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Research article

Static term structure of interest rate construction with tension interpolation splines

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Abstract: Traditional theories of term structure of interest rate consist of four major classical theories, including Pure Expectation Theory, Liquidity Preference Theory, Preferred Habitat Theory and Market Segmentation Theory. However, they cannot be well interpreted by the traditional static term structure of interest rate methods such as polynomial spline and exponential spline. To address problems on low precision and weak stability of traditional methods in constructing static interest rate term structure curve, in this paper, we introduce the tension interpolation spline based on a fourth-order differential equation with local tension parameters calculated by Generalized Reduced Gradient (GRG) algorithm. Our primary focus is to illustrate its better prediction effect and stability with an empirical study conducted using datum of treasury bonds. Then, we divided the datum into intra-sample datum for estimating tension parameters and out-of-sample datum for evaluating their robustness of predicting stochastics collected from Shanghai Stock Exchange on 2nd February, 2019. According to the principle of total least squares and total least absolute deviations, the result shows that the tension interpolation spline model has better precision and stronger stability in prediction of out-of-sample treasury bonds prices compared with the model established by polynomial spline and exponential spline. In addition, it can better explain the Liquidity Preference Theory, which confirms that it is suitable for constructing the static term structure of interest rates in the securities exchange market.

Keywords: discount curves; yield curves; static term structure of interest rate; treasury bonds pricing; Generalized Reduced Gradient; tension interpolation spline
Mathematics Subject Classification: 65D05, 65D17

1. Introduction

The static construction of the term structure of interest rates is the foundation and starting point in quantitative studies on the term structure of interest rates. Many methods have been applied to fit the structure curves. McCulloch proposed using the method of least-squares regression techniques in piecewise polynomial splines, including quadratic polynomial splines in [1] and cubic polynomial splines in [2], to extract the discount function. In [3], Vasicek et al. outlined the approach of exponential spline to fit extract the discount function. Later, in [4], Nelson and Siegel used the differential equation of second-order to construct the parametrically parsimonious model that can characterize more shapes generally associated with yield curves than the exponential spline presented in [5], which has the ability to predict price of the long-term treasury bonds. To minimize the yield errors and price errors, in [6], Svensson added two parameters in the Nelson-Siegel model which can better extract the complex yield curves. Application of B-splines functions to linear regression estimation on interest securities was initially suggested by Shea [7], and proposed by Steeley [8] and Lin and Paxson [9], which was proved to fit sufficiently well. Since then, different approximation functions, especially the tension spline, was introduced to term structure estimation for well properties of local support and shape preservation that parameters hold provide flexible shape control. Barzanti and Corradi [10] presented the approach of tension spline used in a least-squares fitting to bonds, which implied that the parameters can provide smoothness control of the yield curve. Using generalized tension splines for curve construction, Andersen [11] proposed solutions to address convexity and locality issues of cubic interpolations to fit term structure of interest, which can construct best-fit curves on sets of noisy corporate or treasury bills in most general bonds and swap markets in reality. In [12], based on a convex optimization model with relatively simple parameters, the measurement of the interest rate curves for optimizing the term structure of interest rate was proposed. Recently, in [13], some applications of C^2 tension interpolation splines on the construction of interest rate term structure in Chinese financial market were given.

Our purpose of this paper is to make a construction of the static term structure of interest rate including discount curves and yield curves. Combined with the traditional theories of the interest rate term structure, we deliver an estimated pricing for the treasury bonds transacted in Shanghai Stock Exchange with a brand-new method of a kind of C^1 tension spline based on a fourth-order differential equation with local tension parameters $\tau_k, k = 1, 2, \cdots, n-1$ solved by a nonlinear optimization algorithm. With the set of suitable tension parameters minimizing the total squares and total absolute deviations of pricing errors, the tension spline has better accurate prediction effect and smoothness curve than cubic polynomial splines and exponential splines, the representatives of the traditional methods used to build the discount curve and yield curve in the empirical study.

The rest of this paper is organized as follows. Section 2 outlines the basic issues of financial market about the (static) term structure of interest rate curve construction and discusses the interrelationship based on a compatible mathematical model on the theory of the Efficient Market Hypotheses (EMH). In addition, we introduce the traditional theories of the term structure of interest rates for theoretical support. In Section 3, the model of tension splines with local tension parameters $\tau_k, k = 1, 2, \cdots, n-1$ based on the fourth-order differential equation is introduced to apply into the discount curve construction. Later, we discuss the mapping $x \to f$ at knots (x_k, f_k). Section 4 states the reasons of the suitable size of tension splines in constructing Chinese treasury bonds in Shanghai Stock

Exchange and provides a general nonlinear optimization algorithm, Generalized Reduced Gradient (GRG) [14], to estimate the local tension parameters in Section 3. Eventually, feasibility of the tension spline put forward in the paper is proven to be great through empirical study. Our conclusion is given in Section 5.

2. Financial market basics

The term structure of interest rates refers to the quantitative correlation between the interest rates and maturities of the treasury bonds with the same level of risk, which plays a key role in assets pricing, financial production designing, hedging and risk-management of financial instruments. Thus, it is vital to make an exact construction of pricing curves and interest rate curves. In this section, we first interpret the discount function and yield curve construction based on the Efficient Market Hypotheses (EMH) and later introduce the traditional theories of the term structure of interest rate in detail, sustained theory.

2.1. The discount and yield curve construction problem

Consider a risk-free cash flow of one RMB at time t. Its present value at time 0 can be denoted as $D(t)$ known as discount function of t, with a successive mapping $D : \mathbb{R}_+ \to [0, 1]$. It is a monotone decreasing function in an arbitrage free market with the beginning of $D(0) = 1$. However, the discount curve cannot be directly observed from a noisy market but can be deduced from the predictable prices of treasury bonds, which are functions of the discount curve with lower risk.

In fact, the predicable prices of securities can be expressed as a linear combination with a series of certain cash flows and a finite set of points on the discount curve. Given an ordinary coupon, for example, the coupon holder has a privilege to receive an non-random cash flow $c_1, c_2, \cdots, c_K$ at the interest payment days $t_1, t_2, \cdots, t_K (t_1 < t_2 < \cdots < t_K)$. Its estimated present value $\hat{P}_S$ must have,

$$\hat{P}_S = \sum_{i=1}^{K} c_i D(t_i). \tag{2.1}$$

Consider N ordinary coupons extendedly, the k^{th} security's present value can be rewritten as,

$$\hat{P}_{Sk} = \sum_{i=1}^{K} c_{ik} D(t_i), i = 1, 2 \cdots K, k = 1, 2, \cdots, N, \tag{2.2}$$

where c_i^k denotes the cash flow paid at time t_i.

Next, a set of points on the discount curve from Eq (2.2) can be expediently showed in an K-dimensional matrix discount rate vector as follow,

$$D = (D(t_1), D(t_2), \cdots D(t_K))^{\mathrm{T}}$$

at meanwhile, let N-dimensional matrix $\hat{P}$ and $N \times K$-dimensional matrix c denote the vector of $\hat{P} = (\hat{P}_{S1}, \hat{P}_{S2}, \cdots, \hat{P}_{SN})^{\mathrm{T}}$ and $c = \{c_{ik}\}$ including all the cash flows paid from these set of coupons respectively. The Eq (2.2) can be expressed as the following equation based on the starting point for the estimated discount rate vector D,

$$\hat{P} = cD. \tag{2.3}$$

Generally, if $N \geq K$, Eq (2.3) is allowed for a unique solution as well as seek for a set of solutions for $\hat{P}$ making the sum of square pricing errors between formula prices and the real prices at present time minimization. At any other conditions, while $N < K$, it has an infinite set of solutions for $\hat{P}$. Hence, it is necessary to construct a smooth discount rate curve at all continuous time t within a given interval $[a, b]$ covering its subinterval $[t_1, t_K]$ with the method of interpolation spline mentioned in the next chapter.

The yield function $r(t)$ is a basic exponential decreasing factor of $P(t)$ for the reason of the compounding interest. The following invertible transformation (Eq (2.4)) introduces the relationship between discount rate function $P(t)$ and the yield function $r(t)$: $r(t)$ is on a logarithmic transformation of the discount rate function $P(t)$ and $P(t)$ is on an exponential conversion of the yield function $r(t)$.

$$e^{-r(t)t} = D(t) \Leftrightarrow r(t) = -t^{-1} \ln D(t). \tag{2.4}$$

2.2. Traditional theory of the term structure of interest rates

Prior to introducing the traditional theories of the term structure of interest rate, it is necessary for us to introduce a significant hypothesis–Efficient Market Hypothesis (EMH), first officially proposed by Fama in [15], which indicates that if the current prices of assets in the capital market fully reflects all relevant available information, the market will be efficient. More specifically, if managers in the efficient market disclose certain information to all participants in the market, the prices of the securities won't be affected or fluctuate abnormally. It implies that any trader cannot obtain super-normal profit but normal profit on the basis of trading on information revealed in the market.

With the support of EMH, the traditional theories of the term structure of interest rate consisting of Pure Expectation Theory, Liquidity Preference Theory, Preferred Habitat Theory as well as Market Segmentation Theory are introduced.

First, the Pure Expectation Theory proposed by Irving Fisher, indicates that the forward rate fully represents the expected future interest rate, i.e., the geometric mean of the forward rate is equal to the spot rates. Thus, the term structure of interest rate at a certain period can reflect the current market expectations for future short-term interest rates. However, several risks like the reinvestment risk and price risk are not taken into account, rendering that the forward rate cannot be determined.

Subsequently, the Liquidity Preference Theory raised by John R. Hicks, reveals that the holders of long-term bonds will ask for a risk premium that exceeds the expectation for forward rate, which is proportional to the maturity period. As the maturity date increases from short to medium or even long term, will the risk premium increase slowly, making the discount rate function tend to a stable descending exponential function ($D(t) = \eta^t$), where $\eta < 1$. Nevertheless, the assumption that the risk premium increases monotonically with the maturity is inconsistent with the reality.

Furthermore, the Preferred Habitat Theory is similar to the Liquidity Preference Theory. However, the Preferred Habitat Theory believes that only when the investors cash their investment in a short period of time and they are eager to raise long-term loans, will the risk premium increase. Therefore, in reality, the investment institutions will determine the maturities of the bonds according to the characteristic of the liabilities. In addition, they must compensate the investors when they respectively choose diverse bonds with a certain risk premium determined by the investors' risk preferences.

Last but not least, the Market Segmentation Theory is also similar to the Preferred Habitat Theory. The ultimate difference from the Market Segmentation Theory to the Preferred Habitat Theory is that the investors and lenders in the market are not willing to change the investment period to speculate from future spreading on the theory of Market Segmentation while the latter is opposite. In other words, the yield curve is completely determined by the relationship between the supply and demand of securities with different maturities, and the interaction between the supply and demand of securities of different maturities is small. However, the flaw of this theory is that it believes the bond markets with different maturities are totally segmented, which cannot explain the fact that the yields of different bonds tend to change together.

In Section 4, we will combine the foregoing theories with the empirical study of the interest rate term structure to verify whether they are suitable to be explained by observing the shape of the static interest rate term structure curves.

3. Tension interpolation spline basics

Having now discussed major sections in the financial market basics issues, hereon base, we introduce a precise mathematical approach, i.e., splines with local tension parameters based on a fourth-order differential equation, in order to construct the static interest rate term structure curve with lower pricing error than traditional methods like polynomial spline functions and exponential spline functions.

3.1. Construction of Hermite-type interpolation basis functions

At first, we consider a C^1 tension interpolation spline $S(t)$ based on a class of fourth-order ordinary differential equation with the non-uniform tension parameters τ as follows,

$$S^{(4)}(t) - \tau^2 S^{(2)}(t) = 0 \tag{3.1}$$

which on each local interval $[t_k, t_{k+1}]$ is an element of a specific space span, $\{1, t, \sinh \tau t, \cosh \tau t\}$, having been first mentioned by Schweikert [16].

By directly computing the general solution of the differential equation, we obtain the following formula making prerequisites of constructing the Hermite-type interpolation basis functions,

$$S(t) = a_0 + a_1 t + a_2 \cosh \tau t + a_3 \sinh \tau t. \tag{3.2}$$

Then, considering the following four two-point boundary conditions problems,

$$\begin{cases} S(0) = \eta_{i0}, & S(1) = \eta_{i1}, \\ S'(0) = \eta_{i2}, & S'(1) = \eta_{i3}, \end{cases} \tag{3.3}$$

where $i = 0, 1, 2, 3$ and $\eta_{ij} = 1$ for $i = j$, $\eta_{ij} = 0$ for $i \neq j$. It is accessible to obtain the Hermite-type interpolation basis functions $B_i(t, \tau)$, i = 0, 1, 2, 3

$$B_i(t) = a_{i0} + a_{i1} t + a_{i2} \cosh \tau t + a_{i3} \sinh \tau t, \tag{3.4}$$

with

$$\begin{cases} a_{00} = \frac{\tau \sinh \tau - \cosh \tau + 1}{\tau \sinh \tau - 2\cosh \tau + 2}, \\ a_{01} = \frac{-\tau \sinh \tau}{\tau \sinh \tau - 2\cosh \tau + 2}, \\ a_{02} = \frac{-(\cosh \tau - 1)}{\tau \sinh \tau - 2\cosh \tau + 2}, \\ a_{03} = \frac{\sinh \tau}{\tau \sinh \tau - 2\cosh \tau + 2}, \end{cases}$$

and

$$\begin{cases} a_{10} = \frac{-(\cosh\tau-1)}{\tau\sinh\tau-2\cosh\tau+2}, \\ a_{11} = \frac{\tau\sinh\tau}{\tau\sinh\tau-2\cosh\tau+2}, \\ a_{12} = \frac{(\cosh\tau-1)}{\tau\sinh\tau-2\cosh\tau+2}, \\ a_{13} = \frac{-\sinh\tau}{\tau\sinh\tau-2\cosh\tau+2}, \end{cases}$$

and

$$\begin{cases} a_{20} = \frac{\tau\cosh\tau-\sinh\tau}{\tau(\tau\sinh\tau-2\cosh\tau+2)}, \\ a_{21} = \frac{-(\cosh\tau-1)}{\tau(\tau\sinh\tau-2\cosh\tau+2)}, \\ a_{22} = \frac{\sinh\tau-\tau\cosh\tau}{\tau(\tau\sinh\tau-2\cosh\tau+2)}, \\ a_{23} = \frac{\tau\sinh\tau-\cosh\tau+1}{\tau(\tau\sinh\tau-2\cosh\tau+2)}, \end{cases}$$

and

$$\begin{cases} a_{30} = \frac{\sinh\tau-\tau}{\tau(\tau\sinh\tau-2\cosh\tau+2)}, \\ a_{31} = \frac{1-\cosh\tau}{\tau(\tau\sinh\tau-2\cosh\tau+2)}, \\ a_{32} = \frac{\tau-\sinh\tau}{\tau(\tau\sinh\tau-2\cosh\tau+2)}, \\ a_{33} = \frac{\cosh\tau-1}{\tau(\tau\sinh\tau-2\cosh\tau+2)}. \end{cases}$$

3.2. Construction of C^1 interpolation basics functions

Given a data series $\{(x_k, f_k), k = 1, 2, \cdots, n\}$, we set n knots at (x_k, f_k), where $a = x_1 < x_2 \cdots \in< x_n <= b$ is a partition of interval $[a, b]$. f_k can be economically describe by the x_k in Nelson-Siegel model given by Svensson in [6], i.e.,

$$\ln f_k = -x_k\left[\beta_{0t} + \beta_{1t}\left(\frac{1-e^{-\lambda_t x_k}}{\lambda_t x_k}\right) + \beta_{2t}\left(\frac{1-e^{-\lambda_t x_k}}{\lambda_t x_k} - e^{-\lambda_t x_k}\right)\right], \tag{3.5}$$

where β_{0t} represents a long term factor used to measure bond interest rates, β_{1t} is a slope factor of short-term and long-term spread measuring short-term changes in bond interest rates, β_{2t} is a curvature factor representing the medium-term factors and λ_t is a restraining factor that controls the rate of decay.

Using the new four Hermite-type interpolation basis functions $B_i(t, \tau_k)$, i = 0, 1, 2, 3 given in Eq (3.4), we construct a kind of C^1 tension interpolation spline with a specific local tension parameter $\tau_k, k = 1, 2, 3 \ldots, n-1$ in different partitions of interval $[a, b]$ as follow,

$$S_k(x) = f_k B_0(t, \tau_k) + f_{k+1} B_1(t, \tau_k) + h_k d_k B_2(t, \tau_k) + h_k d_{k+1} B_3(t, \tau_k), \tag{3.6}$$

where $h_k = x_{k+1} - x_k$, $t = (x - x_k)/h_k$. d_k are the first derivative values at x_k respectively, which should be estimated by some approach methods beforehand. In our paper, the method we used is described as follow,

$$\begin{cases} d_1 = \Delta_1 - \frac{h_1}{h1+h2}(\Delta_2 - \Delta_1), \\ d_k = \frac{h_k}{h_{k-1}+h_k}\Delta_{k-1} + \frac{h_{k-1}}{h_{k-1}+h_k}\Delta_k, k = 2, 3, \cdots, n-1, \\ d_n = \Delta_{n-1} + \frac{h_{n-1}}{h_{n-2}+h_{n-1}}(\Delta_{n-1} - \Delta_{n-2}), \end{cases} \tag{3.7}$$

where $\Delta_i = (f_{k+1} - f_k)/h_k$, $k = 1, 2, 3\ldots, n-1$.

In addition, the tension spline $S(x)$ Eq (3.6) above is used to represent the discount function $D(t)$, where x_k is equivalent to t representing the maturities of treasury bonds and the local tension parameters $\tau_t, k = 1, 2, 3\ldots, n-1$ are used to deviation adjustment. When τ_t is inappropriate, we will obtain a low precision term structure curve of interest rates, and to get a more ideal result, we need to precisely estimate the tension parameters by an optimal path algorithm.

4. Estimation procedure

In order to construct the static term structure of interest rate precisely and calculate the parameters of the static interest rate term structure curve with the minimum price error, we use Eq (4.1).

$$\begin{aligned} &\underset{\tau_1,\tau_2,\ldots,\tau_{n-1},f_1,f_2,\ldots,f_n\in R}{\arg\min} \quad \textstyle\sum_{k=1}^{N}(\hat{P}_{Sk} - P_{Sk})^2, \\ &s.t. \quad 0 \le f_1, f_2, \ldots, f_n \le 1. \end{aligned} \tag{4.1}$$

By introducing relaxation variables, Eq (4.1) can be written as

$$\begin{aligned} &\underset{\tau_1,\ldots,\tau_{n-1},f_1,\ldots,f_n\in R,\varepsilon_1,\ldots,\varepsilon_{2n}\in R^+}{\arg\min} \quad \textstyle\sum_{k=1}^{N}(\hat{P}_{Sk} - P_{Sk})^2, \\ &s.t. \quad f_1 - \varepsilon_1 = 0, \ldots, f_n - \varepsilon_n = 0, \\ &\qquad f_1 + \varepsilon_{n+1} = 1, \ldots, f_n + \varepsilon_{2n} = 1. \end{aligned} \tag{4.2}$$

In this section, we use the GRG method [14] to solve the nonlinear programming problems and receive tension parameters with the minimum error in each interval $[x_k, x_{k+1}]$. To make it clearer, the algorithm of GRG algorithm and its usage in the parametric estimation as well as the empirical research are shown in this section.

4.1. Generalized Reduced Gradient algorithm

Equation (4.2) can be summarized as a general nonlinear optimization problem,

$$\begin{aligned} &\underset{x\in R^n}{\arg\min}\, f(x), \\ &s.t. \;\; c(x) = 0, \end{aligned} \tag{4.3}$$

where $c(x) = (c_1(x), \ldots, c_m(x))^T$. The variable x can be decomposed as

$$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}, \tag{4.4}$$

where $x_B \in R^m$, $x_N \in R^{n-m}$ and then the constraint condition can be easily written as

$$c(x_B, x_N) = 0. \tag{4.5}$$

Suppose we can get $x_B = \phi(x_N)$ from Eq (4.5), then Eq (4.3) is equivalent to the following unconstrained optimazation problem,

$$\underset{x_N\in R^{n-m}}{\arg\min}\, f(x_B, x_N) = f(\phi(x_N), x_N) = \hat{f}(x_N). \tag{4.6}$$

The reduced gradient on Lagrangian space can be expressed as

$$\hat{g}(x_N) = \frac{\partial}{\partial x_B}[f(x) - \lambda^T c(x)], \tag{4.7}$$

where λ satisfies

$$\frac{\partial f(x)}{\partial x_B} = \frac{\partial c^T(x)}{\partial x_B}\lambda. \tag{4.8}$$

We can construct search direction of Eq (4.6) using the reduced gradient

$$\bar{d}_k = -\hat{g}_k. \tag{4.9}$$

Based on the above conditions, we summarize the GRG algorithm to solve the general nonlinear optimization problem in Algorithm 1.

Algorithm 1 Framework of GRG algorithm.

1: Setting a feasible point x_1, $\epsilon \geq 0, \bar{\epsilon} \geq 0$, a positive integer M, and k: = 1 to start this algorithm;

2: Calculating

$$\frac{\partial c(x_k)^T}{\partial x} = \begin{bmatrix} A_B \\ A_N \end{bmatrix}, \tag{4.10}$$

where the division makes A_B is non-singular; And by solving λ from Eq (4.8), we can derive $\hat{g}_k$ from Eq (4.7);

3: The program would be terminated if $\|\hat{g}_k\| \leq \epsilon$. Otherwise setting the direction $\bar{d}_k = -\hat{g}_k$ and search step $\alpha = \alpha_k^{(0)} > 0$;

4: Set $j := 0$ and update the old x_k,

$$\begin{aligned} x_N &= (x_k)_N + \alpha \bar{d}_k, \\ x_B &= (x_k)_B; \end{aligned} \tag{4.11}$$

5: Updating

$$x_B = x_B - A_B^{-T} c(x_B, x_N), \tag{4.12}$$

and calculating $c(x_B, x_N)$. If $\|c(x_B, x_N)\| \leq \bar{\epsilon}$, turn to Step 7. Else $j = j + 1$, turn to Step 5 if $j < M$;

6: $\alpha := \frac{\alpha}{2}$ and then turn to Step 4;

7: Turn to Step 6 if the Wolfe line search condition

$$\hat{f}((x_k)_N + \alpha_k \bar{d}_k) \leq \hat{f}((x_k)_N) + \beta \alpha_k \bar{d}_k^T \hat{g}_k, \beta \in (0, 1)$$

is a positive constant, which can guarantee the generated iterates converge to a KKT point of the original optimization problem, else set $x_{k+1} = (x_B, x_N), k := k + 1$ and turn to Step 2.

4.2. Datum and parameters setting

Having introduced the construction method of the static interest rate term structure and the algorithm estimation of the parameters under ordinary conditions (for any numbers of tension spline), in this part, we determine the splines quantities so as to make a comparably good result in the degree of fitting and smoothness of the curve under the specific case in Chinese treasury market. Additionally, the larger quantities of the splines, the better fitting results we may obtain, which was measured by the residual variance. However, it will worsen the curve smoothness. On the contrary, the less spline quantity, the smoother the curve we can construct with less estimated parameters.

The datum tested in this paper is collected from the Shanghai Stock Exchange Market, consisting of Zero-coupon bonds and Interest coupons. After ten continuous observations over the period from 24$^{\text{th}}$ April, 2019 to 10$^{\text{th}}$ May, 2019, we have obtained about 120 coupons per day with different maturity periods. Considering investors' attitude towards prediction of the forward interest rate, they can predict the trend of short-term interest rate with much more information if needed. Differently, for a long-term interest rate prediction, such as 22 years, 25 years or even 30 years term, they have no substantial difference in investors' views, which means that they are located in the relatively smooth curve of forward expected interest rates after a long term. Besides, Schaefer [17] proposed that the estimations become unreliable once they pass the maximum maturity bonds. Thus, the selected coupons are all within the maturity period of 18 years (maximum bond maturity), i.e., the edges of the approximation space can be set at $a = 0$ and $b = 18$.

Given that the number of coupon bonds on the Shanghai stock exchange whose maturities are less than 18 years is about 90, in order to reduce the parameters quantities, avoid bias and efficiency caused by accepting an under parameterized function while preserving consistency [18], we determine that each functional segment contains several sample bonds in a relatively isometric interval at a time series 0, 6, 12, 18 or so, with short-term coupons up to six years, medium-term coupons up to twelve years and long-term coupons up to eighteen years. Thus, the quantity of splines is set as 3, i.e., the number of knots n in Eq (3.6) is four. Moreover, it is three tension parameters that control the short-term discount rate curve, medium-term discount rate curve and long-term discount rate curve. In this division, it places comparatively equal quantities of bonds in each section and maximizes degrees of freedom over individual segment of the discount function [8]. Besides, the parameters to be estimated are reduced. Finally, as a supplement, and taking the Chinese treasury bonds exchange market as an example, the short-term coupons are scarce while the medium-term coupons are abundant in the Chinese coupon market with the highest liquidity and long-term coupons next.

4.3. Estimated parameters and interest rates

Given that $n = 4$, there are three approximation tension splines to estimate for each date. Based on the principle of least squares, the results of the vectors of tension parameters, which are computed by Eqs (3.5)–(3.7) and the GRG algorithm are shown in Table 1. This method can be useful for other researchers.

Figures 1 and 2 show the rates of return and discount rates of securities evolution for maturities of between 0 and 18 years over the sample period. The discount curves moving in a parallel style (Figure 1) appears to downward slope with positive values between 0 and 1. For the spot rates at the points of yield curves, they vary along a U-shaped curve with the increase of maturities, which is fluctuant in the short-term interval and flat in the medium-term and long-term intervals. Remarkably, the future yield rates are non-negative and can easily verify that the yield rates at the unreliable times over 18 years are also positive.

Table 1. Tension parameters.

Date	Sample size	τ_1	τ_2	τ_3
04-24-19	86	0.0063	-2.8393	14.4019
04-25-19	86	0.5142	15.9617	35.1120
04-26-19	86	-1.6278	16.9370	35.1120
04-29-19	86	-0.8455	3.5942	25.0116
04-30-19	86	0.5222	2.3199	22.9990
05-06-19	86	0.8288	36.9522	24.4340
05-07-19	86	0.6530	38.4316	34.1013
05-08-19	86	0.7240	13.5436	38.2396
05-09-19	86	0.1986	29.1294	27.3604
05-10-19	86	0.1102	7.8612	29.7620

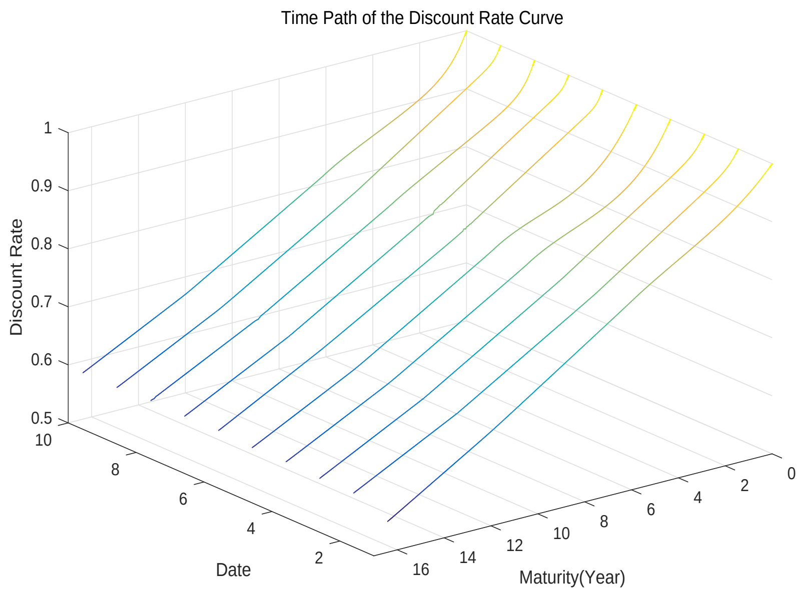

Figure 1. Time path of the discount rate curve.

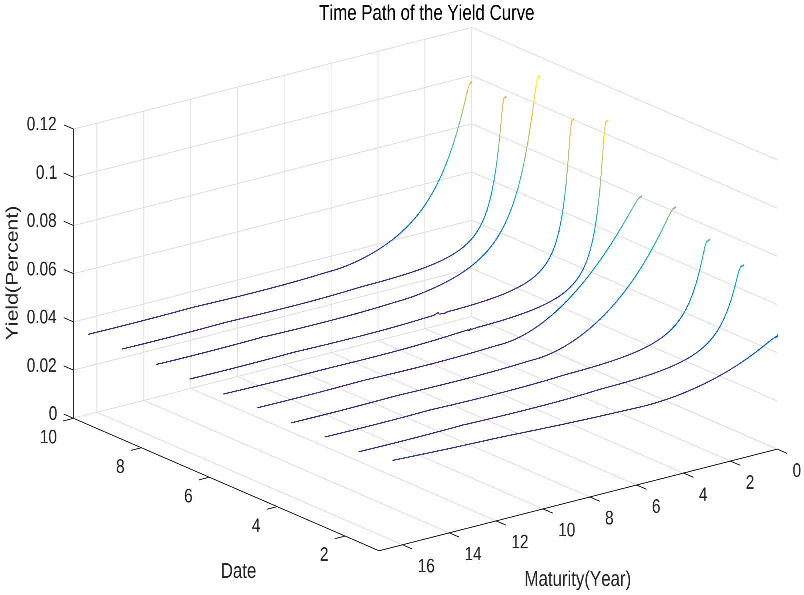

Figure 2. Time path of the yield curve.

Such indices are used to evaluate the precision of these method estimations, which can be measured by computing the mean of total squares of pricing estimated errors, standard deviation and the maximum (minimum) value of square pricing errors. Take a stochastic settlement date (2^{nd} February, 2019) as an example. Tables 2 and 3 show that these indices are based on minimum total squares of pricing errors and total absolute deviations, respectively. In Table 2, it is apparent that the mean of the total squares of pricing estimation errors equal to 10.53 is a little bit larger than the mean of the polynomial spline and exponential spline, with the same conclusion in standard deviation. In particular, the maximum value of square of pricing errors is 158.41, which is significantly a bit smaller than polynomial splines but larger than exponential splines. Measured to only two decimal places, their minimums can be recognized as an identical value 0.00. The similar situation goes in Table 3 based on total absolute deviations minimization. On the contrary, it has the best standard deviation and maximum absolute pricing estimated errors.

Table 2. Model descriptive statistics based on total least squares in intra-samples for estimation.

Types of spline	Mean	Std.	Max	Min
polynomial spline	9.76	24.23	163.88	0.00
exponential spline	10.13	23.76	147.71	0.00
tension spline	10.53	26.53	158.41	0.00

Table 3. Model descriptive statistics based on total least absolute deviations in intra-samples for estimation.

Types of spline	Mean	Std.	Max	Min
polynomial spline	1.97	2.72	13.08	0.00
exponential spline	1.98	3.09	15.72	0.00
tension spline	2.05	2.59	12.67	0.02

Figures 3 to 4 and 5 to 6 exhibit two examples of yield curves of future yield rate and discount rate curves with different types of splines. For yield curves, compared with polynomial splines (light blue curve) and exponential splines (red curve), the interest rate estimated by tension splines (dark blue curve) has more decreasing fluctuation ranged from maturity at zero year to six years approximately as a short-term interval, whereas the other monotonously decrease progressively to about four years. When the maturity is over about six years (mid-term and long-term), all of them turns flat, especially the tension spline. Here, a similar conclusion is generated regarding yield curves constructed by tension splines in Figure 4. Differently, however, for polynomial splines and exponential splines, the constructed yield curves are like U-shape curves with a critical points at about (6, 0.034) and (11, 0.031) which is relatively flatter than the tension spline at the short-term interval but more fluctuated at the mid-term and long-term interval. Considering the situation that Chinese treasury bonds have less circulation and issuance of frequency but take medium and long-term treasury bonds as the major types, the construction of tension splines can better explain the condition of the Liquidity Preference Theory. In accordance with this theory, it is the liquidity contacted with maturities that plays the essential part in the rate of return. Section 4.2 has introduced that the medium-term coupons have high liquidity and long-term coupons compared with low liquidity. For the short-term coupons in Shanghai stock exchange, they have a poor liquidity but a high rate of return as compensation, which will reduce as the maturities rising to medium-term with rise liquidity. Identically, the long-term coupons receive a comparably higher yield than the medium-term coupons due to their lower liquidity. In particular, it is of gradual smoothness that the yield curve from the middle to the end (from the medium to the long-term maturity) constructed by tension splines verifies the phenomenon that the risk premium will rise slowly, i.e., the slope of the curve finally tends to zero.

For discount curves, it is easy to observe that the discount rate is fluctuating and declining as the maturity increases and discount rate is 1 at the maturity time of zero, which meets the requirement of left points ($D(0) = 1$). Figure 5 shows us that the discount function constructed by tension splines is not a 'stable straight line' with a larger fluctuation at the short-term interval $[0, 6]$. Compared with polynomial spline and exponential spline, the discount function tends to a relatively stable line after the maturity time of inflection point 6. Thus, this can also well interpret the theory of Liquidity Preference that the high liquidity makes sum of all the cash flows paid at different times a relatively fixed present value.

Having estimated three tension parameters in this model, we need to appraise its stability for predicting a specific set of securities prices on the same settlement day, where the day is 2$^{\text{nd}}$ February, 2019 in this paper. In Tables 4 and 5, it is easy to obtain that not only based on minimum absolute sum of deviations but also on minimum square sum of deviations, the mean of square (absolute) pricing errors and its standard deviation are notably smaller than polynomial splines and exponential splines but slightly inferior in maximum and minimum of pricing errors. By comprehensive consideration, it means that tension splines based on a fourth-order differential equation with local tension parameters, has a superior prediction and better stability than traditional method of constructing static term structures of interested rates like polynomial splines and exponential splines, which implies that it is more suitable to apply in a treasury bonds market.

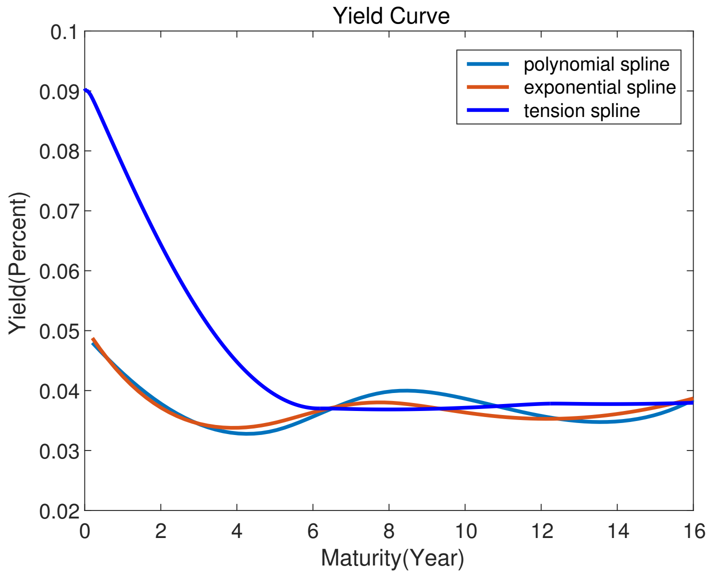

Figure 3. Yield curves constructed based on total least squares.

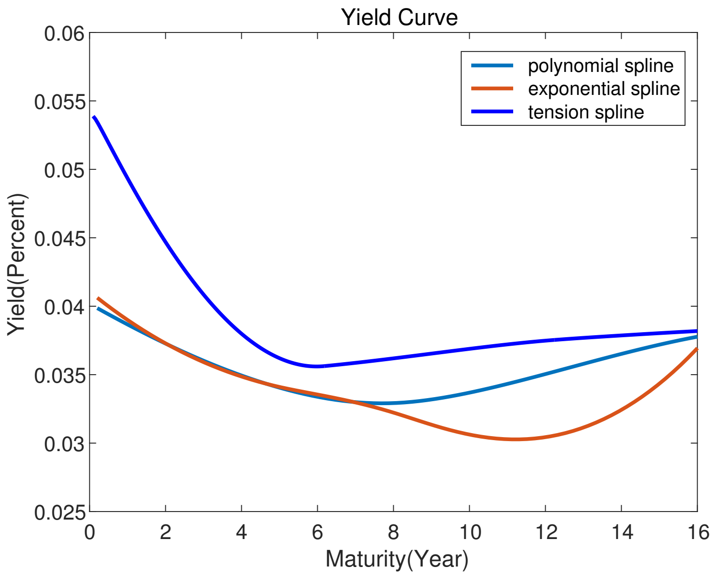

Figure 4. Yield curves constructed based on total least absolute deviations.

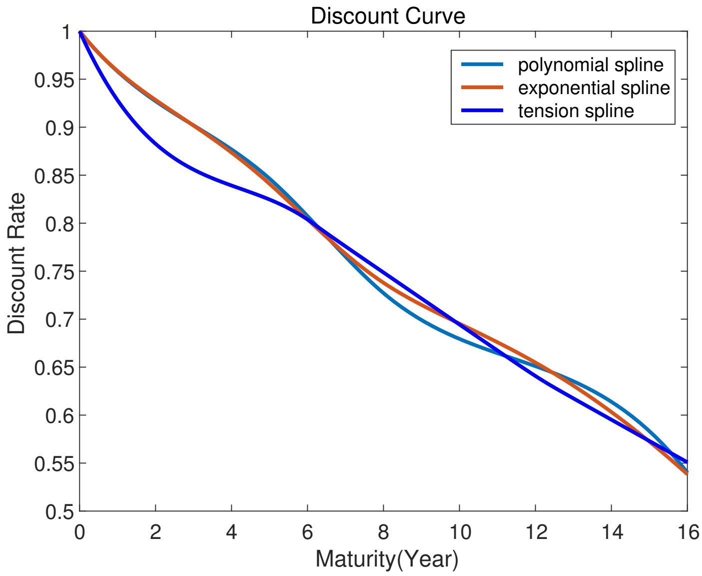

Figure 5. Discount curves constructed based on total least squares.

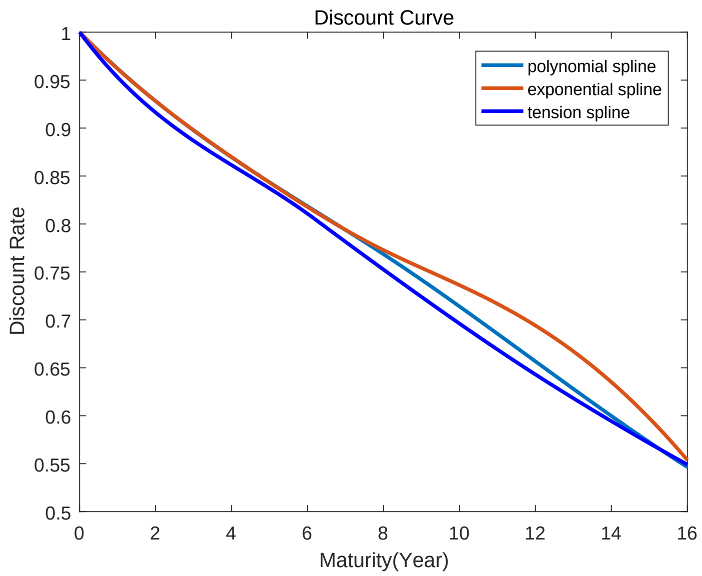

Figure 6. Discount curves constructed based on total least absolute deviations.

Table 4. Model descriptive statistics based on total least squares in out-of-samples for prediction.

Types of spline	Mean	Std.	Max	Min
polynomial spline	16.96	47.90	193.63	0.02
exponential spline	15.64	42.16	170.47	0.01
tension spline	14.91	42.63	172.91	0.03

Table 5. Model descriptive statistics based on total least absolute deviations in out-of-samples for prediction.

Types of spline	Mean	Std.	Max	Min
polynomial spline	2.42	3.31	13.78	0.18
exponential spline	3.14	4.36	16.91	0.20
tension spline	2.21	3.15	13.25	0.18

Figure 7 shows an example of discount curves with tension splines and the Nelson-Siegel model. The datum in this compare experiment is collected from the Shanghai Stock Exchange Market on 21st January, 2005. The mean of total least square error of securities price with tension spline is 1.36, which is a bit smaller than Nelson-Siegel model whose mean of total least square error is 1.37.

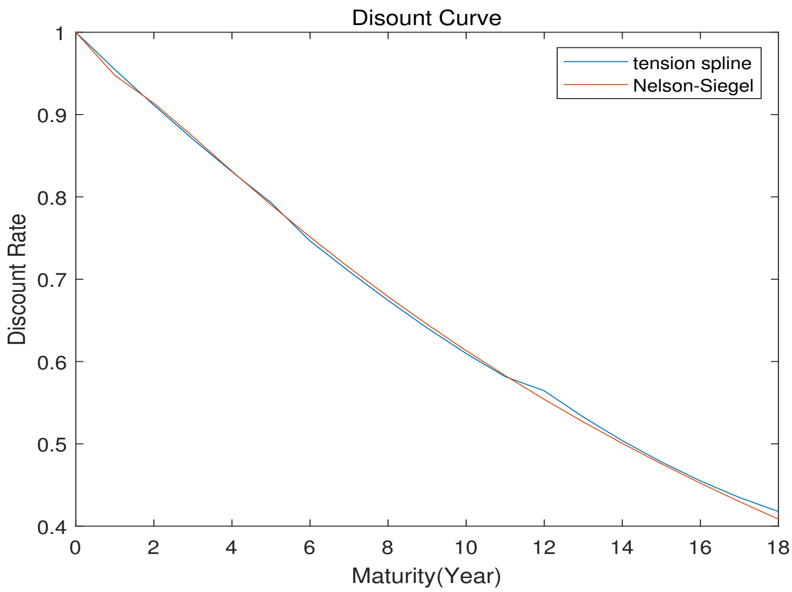

Figure 7. Discount curves constructed based on total least squares.

5. Conclusions

In this work, we present a method of estimating discount functions and yield functions using tension splines based on a fourth-order differential equation with local tension parameters that were first defined by Schweikert and verify that the pricing error is less in the prediction of coupon prices. Besides, we outline the spline approximation method and traditional theories of the term structure of interest rates. Comprehensive details of the form of tension splines and the tension parameters estimated algorithm, GRG, are given later to establish a definitive procedure for the static term structure of interest rate estimation.

Compared with the traditional construction method, it is found by practical research that the calculation of the mean of minimum total squares (absolute deviations), their standard deviations, and the maximum and minimum, respectively, provides available information during the evaluation of the models' precision and stability. The result is that the tension spline has the best predicting precision and stability among the three methods introduced in this paper and it is also more suitable to explain the Liquidity Preference Theory based on the Efficient Market Hypothesis. The path of the estimated several-date term structure is provided with distinctive tension parameters correspondently, which can be used by other researchers. Through the criteria discussed above, the coupon prices are estimated more accurately and the tension spline based on the fourth-order differential equation is a robust alternative to construct the static interest rate term structure curve.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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