{\gamma},\pounds^{\pi},\pounds^{\varsigma})$.

Let $\pounds^{\gamma\pi\varsigma}_E$ and $\pounds^{\star\gamma\pi\varsigma}_E$ be two svns-uniformities on $\mathcal{X}$. $\pounds^{\gamma\pi\varsigma}_E$ is finer than $\pounds^{\star\gamma\pi\varsigma}_E$ ($\pounds^{\star\gamma\pi\varsigma}_E$ is coarser than $\pounds^{\gamma\pi\varsigma}_E$), indicated by $\pounds^{\star\gamma\pi\varsigma}_E\preceq\pounds^{\gamma\pi\varsigma}_E$ provided

$$\pounds^{\star\gamma}_e(\upsilon_A)\leq\pounds^{\gamma}_E(\upsilon_A),\qquad \pounds^{\star\pi}_e(\upsilon_A)\geq\pounds^{\pi}_e(\upsilon_A),\qquad \pounds^{\star\varsigma}_e(\upsilon_A)\geq\pounds^{\varsigma}_e(\upsilon_A),\quad \forall\ e\in E,\upsilon_A\in(\widehat{\mathcal{X}\times\mathcal{X}},E).$$

Remark 2. *Suppose that $(\mathcal{X},\pounds^{\gamma\pi\varsigma}_E)$ is an svns-uniform space. Then, by using the two conditions ($\pounds_1$) and ($\pounds_2$), we obtain, $\pounds^{\gamma}_e(\widehat{E})=1$, $\pounds^{\pi}_e(\widehat{E})=0$, $\pounds^{\varsigma}_e(\widehat{E})=0$ because $\upsilon_A\preceq\widehat{E}$ for every $e\in E$, $\upsilon_A\in(\widehat{\mathcal{X}\times\mathcal{X}},E)$.*

Theorem 1. *Let $(\mathcal{X}, £_E^{\gamma\pi\varsigma})$ be an svns-uniform space. Define for any $e \in E$, $\upsilon_A \in (\widehat{\mathcal{X} \times \mathcal{X}}, E)$.*

$$(£^{\gamma}_{st})_e(\upsilon_A) = \bigvee\{£^{\gamma}_e((\mu_B)) : \ \mu_B \sqcap \widehat{E_\varrho} \preceq \upsilon_A, \ \varrho \in \zeta\},$$

$$(£^{\pi}_{st})_e(\upsilon_A) = \bigwedge\{£^{\pi}_e((\mu_B)) : \ \mu_B \sqcap \widehat{E_\varrho} \preceq \upsilon_A, \ \varrho \in \zeta\},$$

$$(£^{\varsigma}_{st})_e(\upsilon_A) = \bigwedge\{£^{\varsigma}_e((\mu_B)) : \ \mu_B \sqcap \widehat{E_\varrho} \preceq \upsilon_A, \ \varrho \in \zeta\}.$$

Then, $(£^{\gamma\pi\varsigma}_{st})_E$ is the coarsest ssvns-uniformity which is finer than $£^{\gamma\pi\varsigma}_E$.

Proof. ($£_1$) There exists $\upsilon_A \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$ such that $£^{\gamma}_e(\upsilon_A) = 1$, $£^{\pi}_e(\upsilon_A) = 0$, $£^{\varsigma}_e(\upsilon_A) = 0$ for every $e \in \mathrm{E}$. Since $\upsilon_A \sqcap \widehat{E_\varrho} \preceq \upsilon_A$, *then* $(£^{\gamma}_{st})_e(\upsilon_A) = 1$, $(£^{\pi}_{st})_e(\upsilon_A) = 0$, $(£^{\varsigma}_{st})_e(\upsilon_A) = 0$.

($£_2$) Direct from the definition.

($£_3$) Let there exist $(\upsilon_1)_A, (\upsilon_2)_B \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$ such that for every $e \in \mathrm{E}$,

$$(£^{\gamma}_{st})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \ngeq (£^{\gamma}_{st})_e((\upsilon_1)_A) \wedge (£^{\gamma}_{st})_e((\upsilon_2)_B),$$

$$(£^{\pi}_{st})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \nleq (£^{\pi}_{st})_e((\upsilon_1)_A) \vee (£^{\pi}_{st})_e((\upsilon_2)_B),$$

$$(£^{\varsigma}_{st})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \nleq (£^{\varsigma}_{st})_e((\upsilon_1)_A) \vee (£^{\varsigma}_{st})_e((\upsilon_2)_B).$$

By using the definition of $(£^{\gamma\pi\varsigma}_{st})_{\mathrm{E}}$, then there exists $(\mu_1)_C, (\mu_2)_D \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$, $\varrho_1, \varrho_2 \in \zeta$ with $(\mu_1)_C \sqcap \widehat{E_{\varrho_1}} \preceq (\upsilon_1)_A$, $(\mu_2)_D \sqcap \widehat{E_{\varrho_2}} \preceq (\upsilon_2)_B$ such that

$$(£^{\gamma}_{st})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \ngeq £^{\gamma}_e((\mu_1)_C) \wedge £^{\gamma}_e((\mu_2)_D),$$

$$(£^{\pi}_{st})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \nleq £^{\pi}_e((\mu_1)_C) \vee £^{\pi}_e((\mu_2)_D),$$

$$(£^{\varsigma}_{st})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \nleq £^{\varsigma}_e((\mu_1)_C) \vee (£^{\varsigma}_e((\mu_2)_D).$$

Otherwise, $(\mu_1)_C \sqcap (\mu_2)_D \sqcap \widehat{E_{\varrho_1}} \sqcap \widehat{E_{\varrho_2}} \preceq (\upsilon_1)_A \sqcap (\upsilon_2)_B$. Then, we have

$$(£^{\gamma}_{st})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \geq £^{\gamma}_e((\mu_1)_C \sqcap (\mu_2)_D) \geq £^{\gamma}_e((\mu_1)_C) \wedge £^{\gamma}_e((\mu_2)_D),$$

$$(£^{\pi}_{st})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \leq £^{\pi}_e((\mu_1)_C \sqcap (\mu_2)_D) \leq £^{\pi}_e((\mu_1)_C) \vee £^{\pi}_e((\mu_2)_D),$$

$$(£^{\varsigma}_{st})_e((\upsilon_1)_A \sqcap (\upsilon_2)_B) \leq £^{\varsigma}_e((\mu_1)_C \sqcap (\mu_2)_D) \leq £^{\varsigma}_e((\mu_1)_C) \vee (£^{\varsigma}_e((\mu_2)_D).$$

This is a contradiction. Consequently, ($£_3$) holds.

($£_4$) Direct from the definition.

($£_5$) Let

$$(£^{\gamma}_{st})_e(\upsilon^s_A) \ngeq (£^{\gamma}_{st})_e(\upsilon_A), \qquad (£^{\pi}_{st})_e(\upsilon^s_A) \nleq (£^{\pi}_{st})_e(\upsilon_A), \qquad (£^{\varsigma}_{st})_e(\upsilon^s_A) \nleq (£^{\varsigma}_{st})_e(\upsilon_A),$$

$\forall$, $e \in \mathrm{E}$, $\upsilon_A \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$. By using the definition of $(£^{\gamma\pi\varsigma}_{st})_{\mathrm{E}}$, there exists $\mu_B \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$, $\varrho \in \zeta$ with $\mu_B \sqcap \widehat{E_\varrho} \preceq \upsilon_A$, such that

$$(£^{\gamma}_{st})_e(\upsilon^s_A) \ngeq £^{\gamma}_e(\mu_B), \qquad (£^{\pi}_{st})_e(\upsilon^s_A) \nleq £^{\pi}_e(\mu_B), \qquad (£^{\varsigma}_{st})_e(\upsilon^s_A) \nleq £^{\varsigma}_e(\mu_B).$$

Since $£^{\gamma\pi\varsigma}_{\mathrm{E}}$ is svns-uniformity, then

$$£^{\gamma}_e(\mu_B) \leq £^{\gamma}_e(\mu^s_B), \qquad £^{\pi}_e(\mu_B) \geq £^{\pi}_e(\mu^s_B), \qquad £^{\varsigma}_e(\mu_B) \geq £^{\varsigma}_e(\mu^s_B),$$

It follows that

$$(\pounds^{\gamma}_{st})_e(\upsilon^s_A) \not\geq \pounds^{\gamma}_e(\mu^s_B), \qquad (\pounds^{\pi}_{st})_e(\upsilon^s_A) \not\leq \pounds^{\pi}_e(\mu^s_B), \qquad (\pounds^{\varsigma}_{st})_e(\upsilon^s_A) \not\leq \pounds^{\varsigma}_e(\mu^s_B).$$

On the other hand, $\mu^s_B \sqcap \widehat{E_\varrho} \leq \upsilon^s_A$. Hence, for each $e \in \mathrm{E}$

$$(\pounds^{\gamma}_{st})_e(\upsilon^s_A) \geq \pounds^{\gamma}_e(\mu^s_B), \qquad (\pounds^{\pi}_{st})_e(\upsilon^s_A) \leq \pounds^{\pi}_e(\mu^s_B), \qquad (\pounds^{\varsigma}_{st})_e(\upsilon^s_A) \leq \pounds^{\varsigma}_e(\mu^s_B).$$

This is a contradiction. Therefore, $(\pounds_5)$ holds.

$(\pounds_6)$ Suppose that

$$(\pounds^{\gamma}_{st})_e(\upsilon_A) \not\leq \bigvee\{(\pounds^{\gamma}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \leq \upsilon_A\},$$
$$(\pounds^{\pi}_{st})_e(\upsilon_A) \not\geq \bigwedge\{(\pounds^{\pi}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \leq \upsilon_A\},$$
$$(\pounds^{\varsigma}_{st})_e(\upsilon_A) \not\geq \bigwedge\{(\pounds^{\varsigma}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \leq \upsilon_A\}.$$

for any $\upsilon_A \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$. From the definition of $(\pounds^{\gamma\pi\varsigma}_{st})_{\mathrm{E}}$, there exists $\mu_B \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$, $\varrho \in \zeta$ with $\mu_B \sqcap \widehat{E_\varrho} \leq \upsilon_A$ such that

$$\pounds^{\gamma}_e(\mu_B) \not\leq \bigvee\{(\pounds^{\gamma}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \leq \upsilon_A\},$$
$$\pounds^{\pi}_e(\mu_B) \not\geq \bigwedge\{(\pounds^{\pi}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \leq \upsilon_A\},$$
$$\pounds^{\varsigma}_e(\mu_B) \not\geq \bigwedge\{(\pounds^{\varsigma}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \leq \upsilon_A\}.$$

Since $\pounds^{\gamma\pi\varsigma}_{\mathrm{E}}$ is svns-uniformity on $\mathcal{X}$, then

$$\pounds^{\gamma}_e(\mu_B) \leq \bigvee\{\pounds^{\gamma}_e(\sigma_D) : \ \sigma_D \circ \sigma_D \leq \mu_B\},$$
$$\pounds^{\pi}_e(\mu_B) \geq \bigwedge\{\pounds^{\pi}_e(\sigma_D) : \ \sigma_D \circ \sigma_D \leq \mu_B\},$$
$$\pounds^{\varsigma}_e(\mu_B) \geq \bigwedge\{\pounds^{\varsigma}_e(\sigma_D) : \ \sigma_D \circ \sigma_D \leq \mu_B\}.$$

That means, there is $\sigma_D \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$ such that $\sigma_D \sqcap \sigma_D \leq \mu_B$ and that

$$\pounds^{\gamma}_e(\sigma_D) \not\leq \bigvee\{(\pounds^{\gamma}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \leq \upsilon_A\},$$
$$\pounds^{\pi}_e(\sigma_D) \not\geq \bigwedge\{(\pounds^{\pi}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \leq \upsilon_A\},$$
$$\pounds^{\varsigma}_e(\sigma_D) \not\geq \bigwedge\{(\pounds^{\varsigma}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \leq \upsilon_A\}.$$

On the other hand,

$$(\sigma_D \sqcap \widehat{E_\varrho}) \circ (\sigma_D \sqcap \widehat{E_\varrho}) \leq (\sigma_D \circ \sigma_D) \sqcap \widehat{E_\varrho} \leq \mu_B \sqcap \widehat{E_\varrho} \leq \upsilon_A,$$

which means that there is $(\upsilon_1)_C = \sigma_D \sqcap \widehat{E_\varrho}$ with $(\upsilon_1)_C \circ (\upsilon_1)_C \leq \upsilon_A$,

$$\pounds^{\gamma}_e(\sigma_D) \leq (\pounds^{\gamma}_{st})_e((\upsilon_1)_C) \leq \bigvee\{(\pounds^{\gamma}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \leq \upsilon_A\},$$
$$\pounds^{\pi}_e(\sigma_D) \geq (\pounds^{\pi}_{st})_e((\upsilon_1)_C) \geq \bigwedge\{(\pounds^{\pi}_{st})_e((\upsilon_1)_C) : \ (\upsilon_1)_C \circ (\upsilon_1)_C \leq \upsilon_A\},$$

$$£_e^{\varsigma}(\sigma_D) \geq (£_{st}^{\varsigma})_e((\upsilon_1)_C) \geq \bigwedge\{(£_{st}^{\varsigma})_e((\upsilon_1)_C) :\ (\upsilon_1)_C \circ (\upsilon_1)_C \preceq \upsilon_A\}.$$

It is a contradiction. Thus, $(£_6)$ holds.

$(£_{st})$ Since $\widehat{E_\varrho} \sqcap \widehat{E_1} = \widehat{E_\varrho}$ for each $\varrho \in \zeta$, then $(£_{st}^{\gamma})_{\widehat{E}} = 1$, $(£_{st}^{\pi})_{\widehat{E}} = 0$ and $(£_{st}^{\varsigma})_{\widehat{E}} = 0$. Therefore, $(£_{st}^{\gamma\pi\varsigma})_e$ is stratified.

For each $\upsilon_A \in (\widehat{\mathcal{X}\times\mathcal{X}}, \mathrm{E})$, $\upsilon_A \sqcap \widehat{E_1} = \upsilon_A$, we have for each $e \in \mathrm{E}$

$$(£_{st}^{\gamma})_e(\upsilon_A) \geq £_e^{\gamma}(\upsilon_A), \qquad (£_{st}^{\pi})_e(\upsilon_A) \leq £_e^{\pi}(\upsilon_A), \qquad (£_{st}^{\varsigma})_e(\upsilon_A) \leq £_e^{\varsigma}(\upsilon_A).$$

Hence, $(£_{st}^{\gamma\pi\varsigma})_{\mathrm{E}}$ is finer than $£_{\mathrm{E}}^{\gamma\pi\varsigma}$.

Finally, consider $£_{\mathrm{E}}^{\star\gamma\pi\varsigma}$ is an ssvns-uniformity finer than $£_{\mathrm{E}}^{\gamma\pi\varsigma}$. Let there exists $\upsilon_A \in (\widehat{\mathcal{X}\times\mathcal{X}}, \mathrm{E})$ such that

$$(£_{st}^{\gamma})_e(\upsilon_A) \nleq £_e^{\star\gamma}(\upsilon_A), \qquad (£_{st}^{\pi})_e(\upsilon_A) \ngeq £_e^{\star\pi}(\upsilon_A), \qquad (£_{st}^{\varsigma})_e(\upsilon_A) \ngeq £_e^{\star\varsigma}(\upsilon_A).$$

From the definition of $\{(£_{st}^{\gamma})_e(\upsilon_A), (£_{st}^{\pi})_e(\upsilon_A), (£_{st}^{\varsigma})_e(\upsilon_A)\}$, there exists $\mu_B \in (\widehat{\mathcal{X}\times\mathcal{X}}, \mathrm{E})$, $\varrho \in \zeta$ with $\mu_B \sqcap \widehat{E_\varrho} \preceq \upsilon_A$ and

$$£_e^{\gamma}(\mu_B) \nleq £_e^{\star\gamma}(\upsilon_A), \qquad £_e^{\pi}(\mu_B) \ngeq £_e^{\star\pi}(\upsilon_A), \qquad £_e^{\varsigma}(\mu_B) \ngeq £_e^{\star\varsigma}(\upsilon_A).$$

Since $£_{\mathrm{E}}^{\star\gamma\pi\varsigma}$ is stratified, then

$$£_e^{\gamma}(\mu_B) \leq £_e^{\star\gamma}(\mu_B) = £_e^{\star\gamma}(\mu_B) \wedge £_e^{\star\gamma}(\widehat{E_\varrho}) \leq £_e^{\star\gamma}(\mu_B \sqcap \widehat{E_\varrho}) \leq £_e^{\gamma}(\upsilon_A),$$

$$£_e^{\pi}(\mu_B) \geq £_e^{\star\pi}(\mu_B) = £_e^{\star\pi}(\mu_B) \vee £_e^{\star\pi}(\widehat{E_\varrho}) \geq £_e^{\star\pi}(\mu_B \sqcup \widehat{E_\varrho}) \geq £_e^{\pi}(\upsilon_A),$$

$$£_e^{\varsigma}(\mu_B) \geq £_e^{\star\varsigma}(\mu_B) = £_e^{\star\varsigma}(\mu_B) \vee £_e^{\star\varsigma}(\widehat{E_\varrho}) \geq £_e^{\star\varsigma}(\mu_B \sqcup \widehat{E_\varrho}) \geq £_e^{\varsigma}(\upsilon_A).$$

It is a contradiction. Hence,

$$(£_{st}^{\gamma})_e(\upsilon_A) \leq £_e^{\star\gamma}(\upsilon_A), \qquad (£_{st}^{\pi})_e(\upsilon_A) \geq £_e^{\star\pi}(\upsilon_A), \qquad (£_{st}^{\varsigma})_e(\upsilon_A) \geq £_e^{\star\varsigma}(\upsilon_A),$$

for each $\upsilon_A \in (\widehat{\mathcal{X}\times\mathcal{X}}, \mathrm{E})$, $e \in \mathrm{E}$. Hence, $(£_{st}^{\gamma\pi\varsigma})_{\mathrm{E}}$ is the coarsest ssvns-uniformity which is finer than $£_{\mathrm{E}}^{\gamma\pi\varsigma}$. □

Remark 3. *Let $\hbar^{\gamma}, \hbar^{\pi}, \hbar^{\varsigma} : E \to \zeta^{(\widehat{\mathcal{X}\times\mathcal{X}}, E)}$ be a mapping and $\upsilon_A \in (\widehat{\mathcal{X}\times\mathcal{X}}, E)$. Let us define $\langle\hbar_e^{\gamma}\rangle$, $\langle\hbar_e^{\pi}\rangle$ and $\langle\hbar_e^{\varsigma}\rangle$ as follows for each $e \in E$:*

$$\langle\hbar_e^{\gamma}\rangle(\upsilon_A) = \bigvee_{\upsilon_A \preceq \upsilon_B} \hbar_e^{\gamma}(\upsilon_B), \quad \langle\hbar_e^{\pi}\rangle(\upsilon_A) = \bigwedge_{\upsilon_A \preceq \upsilon_B} \hbar_e^{\pi}(\upsilon_B), \quad \langle\hbar_e^{\varsigma}\rangle(\upsilon_A) = \bigwedge_{\upsilon_A \preceq \upsilon_B} \hbar_e^{\varsigma}(\upsilon_B).$$

Definition 9. *A mappings $\hbar^{\gamma}, \hbar^{\pi}, \hbar^{\varsigma} : E \to \zeta^{(\widehat{\mathcal{X}\times\mathcal{X}}, E)}$ is called a svns-uniform base on $\mathcal{X}$ if it meets the next criteria:*

$(\hbar_1)$ There exists $\upsilon_A \in (\widehat{\mathcal{X}\times\mathcal{X}}, E)$ such that $\hbar_e^{\gamma}(\upsilon_A) = 1$, $\hbar_e^{\pi}(\upsilon_A) = 0$, $\hbar_e^{\varsigma}(\upsilon_A) = 0$, for all $e \in E$,
$(\hbar_2)$ for each $\upsilon_A, \mu_B \in (\widehat{\mathcal{X}\times\mathcal{X}}, E)$, $e \in E$, such that

$$\langle\hbar_e^{\gamma}\rangle(\upsilon_A \sqcap \mu_B) \geq \hbar_e^{\gamma}(\upsilon_A) \wedge \hbar_e^{\gamma}(\mu_B), \qquad \langle\hbar_e^{\pi}\rangle(\upsilon_A \sqcap \mu_B) \leq \hbar_e^{\pi}(\upsilon_A) \vee \hbar_e^{\pi}(\mu_B),$$

$$\langle\hbar_e^{\varsigma}\rangle(\upsilon_A \sqcap \mu_B) \leq \hbar_e^{\varsigma}(\upsilon_A) \vee \hbar_e^{\varsigma}(\mu_B),$$

($\hbar_3$) *If* $(\top)_A \not\leq \upsilon_B$, *then* $\hbar^{\gamma}_e(\upsilon_B) = 0$, $\hbar^{\pi}_e(\upsilon_B) = 1$, $\hbar^{\varsigma}_e(\upsilon_B) = 1$.
($\hbar_4$) *For every* $\upsilon_A \in (\widehat{\mathcal{X}\times\mathcal{X}}, E)$, $\langle\hbar^{\gamma}_e\rangle(\upsilon^s_A) \geq \hbar^{\gamma}_e(\upsilon_A)$, $\langle\hbar^{\pi}_e\rangle(\upsilon^s_A) \leq \hbar^{\pi}_e(\upsilon_A)$ *and* $\langle\hbar^{\varsigma}_e\rangle(\upsilon^s_A) \leq \hbar^{\varsigma}_e(\upsilon_A)$,
($\hbar_5$) *For every* $\upsilon_A \in (\widehat{\mathcal{X}\times\mathcal{X}}, E)$,

$$\bigvee\{\hbar^{\gamma}_e(\mu_B):\ (\mu_B \circ \mu_B) \leq \upsilon_A\} \geq \hbar^{\gamma}_e(\upsilon_A),; \qquad \bigwedge\{\hbar^{\pi}_e(\mu_B):\ (\mu_B \circ \mu_B) \leq \upsilon_A\} \leq \hbar^{\pi}_e(\upsilon_A),$$

$$\bigwedge\{\hbar^{\varsigma}_e(\mu_B):\ (\mu_B \circ \mu_B) \leq \upsilon_A\} \leq \hbar^{\varsigma}_e(\upsilon_A).$$

A svns-uniform base $(\hbar^{\gamma}, \hbar^{\pi}, \hbar^{\varsigma})$ *is said to be stratified if and only if* $(\hbar^{\gamma}, \hbar^{\pi}, \hbar^{\varsigma})$ *satisfies*
($\hbar_{st}$) $\hbar^{\gamma}_e(\widehat{E_\varrho}) = 1$, $\hbar^{\pi}_e(\widehat{E_\varrho}) = 0$, $\hbar^{\varsigma}_e(\widehat{E_\varrho}) = 0$, $\forall\ \varrho \in \zeta$, $e \in E$.

In this case $(\hbar^{\gamma}, \hbar^{\pi}, \hbar^{\varsigma})$ *is stratified single-valued neutrosophic soft uniform base (for short, ssvns-uniform base). Sometimes, we will write* $\hbar^{\gamma\pi\varsigma}_E$ *for* $(\hbar^{\gamma}, \hbar^{\pi}, \hbar^{\varsigma})$.

Let $\hbar^{\gamma\pi\varsigma}_E$ *and* $\hbar^{\star\gamma\pi\varsigma}_E$ *be two svns-uniform bases on* $\mathcal{X}$. *Then,* $\hbar^{\gamma\pi\varsigma}_E$ *is finer than* $\hbar^{\star\gamma\pi\varsigma}_E$ *(*$\hbar^{\star\gamma\pi\varsigma}_E$ *is coarser than* $\hbar^{\gamma\pi\varsigma}_E$*), denoted by* $\hbar^{\star\gamma\pi\varsigma}_E \preceq \hbar^{\gamma\pi\varsigma}_E$ *provided*

$$\langle\hbar^{\star\gamma}_e\rangle(\upsilon_A) \leq \langle\hbar^{\gamma}_e\rangle(\upsilon_A),\quad \langle\hbar^{\star\pi}_e\rangle(\upsilon_A) \geq \langle\hbar^{\pi}_e\rangle(\upsilon_A),\quad \langle\hbar^{\star\varsigma}_e\rangle(\upsilon_A) \geq \langle\hbar^{\varsigma}_e\rangle(\upsilon_A),$$

for each $e \in E$, $\upsilon_A \in (\widehat{\mathcal{X}\times\mathcal{X}}, E)$. *Obviously, all svns-uniformity* $\pounds^{\gamma\pi\varsigma}_E$ *on* $\mathcal{X}$ *is a svns- uniform base with* $\langle\pounds^{\gamma\pi\varsigma}_E\rangle = \pounds^{\gamma\pi\varsigma}_E$.

Theorem 2. *Let* $\hbar^{\gamma\pi\varsigma}_E$ *be a svns-uniform base on* $\mathcal{X}$, *define the mappings* $\hbar^{\gamma}, \hbar^{\pi}, \hbar^{\varsigma} : E \to \zeta^{(\widehat{\mathcal{X}\times\mathcal{X}}, E)}$, *for any* $\upsilon_A \in (\widehat{\mathcal{X}\times\mathcal{X}}, E)$, $e \in E$ *as follows:*

$$(\hbar^{\gamma}_{st})_e(\upsilon_A) = \bigvee\{\hbar^{\gamma}_e((\mu_B)):\ \mu_B \sqcap \widehat{E_\varrho} \leq \upsilon_A,\ \varrho \in \zeta\},$$

$$(\hbar^{\pi}_{st})_e(\upsilon_A) = \bigwedge\{\hbar^{\pi}_e((\mu_B)):\ \mu_B \sqcap \widehat{E_\varrho} \leq \upsilon_A,\ \varrho \in \zeta\},$$

$$(\hbar^{\varsigma}_{st})_e(\upsilon_A) = \bigwedge\{\hbar^{\varsigma}_e((\mu_B)):\ \mu_B \sqcap \widehat{E_\varrho} \leq \upsilon_A,\ \varrho \in \zeta\}.$$

Then,
(1) $(\hbar^{\gamma\pi\varsigma}_{st})_E$ *is the coarsest ssvns-uniform base which is finer than* $\hbar^{\gamma\pi\varsigma}_E$,
(2) $\langle(\hbar^{\gamma\pi\varsigma}_{st})_E\rangle = \langle\hbar^{\gamma\pi\varsigma}_E\rangle_{st}$.

Proof. (1) Similar to Theorem 1.

(2) It becomes clear to us from (1), that

$$\langle\hbar^{\gamma}_{\mathrm{E}}\rangle_{st} \leq \langle(\hbar^{\gamma}_{st})_{\mathrm{E}}\rangle,\qquad \langle\hbar^{\pi}_{\mathrm{E}}\rangle_{st} \geq \langle(\hbar^{\pi}_{st})_{\mathrm{E}}\rangle,\qquad \langle\hbar^{\gamma}_{\mathrm{E}}\rangle_{st} \geq \langle(\hbar^{\gamma}_{st})_{\mathrm{E}}\rangle.$$

Conversely, let

$$\langle(\hbar^{\gamma}_{st})_e\rangle(\upsilon_A) \not\leq \langle\hbar^{\gamma}_e\rangle_{st}(\upsilon_A),\qquad \langle(\hbar^{\pi}_{st})_e\rangle(\upsilon_A) \not\geq \langle\hbar^{\pi}_e\rangle_{st}(\upsilon_A),\qquad \langle(\hbar^{\varsigma}_{st})_e\rangle(\upsilon_A) \not\geq \langle\hbar^{\varsigma}_e\rangle_{st}(\upsilon_A),$$

for some $\upsilon_A \in (\widehat{\mathcal{X}\times\mathcal{X}}, \mathrm{E})$. By the concept of $\langle(\hbar^{\gamma\pi\varsigma}_{st})_{\mathrm{E}}\rangle$, there exists $\mu_B \in (\widehat{\mathcal{X}\times\mathcal{X}}, \mathrm{E})$ with $\mu_B \leq \upsilon_A$ such that

$$(\hbar^{\gamma}_{st})_e(\mu_B) \not\leq \langle\hbar^{\gamma}_e\rangle_{st}(\upsilon_A),\qquad (\hbar^{\pi}_{st})_e(\mu_B) \not\geq \langle\hbar^{\pi}_e\rangle_{st}(\upsilon_A),\qquad (\hbar^{\varsigma}_{st})_e(\mu_B) \not\geq \langle\hbar^{\varsigma}_e\rangle_{st}(\upsilon_A).$$

By the concept of $\langle\hbar^{\gamma\pi\varsigma}_{\mathrm{E}}\rangle_{st}$, there exists $\sigma_C \in (\widehat{\mathcal{X}\times\mathcal{X}}, \mathrm{E})$, $\varrho \in \zeta$ with $\sigma_C \sqcap \widehat{E_\varrho} \leq \mu_B$ such that

$$\hbar^{\gamma}_e(\sigma_C) \not\leq \langle\hbar^{\gamma}_e\rangle_{st}(\upsilon_A),\qquad \hbar^{\pi}_e(\sigma_C) \not\geq \langle\hbar^{\pi}_e\rangle_{st}(\upsilon_A),\qquad \hbar^{\varsigma}_e(\sigma_C) \not\geq \langle\hbar^{\varsigma}_e\rangle_{st}(\upsilon_A).$$

On the other hand, $\sigma_C \sqcap \widehat{E_\varrho} \preceq \upsilon_A$ implies that

$$\langle\hbar_e^\gamma\rangle_{st}(\upsilon_A) \geq \langle\hbar_e^\gamma\rangle(\sigma_C) \geq \hbar_e^\gamma(\sigma_C), \qquad \langle\hbar_e^\pi\rangle_{st}(\upsilon_A) \leq \langle\hbar_e^\pi\rangle(\sigma_C) \leq \hbar_e^\gamma(\sigma_C),$$

$$\langle\hbar_e^\varsigma\rangle_{st}(\upsilon_A) \leq \langle\hbar_e^\varsigma\rangle(\sigma_C) \leq \hbar_e^\gamma(\sigma_C).$$

It is a contradiction. Hence, $\langle\hbar_e^\gamma\rangle_{st}(\upsilon_A) \geq \langle(\hbar_{st}^\gamma)_e\rangle(\upsilon_A)$, $\langle\hbar_e^\pi\rangle_{st}(\upsilon_A) \leq \langle(\hbar_{st}^\pi)_e\rangle(\upsilon_A)$, $\langle\hbar_e^\varsigma\rangle_{st}(\upsilon_A) \leq \langle(\hbar_{st}^\varsigma)_e\rangle(\upsilon_A)$, and $\langle(\hbar_{st}^{\gamma\pi\varsigma})_{\mathrm{E}}\rangle = \langle\hbar_{\mathrm{E}}^{\gamma\pi\varsigma}\rangle_{st}$. □

Theorem 3. *Let $(\mathcal{X}, \pounds_E^{\gamma\pi\varsigma})$ be an svns-uniform space. For all $f_B \in \widehat{(\mathcal{X}, E)}$ and $\upsilon_A \in \widehat{(\mathcal{X}\times\mathcal{X}, E)}$, the image $\upsilon_A[f_B]$ of f_B with respect to υ_A is the svns of $\mathcal{X}$ defined by*

$$(\upsilon_e[f_e])(x) = \bigvee_{y\in\mathcal{X}}[f_e(y) \wedge \upsilon_e(y, x)], \forall,\ e \in A\cap B \text{ and } \ x \in \mathcal{X}.$$

For $f_C, (f_D)_j \in \widehat{(\mathcal{X}, E)}$, $\upsilon_A, \mu_B \in \widehat{(\mathcal{X}\times\mathcal{X}, E)}$, we have:
(1) $f_C \preceq \upsilon_A[f_C]$ *whenever* $\pounds_e^\gamma(\upsilon_A) > 0$, $\pounds_e^\pi(\upsilon_A) < 1$, $\pounds_e^\varsigma(\upsilon_A) < 1$,
(2) $\upsilon_A \preceq \upsilon_A \circ \upsilon_A$ *whenever* $\pounds_e^\gamma(\upsilon_A) > 0$, $\pounds_e^\pi(\upsilon_A) < 1$, $\pounds_e^\varsigma(\upsilon_A) < 1$,
(3) $(\mu_B \circ \upsilon_A)[f_C] = \mu_B[\upsilon_A[f_C]]$,
(4) $\upsilon_A[\bigsqcup_j (f_D)_j] = \bigsqcup_j \upsilon_A[(f_D)_j]$,
(5) $(\upsilon_A \sqcap \mu_B)[(f_D)_1 \sqcap (f_D)_2] \preceq \upsilon_A[(f_D)_1] \sqcap \mu_B[(f_D)_2]$,
(6) $(\upsilon_A \sqcup \mu_B)[(f_D)_1 \sqcup (f_D)_2] \preceq \upsilon_A[(f_D)_1] \sqcup \mu_B[(f_D)_2]$,
(7) $\upsilon_A[(\upsilon_A^s[f_C])^c] \preceq f_C^c$.

Proof. Obvious. □

Theorem 4. *Let $\hbar_E^{\gamma\pi\varsigma}$ be a svns-uniform base on $\mathcal{X}$. define the operator $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}} : E \times \widehat{(\mathcal{X}, E)} \times \zeta_0 \to \widehat{(\mathcal{X}, E)}$ as next for every $e \in E$, $r \in \zeta$, $f_B \in \widehat{(\mathcal{X}, E)}$,*

$$\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_B, r) = \bigsqcup\{\mathcal{R}_C : \upsilon_A[\mathcal{R}_C] \preceq f_B, \hbar_e^\gamma(\upsilon_A) \geq r,\ \hbar_e^\pi(\upsilon_A) \leq 1 - r,\ \hbar_e^\varsigma(\upsilon_A) \leq 1 - r\}.$$

Then, $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}$ is an svnsi-operator on $\mathcal{X}$.

Proof. ($\mathtt{I}_1$) Since $\widehat{E} = \upsilon_E[\widehat{E}]$, for all $\hbar_e^\gamma(\upsilon_E) \geq r$, $\hbar_e^\pi(\upsilon_E) \leq 1 - r$, $\hbar_e^\varsigma(\upsilon_E) \leq 1 - r$, then $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, \widehat{E}, r) = \widehat{E}$.

($\mathtt{I}_2$) Whenever $\mathcal{R}_C \preceq \upsilon_A[\mathcal{R}_C] \preceq f_B$, $\forall\ \hbar_e^\gamma(\upsilon_A) \geq r$, $\hbar_e^\pi(\upsilon_A) \leq 1 - r$, $\hbar_e^\varsigma(\upsilon_A) \leq 1 - r$, we get that $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_B, r) \preceq f_B$ for all $f_B \in \widehat{(\mathcal{X}, \mathrm{E})}$.

($\mathtt{I}_3$) Clearly, $\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_B, r) \preceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, \mathcal{R}_D, s)$ for every $f_B \preceq \mathcal{R}_D$, $f_B, \mathcal{R}_D \in \widehat{(\mathcal{X}, \mathrm{E})}$ and $r \leq s$.

($\mathtt{I}_4$) Assume that

$$\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, (f_C)_1, r) \sqcap \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, (f_C)_2, s) \npreceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, (f_C)_1 \sqcap (f_C)_2, r \wedge s).$$

Then, there exists $(\mathcal{R}_D)_1, (\mathcal{R}_D)_2 \in \widehat{(\mathcal{X}, \mathrm{E})}$ with $\upsilon_A[(\mathcal{R}_D)_1] \preceq (f_C)_1$ $\mu_B[(\mathcal{R}_D)_2] \preceq (f_C)_2$ and

$$\hbar_e^\gamma(\upsilon_A) \geq r, \qquad \hbar_e^\pi(\upsilon_A) \leq 1 - r, \qquad \hbar_e^\varsigma(\upsilon_A) \leq 1 - r,$$

$$\hbar_e^\gamma(\mu_B) \geq s, \qquad \hbar_e^\pi(\mu_B) \leq 1 - s, \qquad \hbar_e^\varsigma(\mu_B) \leq 1 - s,$$

such that

$$(\mathcal{R}_D)_1 \sqcap (\mathcal{R}_D)_2 \npreceq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(\widehat{e}, (f_C)_1 \sqcap (f_C)_2, r \wedge s).$$

Since

$$\hbar_e^{\gamma}(\upsilon_A \sqcap \mu_B) \geq \hbar_e^{\gamma}(\upsilon_A) \wedge \hbar_e^{\gamma}(\mu_B), \qquad \hbar_e^{\pi}(\upsilon_A \sqcap \mu_B) \leq \hbar_e^{\pi}(\upsilon_A) \vee \hbar_e^{\pi}(\mu_B),$$
$$\hbar_e^{\varsigma}(\upsilon_A \sqcap \mu_B) \leq \hbar_e^{\varsigma}(\upsilon_A) \vee \hbar_e^{\varsigma}(\mu_B),$$

we get than

$$(\upsilon_A \sqcap \mu_B)[(\mathcal{R}_D)_1 \sqcap (\mathcal{R}_D)_2] \leq \upsilon_A[(\mathcal{R}_D)_1] \sqcap \mu_B[(\mathcal{R}_D)_2] \leq (f_C)_1 \sqcap (f_C)_2.$$

Then,

$$(\mathcal{R}_D)_1 \sqcap (\mathcal{R}_D)_2 \leq \mathtt{I}_{\hbar\gamma\pi\varsigma}(\widehat{e}, (f_C)_1 \sqcap (f_C)_2, r \wedge s).$$

This is a contradiction. Consequently, $(\mathtt{I}_4)$ holds.

$(\mathtt{I}_5)$ Assume that $\mathtt{I}_{\hbar\gamma\pi\varsigma}(e, f_C, r) \not\leq \mathtt{I}_{\hbar\gamma\pi\varsigma}(e, \mathtt{I}_{\hbar\gamma\pi\varsigma}(e, f_C, r), r)$. By using the definition of $\mathtt{I}_{\hbar\gamma\pi\varsigma}(e, f_C, r)$, there exists $\upsilon_A \in (\widehat{\mathcal{X} \times \mathcal{X}, \mathrm{E}})$ and $\mathcal{R}_D \in (\widehat{\mathcal{X}, \mathrm{E}})$, such that

$$\hbar_e^{\gamma}(\upsilon_A) \geq r,\ \hbar_e^{\pi}(\upsilon_A) \leq 1 - r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1 - r,\ \upsilon_A[\mathcal{R}_D] \leq f_C,$$

and $\mathcal{R}_D \not\leq \mathtt{I}_{\hbar\gamma\pi\varsigma}(e, \mathtt{I}_{\hbar\gamma\pi\varsigma}(e, f_C, r), r)$. Otherwise, since

$$\bigvee\{\hbar_e^{\gamma}(\mu_B) : \ \mu_B \circ \mu_B \leq \upsilon_A\} \geq \hbar_e^{\gamma}(\upsilon_A) \geq r,$$

$$\bigwedge\{\hbar_e^{\pi}(\mu_B) : \ \mu_B \circ \mu_B \leq \upsilon_A\} \leq \hbar_e^{\pi}(\upsilon_A) \leq 1 - r,$$
$$\bigwedge\{\hbar_e^{\varsigma}(\mu_B) : \ \mu_B \circ \mu_B \leq \upsilon_A\} \leq \hbar_e^{\varsigma}(\upsilon_A) \leq 1 - r,$$

there exists $\mu_B \in (\widehat{\mathcal{X} \times \mathcal{X}, \mathrm{E}})$ with $\mu_B \circ \mu_B \leq \upsilon_A$ such that

$$\hbar_e^{\gamma}(\mu_B) \geq r, \quad \hbar_e^{\pi}(\mu_B) \leq 1 - r, \quad \hbar_e^{\varsigma}(\mu_B) \leq 1 - r, \qquad \mu_B[\mu_B[\mathcal{R}_D]] \leq \upsilon_A[\mathcal{R}_D] \leq f_C.$$

By using the definition of $\mathtt{I}_{\hbar\gamma\pi\varsigma}(e, f_C, r)$, we obtain $\mu_B[\mathcal{R}_D] \leq \mathtt{I}_{\hbar\gamma\pi\varsigma}(e, f_C, r)$. By the concept of $\mathtt{I}_{\hbar\gamma\pi\varsigma}(e, \mathtt{I}_{\hbar\gamma\pi\varsigma}(e, f_C, r), r)$, it follows that

$$\mathcal{R}_D \leq \mathtt{I}_{\hbar\gamma\pi\varsigma}(e, \mathtt{I}_{\hbar\gamma\pi\varsigma}(e, f_C, r), r).$$

This is a contradiction. Consequently, $(\mathtt{I}_5)$ holds. □

Theorem 5. *Let $\hbar_E^{\gamma\pi\varsigma}$ be a svns-uniform base on $\mathcal{X}$. Define the operator $\mathsf{C}_{\hbar\gamma\pi\varsigma} : E \times (\widehat{\mathcal{X}, E}) \times \zeta_0 \to (\widehat{\mathcal{X}, E})$ as next for every $e \in E$, $f_B \in (\widehat{\mathcal{X}, E})$, $r \in \zeta$,*

$$\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_B, r) = \bigsqcap\{\upsilon_A^s[f_B] : \hbar_e^{\gamma}(\upsilon_A) \geq r,\ \hbar_e^{\pi}(\upsilon_A) \leq 1 - r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1 - r\},$$

Then, $\mathsf{C}_{\hbar\gamma\pi\varsigma}$ is a svnsc-operator on $\mathcal{X}$.

Proof. (C_1) Since $\widehat{\Phi} = \upsilon_A[\widehat{\Phi}]$, for all $\hbar_e^{\gamma}(\upsilon_A) \geq r,\ \hbar_e^{\pi}(\upsilon_A) \leq 1 - r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1 - r$, then $\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, \widehat{\Phi}, r) = \widehat{\Phi}$.

(C_2) Whenever $\mathcal{R}_C \leq \upsilon_A[\mathcal{R}_C] \leq f_B$, for all $\hbar_e^{\gamma}(\upsilon_A) \geq r,\ \hbar_e^{\pi}(\upsilon_A) \leq 1 - r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1 - r$, we get that $\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_B, r) \geq f_B$ for each $f_B \in (\widehat{\mathcal{X}, \mathrm{E}})$.

(C_3) It is established that $\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_B, r) \leq \mathsf{C}_{\hbar\gamma\pi\varsigma}(e, \mathcal{R}_D, s)$ for every $f_B \leq \mathcal{R}_D$, $f_B, \mathcal{R}_D \in (\widehat{\mathcal{X}, \mathrm{E}})$ and $r \leq s$.

(C_4) Assume that

$$\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C, r) \sqcup \mathsf{C}_{\hbar\gamma\pi\varsigma}(e, \mathcal{R}_D, s) \not\geq \mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C \sqcup \mathcal{R}_D, r \wedge s).$$

Then, there exists $r, s \in \zeta_0$, $\upsilon_A, \mu_B \in (\widehat{\mathcal{X}\times\mathcal{X}}, \mathrm{E})$ with

$$\hbar_e^{\gamma}(\upsilon_A) \geq r \wedge s, \quad \hbar_e^{\pi}(\upsilon_A) \leq 1-(r \wedge s), \quad \hbar_e^{\varsigma}(\upsilon_A) \leq 1-(r \wedge s),$$

$$\hbar_e^{\gamma}(\mu_B) \geq s \wedge s, \quad \hbar_e^{\pi}(\mu_B) \leq 1-(s \wedge s), \quad \hbar_e^{\varsigma}(\mu_B) \leq 1-(r \wedge s),$$

such that

$$\upsilon_A^s[f_C] \sqcup \mu_B^s[\mathcal{R}_D] \not\geq \mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C \sqcup \mathcal{R}_D, r \wedge s).$$

Since $\hbar_e^{\gamma}(\upsilon_A \sqcup \mu_B) \geq \hbar_e^{\gamma}(\upsilon_A) \sqcap \hbar_e^{\gamma}(\mu_B) \geq r \wedge s$, $\hbar_e^{\pi}(\upsilon_A \sqcup \mu_B) \leq \hbar_e^{\pi}(\upsilon_A) \sqcup \hbar_e^{\pi}(\mu_B) \leq 1-(r \vee s)$, $\hbar_e^{\varsigma}(\upsilon_A \sqcup \mu_B) \leq \hbar_e^{\varsigma}(\upsilon_A) \sqcup \hbar_e^{\varsigma}(\mu_B) \leq 1-(r \vee s)$ and $(\upsilon_A \sqcup \mu_B)^s[f_C \sqcup \mathcal{R}_D] \leq \upsilon_A^s[f_C] \sqcup \mu_B^s[\mathcal{R}_D]$, then $\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C \sqcup \mathcal{R}_D, r \wedge s) \leq \upsilon_A^s[f_C] \sqcup \mu_B^s[\mathcal{R}_D]$. It is a contradiction. Thus, (C_4) holds.

(C_5) Assume that there exists $r \in \zeta_0$, $e \in \mathrm{E}$, $f_C \in (\widehat{\mathcal{X}, \mathrm{E}})$, such that

$$\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C, r) \not\geq \mathsf{C}_{\hbar\gamma\pi\varsigma}(e, \mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C, r), r).$$

Using the concept of $\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C, r)$, there exist $\upsilon_A \in (\widehat{\mathcal{X}\times\mathcal{X}}, \mathrm{E})$ with

$$\hbar_e^{\gamma}(\upsilon_A) \geq r, \quad \hbar_e^{\pi}(\upsilon_A) \leq 1-r, \quad \hbar_e^{\varsigma}(\upsilon_A) \leq 1-r,$$

such that $\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, \mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C, r), r) \not\leq \upsilon_A^s[f_C]$. Otherwise, from $(\hbar_5)$, we have

$$\bigvee\{\hbar_e^{\gamma}(\mu_B): (\mu_B \circ \mu_B) \leq \upsilon_A\} \geq \hbar_e^{\gamma}(\upsilon_A) \geq r, \quad \bigwedge\{\hbar_e^{\pi}(\mu_B): (\mu_B \circ \mu_B) \leq \upsilon_A\} \leq \hbar_e^{\pi}(\upsilon_A) \leq 1-r,$$

$$\bigwedge\{\hbar_e^{\varsigma}(\mu_B): (\mu_B \circ \mu_B) \leq \upsilon_A\} \leq \hbar_e^{\varsigma}(\upsilon_A) \leq 1-r,$$

which leads to the existence of $\mu_B \in (\widehat{\mathcal{X}\times\mathcal{X}}, \mathrm{E})$ with $\mu_B \circ \mu_B \leq \upsilon_A$ and

$$\hbar_e^{\gamma}(\mu_B) \geq r, \quad \hbar_e^{\pi}(\mu_B) \leq 1-r, \quad \hbar_e^{\varsigma}(\mu_B) \leq 1-r.$$

It follows that

$$\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, \mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C, r), r) \leq \mu_B^s[\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C, r)] \leq \mu_B^s[\mu_B^s[f_C]] \leq \upsilon_A^s[f_C].$$

It is a contradiction. Thus, (C_5) holds.

(C_6) We want, for each $e \in \mathrm{E}$, $f_C \in (\widehat{\mathcal{X}, \mathrm{E}})$, $r \in \zeta_0$, to verify that $\mathsf{C}_{\hbar\gamma\pi\varsigma}(e, f_C, r) = (\mathtt{I}_{\hbar\gamma\pi\varsigma}(e, f_C^c, r))^c$. This means that we need to prove it:

$$\bigsqcap\{\upsilon_A^s[f_C] : \hbar_e^{\gamma}(\upsilon_A) \geq r, \hbar_e^{\pi}(\upsilon_A) \leq 1-r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1-r\}$$
$$= \bigsqcap\{\mathcal{R}_D^c : \upsilon_A[\mathcal{R}_D] \leq f_C^c, \hbar_e^{\gamma}(\upsilon_A) \geq r,\ \hbar_e^{\pi}(\upsilon_A) \leq 1-r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1-r\}.$$

Since $\upsilon_A[(\upsilon_A^s[f_C])^c] \leq f_C^c$, from (7) in Theorem 3, we obtain

$$\bigsqcap\{\upsilon_A^s[f_C] : \hbar_e^{\gamma}(\upsilon_A) \geq r, \hbar_e^{\pi}(\upsilon_A) \leq 1-r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1-r\}$$
$$= \bigsqcap\{\mathcal{R}_D^c : \upsilon_A[\mathcal{R}_D] \leq f_C^c, \hbar_e^{\gamma}(\upsilon_A) \geq r,\ \hbar_e^{\pi}(\upsilon_A) \leq 1-r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1-r\}.$$

Since $(\upsilon_A[\mathcal{R}_D])^c \geq f_C$, we obtain $\upsilon_A^s[f_C] \leq \upsilon_A^s[(\upsilon_A[\mathcal{R}_D])^c]$. Then,

$$\bigsqcap\{\upsilon_A^s[f_C] : \hbar_e^{\gamma}(\upsilon_A) \geq r, \hbar_e^{\pi}(\upsilon_A) \leq 1-r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1-r\}$$
$$= \bigsqcap\{\mathcal{R}_D^c : \upsilon_A[\mathcal{R}_D] \leq f_C^c, \hbar_e^{\gamma}(\upsilon_A) \geq r,\ \hbar_e^{\pi}(\upsilon_A) \leq 1-r,\ \hbar_e^{\varsigma}(\upsilon_A) \leq 1-r\}.$$

Thus, (C_6) holds. □

4. Single-valued neutrosophic soft topologies induced by a single-valued neutrosophic soft uniformity

In this section, we study several single-valued neutrosophic soft topologies induced by a single-valued neutrosophic soft uniform structure. We have proved that single-valued neutrosophic soft uniform base and single-valued neutrosophic soft uniform space are single-valued neutrosophic soft topological spaces.

Theorem 6. *Let $\hbar_E^{\gamma\pi\varsigma}$ be an svns-uniform base on $\mathcal{X}$, define the mappings $\mathfrak{T}_\hbar^\gamma : E \to \zeta^{\widehat{(\mathcal{X},E)}}$, $\mathfrak{T}_\hbar^\pi : E \to \zeta^{\widehat{(\mathcal{X},E)}}$, $\mathfrak{T}_\hbar^\varsigma : E \to \zeta^{\widehat{(\mathcal{X},E)}}$ as follows for each $e \in E$, $r \in \zeta_0$, $f_A \in \widehat{(\mathcal{X},E)}$,*

$$(\mathfrak{T}_\hbar^\gamma)_e(f_A) = \bigvee\{r : \quad f_A \leq I_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r)\},$$

$$(\mathfrak{T}_\hbar^\pi)_e(f_A) = \bigwedge\{1 - r : \quad f_A \leq I_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r)\},$$

$$(\mathfrak{T}_\hbar^\varsigma)_e(f_A) = \bigwedge\{1 - r : \quad f_A \leq I_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r)\}.$$

Then, $\mathfrak{T}_\hbar^{\gamma\pi\varsigma}$ is an svnst on $\mathcal{X}$.

Proof. $(\mathfrak{T}_1)$ Since $\mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, \widehat{E}, r) = \widehat{E}$ and $\mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, \widehat{\Phi}, r) = \widehat{\Phi}$ for each $r \in \zeta_0$, $e \in$ E, then

$$\mathfrak{T}_e^\gamma(\widehat{\Phi}) = 1, \quad \mathfrak{T}_e^\pi(\widehat{\Phi}) = 0, \quad = \mathfrak{T}_e^\varsigma(\widehat{\Phi}) = 0,$$

$$\mathfrak{T}_e^\gamma(\widehat{E}) = 1, \quad \mathfrak{T}_e^\pi(\widehat{E}) = 0, \quad = \mathfrak{T}_e^\varsigma(\widehat{E}) = 0.$$

$(\mathfrak{T}_2)$ To prove the second condition, we follow as follows:

$$\begin{aligned}(\mathfrak{T}_\hbar^\gamma)_e(f_A) \quad &\wedge(\mathfrak{T}_\hbar^\gamma)_e(g_B) = \bigvee\{r \mid f_A \leq \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r)\} \wedge \bigvee\{s \mid g_B \leq \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, g_B, s)\}\\
&\leq \bigvee\{r \wedge s \mid f_A \sqcap g_B \leq \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r) \sqcap \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, g_B, s)\}\\
&\leq \bigvee\{r \wedge s \mid f_A \sqcap g_B \leq \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A \sqcap g_B, r \wedge s)\}\\
&\leq (\mathfrak{T}_\hbar^\gamma)_e(f_A \sqcap g_B),\end{aligned}$$

$$\begin{aligned}(\mathfrak{T}_\hbar^\pi)_e(f_A) \vee (\mathfrak{T}_\hbar^\pi)_e(g_B) \quad &= \bigwedge\{1 - r \mid f_A \leq \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r)\} \vee \bigwedge\{1 - s \mid g_B \leq \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, g_B, s)\}\\
&\geq \bigwedge\{1 - r \vee 1 - s) \mid f_A \sqcup g_B \leq \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r) \sqcup \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, g_B, s)\}\\
&\geq \bigwedge\{1 - (r \wedge s) \mid f_A \sqcup g_B \leq \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A \sqcup g_B, r \wedge s)\}\\
&\geq \bigwedge\{1 - (r \wedge s) \mid f_A \sqcap g_B \leq \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A \sqcap g_B, r \wedge s)\}\\
&\geq (\mathfrak{T}_\hbar^\pi)_e(f_A \sqcap g_B),\end{aligned}$$

$$\begin{aligned}(\mathfrak{T}_\hbar^\varsigma)_e(f_A) \vee (\mathfrak{T}_\hbar^\varsigma)_e(g_B) \quad &= \bigwedge\{1 - r \mid f_A \leq \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r)\} \vee \bigwedge\{1 - s \mid g_B \leq \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, g_B, s)\}\\
&\geq \bigwedge\{1 - r \vee 1 - s) \mid f_A \sqcup g_B \leq \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r) \sqcup \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, g_B, s)\}\\
&\geq \bigwedge\{1 - (r \wedge s) \mid f_A \sqcup g_B \leq \mathrm{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A \sqcup g_B, r \wedge s)\}\end{aligned}$$

$$\geq \bigwedge\{1-(r\wedge s) \mid f_A \sqcap g_B \leq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A \sqcap g_B, r\wedge s)\}$$
$$\geq (\mathfrak{T}^{\varsigma}_{\hbar})_e(f_A \sqcap g_B).$$

($\mathfrak{T}_3$) Assume that there exists a collection $\{(f_A)_j : j\in\Gamma\}$ such that

$$\mathfrak{T}^{\gamma}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \ngeq \bigwedge_{j\in\Gamma}\mathfrak{T}^{\gamma}_e((f_A)_j), \qquad \mathfrak{T}^{\pi}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \nleq \bigvee_{j\in\Gamma}\mathfrak{T}^{\pi}_e((f_A)_j),$$

$$\mathfrak{T}^{\varsigma}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \nleq \bigvee_{j\in\Gamma}\mathfrak{T}^{\varsigma}_e((f_A)_j).$$

For every $j\in\Gamma$, there exist $r_j \in \zeta_0$ such that $(f_A)_j \leq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, f_A, r)$ and that

$$\mathfrak{T}^{\gamma}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \ngeq \bigwedge_{j\in\Gamma} r_j, \qquad \mathfrak{T}^{\pi}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \nleq \bigvee_{j\in\Gamma}(1-r)_j, \qquad \mathfrak{T}^{\varsigma}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \nleq \bigvee_{j\in\Gamma}(1-r)_j.$$

Putting $r = \bigwedge_{j\in\Gamma} r_j$ and $1-r = \bigvee_{j\in\Gamma}(1-r)_j$, from Theorem 4, we get that

$$\bigsqcup_{j\in\Gamma}(f_A)_j \leq \bigsqcup_{j\in\Gamma}\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e,(f_A)_j, r_j) \leq \bigsqcup_{j\in\Gamma}\mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e,(f_A)_j, r) \leq \mathtt{I}_{\hbar^{\gamma\pi\varsigma}}(e, \bigsqcup_{j\in\Gamma}(f_A)_j, r).$$

It follows that

$$\mathfrak{T}^{\gamma}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \geq \bigwedge_{j\in\Gamma} r_j = r, \qquad \mathfrak{T}^{\pi}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \leq \bigvee_{j\in\Gamma}(1-r)_j = 1-r,$$

$$\mathfrak{T}^{\varsigma}_e(\bigsqcup_{j\in\Gamma}(f_A)_j) \leq \bigvee_{j\in\Gamma}(1-r)_j = 1-r.$$

It is a contradiction. Thus, $\mathfrak{T}_3$ holds. □

Definition 10. *Let $f_B \in \widehat{(\mathcal{X}, E)}$ and $\upsilon_A \in (\widehat{\mathcal{X}\times\mathcal{X}, E})$. Define $\upsilon_A^{f_B} \in (\widehat{\mathcal{X}\times\mathcal{X}, E})$, for each $e\in A\cap B$ related with f_B by*

$$(\upsilon_A^{f_B})_e(x,y) = \begin{cases} \langle 1,0,0\rangle, & \text{if } x=y,\\ \gamma_{f_e(x)\wedge f_e(y)}, \pi_{f_e(x)\vee f_e(y)}, \varsigma_{f_e(x)\vee f_e(y)}, & \text{otherwise.}\end{cases}$$

Theorem 7. *Let $(\mathcal{X}, £^{\gamma\pi\varsigma}_E)$ be an svns-uniform space, define the mappings $\mathfrak{T}^{*\gamma}_£, \mathfrak{T}^{*\pi}_£, \mathfrak{T}^{*\varsigma}_£ : E \to \zeta^{\widehat{(\mathcal{X},E)}}$ as follows:*

$$(\mathfrak{T}^{*\gamma}_£)_e(f_B) = \begin{cases} 1, & \text{if } f_B = \widehat{\Phi},\\ £^{\gamma}_e(\upsilon_A^{f_B}), & \text{if } f_B \in \widehat{(\mathcal{X},E)} - \widehat{\Phi},\end{cases}$$

$$(\mathfrak{T}^{*\pi}_£)_e(f_B) = \begin{cases} 0, & \text{if } f_B = \widehat{\Phi},\\ £^{\pi}_e(\upsilon_A^{f_B}), & \text{if } f_B \in \widehat{(\mathcal{X},E)} - \widehat{\Phi},\end{cases}$$

$$(\mathfrak{T}^{*\varsigma}_£)_e(f_B) = \begin{cases} 0, & \text{if } f_B = \widehat{\Phi},\\ £^{\varsigma}_e(\upsilon_A^{f_B}), & \text{if } f_B \in \widehat{(\mathcal{X},E)} - \widehat{\Phi}.\end{cases}$$

Then, $\mathfrak{T}^{*\gamma\pi\varsigma}_£$ is an svnst on $\mathcal{X}$.

Proof. $(\mathfrak{T}_1)$ $(\mathfrak{T}^{*\gamma}_{£})_e(\widehat{\Phi}) = 1$, $(\mathfrak{T}^{*\pi}_{£})_e(\widehat{\Phi}) = 0$, $(\mathfrak{T}^{*\varsigma}_{£})_e(\widehat{\Phi}) = 0$ and $(\mathfrak{T}^{*\gamma}_{£})_e(\widehat{E}) = £^{\gamma}_{e}(\upsilon^{\widehat{E}}_{e}) = 1$, $(\mathfrak{T}^{*\pi}_{£})_e(\widehat{E}) = £^{\pi}_{e}(\upsilon^{\widehat{E}}_{e}) = 0$, $(\mathfrak{T}^{*\varsigma}_{£})_e(\widehat{E}) = £^{\varsigma}_{e}(\upsilon^{\widehat{E}}_{e}) = 0$.

$(\mathfrak{T}_2)$ Since $\upsilon^{f_B}_{A} \sqcap \upsilon^{\mathcal{R}_C}_{A} = \upsilon^{f_B \sqcap \mathcal{R}_C}_{A}$ for every $f_B, \mathcal{R}_C \in \widehat{(\mathcal{X}, \mathrm{E})}$, by $(£_3)$, we have

$$£^{\gamma}_{e}(\upsilon^{f_B \sqcap \mathcal{R}_C}_{A}) = £^{\gamma}_{e}(\upsilon^{f_B}_{A} \sqcap \upsilon^{\mathcal{R}_C}_{A}) \geq £^{\gamma}_{e}(\upsilon^{f_B}_{A}) \wedge £^{\gamma}_{e}(\upsilon^{\mathcal{R}_C}_{A}),$$
$$£^{\pi}_{e}(\upsilon^{f_B \sqcap \mathcal{R}_C}_{A}) = £^{\pi}_{e}(\upsilon^{f_B}_{A} \sqcap \upsilon^{\mathcal{R}_C}_{A}) \leq £^{\pi}_{e}(\upsilon^{f_B}_{A}) \vee £^{\pi}_{e}(\upsilon^{\mathcal{R}_C}_{A}),$$
$$£^{\varsigma}_{e}(\upsilon^{f_B \sqcap \mathcal{R}_C}_{A}) = £^{\varsigma}_{e}(\upsilon^{f_B}_{A} \sqcap \upsilon^{\mathcal{R}_C}_{A}) \leq £^{\varsigma}_{e}(\upsilon^{f_B}_{A}) \vee £^{\varsigma}_{e}(\upsilon^{\mathcal{R}_C}_{A}).$$

Thus,

$$(\mathfrak{T}^{*\gamma}_{£})_e(f_B \sqcap \mathcal{R}_C) = £^{\gamma}_{e}(\upsilon^{f_B \sqcap \mathcal{R}_C}_{A}) \geq £^{\gamma}_{e}(\upsilon^{f_B}_{A}) \wedge £^{\gamma}_{e}(\upsilon^{\mathcal{R}_C}_{A}) = (\mathfrak{T}^{*\gamma}_{£})_e(f_B) \wedge (\mathfrak{T}^{*\gamma}_{£})_e(\mathcal{R}_C),$$
$$(\mathfrak{T}^{*\pi}_{£})_e(f_B \sqcap \mathcal{R}_C) = £^{\pi}_{e}(\upsilon^{f_B \sqcap \mathcal{R}_C}_{A}) \leq £^{\pi}_{e}(\upsilon^{f_B}_{A}) \vee £^{\pi}_{e}(\upsilon^{\mathcal{R}_C}_{A}) = (\mathfrak{T}^{*\pi}_{£})_e(f_B) \vee (\mathfrak{T}^{*\pi}_{£})_e(\mathcal{R}_C),$$
$$(\mathfrak{T}^{*\varsigma}_{£})_e(f_B \sqcap \mathcal{R}_C) = £^{\varsigma}_{e}(\upsilon^{f_B \sqcap \mathcal{R}_C}_{A}) \leq £^{\varsigma}_{e}(\upsilon^{f_B}_{A}) \vee £^{\varsigma}_{e}(\upsilon^{\mathcal{R}_C}_{A}) = (\mathfrak{T}^{*\varsigma}_{£})_e(f_B) \vee (\mathfrak{T}^{*\varsigma}_{£})_e(\mathcal{R}_C).$$

$(\mathfrak{T}_3)$ Similar to the proof in $(\mathfrak{T}_3)$ from Theorem 6. □

Theorem 8. *Let $(\mathcal{X}, £^{\gamma\pi\varsigma}_{E})$ be a svns-uniform space, define the mappings $\mathfrak{T}^{\star\star\gamma}_{£}, \mathfrak{T}^{\star\star\pi}_{£}, \mathfrak{T}^{\star\star\varsigma}_{£} : E \to \zeta^{\widehat{(\mathcal{X},E)}}$ as follows:*

$$(\mathfrak{T}^{\star\star\gamma}_{£})_e(f_B) = \bigwedge_{x \in \mathcal{X}} \left[(f_e)^c(x) \vee \bigvee_{\upsilon_A[x] \leq f_A} £^{\gamma}_{e}(\upsilon_A) \right],$$
$$(\mathfrak{T}^{\star\star\pi}_{£})_e(f_B) = \bigvee_{x \in \mathcal{X}} \left[(f_e)^c(x) \wedge \bigvee_{\upsilon_A[x] \leq f_A} £^{\pi}_{e}(\upsilon_A) \right],$$
$$(\mathfrak{T}^{\star\star\varsigma}_{£})_e(f_B) = \bigvee_{x \in \mathcal{X}} \left[(f_e)^c(x) \wedge \bigvee_{\upsilon_A[x] \leq f_A} £^{\varsigma}_{e}(\upsilon_A) \right].$$

Then, $\mathfrak{T}^{\star\star\gamma\pi\varsigma}_{£}$ is an svnst on $\mathcal{X}$, where $(\upsilon_A[x])(y) \leq \upsilon_A(y, x)$ for all $e \in A$.

Proof. $(\mathfrak{T}_1)$ Obvious.

$(\mathfrak{T}_2)$ Assume that

$$\bigvee_{\upsilon_A[x] \leq (f_D)_1} £^{\gamma}_{e}(\upsilon_A) \wedge \bigvee_{\mu_B[x] \leq (f_D)_2} £^{\gamma}_{e}(\mu_B) \not\leq \bigvee_{\kappa_C[x] \leq (f_D)_1 \sqcap (f_D)_2} £^{\gamma}_{e}(\kappa_C),$$
$$\bigwedge_{\upsilon_A[x] \leq (f_D)_1} £^{\pi}_{e}(\upsilon_A) \vee \bigwedge_{\mu_B[x] \leq (f_D)_2} £^{\pi}_{e}(\mu_B) \not\geq \bigwedge_{\kappa_C[x] \leq (f_D)_1 \sqcap (f_D)_2} £^{\pi}_{e}(\kappa_C),$$
$$\bigwedge_{\upsilon_A[x] \leq (f_D)_1} £^{\varsigma}_{e}(\upsilon_A) \vee \bigwedge_{\mu_B[x] \leq (f_D)_2} £^{\varsigma}_{e}(\mu_B) \not\geq \bigwedge_{\kappa_C[x] \leq (f_D)_1 \sqcap (f_D)_2} £^{\varsigma}_{e}(\kappa_C).$$

Then, there exists υ_A, μ_B with $\upsilon_A[x] \leq (f_D)_1$, $\mu_B[x] \leq (f_D)_2$ such that

$$£^{\gamma}_{e}(\upsilon_A) \wedge £^{\gamma}_{e}(\mu_B) \not\leq \bigvee_{\kappa_C[x] \leq (f_D)_1 \sqcap (f_D)_2} £^{\pi}_{e}(\kappa_C), \qquad £^{\pi}_{e}(\upsilon_A) \vee £^{\pi}_{e}(\mu_B) \not\geq \bigwedge_{\kappa_C[x] \leq (f_D)_1 \sqcap (f_D)_2} £^{\pi}_{e}(\kappa_A),$$

$$£^{\varsigma}_{e}(\upsilon_{A}) \vee £^{\varsigma}_{e}(\mu_{B}) \ngeq \bigwedge_{\kappa_C[x]\preceq(f_D)_1\sqcap(f_D)_2} £^{\varsigma}_{e}(\kappa_{C}).$$

This results in $(\upsilon_{A} \sqcap \mu_{B})[x] \preceq (f_{D})_{1} \sqcap (f_{D})_{2}$ such that

$$\bigvee_{\kappa_C[x]\preceq(f_D)_1\sqcap(f_D)_2} £^{\gamma}_{e}(\kappa_{C}) \geq £^{\gamma}_{e}(\upsilon_{A} \sqcap \mu_{B}) \geq £^{\gamma}_{e}(\upsilon_{A}) \wedge £^{\gamma}_{e}(\mu_{B}),$$

$$\bigwedge_{\kappa_C[x]\preceq(f_D)_1\sqcap(f_D)_2} £^{\pi}_{e}(\kappa_{C}) \leq £^{\pi}_{e}(\upsilon_{A} \sqcap \mu_{B}) \leq £^{\pi}_{e}(\upsilon_{A}) \vee £^{\pi}_{e}(\mu_{B}),$$

$$\bigwedge_{\kappa_C[x]\preceq(f_D)_1\sqcap(f_D)_2} £^{\varsigma}_{e}(\kappa_{C}) \leq £^{\varsigma}_{e}(\upsilon_{A} \sqcap \mu_{B}) \leq £^{\varsigma}_{e}(\upsilon_{A}) \vee £^{\varsigma}_{e}(\mu_{B}).$$

It is a contradiction. Thus,

$$\begin{aligned}
&(\mathfrak{T}^{\star\star\gamma}_{£})_e((f_D)_1) \wedge (\mathfrak{T}^{\star\star\gamma}_{£})_e((f_D)_2)\\
&= \left(\bigwedge_{x\in\mathcal{X}}\left[(f^c_e)_1(x) \vee \bigvee_{\upsilon_A[x]\preceq(f_D)_1} £^{\gamma}_{e}(\upsilon_A)\right]\right) \wedge \left(\bigwedge_{x\in\mathcal{X}}\left[(f^c_e)_2(x) \vee \bigvee_{\mu_B[x]\preceq(f_D)_2} £^{\gamma}_{e}(\mu_B)\right]\right)\\
&\leq \left(\bigwedge_{x\in\mathcal{X}}\left[(f^c_e)_1(x) \vee \bigvee_{\upsilon_A[x]\preceq(f_D)_1} £^{\gamma}_{e}(\upsilon_A)\right] \wedge \left[(f^c_e)_2(x) \vee \bigvee_{\mu_B[x]\preceq(f_D)_2} £^{\gamma}_{e}(\mu_B)\right]\right)\\
&\leq \bigwedge_{x\in\mathcal{X}}\left[((f^c_e)_1 \sqcup (f^c_e)_2)(x) \vee \bigvee_{\upsilon_A[x]\preceq(f_D)_1} £^{\gamma}_{e}(\upsilon_A) \wedge \bigvee_{\mu_B[x]\preceq(f_D)_2} £^{\gamma}_{e}(\mu_B)\right]\\
&\leq \bigwedge_{x\in\mathcal{X}}\left[((f^c_e)_1 \sqcup (f^c_e)_2)(x) \vee \bigvee_{(\upsilon_A\sqcap\mu_B)[x]\preceq(f_D)_1\sqcap(f_D)_2} £^{\gamma}_{e}(\upsilon_A \sqcap \mu_B)\right]\\
&\leq (\mathfrak{T}^{\star\star\gamma}_{£})_e((f_D)_1 \sqcap (f_D)_2),
\end{aligned}$$

$$\begin{aligned}
&(\mathfrak{T}^{\star\star\pi}_{£})_e((f_D)_1) \vee (\mathfrak{T}^{\star\pi}_{£})_e((f_D)_2)\\
&= \left(\bigvee_{x\in\mathcal{X}}\left[(f^c_e)_1(x) \wedge \bigvee_{\upsilon_A[x]\preceq(f_D)_1} £^{\pi}_{e}(\upsilon_A)\right]\right) \vee \left(\bigvee_{x\in\mathcal{X}}\left[(f^c_e)_2(x) \wedge \bigvee_{\mu_B[x]\preceq(f_D)_2} £^{\pi}_{e}(\mu_B)\right]\right)\\
&\geq \bigvee_{x\in\mathcal{X}}\left(\left[(f^c_e)_1(x) \wedge \bigvee_{\upsilon_A[x]\preceq(f_D)_1} £^{\pi}_{e}(\upsilon_A)\right] \vee \left[(f^c_e)_2(x) \wedge \bigvee_{\mu_B[x]\preceq(f_D)_2} £^{\pi}_{e}(\mu_B)\right]\right)\\
&\geq \bigvee_{x\in\mathcal{X}}\left[((f^c_e)_1 \sqcap (f^c_e)_2)(x) \wedge \bigvee_{\upsilon_A[x]\preceq(f_D)_1} £^{\pi}_{e}(\upsilon_A) \vee \bigvee_{\mu_B[x]\preceq(f_D)_2} £^{\pi}_{e}(\mu_B)\right]\\
&\geq \bigvee_{x\in\mathcal{X}}\left[((f^c_e)_1 \sqcap (f^c_e)_2)(x) \wedge \bigvee_{(\upsilon_A\sqcap\mu_B)[x]\preceq(f_D)_1\sqcap(f_D)_2} £^{\pi}_{e}(\upsilon_A \sqcap \mu_B)\right]\\
&\geq (\mathfrak{T}^{\star\star\pi}_{£})_e((f_D)_1 \sqcap (f_D)_2).
\end{aligned}$$

Likewise, we can establish through a similar line of reasoning that

$$(\mathfrak{T}_{£}^{\star\star\varsigma})_e((f_D)_1) \vee (\mathfrak{T}_{£}^{\star\star\varsigma})_e((f_D)_2) \geq (\mathfrak{T}_{£}^{\star\star\varsigma})_e((f_D)_1 \sqcap (f_D)_2).$$

($\mathfrak{T}_3$) For $e \in$ E

$$\begin{aligned}
(\mathfrak{T}_{£}^{\star\star\gamma})_e\left(\bigvee_{j\in\Gamma}(f_B)_j\right) &= \bigwedge_{x\in\mathcal{X}}\left[\left(\bigvee_{j\in\Gamma}(f_e)_j\right)^c(x)\right] \vee \left[\bigvee_{\upsilon_A[x]\leq\bigsqcup_j(f_B)_j} £_e^{\gamma}(\upsilon_A)\right] \\
&= \bigwedge_{x\in\mathcal{X}}\left[\bigwedge_{j\in\Gamma}(f_e^c)_j(x) \vee \bigvee_{\upsilon_A[x]\leq\bigsqcup_j(f_B)_j} £_e^{\gamma}(\upsilon_A)\right] \\
&= \bigwedge_{j\in\Gamma}\left[\bigwedge_{x\in\mathcal{X}}(f_e^c)_j(x) \vee \bigvee_{\upsilon_A[x]\leq\bigsqcup_j(f_B)_j} £_e^{\gamma}(\upsilon_A)\right] \\
&\geq \bigwedge_{j\in\Gamma}\left[\bigwedge_{x\in\mathcal{X}}(f_e^c)_j(x) \vee \bigvee_{\upsilon_A[x]\leq(f_B)_j} £_e^{\gamma}(\upsilon_A)\right] \\
&= \bigwedge_{j\in\Gamma}(\mathfrak{T}_{£}^{\star\star\gamma})_e((f_B)_j),
\end{aligned}$$

$$\begin{aligned}
(\mathfrak{T}_{£}^{\star\star\pi})_e\left(\bigvee_{j\in\Gamma}(f_B)_j\right) &= \bigvee_{x\in\mathcal{X}}\left[\left(\bigvee_{j\in\Gamma}(f_e)_j\right)^c(x)\right] \wedge \left[\bigvee_{\upsilon_A[x]\leq\bigsqcup_j(f_B)_j} £_e^{\pi}(\upsilon_A)\right] \\
&= \bigvee_{x\in\mathcal{X}}\left[\bigwedge_{j\in\Gamma}(f_e^c)_j(x) \wedge \bigvee_{\upsilon_A[x]\leq\bigsqcup_j(f_B)_j} £_e^{\pi}(\upsilon_A)\right] \\
&\leq \bigvee_{x\in\mathcal{X}}\left[\bigvee_{j\in\Gamma}(f_e^c)_j(x) \wedge \bigvee_{\upsilon_A[x]\leq\bigsqcup_j(f_B)_j} £_e^{\pi}(\upsilon_A)\right] \\
&= \bigvee_{j\in\Gamma}\left[\bigvee_{x\in\mathcal{X}}(f_e^c)_j(x) \wedge \bigvee_{\upsilon_A[x]\leq\bigsqcup_j(f_B)_j} £_e^{\pi}(\upsilon_A)\right] \\
&\leq \bigvee_{j\in\Gamma}\left[\bigvee_{x\in\mathcal{X}}(f_e^c)_j(x) \wedge \bigvee_{\upsilon_A[x]\leq(f_B)_j} £_e^{\pi}(\upsilon_A)\right] \\
&= \bigvee_{j\in\Gamma}(\mathfrak{T}_{£}^{\star\star\pi})_e((f_B)_j).
\end{aligned}$$

In a similar vein, we can demonstrate through a parallel line of reasoning that

$$(\mathfrak{T}_{\pounds}^{\star\star\varsigma})_e\left(\bigvee_{j\in\Gamma}(f_B)_j\right) \le \bigvee_{j\in\Gamma}(\mathfrak{T}_{\pounds}^{\star\star\varsigma})_e((f_B)_j).$$

Therefore, $\mathfrak{T}_{\pounds}^{\star\star\gamma\pi\varsigma}$ is an svnst on $\mathcal{X}$. □

5. Single-valued neutrosophic soft uniformly continuous mappings

In this section, we obtain crucial results in introducing and characterizing single-valued neutrosophic soft uniformly continuous, on single-valued neutrosophic soft uniformly topological spaces. Moreover, the relationship between single-valued neutrosophic soft uniformly continuous and single-valued neutrosophic soft continuous is studied.

Definition 11. *Let $(\mathcal{X}, \pounds_E^{\gamma\pi\varsigma})$ and $(\mathcal{G}, \pounds_R^{\star\gamma\pi\varsigma})$ be two svns-uniform spaces and $\psi : \mathcal{X} \to \mathcal{G}$ and $\vartheta : E \to \mathcal{R}$ be two mappings. Then, an svns-map $\psi_\vartheta : (\widehat{\mathcal{X}\times\mathcal{X}}, E) \to (\widehat{\mathcal{G}\times\mathcal{G}}, \mathcal{R})$ is called single-valued neutrosophic soft uniformly continuous (svns-uniformly continuous) if*

$$\pounds_e^{\gamma}((\psi\times\psi)_\vartheta^{-1}(\mu_B)) \ge \pounds_{\vartheta(e)}^{\star\gamma}(\mu_B), \quad \pounds_e^{\pi}((\psi\times\psi)_\vartheta^{-1}(\mu_B)) \le \pounds_{\vartheta(e)}^{\star\pi}(\mu_B),$$

$$\pounds_e^{\varsigma}((\psi\times\psi)_\vartheta^{-1}(\mu_B)) \le \pounds_{\vartheta(e)}^{\star\varsigma}(\mu_B),$$

for each $\mu_B \in (\widehat{\mathcal{G}\times\mathcal{G}}, \mathcal{R})$, $e \in E$.

Proposition 1. *Let $(\mathcal{X}, \pounds_E^{\gamma\pi\varsigma})$ and $(\mathcal{G}, \mathcal{F}_R^{\gamma\pi\varsigma})$ be svns-uniform spaces. If $\psi_\vartheta : (\mathcal{X}, \pounds^{\gamma\pi\varsigma}) \to (\mathcal{G}, \mathcal{F}^{\gamma\pi\varsigma})$ is svns-uniformly continuous, then $\psi_\vartheta : (\mathcal{X}, (\pounds_{st}^{\gamma\pi\varsigma}) \to (\mathcal{G}, \mathcal{F}_{st}^{\gamma\pi\varsigma})$ is svns-uniformly continuous.*

Proof. To prove this theorem, we need to prove that

$$(\pounds_{st}^{\gamma})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \ge (\mathcal{F}_{st}^{\gamma})_{\vartheta(e)}(\upsilon_A), \qquad (\pounds_{st}^{\pi})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \le (\mathcal{F}_{st}^{\pi})_{\vartheta(e)}(\upsilon_A),$$

$$(\pounds_{st}^{\varsigma})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \le (\mathcal{F}_{st}^{\varsigma})_{\vartheta(e)}(\upsilon_A),$$

for each $\upsilon_A \in (\widehat{\mathcal{G}\times\mathcal{G}}, \mathcal{R})$, $e \in$ E.

Assume that

$$(\pounds_{st}^{\gamma})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \ngeq (\mathcal{F}_{st}^{\gamma})_{\vartheta(e)}(\upsilon_A), \qquad (\pounds_{st}^{\pi})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A) \nleq (\mathcal{F}_{st}^{\pi})_{\vartheta(e)}(\upsilon_A),$$

$$(\pounds_{st}^{\varsigma})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \nleq (\mathcal{F}_{st}^{\varsigma})_{\vartheta(e)}(\upsilon_A).$$

From the concept of $(\mathcal{F}_{st}^{\star\gamma\pi\varsigma})_{\vartheta(e)}(\upsilon_A)$, there exists $\mu_B \in (\widehat{\mathcal{G}\times\mathcal{G}}, \mathcal{R})$, $e \in$ E, $\varrho \in \zeta$ with $\mu_B \sqcap \widehat{E_\varrho} \le \upsilon_A$ such that

$$(\pounds_{st}^{\gamma})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \ngeq \mathcal{F}_{\vartheta(e)}^{\gamma}(\mu_B), \qquad (\pounds_{st}^{\pi})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \nleq \mathcal{F}_{\vartheta(e)}^{\pi}(\mu_B),$$

$$(\pounds_{st}^{\varsigma})_e((\psi\times\psi)_\vartheta^{-1}(\upsilon_A)) \nleq \mathcal{F}_{\vartheta(e)}^{\varsigma}(\mu_B).$$

Since $\psi_\vartheta : (\mathcal{X}, (\pounds_{st}^{\gamma\pi\varsigma}) \to (\mathcal{G}, \mathcal{F}_{st}^{\gamma\pi\varsigma})$ is svns-uniformly continuous,

$$\pounds_e^{\gamma}((\psi\times\psi)_\vartheta^{-1}(\mu_B)) \ge \mathcal{F}_{\vartheta(e)}^{\gamma}(\mu_B), \qquad \pounds_e^{\pi}((\psi\times\psi)_\vartheta^{-1}(\mu_B)) \le \mathcal{F}_{\vartheta(e)}^{\pi}(\mu_B),$$

$$£^{\varsigma}_{e}((\psi\times\psi)^{-1}_{\vartheta}(\mu_{B}))\leq\mathcal{F}^{\varsigma}_{\vartheta(e)}(\mu_{B}).$$

From the concept of $£^{\gamma\pi\varsigma}_{e}((\psi\times\psi)^{-1}_{\vartheta}(\upsilon_{A}))$, we get

$$(£^{\gamma}_{st})_{e}((\psi\times\psi)^{-1}_{\vartheta}(\upsilon_{A}))\geq £^{\gamma}_{e}((\psi\times\psi)^{-1}_{\vartheta}(\mu_{B}))\geq\mathcal{F}^{\gamma}_{\vartheta(e)}(\mu_{B}),$$

$$(£^{\pi}_{st})_{e}((\psi\times\psi)^{-1}_{\vartheta}(\upsilon_{A}))\leq £^{\pi}_{e}((\psi\times\psi)^{-1}_{\vartheta}(\mu_{B}))\leq\mathcal{F}^{\pi}_{\vartheta(e)}(\mu_{B}),$$

$$(£^{\varsigma}_{st})_{e}((\psi\times\psi)^{-1}_{\vartheta}(\upsilon_{A}))\leq £^{\pi}_{e}((\psi\times\psi)^{-1}_{\vartheta}(\mu_{B}))\leq\mathcal{F}^{\varsigma}_{\vartheta(e)}(\mu_{B}).$$

This is a conflict with the hypothesis. □

Proposition 2. *Let $\psi:\mathcal{X}\to\mathcal{G}$, and $\vartheta: E\to\mathcal{R}$ be two mappings, and let $f_{D}\in\widehat{(\mathcal{X},E)}$, $\upsilon_{A},\mu_{B},\kappa_{C}\in\widehat{(\mathcal{G}\times\mathcal{G},\mathcal{R})}$. Then, the following results hold in general:*
(1) $\psi^{-1}_{\vartheta}(\upsilon_{A}[\psi_{\vartheta}(f_{D})])=((\psi\times\psi)^{-1}_{\vartheta}(\upsilon_{A}))[f_{D}]$,
(2) $((\psi\times\psi)^{-1}_{\vartheta}(\upsilon^{s}_{A}))[f_{D}]=((\psi\times\psi)^{-1}_{\vartheta}(\upsilon_{A}))^{s}[f_{D}]$,
(3) $(\psi\times\psi)^{-1}_{\vartheta}(\upsilon_{A}\sqcap\mu_{B})=(\psi\times\psi)^{-1}_{\vartheta}(\upsilon_{A})\sqcap(\psi\times\psi)^{-1}_{\vartheta}(\mu_{B})$,
(4) $(\psi\times\psi)^{-1}_{\vartheta}(\upsilon_{A})\circ(\psi\times\psi)^{-1}_{\vartheta}(\upsilon_{A})\leq(\psi\times\psi)^{-1}_{\vartheta}(\upsilon_{A}\circ\upsilon_{A})$.

Proof. (1) For $\omega\in\psi(\mathrm{E})$, we get that

$$\begin{aligned}\psi^{-1}_{\vartheta}(\upsilon_{\omega}[\psi_{\vartheta}(f_{\vartheta^{-1}(\omega)})])(x) &= \psi^{-1}_{\vartheta}(\upsilon_{\omega}[(\psi(f))_{\omega}])(x)=(\upsilon_{\omega}[(\psi(f))_{\omega}])(\psi(x))\\ &= \bigvee_{y\in\mathcal{G}}[(\psi(f))_{\omega}(y)\wedge\upsilon_{\omega}(y,\psi(x))]\\ &= \bigvee_{z\in\mathcal{X}}[(\psi(f))_{\omega}(\psi(z))\wedge\upsilon_{\omega}(\psi(z),\psi(x))]\\ &= \bigvee_{z\in\mathcal{X}}[(f_{\vartheta^{-1}(\omega)})(z)\wedge(\psi\times\psi)^{-1}(\upsilon_{\omega}(z,x))]\\ &= (\psi\times\psi)^{-1}_{\vartheta}(\upsilon_{\omega}[(f_{\vartheta^{-1}(\omega)}])(x).\end{aligned}$$

(2) For $\omega\in\psi(\mathrm{E})$, we have

$$\begin{aligned}((\psi\times\psi)^{-1}_{\vartheta}(\upsilon^{s}_{\omega}))[f_{\vartheta^{-1}(\omega)}](x) &= \bigvee_{z\in\mathcal{X}}[(f_{\vartheta^{-1}(\omega)})(z)\wedge((\psi\times\psi^{-1})_{\vartheta}(\upsilon^{s}_{\omega}))(z,x)]\\ &= \bigvee_{z\in\mathcal{X}}[(f_{\vartheta^{-1}(\omega)})(z)\wedge\upsilon^{s}_{\omega}(\psi(z),\psi(x)]\\ &= \bigvee_{z\in\mathcal{X}}[(f_{\vartheta^{-1}(\omega)})(z)\wedge\upsilon_{\omega}(\psi(x),\psi(z)]\\ &= \bigvee_{z\in\mathcal{X}}[(f_{\vartheta^{-1}(\omega)})(z)\wedge((\psi\times\psi)^{-1}_{\vartheta}(\upsilon_{\omega}))(x,z)]\\ &= \bigvee_{z\in\mathcal{X}}[(f_{\vartheta^{-1}(\omega)})(z)\wedge((\psi\times\psi)^{-1}_{\vartheta}(\upsilon_{\omega}))^{s}(z,x)]\\ &= ((\psi\times\psi)^{-1}_{\vartheta}(\upsilon_{\omega}))^{s}[(f_{\vartheta^{-1}(\omega)}](x).\end{aligned}$$

(3) Direct.

(4) For $\omega \in \psi(\mathrm{E})$, we have

$$\begin{aligned}((\psi\times\psi)_{\vartheta}^{-1}(\upsilon_{\omega})\circ(\psi\times\psi)_{\vartheta}^{-1}(\upsilon_{\omega}))(x_1,x_2) &= \bigvee_{z\in\mathcal{X}}[(\psi\times\psi)_{\vartheta}^{-1}(\upsilon_{\omega})(x_1,z)\wedge(\psi\times\psi)_{\vartheta}^{-1}(\upsilon_{\omega})(z,x_2)]\\ &= \bigvee_{z\in\mathcal{X}}[\upsilon_{\omega}(\psi(x_1),\psi(z))\wedge\upsilon_{\omega}(\psi(z),\psi(x_2))]\\ &\leq \bigvee_{z\in\mathcal{X}}[\upsilon_{\omega}(\psi(x_1),y)\wedge\upsilon_{\omega}(y,\psi(x_2))]\\ &= (\upsilon_{\omega}\circ\upsilon_{\omega})(\psi(x_1),\psi(x_2)) = (\psi\times\psi)_{\vartheta}^{-1}(\upsilon_{\omega}\circ\upsilon_{\omega})(x_1,x_2).\end{aligned}$$

□

Theorem 9. *Let $(\mathcal{X},\pounds^{\gamma\pi\varsigma})$ and $(\mathcal{G},\mathcal{F}^{\gamma\pi\varsigma})$ be svns-uniform spaces, $\psi_{\vartheta} : (\widehat{\mathcal{X}\times\mathcal{X},E}) \to (\widehat{\mathcal{G}\times\mathcal{G},\mathcal{R}})$ be svns-uniformly continuous. Then, the following results hold in general.*
(1) $\psi_{\vartheta}^{-1}(\mathcal{I}_{\mathcal{F}^{\gamma\pi\varsigma}}(\omega,f_C,r)) \leq \mathcal{I}_{\pounds^{\gamma\pi\varsigma}}(\vartheta^{-1}(\omega),\psi_{\vartheta}^{-1}(f_C),r))$, for each $f_C \in (\widehat{\mathcal{G},\mathcal{R}})$, $r\in\xi$, $\omega\in\mathcal{R}$,
(2) $\mathcal{C}_{\pounds^{\gamma\pi\varsigma}}(\vartheta^{-1}(\omega),\psi_{\vartheta}^{-1}(f_C),r)) \leq \psi_{\vartheta}^{-1}(\mathcal{C}_{\mathcal{F}^{\gamma\pi\varsigma}}(\omega,f_C,r))$, for each $f_C \in (\widehat{\mathcal{G},\mathcal{R}})$, $r\in\xi$, $\omega\in\mathcal{R}$,
(3) $\psi_{\vartheta}(\mathcal{C}_{\pounds^{\gamma\pi\varsigma}}(e,g_D,r)) \leq \mathcal{C}_{\mathcal{F}^{\gamma\pi\varsigma}}(\vartheta(\omega),\psi_{\vartheta}(g_D),r))$, for each $g_D \in (\widehat{\mathcal{X},\mathrm{E}})$, $r\in\xi$, $e\in\mathrm{E}$.

Proof. (1) For each $\upsilon_A, \in (\widehat{\mathcal{G}\times\mathcal{G},\mathcal{R}})$ and $f_C, g_D \in (\widehat{\mathcal{G},\mathcal{R}})$, from Proposition 2, $\upsilon_A[f_C] \leq g_D$ implies that

$$((\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A))[\psi_{\vartheta}^{-1}(f_C)] = \psi_{\vartheta}^{-1}(\upsilon_A[\psi_{\vartheta}(\psi_{\vartheta}^{-1}(f_C))]) \leq \psi_{\vartheta}^{-1}(\upsilon_A[f_C]) \leq \psi_{\vartheta}^{-1}(g_D).$$

Since

$$\pounds^{\gamma}_{\vartheta^{-1}(\omega)}(\mu_B) \geq \mathcal{F}^{\gamma}_{\omega}(\upsilon_A),\quad \pounds^{\pi}_{\vartheta^{-1}(\omega)}(\mu_B) \leq \mathcal{F}^{\pi}_{\omega}(\upsilon_A),\quad \pounds^{\varsigma}_{\vartheta^{-1}(\omega)}(\mu_B) \leq \mathcal{F}^{\varsigma}_{\omega}(\upsilon_A),$$

for every $\mu_B \in (\psi\times\psi)_{\vartheta}^{-1}(\upsilon_A)$, we obtain

$$\begin{aligned}&\psi_{\vartheta}^{-1}(\mathrm{I}_{\mathcal{F}^{\gamma\pi\varsigma}}(\omega,f_C,r))\\ &= \psi_{\vartheta}^{-1}(\bigsqcup\{g_D \in (\widehat{\mathcal{G},\mathcal{R}}) :\ \upsilon_A[g_D] \leq f_C, \mathcal{F}^{\gamma}(\upsilon_A) \geq r, \mathcal{F}^{\pi}_{\omega}(\upsilon_A) \leq 1-r, \mathcal{F}^{\varsigma}_{\omega}(\upsilon_A) \leq 1-r\})\\ &= \bigsqcup\{\psi_{\vartheta}^{-1}(g_D) \in (\widehat{\mathcal{X},\mathrm{E}}) :\ \upsilon_A[g_D]\\ &\leq f_C, \mathcal{F}^{\gamma}(\upsilon_A) \geq r, \mathcal{F}^{\pi}_{\omega}(\upsilon_A) \leq 1-r, \mathcal{F}^{\varsigma}_{\omega}(\upsilon_A) \leq 1-r\}\\ &\leq \bigsqcup\{\psi_{\vartheta}^{-1}(f_C) \in (\widehat{\mathcal{X},\mathrm{E}}) :\ \mu_B[\psi_{\vartheta}^{-1}(g_D)]\\ &\leq \psi_{\vartheta}^{-1}(f_C), \pounds^{\gamma}_{\vartheta^{-1}(\omega)}(\mu_B) \geq r, \pounds^{\pi}_{\vartheta^{-1}(\omega)}(\mu_B) \leq 1-r, \pounds^{\varsigma}_{\vartheta^{-1}(\omega)}(\mu_B) \leq 1-r\}\\ &\leq \mathrm{I}_{\pounds^{\gamma\pi\varsigma}}(\vartheta^{-1}(\omega),\psi_{\vartheta}^{-1}(f_C),r)).\end{aligned}$$

In a similar vein, we can demonstrate (2) and (3) through a parallel line of reasoning. □

Theorem 10. *Let $(\mathcal{X},\pounds_E^{\gamma\pi\varsigma})$ and $(\mathcal{G},\mathcal{F}_{\mathcal{R}}^{\gamma\pi\varsigma})$ be svns-uniform spaces, and $\psi_{\vartheta} : (\widehat{\mathcal{X},E}) \to (\widehat{\mathcal{G},\mathcal{R}})$ an injective svns-uniformly continuous. Then, $\psi_{\vartheta} : (\mathcal{X},\mathfrak{T}_{\pounds}^{\star\gamma\pi\varsigma}) \to (\mathcal{G},\mathfrak{T}_{\mathcal{F}}^{\star\gamma\pi\varsigma})$ is svns-continuous.*

Proof. Since ψ_{ϑ} injective and by applying Theorem 4, we get that:
For each $\upsilon_A, \in (\widehat{\mathcal{G}\times\mathcal{G},\mathcal{R}})$ and $f_B \in (\widehat{\mathcal{G},\mathcal{R}})$, $\omega \in A\cap B$. Then,

$$((\psi\times\psi)_{\vartheta}^{-1}((\upsilon_A^{f_B})_{\omega}))(x_1,x_2) = (\upsilon_A^{f_B})_{\omega}(\psi(x_1),\psi(x_2))$$

$$= \begin{cases} 1, & \text{if } \psi(x_1) = \psi(x_2), \\ f_\omega(\psi(x_1)) \wedge f_\omega(\psi(x_2)), & \text{if } \psi(x_1) \neq \psi(x_2), \end{cases}$$
$$= \begin{cases} 1, & \text{if } \psi(x_1) = \psi(x_2), \\ \psi_\vartheta^{-1}(f_\omega)(x_1) \wedge \psi_\vartheta^{-1}(f_\omega)(x_2), & \text{if } \psi(x_1) \neq \psi(x_2), \end{cases}$$
$$= \left(\upsilon_{\vartheta^{-1}(A)}^{\psi_\vartheta^{-1}(f_B)}\right)_{\vartheta^{-1}(\omega)} (x_1, x_2).$$

Therefore, $\forall\ e \in$ E

$$(\mathfrak{T}_{£}^{\star\gamma})_e(\psi_\vartheta^{-1}(f_B)) = £_e^\gamma(\upsilon_{\vartheta^{-1}(A)}^{\psi_\vartheta^{-1}(f_B)}) = £_e^\gamma((\psi \times \psi)_\vartheta^{-1}(\upsilon_A^{f_B})) \geq \mathcal{F}_{\vartheta(e)}^\gamma(\upsilon_A^{f_B}) = (\mathfrak{T}_{\mathcal{F}}^{\star\gamma})_{\vartheta(e)}(f_B)$$

$$(\mathfrak{T}_{£}^{\star\pi})_e(\psi_\vartheta^{-1}(f_B)) = £_e^\pi(\upsilon_{\vartheta^{-1}(A)}^{\psi_\vartheta^{-1}(f_B)}) = £_e^\pi((\psi \times \psi)_\vartheta^{-1}(\upsilon_A^{f_B})) \leq \mathcal{F}_{\vartheta(e)}^\pi(\upsilon_A^{f_B}) = (\mathfrak{T}_{\mathcal{F}}^{\star\pi})_{\vartheta(e)}(f_B)$$

$$(\mathfrak{T}_{£}^{\star\varsigma})_e(\psi_\vartheta^{-1}(f_B)) = £_e^\varsigma(\upsilon_{\vartheta^{-1}(A)}^{\psi_\vartheta^{-1}(f_B)}) = £_e^\varsigma((\psi \times \psi)_\vartheta^{-1}(\upsilon_A^{f_B})) \leq \mathcal{F}_{\vartheta(e)}^\varsigma(\upsilon_A^{f_B}) = (\mathfrak{T}_{\mathcal{F}}^{\star\varsigma})_{\vartheta(e)}(f_B).$$

□

Theorem 11. *Let $(\mathcal{X}, £_E^{\gamma\pi\varsigma})$ and $(\mathcal{G}, \mathcal{F}_{\mathcal{R}}^{\gamma\pi\varsigma})$ be two svns-uniform spaces and $\psi_\vartheta : \widehat{(\mathcal{X}, E)} \to \widehat{(\mathcal{G}, \mathcal{R})}$ be an svns-uniformly continuous mapping. Then, $\psi_\vartheta : (\mathcal{X}, \mathfrak{T}_{£}^{\star\star\gamma\pi\varsigma}) \to (\mathcal{G}, \mathfrak{T}_{\mathcal{F}}^{\star\star\gamma\pi\varsigma})$ is svns-continuous.*

Proof. Initially, it is clear that $\psi_\vartheta^{-1}(\upsilon_A[\psi(x)]) = (\psi \times \psi)_\vartheta^{-1}(\upsilon_A[x])$ from that:

$$[\psi_\vartheta^{-1}(\upsilon_A[\psi(x)])](z) = (\upsilon_A[\psi(x)])(\psi(z)) = \upsilon_A(\psi(z), \psi(x)) = ((\psi \times \psi)_\vartheta^{-1}(\upsilon_A))(z, x)$$
$$= [((\psi \times \psi)_\vartheta^{-1}(\upsilon_A))[x]](z).$$

Thus, $\upsilon_A[\psi(x)] \leq f_B$ implies that $\psi_\vartheta^{-1}(\upsilon_A[\psi(x)]) = ((\psi \times \psi)_\vartheta^{-1}(\upsilon_A))[x] \leq \psi_\vartheta^{-1}(f_B)$. By applying Theorem 8, we obtain

$$(\mathfrak{T}_{\mathcal{F}}^{\star\star\gamma})_\omega(f_B) = \bigwedge_y \left[(f_B^c)(y) \vee \bigvee_{\upsilon_A[y] \leq f_B} \mathcal{F}_\omega^\gamma(\upsilon_A)\right] \leq \bigwedge_x \left[f_B^c(\psi(x)) \vee \bigvee_{\upsilon_A[\psi(x)] \leq f_B} \mathcal{F}_\omega^\gamma(\upsilon_A)\right]$$
$$\leq \bigwedge_x \left[(\psi_\vartheta^{-1}(f_B))^c(x) \vee \bigvee_{((\psi\times\psi)_\vartheta^{-1}(\upsilon_A))[x] \leq \psi_\vartheta^{-1}(f_B)} £_{\vartheta^{-1}(\omega)}^\gamma((\psi \times \psi)_\vartheta^{-1}(\upsilon_A))\right]$$
$$\leq (\mathfrak{T}_{£}^{\star\star\gamma})_{\vartheta^{-1}(\omega)}(\psi_\vartheta^{-1}(f_B)),$$

$$(\mathfrak{T}_{£}^{\star\star\pi})_\omega(f_B) = \bigvee_y \left[(f_B^c)(y) \wedge \bigvee_{\upsilon_A[y] \leq f_B} \mathcal{F}_\omega^\pi(\upsilon_A)\right] \geq \bigvee_x \left[f_B^c(\psi(x)) \wedge \bigvee_{\upsilon_A[\psi(x)] \leq f_B} \mathcal{F}_\omega^\pi(\upsilon_A)\right]$$
$$\geq \bigvee_x \left[(\psi_\vartheta^{-1}(f_B))^c(x) \wedge \bigvee_{((\psi\times\psi)_\vartheta^{-1}(\upsilon_A))[x] \leq \psi_\vartheta^{-1}(f_B)} £_{\vartheta^{-1}(\omega)}^\pi((\psi \times \psi)_\vartheta^{-1}(\upsilon_A))\right]$$
$$\geq (\mathfrak{T}_{£}^{\star\star\pi})_{\vartheta^{-1}(\omega)}(\psi_\vartheta^{-1}(f_B)),$$

Likewise, we can establish through a similar line of reasoning that $(\mathfrak{T}_{£}^{\star\star\varsigma})_\omega(f_B) \geq (\mathfrak{T}_{£}^{\star\star\varsigma})_{\vartheta^{-1}(\omega)}(\psi_\vartheta^{-1}(f_B))$. □

Theorem 12. *Let* $\{(\mathcal{X}_j, (\pounds^{\gamma\pi\varsigma}_j)_{E_j}) : j \in \Gamma\}$ *be a family of svns- uniform spaces and, for all* $j \in \Gamma$, $\psi_j : \mathcal{X} \to \mathcal{X}_j$, *and* $\vartheta_j : E \to E_j$ *are mappings. Define* $\pounds^\gamma : E \to \zeta^{(\widehat{\mathcal{X}\times\mathcal{X}},E)}$, $\pounds^\pi : E \to \zeta^{(\widehat{\mathcal{X}\times\mathcal{X}},E)}$ *and* $\pounds^\varsigma : E \to \zeta^{(\widehat{\mathcal{X}\times\mathcal{X}},E)}$ *on* $\mathcal{X}$ *by:*

$$\pounds^\gamma_e(\upsilon_A) = \bigvee\left[\bigwedge_{j=1}^{n}((\pounds^\gamma_{\omega_j})_{\vartheta_{\omega_j}(e)})((\mu_B)_{\omega_j}) \mid \upsilon_A \geq \sqcap_{j=1}^{n}(\psi_{\omega_j}\times\psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j})\right],$$

$$\pounds^\pi_e(\upsilon_A) = \bigwedge\left[\bigvee_{j=1}^{n}((\pounds^\pi_{\omega_j})_{\vartheta_{\omega_j}(e)})((\mu_B)_{\omega_j}) \mid \upsilon_A \geq \sqcap_{j=1}^{n}(\psi_{\omega_j}\times\psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j})\right],$$

$$\pounds^\varsigma_e(\upsilon_A) = \bigwedge\left[\bigvee_{j=1}^{n}((\pounds^\varsigma_{\omega_j})_{\vartheta_{\omega_j}(e)})((\mu_B)_{\omega_j}) \mid \upsilon_A \geq \sqcap_{j=1}^{n}(\psi_{\omega_j}\times\psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j})\right].$$

where $\bigvee$ *is taken over all finite subsets* $\Omega = \{\omega_1, \omega_2, ..., \omega_n\} \subseteq \Gamma$. *Then,*
(1) $\pounds^{\gamma\pi\varsigma}$ *is the coarsest svns-uniformity on* $\mathcal{X}$ *for which all* $\{(\psi_\vartheta)_j : j \in \Gamma\}$ *are svns-uniformly continuous.*
(2) *A map* $\psi_\vartheta : (\mathcal{X}^\star, \pounds^{\gamma\pi\varsigma}_{\mathcal{R}}) \to (\mathcal{X}, \pounds^{\gamma\pi\varsigma}_E)$ *is svns-uniformly continuous if for all* $j \in \Gamma$, $(\psi_\vartheta)_j \circ \psi_\vartheta : (\mathcal{X}^\star, \pounds^{\gamma\pi\varsigma}_{\mathcal{R}}) \to (\mathcal{X}_j, (\pounds^{\gamma\pi\varsigma}_j)_{E_j})$ *is svns-uniformly continuous.*

Proof. (1) Initially, we indication that $\pounds^{\gamma\pi\varsigma}$ is an svns-uniformity on $\mathcal{X}$ for which all $\{(\psi_\vartheta)_j : j \in \Gamma\}$ are svns-uniformly continuous.

($\pounds_1$) For every $\omega_j \in \Omega$, there exists $(\upsilon_A)_{\omega_j} \in (\mathcal{X}_{\omega_j} \times \widehat{\mathcal{X}}_{\omega_j}, \mathrm{E}_{\omega_j})$ such that, for $e \in \mathrm{E}$, we obtain that

$$(\pounds^\gamma_{\omega_j})_{\vartheta_{\omega_j}(e)})((\mu_B)_{\omega_j}) = 1, \quad (\pounds^\pi_{\omega_j})_{\vartheta_{\omega_j}(e)})((\mu_B)_{\omega_j}) = 0, \quad (\pounds^\varsigma_{\omega_j})_{\vartheta_{\omega_j}(e)})((\mu_B)_{\omega_j}) = 0.$$

Put $(\psi_{\omega_j}\times\psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j}) = \upsilon_A$. Then, $\pounds^\gamma_e(\upsilon_A) = 1$, $\pounds^\pi_e(\upsilon_A) = 0$ and $\pounds^\varsigma_e(\upsilon_A) = 0$.

($\pounds_2$) It is obvious from the definition of $\pounds^{\gamma\pi\varsigma}$.

($\pounds_3$) For all limited subsets $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$, $T = \{t_1, t_2, ..., t_m\}$ of Γ such that

$$\sqcap_{j=1}^{n}(\psi_{\omega_j}\times\psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\upsilon_A)_{\omega_j}) \leq \upsilon_A, \quad \sqcap_{j=1}^{m}(\psi_{t_j}\times\psi_{t_j})^{-1}_{\vartheta_{t_j}}((\mu_B)_{t_j}) \leq \mu_B$$

we have

$$\sqcap_{j=1}^{m}(\psi_{t_j}\times\psi_{t_j})^{-1}_{\vartheta_{t_j}}((\mu_B)_{t_j}) \sqcap \sqcap_{j=1}^{n}(\psi_{\omega_j}\times\psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\upsilon_A)_{\omega_j}) \leq \mu_B \sqcap \upsilon_A.$$

Moreover, for all $\omega \in \Omega \cap T$ we have

$$(\psi_\omega\times\psi_\omega)^{-1}_{\vartheta_\omega}((\mu_B)_\omega) \sqcap (\psi_\omega\times\psi_\omega)^{-1}_{\vartheta_\omega}((\upsilon_A)_\omega) = (\psi_\omega\times\psi_\omega)^{-1}_{\vartheta_\omega}((\mu_B)_\omega \sqcap (\upsilon_A)_\omega).$$

Put $(\psi_{m_j}\times\psi_{m_j})^{-1}_{\vartheta_{m_j}}((\mathcal{W}_C)_{m_j}) \leq \mu_B \sqcap \upsilon_A$, where

$$\gamma_{(\mathcal{W}_C)m_j}(x) = \begin{cases} \gamma_{(\upsilon_A)m_j}(x), & \text{if } m_j \in \Omega - (\Omega \cap T), \\ \gamma_{(\mu_B)m_j}(x), & \text{if } m_j \in \Omega - (\Omega \cap T), \\ \gamma_{(\upsilon_A)m_j}(x) \cap \gamma_{(\mu_B)m_j}(x), & \text{if } m_j \in \Omega \cap T, \end{cases}$$

$$\pi_{(\mathcal{W}_C)_{m_j}}(x)=\begin{cases}\pi_{(\upsilon_A)_{m_j}}(x), & \text{if } m_j\in\Omega-(\Omega\cap T),\\ \pi_{(\mu_B)_{m_j}}(x), & \text{if } m_j\in\Omega-(\Omega\cap T),\\ \pi_{(\upsilon_A)_{m_j}}(x)\cup\pi_{(\mu_B)_{m_j}}(x), & \text{if } m_j\in\Omega\cap T,\end{cases}$$

$$\varsigma_{(\mathcal{W}_C)_{m_j}}(x)=\begin{cases}\varsigma_{(\upsilon_A)_{m_j}}(x), & \text{if } m_j\in\Omega-(\Omega\cap T),\\ \varsigma_{(\mu_B)_{m_j}}(x), & \text{if } m_j\in\Omega-(\Omega\cap T),\\ \varsigma_{(\upsilon_A)_{m_j}}(x)\cup\varsigma_{(\mu_B)_{m_j}}(x), & \text{if } m_j\in\Omega\cap T,\end{cases}$$

Therefore, we obtain

$$\pounds^{\gamma}_e(\upsilon_A\sqcap\mu_B)\geq\bigwedge_{j\in\Omega\cup T}(\pounds^{\gamma}_j)_{\vartheta_j(e)}((\mathcal{W}_C)_j)\geq\left[\bigwedge_{j=1}^{n}(\pounds^{\gamma}_{\omega_j})_{\vartheta_{\omega_j}(e)}((\upsilon_A)_{\omega_j})\right]\wedge\left[\bigwedge_{j=1}^{m}(\pounds^{\gamma}_{t_j})_{\vartheta_{t_j}(e)}((\mu_B)_{t_j})\right]$$

$$\pounds^{\pi}_e(\upsilon_A\sqcap\mu_B)\leq\bigvee_{j\in\Omega\cup T}(\pounds^{\pi}_j)_{\vartheta_j(e)}((\mathcal{W}_C)_j)\leq\left[\bigvee_{j=1}^{n}(\pounds^{\pi}_{\omega_j})_{\vartheta_{\omega_j}(e)}((\upsilon_A)_{\omega_j})\right]\vee\left[\bigvee_{j=1}^{m}(\pounds^{\pi}_{t_j})_{\vartheta_{t_j}(e)}((\mu_B)_{t_j})\right],$$

$$\pounds^{\varsigma}_e(\upsilon_A\sqcap\mu_B)\leq\bigvee_{j\in\Omega\cup T}(\pounds^{\varsigma}_j)_{\vartheta_j(e)}((\mathcal{W}_C)_j)\leq\left[\bigvee_{j=1}^{n}(\pounds^{\varsigma}_{\omega_j})_{\vartheta_{\omega_j}(e)}((\upsilon_A)_{\omega_j})\right]\vee\left[\bigvee_{j=1}^{m}(\pounds^{\varsigma}_{t_j})_{\vartheta_{t_j}(e)}((\mu_B)_{t_j})\right].$$

Taking the supremum on the families $\sqcap_{j=1}^{n}(\psi_{\omega_j}\times\psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\upsilon_A)_{\omega_j})\leq\upsilon_A$ and $\sqcap_{j=1}^{m}(\psi_{t_j}\times\psi_{t_j})^{-1}_{\vartheta_{t_j}}((\mu_B)_{t_j})\leq\mu_B$ we obtain

$$\pounds^{\gamma}_e(\upsilon_A\sqcap\mu_B)\geq\pounds^{\gamma}_e(\upsilon_A)\wedge\pounds^{\gamma}_e(\mu_B),\qquad \pounds^{\pi}_e(\upsilon_A\sqcap\mu_B)\leq\pounds^{\pi}_e(\upsilon_A)\vee\pounds^{\pi}_e(\mu_B),$$
$$\pounds^{\varsigma}_e(\upsilon_A\sqcap\mu_B)\leq\pounds^{\varsigma}_e(\upsilon_A)\vee\pounds^{\varsigma}_e(\mu_B),\quad\forall\, e\in\mathrm{E}.$$

($\pounds_4$) If $\pounds^{\gamma}_e(\upsilon_A)\neq 0$, $\pounds^{\pi}_e(\upsilon_A)\neq 1$ and $\pounds^{\varsigma}_e(\upsilon_A)\neq 1$, then there exists $\Omega=\{\omega_1,\omega_2,...,\omega_p\}$ of Γ with $\sqcap_{j=1}^{p}(\psi_{\omega_j}\times\psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j})\leq\upsilon_A$ such that

$$\pounds^{\gamma}_e(\upsilon_A)\geq\bigwedge_{j=1}^{p}(\pounds^{\gamma}_{\omega_j})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j})\neq 0,\qquad \pounds^{\pi}_e(\upsilon_A)\leq\bigvee_{j=1}^{p}(\pounds^{\pi}_{\omega_j})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j})\neq 1,$$

$$\pounds^{\pi}_e(\upsilon_A)\leq\bigvee_{j=1}^{p}(\pounds^{\pi}_{\omega_j})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j})\neq 1.$$

Since, $(\pounds^{\gamma}_{\omega_j})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j})\neq 0$, $(\pounds^{\pi}_{\omega_j})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j})\neq 1$, $(\pounds^{\varsigma}_{\omega_j})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j})\neq 0\ \forall\ \omega_j\in\Omega$, then $(\top)_C\nleq(\upsilon_B)_{\omega_j}$. Thus,

$$(\top)_C\leq(\psi_{\omega_j}\times\psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\top)_C)\leq\sqcap_{j=1}^{p}(\psi_{\omega_j}\times\psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j})\leq\upsilon_A.$$

($\pounds_5$) Assume that $\pounds^{\gamma}_e(\upsilon^s_A)\ngeq\pounds^{\gamma}_e(\upsilon_A)$, $\pounds^{\pi}_e(\upsilon^s_A)\nleq\pounds^{\pi}_e(\upsilon_A)$ and $\pounds^{\varsigma}_e(\upsilon^s_A)\nleq\pounds^{\varsigma}_e(\upsilon_A)$. From the concept of $\pounds^{\gamma\pi\varsigma}$, there exists $\Omega=\{\omega_1,\omega_2,...,\omega_p\}$ of Γ with $\sqcap_{j=1}^{p}(\psi_{\omega_j}\times\psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j})\leq\upsilon_A$ such that

$$\pounds^{\gamma}_e(\upsilon^s_A)\ngeq\bigwedge_{j=1}^{p}(\pounds^{\gamma}_{\omega_j})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j}),\qquad \pounds^{\pi}_e(\upsilon^s_A)\nleq\bigvee_{j=1}^{p}(\pounds^{\pi}_{\omega_j})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j}),$$

$$\pounds_e^{\varsigma}(\upsilon_A^s) \not\leq \bigvee_{j=1}^{p} (\pounds_{\omega_j}^{\varsigma})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j}).$$

Since $\pounds_{\omega_j}^{\gamma\pi\varsigma}$ is an svns-uniformity on $\mathcal{X}$ for each ω_j

$$(\pounds_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}((\mu_B^s)_{\omega_j}) \geq (\pounds_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j}), \quad (\pounds_{\omega_j}^{\pi})_{\vartheta_{\omega_j}(e)}((\mu_B^s)_{\omega_j}) \leq (\pounds_{\omega_j}^{\pi})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j}),$$

$$(\pounds_{\omega_j}^{\varsigma})_{\vartheta_{\omega_j}(e)}((\mu_B^s)_{\omega_j}) \leq (\pounds_{\omega_j}^{\varsigma})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j}).$$

It follows that

$$\pounds_e^{\gamma}(\upsilon_A^s) \not\geq \bigwedge_{j=1}^{p} (\pounds_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}((\mu_B^s)_{\omega_j}), \quad \pounds_e^{\pi}(\upsilon_A^s) \not\leq \bigvee_{j=1}^{p} (\pounds_{\omega_j}^{\pi})_{\vartheta_{\omega_j}(e)}((\mu_B^s)_{\omega_j}),$$

$$\pounds_e^{\varsigma}(\upsilon_A^s) \not\leq \bigvee_{j=1}^{p} (\pounds_{\omega_j}^{\varsigma})_{\vartheta_{\omega_j}(e)}((\mu_B^s)_{\omega_j}).$$

On the other hand,

$$\sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})_{\vartheta_{\omega_j}}^{-1}((\mu_B^s)_{\omega_j}) = \sqcap_{j=1}^{n}((\psi_{\omega_j} \times \psi_{\omega_j})_{\vartheta_{\omega_j}}^{-1}((\mu_B)_{\omega_j}))^s \leq \upsilon_A^s.$$

Hence,

$$\pounds_e^{\gamma}(\upsilon_A^s) \geq \bigwedge_{j=1}^{p} \pounds_{\vartheta_{\omega_j}(e)}^{\gamma}((\mu_B^s)_{\omega_j}), \quad \pounds_e^{\pi}(\upsilon_A^s) \leq \bigvee_{j=1}^{p} \pounds_{\vartheta_{\omega_j}(e)}^{\pi}((\mu_B^s)_{\omega_j}), \quad \pounds_e^{\varsigma}(\upsilon_A^s) \leq \bigvee_{j=1}^{p} \pounds_{\vartheta_{\omega_j}(e)}^{\varsigma}((\mu_B^s)_{\omega_j}).$$

It is a contradiction. Hence, ($\pounds_5$) holds.
($\pounds_6$) Suppose that for each $\upsilon_A \in (\widehat{\mathcal{X} \times \mathcal{X}}, \mathrm{E})$

$$\pounds_e^{\gamma}(\upsilon_A) \not\leq \bigvee\{\pounds_e^{\gamma}((\upsilon_A)_1) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \leq \upsilon_A\}, \quad \pounds_e^{\pi}(\upsilon_A) \not\geq \bigwedge\{\pounds_e^{\pi}((\upsilon_A)_1)) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \leq \upsilon_A\},$$

$$\pounds_e^{\varsigma}(\upsilon_A) \not\geq \bigwedge\{\pounds_e^{\varsigma}((\upsilon_A)_1)) \mid (\upsilon_A)_1) \circ (\upsilon_A)_1) \leq \upsilon_A\}.$$

By the concept of $\pounds^{\gamma\pi\varsigma}$, there exists $\Omega = \{\omega_1, \omega_2, ..., \omega_p\}$ of Γ with $\sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})_{\vartheta_{\omega_j}}^{-1}((\mu_B)_{\omega_j}) \leq \upsilon_A$ such that

$$\bigwedge_{j=1}^{p}(\pounds_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}(\mu_{\omega_j}) \not\leq \bigvee\{\pounds_e^{\gamma}((\upsilon_A)_1) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \leq \upsilon_A\},$$

$$\bigvee_{j=1}^{p}(\pounds_{\omega_j}^{\pi})_{\vartheta_{\omega_j}(e)}(\mu_{\omega_j}) \not\geq \bigwedge\{\pounds_e^{\pi}((\upsilon_A)_1)) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \leq \upsilon_A\},$$

$$\bigvee_{j=1}^{p}(\pounds_{\varsigma_j}^{\pi})_{\vartheta_{\omega_j}(e)}(\mu_{\omega_j}) \not\geq \bigwedge\{\pounds_e^{\varsigma}((\upsilon_A)_1)) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \leq \upsilon_A\}.$$

Since $\pounds_{\omega_j}^{\gamma\pi\varsigma}$ is *svns- uniformity* on $\mathcal{X}_{\omega_j}$ for each $\omega_j \in \Omega$

$$(\pounds_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}((\mu_B)_j) \not\leq \bigvee\{(\pounds_{\omega_j}^{\gamma})_{\vartheta_{\omega_j}(e)}(\mathcal{W}_C) \mid \mathcal{W}_C \circ \mathcal{W}_C \leq \mu_B\},$$

$$(\pounds^{\pi}_{\omega_j})_{\vartheta_{\omega_j(e)}}((\mu_B)_j) \ngeq \bigwedge\{(\pounds^{\pi}_{\omega_j})_{\vartheta_{\omega_j(e)}}(\mathcal{W}_C) \mid \mathcal{W}_C \circ \mathcal{W}_C \leq \mu_B\},$$

$$(\pounds^{\varsigma}_{\omega_j})_{\vartheta_{\omega_j(e)}}((\mu_B)_j) \ngeq \bigwedge\{(\pounds^{\varsigma}_{\omega_j})_{\vartheta_{\omega_j(e)}}(\mathcal{W}_C) \mid \mathcal{W}_C \circ \mathcal{W}_C \leq \mu_B\}.$$

Thus, there exists $\mathcal{W}_C \in (\mathcal{X}_{\omega_j} \widehat{\times \mathcal{X}_{\omega_j}}, \mathrm{E}_{\omega_j})$, $\mathcal{W}_C \circ \mathcal{W}_C \leq \upsilon_A$ such that

$$\bigwedge_{j=1}^{p}(\pounds^{\gamma}_{\omega_j})_{\vartheta_{\omega_j(e)}}(\mathcal{W}_C) \nleq \bigvee\{\pounds^{\gamma}_e((\upsilon_A)_1) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \leq \upsilon_A\},$$

$$\bigvee_{j=1}^{p}(\pounds^{\pi}_{\omega_j})_{\vartheta_{\omega_j(e)}}(\mathcal{W}_C) \ngeq \bigwedge\{\pounds^{\pi}_e((\upsilon_A)_1)) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \leq \upsilon_A\},$$

$$\bigvee_{j=1}^{p}(\pounds^{\pi}_{\varsigma_j})_{\vartheta_{\omega_j(e)}}(\mathcal{W}_C) \ngeq \bigwedge\{\pounds^{\varsigma}_e((\upsilon_A)_1)) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \leq \upsilon_A\}.$$

On the other hand,

$$\begin{aligned}\sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}(\mathcal{W}_C) \circ \sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}(\mathcal{W}_C) &\leq \sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}(\mathcal{W}_C \circ \mathcal{W}_C)\\ &\leq \sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j}) \leq \upsilon_A.\end{aligned}$$

Therefore, we have

$$\bigvee\{\pounds^{\gamma}_e((\upsilon_A)_1) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \leq \upsilon_A\} \geq \pounds^{\gamma}_e((\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}(\mathcal{W}_C)) \geq (\pounds^{\gamma}_{\omega_j})_{\vartheta_{\omega_j(e)}}(\mathcal{W}_C),$$

$$\bigwedge\{\pounds^{\pi}_e((\upsilon_A)_1) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \leq \upsilon_A\} \leq \pounds^{\pi}_e((\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}(\mathcal{W}_C)) \leq (\pounds^{\pi}_{\omega_j})_{\vartheta_{\omega_j(e)}}(\mathcal{W}_C),$$

$$\bigwedge\{\pounds^{\varsigma}_e((\upsilon_A)_1) \mid (\upsilon_A)_1 \circ (\upsilon_A)_1 \leq \upsilon_A\} \leq \pounds^{\varsigma}_e((\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}(\mathcal{W}_C)) \leq (\pounds^{\varsigma}_{\omega_j})_{\vartheta_{\omega_j(e)}}(\mathcal{W}_C).$$

It is a contradiction. Hence, $\pounds_6$ holds.

Next by the concept of $\pounds^{\gamma\pi\varsigma}$ it is easily proved that, for each $j \in \Gamma$

$$\pounds^{\gamma}_e((\psi_j \times \psi_j)^{-1}_{\vartheta_j}(\mu_B)) \geq (\pounds^{\gamma}_j)_{\vartheta_{j(e)}}(\mu_B), \qquad \pounds^{\pi}_e((\psi_j \times \psi_j)^{-1}_{\vartheta_j}(\mu_B)) \leq (\pounds^{\pi}_j)_{\vartheta_{j(e)}}(\mu_B),$$

$$\pounds^{\varsigma}_e((\psi_j \times \psi_j)^{-1}_{\vartheta_j}(\mu_B)) \leq (\pounds^{\varsigma}_j)_{\vartheta_{j(e)}}(\mu_B),\ \forall\ \mu_B \in (\mathcal{X}_j \widehat{\times \mathcal{X}_j}, \mathrm{E}_j).$$

Thus, $(\psi_j)_{\vartheta_j} : \mathcal{X} \to \mathcal{X}_j$ is svns-uniformly continuous.

Lastly, let us say that $\pounds^{\star\gamma\pi\varsigma}$ is an svns-uniformity on $\mathcal{X}$ and $(\psi_j)_{\vartheta_j} : (\mathcal{X}, \pounds^{\star\gamma\pi\varsigma}) \to (\mathcal{X}_j, \pounds^{\gamma\pi\varsigma}_j)$ is svns-uniformly continuous, that is, for every $j \in \Gamma$ and $(\mu_B)_j \in (\mathcal{X}_j \widehat{\times \mathcal{X}_j}, \mathrm{E}_j)$,

$$\pounds^{\star\gamma}_e((\psi_j \times \psi_j)^{-1}_{\vartheta_j}((\mu_B)_j)) \geq (\pounds^{\gamma}_j)_{\vartheta_{j(e)}}((\mu_B)_j), \qquad \pounds^{\star\pi}_e((\psi_j \times \psi_j)^{-1}_{\vartheta_j}((\mu_B)_j)) \leq (\pounds^{\pi}_j)_{\vartheta_{j(e)}}((\mu_B)_j),$$

$$\pounds^{\star\varsigma}_e((\psi_j \times \psi_j)^{-1}_{\vartheta_j}((\mu_B)_j)) \leq (\pounds^{\varsigma}_j)_{\vartheta_{j(e)}}((\mu_B)_j).$$

For every finite subset $\Omega = \{\omega_1, \omega_2, ..., \omega_p\}$ of Γ with $\sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j}) \leq \upsilon_A$, we have

$$\begin{aligned}
£^{\gamma}_{e}(\upsilon_A) &= \bigvee \left[\bigwedge_{j=1}^{p} (£^{\gamma}_{j})_{\vartheta_{\omega_j}(e)}((\mu_B)_{\omega_j}) \mid \sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j}) \leq \upsilon_A \right] \\
&\leq \bigvee \left[\bigwedge_{j=1}^{p} (£^{\star\gamma}_{e}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j}) \mid \sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j}) \leq \upsilon_A \right] \\
&\leq \bigvee \left[\bigwedge_{j=1}^{p} (£^{\star\gamma}_{e}(\sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j})) \mid \sqcap_{j=1}^{p}(\psi_{\omega_j} \times \psi_{\omega_j})^{-1}_{\vartheta_{\omega_j}}((\mu_B)_{\omega_j}) \leq \upsilon_A \right] \\
&\leq £^{\star\gamma}_{e}(\upsilon_A).
\end{aligned}$$

In a similar vein, we can demonstrate through a parallel line of reasoning that $£^{\pi}_{e}(\upsilon_A) \geq £^{\star\pi}_{e}(\upsilon_A)$ and $£^{\varsigma}_{e}(\upsilon_A) \geq £^{\star\varsigma}_{e}(\upsilon_A)$.

(2) It can be easily proved. □

6. Conclusions

Many scientists have studied the soft set theory and easily applied it to many problems in social life. In the present work, we defined the single-valued neutrosophic soft uniform spaces and single-valued neutrosophic soft uniform bases. The relationships between them were also investigated. Next, the relationship among single-valued neutrosophic soft uniformities, single-valued neutrosophic soft topologies, and single-valued neutrosophic soft interior operators were introduced and studied. Finally, we proved crucial results in introducing and characterizing single-valued neutrosophic soft uniformly continuous, on single-valued neutrosophic soft uniformly topological spaces. Moreover, the relationship between single-valued neutrosophic soft uniformly continuous and single-valued neutrosophic soft continuous was studied. This paper can form the theoretical basis for further applications of single-valued neutrosophic soft topology, potentially leading to the development of other scientific areas.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

Acknowledgments

The authors would like to extend their sincere appreciation to Supporting Project number (RSPD2023R860) King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
2. K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.*, **20** (1986), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3
3. Z. Pawlak, Rough sets, *Int. J. Comput. Inform. Sci.*, **11** (1982), 341–356. https://doi.org/10.1007/BF01001956
4. F. Smarandache, *A unifying field in logics: Neutrosophic logic*, Rehoboth: American Research Press, 1999.
5. D. Molodtsov, Soft set theory-first result, *Comput. Math. Appl.*, **37** (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5
6. P. K. Maji, R. Biswas, A. R. Roy, Fuzzy soft sets, *J. Fuzzy Math.*, **9** (2001), 589–602.
7. P. K. Maji, Neutrosophic soft set, *Ann. Fuzzy Math. Inf.*, **5** (2013), 157–168.
8. B. Ahmad, A. Kharal, On fuzzy soft sets, *Adv. Fuzzy Syst.*, **2009** (2009), 586507. https://doi.org/10.1155/2009/586507
9. H. Wang, F. Smarandache, Y. Q. Zhang, R. Sunderraman, Single valued neutrosophic sets, *Multispace Multistruct*, **4** (2010), 410–413.
10. Y. M. Saber, F. Alsharari, F. Smarandache, On Single-valued neutrosophic ideals in Šostak sense, *Symmetry*, **12** (2020), 193. https://doi.org/10.3390/sym12020193
11. Y. M. Saber, F. Alsharari, F. Smarandache, A. Abdel-Sattar, Connectedness and stratification of single-valued neutrosophic topological spaces, *Symmetry*, **12** (2020), 1464. https://doi.org/10.3390/sym12091464
12. F. Alsharari, F. Smarandache, Y. M. Saber, Compactness on single-valued neutrosophic ideal topological spaces, *Neutrosophic Sets Syst.*, **41** (2021), 127–145.
13. Y. M. Saber, F. Alsharari, F. Smarandache, A. Abdel-Sattar, On single valued neutrosophic regularity spaces, *Comput. Model. Eng. Sci.*, **130** (2022), 1625–1648. https://doi.org/10.32604/cmes.2022.017782
14. D. Sasirekha, P. Senthilkumar, Determining the Best Plastic Recycling Technology Using the MABAC Method in a Single-Valued Neutrosophic Fuzzy Approach, *Neutrosophic Sets Syst.*, **58** (2023), 194–210
15. F. Masri, M. Zeina, O. Zeitouny, Some Single Valued Neutrosophic Queueing Systems with Maple Code, *Neutrosophic Sets Syst.*, **53** (2023), 251–273. https://doi.org/10.5281/zenodo.7536023
16. Y. M. Saber, F. Alsharari, F. Smarandache, An Introduction to Single-Valued Neutrosophic Soft Topological Structure, *Soft Comput.*, **26** (2022), 7107–7122. https://doi.org/10.1007/s00500-022-07150-4
17. S. Shahzadi , A. Rasool, G. Santos-Garcìa, Methods to find strength of job competition among candidates under single-valued neutrosophic soft model, *Math. Bio. Eng.*, **20** (2023), 4609–4642. https://doi.org/10.3934/mbe.2023214

18. A. Cano, G. Petalcorin, Single-valued Neutrosophic Soft sets in Hyper UP-Algebra, *Eur. J. Pure. Appl. Math.*, **16** (2023), 548–576. https://doi.org/10.29020/nybg.ejpam.v16i1.4637
19. A. Özkan, Ş. Yazgan, S. Kaur, Neutrosophic Soft Generalized b-Closed Sets in Neutrosophic Soft Topological Spaces, *Neutrosophic Sets Syst.*, **56** (2023), 48–69.
20. S. Al-Hijjawi, A. Ahmad, S. Alkhazaleh, A generalized effective neurosophic soft set and its applications, *AIMS Mathematics*, **18** (2023), 29628–29666. https://doi.org/10.3934/math.20231517
21. C. Jana, M. Pal, A Robust Single-Valued Neutrosophic Soft Aggregation Operators in Multi-Criteria Decision Making, *Symmetry*, **11** (2019), 110. https://doi.org/10.3390/sym11010110
22. N. L. A. Mohd Kamal, L. Abdullah, Multi-Valued Neutrosophic Soft Set, *Malays. J. Math. Sci.*, **13** (2019), 153–168.
23. R. Lowen, Fuzzy uniform spaces, *J. Math. Anal. Appl.*, **82** (1981), 370–385. https://doi.org/10.1016/0022-247X(81)90202-X
24. W. Kotzé, Uniform spaces, In: *Mathematics of Fuzzy Sets*, Boston: Springer, 553–580.
25. B. Hutton, Uniformities in fuzzy topological spaces, *J. Math. Anal. Appl.*, **58** (1977), 559–571. https://doi.org/10.1016/0022-247X(77)90192-5
26. S. E. Abbas, I. Ibedou, Fuzzy soft uniform spaces, *Soft Comput.*, **21** (2017), 6073–6083. https://doi.org/10.1007/s00500-016-2327-3
27. J. Dezert, Open Questions to Neutrosophic Inferences, *Multi. Val. Logic.*, **8** (2001), 439–472.

AIMS Press

© 2024 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)