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Research article

Characterizations of normal cancellative monoids

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Abstract: Normal cancellative monoids were introduced to explore the general structure of cancellative monoids, which are innovative and open up new possibilities. Specifically, we pointed out that the Green's relations in a cancellative monoid S are determined by its unitary subgroup U to a great extent. The specific composition of egg boxes in S, derived from the general semigroup theory, can be settled by the subgroups of U. We call a cancellative monoid normal when these subgroups are normal and characterize it as an NCM-system. This NCM-system was created in this article and can be obtained by combining a group and a condensed cancellative monoid. Furthermore, we introduced the concept of torsion extension and proved that a special kind of normal cancellative monoids can be constructed delicately by the outer automorphism groups of given groups and some simplified cancellative monoids.

Keywords: cancellative monoids; Green's relations; normal cancellative monoids; NCM-system
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1. Introduction

A monoid S is called *cancellative* when the cancellative law holds. Precisely, for all $x, y, a \in S$, $ax = ay$ or $xa = ya$ implies that $x = y$. The study of the structure of commutative cancellative monoids has a long and rich history, rising from the theory of integral domains and free abelian monoids. Specifically, factorization theory on the subject has become more and more popular in recent years (see [1–5]). On the other hand, the general theory of non-commutative cancellative monoids has received limited attention, except for the word problem, which is an old and celebrated problem in combinatorial algebra and can often be reduced to the cancellative cases [6]. Baeth and Smertnig extended the classical theory of non-unique factorization to a non-commutative setting [7]. Lawson proved that a class of left cancellative monoids called left Rees monoids can be constructed by the Zappa-Szép product (a generalization of the semidirect product) of a free monoid and a group using self-similar group actions (see [8]). Wazzan defined a new consequence of the generalized general product and investigated some known algebraic properties, including left cancellative [9]. Attention has also been focused on the flatness properties of S systems to study the structure of [left, right] cancellative monoids in the past (see [10–13]).

The motivation of this paper is to make an effort in the study of general cancellative monoids, which we have seen to be rare in the existing literature. The relations and affiliations between different elements in cancellative monoids are not so clear as in groups. However, the Green's relations will reveal some for us. The structure of the paper is as follows. Section 2 provides alternative characterizations of the Green's relations in a cancellative monoid. We then investigate the nature of divisibility in a special class of cancellative monoids called normal cancellative monoids and present a novel construction method for generating such a cancellative monoid. This construction method utilizes a generalized product structure that combines a group and a condensed cancellative monoid, resulting in an NCM-system as described in Section 3. In Section 4, we introduce the concept of torsion extensions, and point out that a special kind of normal cancellative monoids can be constructed by the outer automorphism groups of given groups and some simplified cancellative monoids.

The reader is referred to [14, 15] for notations and terminology not given in this paper.

2. Green's relations

Green's relations characterize the elements of a monoid S in terms of the principal ideals they generate. For elements a and b of S, Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ are defined by

- $a\mathcal{L}b$ if, and only if, $Sa = Sb$, i.e., there exists $x, y \in S$ such that $a = xb$ and $b = ya$.
- $a\mathcal{R}b$ if, and only if, $aS = bS$, i.e., there exists $x, y \in S$ such that $a = bx$ and $b = ay$.
- $a\mathcal{H}b$ if, and only if, $a\mathcal{L}b$ and $a\mathcal{R}b$.
- $\mathcal{D}$ is the smallest equivalence relation containing both $\mathcal{L}$ and $\mathcal{R}$; that is, $a\mathcal{D}b$ if, and only if, there exists c in S such that $a\mathcal{L}c$ and $c\mathcal{R}b$.

These relations above are equivalences on S, so each of them yields a partition of S into equivalence classes. The $\mathcal{L}$-class of a is denoted by L_a (and similarly for the other relations).

Let S be a cancellative monoid with identity element 1. Denote by $U(S)$, or just U, the set of all *units* in S; i.e., $G = \{u \in S : (\exists v \in S)uv = 1\}$. Note that $uv = 1$ implies $vu = 1$ in a cancellative monoid. It is easy to verify that U is a subgroup of S, and we call it the *unitary subgroup* of S.

Proposition 2.1. *The following statements are true for a cancellative monoid S:*

(1) $a\mathcal{L}b$ if, and only if, $a = gb$ for some $g \in U(S)$;

(2) $a\mathcal{R}b$ if, and only if, $a = bh$ for some $h \in U(S)$;

(3) $a\mathcal{D}b$ if, and only if, $a = gbh$ for some $g, h \in U(S)$.

Proof. (1) If $a\mathcal{L}b$, then there exists $x, y \in S$ such that $a = xb$ and $b = ya$. It follows that $a = xb = xya$, and so $xy = 1$ since S satisfies the cancellative law. This shows that $x, y \in U(S)$. Conversely, if $a = gb$ for some $g \in U(S)$, then $b = g^{-1}a$ and, hence, $a\mathcal{L}b$. This proves (1).

(2) It is dual to (1).

(3) If $a\mathcal{D}b$, then there exists $c \in S$ such that $a\mathcal{L}c\mathcal{R}b$. By (1) and (2), $a = gc = gbh$ for some $g, h \in U$. Conversely, if $a = gbh$ for some $g, h \in U$, then by (1) and (2), $a\mathcal{L}\ bh\ \mathcal{R}b$, i.e., $a\mathcal{D}b$. This completes the proof. □

For $a \in S$, put

$$\begin{aligned} G_a &= \{g \in U : ga = ah(\exists h \in U)\}, \\ N_a &= \{h \in U : ga = ah(\exists g \in U)\}. \end{aligned} \tag{2.1}$$

It is easy to check that these are subgroups of U. The following lemma lists some of their properties.

Lemma 2.2. *Let S be a cancellative monoid with $a, b \in S$, then*

(1) The map f_a: $G_a \to N_a$; $g \mapsto h$, where $ga = ha$ $(g, h \in G)$ is an isomorphism;
(2) $a\mathcal{L}b \Rightarrow N_a = N_b$;
(3) $a\mathcal{R}b \Rightarrow G_a = G_b$;
(4) $a\mathcal{H}b \Leftrightarrow a \in bN_a \Leftrightarrow a \in G_a b$;
(5) $D_a = UaU$;
(6) $H_a = aN_a = G_a a$;
(7) $L_a = Ua = UG_a a$;
(8) $R_a = aU = aN_a U$;
(9) $N_{ag} = g^{-1}N_a g$ for all $g \in U$;
(10) $G_{ga} = gG_a g^{-1}$ for all $g \in U$.

Proof. (1) According to formula (2.1) and the cancellativity of S, the map f_a is well defined and a bijection. Now, let $g_1, g_2 \in G_a$, then there exists $h_1, h_2 \in N_a$ such that $g_i a = ah_i$ with $i = 1, 2$. Hence, $g_1 g_2 a = g_1 a h_2 = ah_1 h_2$, i.e., $f_a(g_1 g_2) = f_a(g_1) f_a(g_2)$. Thus, f_a is a homomorphism and an isomorphism.

(2) If $a\mathcal{L}b$, there exists $g' \in U$ such that $a = g'b$ by Proposition 2.1. For any $h \in N_a$, $ah = ga$ for some $g \in U$. Thus, $ah = g'bh = ga \Rightarrow bh = g'^{-1}ga$, which yields that $h \in N_b$ and $N_a \subseteq N_b$. Similarly, we can prove $N_b \subseteq N_a$ and, thus, $N_a = N_b$.

(3) It can be proved in the some way.

(4) $a\mathcal{H}b \Leftrightarrow a = gb = bh$ for some $g, h \in U$, where $g \in G_b = G_a$ and $h \in N_b = N_a$ by (2), which is equivalent to $a \in bN_a$ or $a \in G_a b$.

(5–8) They can be proved by Proposition 2.1 directly.

(9,10) $h \in N_{ag}$ if, and only if, $agh = g'ag$ for some $g' \in U$, which gives that $aghg^{-1} = g'a \Leftrightarrow ghg^{-1} \in N_a \Leftrightarrow h \in g^{-1}N_a g$. We can prove (10) dually. □

For $r \in S$, $D_r = UrU$ and $H_r = rN_r = G_r r$, by noting that

$$H_{rg} = rgN_{rg} = rg \cdot g^{-1}N_r g = rN_r g = G_r rg$$

for $g \in U$, we obtain that the $\mathcal{H}$-classes of S containing in R_r are of the form $rN_r g$ $(g \in U)$. Since S is a cancellative monoid, $rN_r g$ and $rN_r h$ are distinct if, and only if, $N_r g$ and $N_r h$ are distinct right cosets of N_r in U. This shows that the set of $\mathcal{H}$-classes of S contained in R_r is in bijection with the set of right cosets of N_r in U. Similarly, the set of $\mathcal{H}$-classes of S contained in L_r is in bijection with the set of left cosets of G_r in U. The above results provide the specific composition of egg boxes in general semigroup theory when S is cancellative. We use S_0 to denote the set of the non-regular (i.e., not in $U(S)$) elements of S. As the application of Lemma 2.2, we have:

Proposition 2.3. *The following conditions hold for a cancellative monoid S:*

(1) If U is finite, $|S/\mathcal{L}| = |S/\mathcal{R}|$.

(2) If $S/\mathcal{D}$ is finite and $S_0 \neq \emptyset$, U must be infinite.

Proof. (1) We need only to prove that for all $r \in S$, the cardinal number of the set X_r of $\mathcal{H}$-classes of S in R_r is equal to the cardinal number of the set Y_r of $\mathcal{H}$-classes of S in L_r. By Lemma 2.2, we have

$$|X_r| = |U/N_r| \text{ and } |Y_r| = |U/G_r|.$$

On the other hand, G_r is isomorphic to N_r, and $|U/G_r| = |U/N_r|$. Thus, $|X_r| = |Y_r|$, as required.

(2) Assume contrarily that U is finite.

$$D_r = \{grh : g, h \in U\}$$

for all $r \in S$ shows that $|D_r| < +\infty$. Furthermore, by the hypothesis that $|S/\mathcal{D}| < +\infty$, S is a finite cancellative monoid. This implies that S is a group, and $S_0 = \emptyset$. This is a contradiction, inferring that U is infinite. □

3. Normal cancellative monoids

In this section, we consider a special class of cancellative monoids.

Definition 3.1. *Let S be a cancellative monoid and $r \in S$. G_r is called a* ***characteristic subgroup*** *of S in D_r if $H_r = rG_r = G_r r$ and G_r is a normal subgroup of U. In this case, D_r is called a* ***normal $\mathcal{D}$-class*** *of S and G_r is the characteristic subgroup of D_r.*

Note that the characteristic subgroup of $\mathcal{D}$-class D_r is independent of the representative of the $\mathcal{D}$-class by Lemma 2.2.

Proposition 3.2. *If S is a cancellative monoid and $r \in S$, then D_r is a normal $\mathcal{D}$-class of S if, and only if, $G_r = N_r$ and N_r is a normal subgroup of U.*

Proof. We need only to prove the necessity part. Now, let D_r be normal. $H_r = rG_r = rN_r$ implies that for all $g \in G_r$, there exists $h \in N_r$ such that $rg = rh$. However, S is cancellative and $g = h$. Hence, $G_r \subseteq N_r$. Similarly, we can prove $N_r \subseteq G_r$. Thus, $G_r = N_r$ which completes the proof. □

Definition 3.3. *A cancellative monoid S is called a* ***normal*** *cancellative monoid with characteristic subgroup G if all non-regular $\mathcal{D}$-classes of S are all normal and have the same characteristic subgroup G.*

Note that the identity of S is the only idempotent so that $U = D_1$ is the only regular $\mathcal{D}$-class, and $S_0 = S \setminus U$ is the set of non-regular elements in S. Clearly, groups and commutative cancellative monoids are all normal cancellative monoids. The following example gives another kind of normal cancellative monoid.

Example 3.4. *Let G be a group and M be a cancellative semigroup without an identity element. Let M^1 be a monoid with an identity adjoined and $S = G \times M^1$ be the direct product of G and M^1. $U(S) = (G, 1)$ is the unitary subgroup. It is easy to check that the $\mathcal{H}$-, $\mathcal{L}$-, $\mathcal{R}$-, $\mathcal{D}$-classes of S are $\{(G, m) : m \in M^1\}$. Hence, S is a normal cancellative monoid with characteristic subgroup $U(S)$.*

There are also cancellative monoids that cannot be normal.

Example 3.5. *Let*

$$S = \left\{ M \in \mathbb{R}^{2\times 2} \,\middle|\, M = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix}, x \in \mathbb{Z}^+, y \in \mathbb{Z}^+, z \in \mathbb{Z} \right\}.$$

S forms a monoid under the matrix multiplication. For any $M \in S$, the determinant $|M| > 0$, M is invertible, which keeps the law of cancellation in S. In fact, S can be embedded into the general linear group $GL(2, \mathbb{R})$. A routine calculation following Proposition 2.1 and Lemma 2.2 can check these facts below:

- $U = \left\{ M \,\middle|\, M = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, z \in \mathbb{Z} \right\}$ *is the unitary subgroup of S which is commutative;*
- $\begin{pmatrix} x_1 & z_1 \\ 0 & y_1 \end{pmatrix} \mathcal{L} \begin{pmatrix} x_2 & z_2 \\ 0 & y_2 \end{pmatrix}$ *if, and only if, $x_1 = x_2, y_1 = y_2$ and $z_1 \equiv z_2 \mod y_1$;*
- $\begin{pmatrix} x_1 & z_1 \\ 0 & y_1 \end{pmatrix} \mathcal{R} \begin{pmatrix} x_2 & z_2 \\ 0 & y_2 \end{pmatrix}$ *if, and only if, $x_1 = x_2, y_1 = y_2$ and $z_1 \equiv z_2 \mod x_1$;*
- *For any $M = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \in S$,*

$$G_M = \left\{ G \in \mathbb{R}^{2\times 2} \,\middle|\, G = \begin{pmatrix} 1 & k\,\mathrm{lcm}(x, y)/y \\ 0 & 1 \end{pmatrix},\ k \in \mathbb{Z} \right\},$$

$$N_M = \left\{ H \in \mathbb{R}^{2\times 2} \,\middle|\, H = \begin{pmatrix} 1 & k\,\mathrm{lcm}(x, y)/x \\ 0 & 1 \end{pmatrix},\ k \in \mathbb{Z} \right\},$$

where $\mathrm{lcm}(x, y)$ means the least common multiple of x and y. D_M is normal if, and only if, $x = y$, and in this case, the characteristic subgroup of D_M is U.

Definition 3.6. *Let S be a cancellative monoid, then S is called **fundamental** if $\mathcal{H} \bigcap (S_0 \times S_0) = \iota_{S_0}$ (the identity relation on S_0).*

Proposition 3.7. *Any fundamental cancellative monoid is normal and the characteristic subgroup is $\{1\}$.*

Proof. Let S be a fundamental cancellative monoid and $r \in S_0$, then

$$G_r r = r N_r = H_r = \{r\}$$

by Lemma 2.2(6), and

$$G_r = \{1\} = N_r$$

since S is cancellative. Thus, S is a normal cancellative monoid with characteristic subgroup $\{1\}$. □

Proposition 3.8. *If S is a normal cancellative monoid with characteristic subgroup G, then the relation*

$$a\rho b \Leftrightarrow a = gb = bh$$

for some $g, h \in G$ is a congruence on S, such that S/ρ is a fundamental cancellative monoid with unitary subgroup U/G and characteristic subgroup $\{1\rho\}$.

Proof. We first show that ρ is a congruence. Let $x_i, y_i \in S$ with $i = 1, 2$ such that $x_1\rho y_1$ and $x_2\rho y_2$. We have

$$x_1 = g_1y_1 = y_1h_1$$

and

$$x_2 = g_2y_2 = y_2h_2$$

for some $g_i, h_i \in G$. Thus,

$$\begin{aligned} x_1x_2 &= g_1y_1g_2y_2 = g_1uy_1y_2 && \text{(for some } u \in G) \\ &= y_1h_1y_2h_2 = y_1y_2vh_2 && \text{(for some } v \in G) \end{aligned}$$

and, hence, $x_1x_2\rho y_1y_2$. This proves that ρ is a congruence on S.

Furthermore, if $a\rho\ b\rho = a\rho\ c\rho$, then $ab = acg$ for some $g \in G$. Hence, $b = cg$ and $b\rho = c\rho$. This shows that S/ρ satisfies the left cancellative law. Dually, S/ρ satisfies the right cancellative law. Thus, S/ρ is a cancellative monoid. We can also check the fact that

$$U(S/\rho) = U/G.$$

Finally, let $a, b \in S_0$ such that $a\rho\ \mathcal{H}\ b\rho$, then by Proposition 2.1 there exists $g, h \in U$ such that $a\rho = b\rho\ g\rho = h\rho\ b\rho$. This means that $a = bgp$ and $a = qhb$ for some $p, q \in G$. Hence, $a\mathcal{H}b$. On the other hand, since $H_b = Gb = H_a$, we have $m \in G$ such that $a = mb$. Thus, $mb = a = qhb$ and $m = qh$ since S is cancellative. Hence, $h = q^{-1}m \in G$ and, similarly, $g \in G$. Clearly, $qh \in G$ and $a\rho = b\rho$. We have now proved that

$$\mathcal{H} \bigcap ((S/\rho)_0 \times (S/\rho)_0 = \iota_{(S/\rho)_0}.$$

Thus, S/ρ is fundamental. □

It is easy to see that $a\rho b \Leftrightarrow a\mathcal{H}b$ when $a, b \in S_0$, by Lemma 2.2. In the remainder of this section, we shall establish the construction theorem of normal cancellative monoids.

Consider

G:	a group with identity e.
M:	a fundamental cancellative monoid with identity 1.
$Aut(G)$:	the group of automorphisms of G.
φ:	a mapping of M into $Aut(G)$ defined by: $m \mapsto \varphi_m$.
$P = (p_{ij})$:	a $M \times M$-matrix with entries in G.

The above quadruple $(G, M; \varphi, P)$ is called an *NCM-system* if the following conditions hold:

(NCM1) φ_1 is the identical mapping on G.

(NCM2) for all $i \in M$, $p_{1i} = p_{i1} = e$.

(NCM3) for all $i, j, k \in M$ and $g \in G$, $p_{i,jk}(g\varphi_{jk})p_{jk} = p_{ij,k}(p_{ij}\varphi_k)(g\varphi_j \circ \varphi_k)$.

Given an NCM-system $(G, M; \varphi, P)$, form the set

$$NCM = NCM(G, M; \varphi, P) = M \times G$$

and define a multiplication by

$$(x, g) \star (y, h) = (xy, p_{xy}(g\varphi_y)h).$$

Clearly, $\star$ is well defined.

Lemma 3.9. *$(NCM, \star)$ is a normal cancellative monoid.*

Proof. Let $(x, g), (y, h), (z, m) \in NCM$, then by (NCM3),

$$\begin{aligned}((x, g) \star (y, h)) \star (z, m) &= (xy, p_{xy}(g\varphi_y)h) \star (z, m)\\ &= (xyz, p_{xy,z}(p_{xy}((g\varphi_y)h)\varphi_z)m)\\ &= (xyz, p_{xy,z}(p_{xy}\varphi_z)(g\varphi_y \circ \varphi_z)(h\varphi_z)m)\\ &= (xyz, p_{x,yz}(g\varphi_{yz})p_{yz}(h\varphi_z)m)\\ &= (x, g) \star (yz, p_{yz}(h\varphi_z)m)\\ &= (x, g) \star ((y, h) \star (z, m)),\end{aligned}$$

and $\star$ satisfies the associative law. Hence, $(S, \star)$ is a semigroup. By (NCM1) and (NCM2), a routine calculation can show that $(1, e)$ is the identity of NCM. Thus, $(NCM, \star)$ is a monoid.

Next, we prove that NCM satisfies the cancellative law. For this, let

$$(x, g), (y, h), (z, m) \in NCM.$$

- If
$$(x, g) \star (y, h) = (x, g) \star (z, m),$$
then $xy = xz$ and
$$p_{xy}(g\varphi_y)h = p_{xz}(g\varphi_z)m.$$
The prior formula implies that $y = z$ since M is a cancellative monoid. From this and the latter formula, we have $h = m$ since G is a group.
- If
$$(x, g) \star (y, h) = (z, m) \star (y, h),$$
then $xy = zy$ and
$$p_{xy}(g\varphi_y)h = p_{zy}(m\varphi_y)h.$$
By the prior formula, we have $x = z$. Moreover, by the latter formula, we have $g\varphi_y = m\varphi_y$, and so $g = m$ since φ_y is an isomorphism.

This shows that NCM satisfies the cancellative law. Therefore, NCM is a cancellative monoid with

$$U(NCM) = U(M) \times G,$$

since for all

$$(x, g), (y, h) \in NCM, \quad (x, g) \star (y, h) = (xy, p_{xy}(g\varphi_y)h) = (1, e),$$

if, and only if, $y = x^{-1}$ and

$$h = (g\varphi_x)^{-1} p_{xy}^{-1}.$$

It remains to verify that $\{1\} \times G$ is the characteristic subgroup of D_r for all $r \in NCM$. Now, let $(x, g) \in NCM$, where x is not a unit. Since M is a fundamental cancellative monoid, by the proof of Proposition 3.7, we have $N_x = G_x = \{1\}$ in M. For any

$$(m, h), (m', h') \in U(NCM),$$

if

$$(m, h) \star (x, g) = (x, g) \star (m', h'),\ \ (mx, p_{mx}(h\varphi_x)g) = (xm', p_{xm'}(g\varphi_{m'})h')$$

gives that $mx = xm'$, which means

$$m = m' = 1,\ \ p_{mx} = p_{xm'} = e,\ \ g\varphi_{m'} = g$$

and

$$G_{(x,g)} \subset \{1\} \times G$$

in NCM. We can also take

$$h' = g^{-1}(h\varphi_x)g$$

or

$$h = (gh'g^{-1})\phi_x^{-1}$$

for any $h, h' \in G$ to keep the product equality; thus,

$$G_{(x,g)} = N_{(x,g)} = \{1\} \times G.$$

On the other hand, it is easy to see that $\{1\} \times G$ is a normal subgroup of $U(NCM)$. We have now proved that NCM is a normal cancellative monoid. □

We now arrive at the main result of this section.

Theorem 3.10. *If $(G, M; \varphi, P)$ is an NCM-system, then $NCM(G, M; \varphi, P)$ is a normal cancellative monoid with the characteristic subgroup isomorphic to G. Conversely, any normal cancellative monoid can be constructed in this way.*

Proof. We need only to verify the second part. Let S be a normal cancellative monoid with identity 1 and the characteristic subgroup G. By Proposition 3.8, S/ρ is a fundamental cancellative monoid. Denote by r_x a fixed representative of the ρ-class of S containing x. Put

$$N = \{r_x :\ x \in S\}$$

with $r_1 = 1$. Define a multiplication on N by

$$r_x \otimes r_y = r_{xy}.$$

It is easy to see that $(N, \otimes)$ is a semigroup isomorphic to S/ρ.

For all $r_x, r_y \in N$, we have a unique $g \in G$ such that $r_x r_y = r_{xy} g$. Now, we define $q_{r_x,r_y} = g$ and form a $N \times N$-matrix $Q = (q_{r_x,r_y})$ with entries $q_{r_x,r_y} \in G$.

By the definition of characteristic subgroup, for all $g \in G$, there exists a unique $h_{r_x} \in G$ such that $gr_x = r_x h_{r_x}$. Now, define

$$\psi_{r_x} : G \to G; \quad g \mapsto h_{r_x}.$$

By Lemma 2.2(1), ψ_{r_x} is an isomorphism of G onto itself. Define a mapping

$$\psi : N \to Aut(G); \quad r_x \mapsto \psi_{r_x}.$$

Now, we can form the quadruple $(N, G; \psi, Q)$. In fact, $(N, G; \psi, Q)$ is an NCM-system. Clearly, ψ_{r_1} is the identical mapping; that is, condition (NCM1) holds. Since $r_x = r_1 r_x = r_x q_{r_1,r_x}$, cancellativity implies $q_{r_1,r_x} = 1$, the identity element of G. It's the same way to show that $q_{r_x,r_1} = 1$. Accordingly, condition (NCM2) holds. Now, let $r_x, r_y, r_z \in N$ and $g \in G$, then

$$\begin{aligned}
[(r_x \otimes r_y) \otimes r_z] q_{(r_x \otimes r_y),r_z} (q_{r_x,r_y} \psi_{r_z})(g\psi_{r_y} \circ \psi_{r_z}) &= (r_x \otimes r_y) r_z [(q_{r_x,r_y}(g\psi_{r_y}))\psi_{r_z}] \\
&= (r_x \otimes r_y) q_{r_x,r_y} (g\psi_{r_y}) r_z \\
&= r_x r_y (g\psi_{r_y}) r_z = r_x g r_y r_z \\
&= r_x g (r_y \otimes r_z) q_{r_y,r_z} \\
&= r_x (r_y \otimes r_z)(g\psi_{(r_y \otimes r_z)}) q_{r_y,r_z} \\
&= [r_x \otimes (r_y \otimes r_z)] q_{r_x,(r_y \otimes r_z)} (g\psi_{(r_y \otimes r_z)}) q_{r_y,r_z}
\end{aligned}$$

and, thus,

$$q_{(r_x \otimes r_y),r_z} (q_{r_x,r_y} \psi_{r_z})(g\psi_{r_y} \psi_{r_z}) = q_{r_x,(r_y \otimes r_z)} (g\psi_{(r_y \otimes r_z)}) q_{r_y,r_z}.$$

This means that condition (NCM3) is satisfied and $(N, G; \psi, Q)$ in fact is an NCM-system.

It remains to verify that the mapping

$$\begin{aligned}
\theta : S &\to NCM(N, G; \psi, Q), \\
s &\mapsto (r_s, h_s),
\end{aligned}$$

where $s = r_s h_s$ for $h_s \in G$, is a semigroup isomorphism. Undoubtedly, θ is well defined and injective. For all $(x, h) \in N \times G$, by the definition of θ, we have $r_{xh} = x$ and, hence, $h_{xh} = h$ since S is a cancellative monoid. This means that $(xh)\theta = (x, h)$. Thus, θ is surjective. Now, let $s, t \in S$, then

$$\begin{aligned}
(s\theta) \star (t\theta) &= (r_s, h_s) \star (r_t, h_t) \\
&= (r_{st}, q_{r_s,r_t} (h_s \psi_{r_t}) h_t) \\
&= (r_{st} q_{r_s,r_t} (h_s \psi_{r_t}) h_t)\theta \\
&= (r_s r_t (h_s \psi_{r_t}) h_t)\theta \\
&= (r_s h_s r_t h_t)\theta \\
&= (st)\theta
\end{aligned}$$

and θ is a homomorphism. We have now proved that θ is a semigroup isomorphism, as required. □

Example 3.11. *Let*

$$S = \left\{ M \in \mathbb{R}^{2\times 2} \middle| M = \begin{pmatrix} x & z \\ 0 & x \end{pmatrix}, x \in \mathbb{Z}^+, z \in \mathbb{Z} \right\}.$$

It has been pointed out in Example 3.5 *that S forms a commutative cancellative monoid under the matrix multiplication.*

$$G = \left\{ M \middle| M = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, z \in \mathbb{Z} \right\}$$

is the unitary subgroup. S is a normal cancellative monoid and the characteristic subgroup is G. In fact, $\begin{pmatrix} x_1 & z_1 \\ 0 & x_1 \end{pmatrix} \mathcal{H} \begin{pmatrix} x_2 & z_2 \\ 0 & x_2 \end{pmatrix}$ if, and only if, $x_1 = x_2$ and $z_1 \equiv z_2 \mod x_1$. We can construct an NCM-system following the proof of Theorem 3.10. *For any*

$$M = \begin{pmatrix} x & a \\ 0 & x \end{pmatrix} \in S,$$

select

$$r_M = \begin{pmatrix} x & \mod (a, x) \\ 0 & x \end{pmatrix}$$

as the fixed representative of the ρ-class of S containing M. In the semigroup $(N, \otimes)$,

$$N = \left\{ M \in \mathbb{R}^{2\times 2} \middle| M = \begin{pmatrix} x & z \\ 0 & x \end{pmatrix}, x \in \mathbb{Z}^+, 0 \le z < x, z \in \mathbb{Z} \right\}$$

and for any $T = \begin{pmatrix} y & b \\ 0 & y \end{pmatrix} \in S$,

$$r_M \otimes r_T = \begin{pmatrix} xy & \mod (ay + bx, xy) \\ 0 & xy \end{pmatrix}.$$

Moreover, ψ_{r_M} is always the identical mapping on G since S is commutative. The entries q_{r_M, r_T} of $N \times N$-matrix $Q = (q_{r_M, r_T})$ satisfy

$$r_M r_T = (r_M \otimes r_T) q_{r_M, r_T}.$$

Conditions (NCM1)–(NCM3) for the NCM-system $(N, G; \psi, Q)$ have also been checked in the proof of Theorem 3.10. *For the isomorphism θ, $(M)\theta = (r_M, g)$, where we can find $g \in G$ such that $M = r_M g$ because $\mod (a, x) \equiv a \mod x$.*

4. The torsion extensions

The previous section shows that the structure of a cancellative monoid S is largely determined by its subgroup consisting of all the units, which is called the unitary subgroup of S. Two kinds of isomorphic subgroups of the unitary subgroup are introduced to investigate the structure of the $\mathcal{D}$-classes of S. It is a difficult problem for us to make it clear how these subgroups affect the overall structural properties of the cancellative monoid unless there are additional special conditions in place. Specifically, when all these subgroups are completely consistent with a normal subgroup of the unitary subgroup, we call S a normal cancellative monoid. We can already construct a normal cancellative monoid by combining a fundamental cancellative monoid with a group using an NCM-system and vice versa. However, NCM-systems are complex and not intuitive enough to some extent. A new and more simple way to characterize a normal cancellative monoid S directly is introduced in this section, when the characteristic subgroup of S has a trivial center.

Suppose that G is a group. For each element $g \in G$, the mapping φ_g defined by $x \mapsto g^{-1}xg$ from G onto G is an *inner automorphism* of G. For any $g, h \in G$, $\varphi_g = \varphi_h$ if, and only if, $g = h$ when the center of G is trivial. All the inner automorphisms consist of a normal subgroup of the automorphism group $Aut(G)$. We denote by $Inn(G)$ the group of inner automorphisms of G. The quotient group $Aut(G)/Inn(G)$ is called the *outer automorphism group* of G. We can recall from the last section the map $f_a \colon G_a \to N_a; g \mapsto h$, where $ga = ah$ $(g, h \in G)$ is an isomorphism.

Lemma 4.1. *Let $a, b \in S$ and $a\mathcal{H}b$. If a, b are in a normal $\mathcal{D}$-class of S with characteristic subgroup G, then*

$$f_a = f_b \circ \varphi_h = \varphi_g \circ f_b,$$

where $a = gb = bh, g, h \in G$.

Proof. It follows from Lemma 2.2 that there exists $g, h \in G$ such that $a = gb = bh$. For any $x \in G$, we have

$$\begin{aligned} a(xf_a) &= xa = xbh \\ &= b(xf_b)h = bhh^{-1}(xf_b)h \\ &= a(x(f_b \circ \varphi_h)). \end{aligned}$$

Consequently, $f_a = f_b \circ \varphi_h$. Similarly, $f_a = \varphi_g \circ f_b$. □

Proposition 4.2. *Let S be a normal cancellative monoid with characteristic subgroup G_0, then the mapping Ψ from $S_1 = S/\rho$ into $Aut(G_0)/Inn(G_0)$ defined by*

$$\Psi : S_1 \to Aut(G_0)/Inn(G_0); \qquad a\rho \mapsto f_a Inn(G_0)$$

is a homomorphism. Moreover, if $a \in G_0$, in this case $f_a = \varphi_a$.

Proof. By Lemma 4.1, it is easy to see that Ψ is well defined. Now, let $a, b \in S$, then for all $g \in G_0$,

$$\begin{aligned} ab(gf_{ab}) &= gab = a(gf_a)b \\ &= ab(g(f_a \circ f_b)); \end{aligned}$$

that is,

$$f_{ab} = f_a \circ f_b. \tag{4.1}$$

Consequently,

$$\begin{aligned} ((a\rho)(b\rho))\Psi &= f_{ab} Inn(G_0) \\ &= f_a Inn(G_0) f_b Inn(G_0) \\ &= (a\rho)\Psi(b\rho)\Psi. \end{aligned}$$

The rest of the proof is trivial. □

Next, we introduce the concept of torsion extensions. Let T be a fundamental cancellative monoid with unitary subgroup G_1. Let G_0 be a group with trivial center. Denote by $\{f_\lambda Inn(G_0) : \lambda \in \Lambda\}$ the outer automorphism group $Aut(G_0)/Inn(G_0)$, where f_λ are fixed elements in $Aut(G_0)$ and the cosets of $Inn(G_0)$ in $Aut(G_0)$ is indexed by Λ. We assume that the representative of the coset $Inn(G_0)$ itself is id_{G_0}, the identity automorphism of G_0. Let Fd: $T \to Aut(G_0)/Inn(G_0)$ be a homomorphism. For $t \in T$ write λ_t for the element of Λ such that $tF = f_{\lambda_t} Inn(G_0)$.

Additionally, we know that the identity element of $Aut(G_0)/Inn(G_0)$ is $Inn(G_0)$. Put

$$TE = TE(T, G_0; F) = T \times G_0$$

and define a multiplication on TE by

$$(t, g) * (s, h) = (ts, n),$$

where n is contained in G_0 such that

$$f_{\lambda_t} \circ \varphi_g \circ f_{\lambda_s} \circ \varphi_h \in f_{\lambda_t} \circ f_{\lambda_s} Inn(G_0) = f_{\lambda_{ts}} Inn(G_0)$$

and

$$f_{\lambda_t} \circ \varphi_g \circ f_{\lambda_s} \circ \varphi_h = f_{\lambda_{ts}} \circ \varphi_n.$$

Proposition 4.3. *$S = (TE, *)$ is a normal cancellative monoid with unitary subgroup $(G_1 \times G_0, *)$ and characteristic subgroup $(1 \times G_0, *)$.*

Proof. First, we show that $(S, *)$ is a cancellative monoid. Now, we need to verify that $*$ is associative. Let $(t, g), (s, h), (x, n) \in S$. Suppose that

$$((t, g) * (s, h)) * (x, n) = (ts, u) * (x, n) = (tsx, v) \tag{4.2}$$

with $u, v \in G_0$ and that

$$(t, g) * ((s, h) * (x, n)) = (t, g) * (sx, k) = (tsx, l) \tag{4.3}$$

with $k, l \in G_0$. It is sufficient for us to show that $v = l$. By Eq (4.2),

$$f_{\lambda_t} \circ \varphi_g \circ f_{\lambda_s} \circ \varphi_h = f_{\lambda_{ts}} \circ \varphi_u \tag{4.4}$$

and

$$f_{\lambda_{ts}} \circ \varphi_u \circ f_{\lambda_x} \circ \varphi_n = f_{\lambda_{tsx}} \circ \varphi_v. \tag{4.5}$$

Composing Eqs (4.4) and (4.5) , we get

$$f_{\lambda_t} \circ \varphi_g \circ f_{\lambda_s} \circ \varphi_h \circ f_{\lambda_x} \circ \varphi_n = f_{\lambda_{tsx}} \circ \varphi_v. \tag{4.6}$$

On the other hand, Eq (4.3) implies that

$$f_{\lambda_s} \circ \varphi_h \circ f_{\lambda_x} \circ \varphi_n = f_{\lambda_{sx}} \circ \varphi_k$$

and

$$f_{\lambda_t} \circ \varphi_g \circ f_{\lambda_{sx}} \circ \varphi_k = f_{\lambda_{tsx}} \circ \varphi_l.$$

Hence, we have

$$f_{\lambda_t}\circ\varphi_g\circ f_{\lambda_s}\circ\varphi_h\circ f_{\lambda_x}\circ\varphi_n = f_{\lambda_{tsx}}\circ\varphi_l. \tag{4.7}$$

Compare Eqs (4.6) and (4.7), and we have

$$f_{\lambda_{tsx}}\circ\varphi_v = f_{\lambda_{tsx}}\circ\varphi_l$$

and $\varphi_v = \varphi_l$; thus $v = l$, as required.

Now, we show that S is cancellative. If

$$(t,g)*(s,h) = (x,n)*(s,h),$$

then $ts = xs$; hence, $t = x$ and

$$f_{\lambda_t}\circ\varphi_g\circ f_{\lambda_s}\circ\varphi_h = f_{\lambda_x}\circ\varphi_n\circ f_{\lambda_s}\circ\varphi_h.$$

Thus, $\varphi_g = \varphi_n$ and $g = n$, so that $(t,g) = (x,n)$. This means that S is right cancellative. Dually, we can show that $(S,*)$ is also left cancellative.

We show next that $(1,e)$ is the identity element of S, where e is the identity element of G_0 and $1F = Inn(G_0)$. Suppose that

$$(t,g)*(1,e) = (t,u)$$

for some $u \in G_0$, then

$$f_{\lambda_t}\circ\varphi_g\circ f_{\lambda_1}\circ\varphi_e = f_{\lambda_t}\circ\varphi_u.$$

Hence, $u = g$ and

$$(t,g)*(1,e) = (t,g).$$

In other words, $(1,e)$ is a left identity of S. Dually, we can prove that $(1,e)$ is a right identity of S. Thus, we have proved that $(S,*)$ is a cancellative monoid with identity $(1,e)$.

If

$$(t,g)*(s,h) = (1,e),$$

then $ts = 1$ and $t,s \in G_1$. Conversely, for all $(u,g) \in G_1\times G_0$, we have

$$(u,g)*(u^{-1},h^{-1}v^{-1}) = (1,e),$$

where $h,v \in G_0$ such that

$$\varphi_g\circ f_{\lambda_{u^{-1}}} = f_{\lambda_{u^{-1}}}\circ\varphi_h$$

and

$$f_{\lambda_u}\circ f_{\lambda_{u^{-1}}} = \varphi_v \in Inn(G_0).$$

Hence, $G_1\times G_0$ is the unitary subgroup of S.

It remains to show that $1\times G_0$ is the characteristic subgroup of S. For all

$$(s,g) \in S_0 = S \setminus (G_1\times G_0),$$

if $(u,h) \in N_{(s,g)}$, then $u \in N_s = \{1\}$, showing that $N_{(s,g)} \subset 1\times G_0$. Similarly, we can obtain $G_{(s,g)} \subset 1\times G_0$. Conversely, for all $(1,u) \in 1\times G_0$, it is easy to check that there exists $(1,v) \in 1\times G_0$ such that

$$(1,u)*(s,g) = (s,g)*(1,v),$$

where $v = g^{-1}t$ and

$$\varphi_u \circ f_{\lambda_s} = f_{\lambda_s} \circ \varphi_t.$$

That is, $1 \times G_0 \subset G_{(s,g)}$. There also exists $(1, h) \in 1{\times}G_0$ such that

$$(s, g) * (1, u) = (1, h) * (s, g),$$

where

$$\varphi_h \circ f_{\lambda_s} = f_{\lambda_s} \circ \varphi_{gug^{-1}}.$$

That is, $1 \times G_0 \subset N_{(s,g)}$. Consequently, $1{\times}G_0$ is the characteristic subgroup of S. □

Definition 4.4. *We shall call the above semigroup $(TE, *)$ the* ***torsion extension*** *of the fundamental cancellative monoid T by the group G_0.*

We conclude this section by proving that any normal cancellative monoid whose characteristic subgroup has a trivial center can be constructed as the torsion extension of a fundamental cancellative monoid by a group. In the remainder of this section, we always assume that S is a normal cancellative monoid with unitary subgroup U and characteristic subgroup G having a trivial center. Let ρ be the same as in Proposition 3.8, in which it was shown that S/ρ is a fundamental cancellative monoid with unitary subgroup U/G.

Recall the semigroup $(N, \otimes)$ in the proof of Theorem 3.10, where r_x is a fixed representative of the ρ-class of S containing x, $N = \{r_x : x \in S\}$ with $r_1 = 1$, $r_x \otimes r_y = r_{xy}$ and $(N, \otimes)$ is isomorphic to S/ρ. It follows from Proposition 4.2 that there exists a homomorphism Ψ from N into the outer automorphism group $Aut(G)/Inn(G)$ such that $r_x \mapsto f_{r_x} Inn(G)$. We can form the the torsion extension $TE(N, G; \Psi)$ of N by G.

Define

$$\Phi : TE \to S;\ \ (r_x, g) \mapsto r_x g.$$

It is routine to check that Φ is a bijection by Lemma 2.2. If

$$(r_x, g) * (r_y, h) = (r_z, u),$$

we claim that $r_z = r_x r_y v$ with some $v \in G$. Otherwise, by the definition of $*$, we have

$$f_{r_x} \circ \varphi_g \circ f_{r_y} \circ \varphi_h = f_{r_z} \circ \varphi_u.$$

Furthermore, for all $k \in G$,

$$\begin{aligned} kr_x gr_y h &= r_x gr_y h(k(f_{r_x} \circ \varphi_g \circ f_{r_y} \circ \varphi_h)) \\ &= kr_x r_y vv^{-1}(gf_{r_y})h = kr_x r_y v \cdot p, \quad \text{(here } p = v^{-1}(gf_{r_y})h) \\ &= r_x r_y vp(k(f_{r_x} \circ f_{r_y} \circ \varphi_v \circ \varphi_p)) \\ &= r_x gr_y h(k(f_{r_z} \circ \varphi_p) \\ &= r_x gr_y h(k(f_{r_z} \circ \varphi_u). \end{aligned}$$

Hence,

$$f_{r_z} \circ \varphi_p = f_{r_z} \circ \varphi_u \ \text{ and } \ u = v^{-1}(gf_{r_y})h.$$

It follows that

$$\begin{aligned} r_z u &= ((r_x, g) * (r_y, h))\Phi \\ &= r_z v^{-1}(g f_{r_y})h = r_x r_y (g f_{r_y})h \\ &= r_x g r_y h = (r_x, g)\Phi(r_y, h)\Phi. \end{aligned}$$

Thus, Φ is actually an isomorphism. Indeed, we have proved the following theorem.

Theorem 4.5. *Let T be a fundamental cancellative monoid and G a group with a trivial center. If F is a homomorphism of T into the outer automorphism group $Aut(G)/Inn(G)$, the torsion extension $(TE(T, G; F), *)$ of T by the group G is a normal cancellative monoid. Conversely, any normal cancellative monoid whose characteristic subgroup has a trivial center is isomorphic to the torsion extension $(TE, *)$ of a fundamental cancellative monoid and a group.*

Example 4.6. *The symmetric group S_4 is the group of permutations of $\{1, 2, 3, 4\}$. The alternating group A_4 is the subgroup of S_n consisting of even permutations. Any 4×4 permutation matrix E_σ can be obtained from identity matrix $E = (e_1, \cdots, e_4)$ by $E_\sigma = (e_{\sigma(1)}, \cdots, e_{\sigma(4)})$ for some $\sigma \in S_4$, where $\{e_1, \cdots, e_4\}$ are the columns of E. E_σ is called even (odd) when σ is even (odd). Let*

$$G = \{E_\tau | \tau \in A_4\} \;\; and \;\; T = \{aE_\sigma | a > 1, a \in \mathbb{Z}, \sigma \in S_4\}.$$

It is easy to verify that $S = G \cup T$ forms a cancellative monoid under the matrix multiplication and the unitary subgroup $U(S) = G$ having a trivial center. For any

$$A = aE_\sigma \in T, \;\; E_\tau \in G, \;\; E_\tau A = A \cdot E_\sigma^{-1} E_\tau E_\sigma = AE_{\tau'},$$

where $E_\sigma^{-1} E_\tau E_\sigma$ is even and coincides with $E_{\tau'}$ for some $\tau' \in A_4$. This shows that S is normal and the characteristic subgroup is G. We can choose a fixed representative of the ρ-class of S containing A as $r_A = aE_{\sigma_A}$, where $\sigma_A = (1)$ if E_σ is even and $\sigma_A = (12)$ if E_σ is odd. S/ρ is isomorphic to the semigroup of $\{1, \pm 2, \pm 3, \cdots\}$ under integer multiplication. For any

$$r_B = bE_{\sigma_B} \in (N, \otimes), \;\; r_A \otimes r_B = r_{AB} = r_{abE_{\sigma_A \sigma_B}},$$

the isomorphism $f_{r_A} \in Aut(G)$ can be written as

$$(E_\tau) f_{r_A} = E_{\sigma_A}^{-1} E_\tau E_{\sigma_A}.$$

Let Ψ be the homomorphism from N into the outer automorphism group $Aut(G)/Inn(G)$ derived from Proposition 4.2, then

$$\Psi(r_A) = \begin{cases} Inn(G), & \text{if } E_{\sigma_A} \text{is even}, \\ f_{(12)} Inn(G), & \text{if } E_{\sigma_A} \text{is odd}, \end{cases}$$

where

$$(E_\tau) f_{(12)} = E_{(12)}^{-1} E_\tau E_{(12)}.$$

The original semigroup S can be rebuilt by the torsion extension $(TE(N, G; \Psi), *)$ following Theorem 4.5.

5. Conclusions

We have made some new progress in the study of the structure of a cancellative monoid S. For any $a \in S$, $\mathcal{H}$-class of a can be obtained by multiplying a on the left or right side of certain subgroups of $U(S)$, which is the unitary subgroup of S. When these subgroups are all the same normal subgroup of $U(S)$, we can construct S under a uniform mode and we call S a normal cancellative monoid. Furthermore, if the normal subgroup has a trivial center, S can be characterized in a natural and intuitive way. It is worth paying attention to further related research on the use of these methods to characterize other types of monoids. How to find the correlation between characteristic subgroups of distinct normal $\mathcal{D}$-classes when S is not normal remains an open question.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflict of interest.

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