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Research article

Hilfer fractional quantum system with Riemann-Liouville fractional derivatives and integrals in boundary conditions

Donny Passary[1,*], **Sotiris K. Ntouyas**[2] **and Jessada Tariboon**[1]

[1] Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand

[2] Department of Mathematics, University of Ioannina, Ioannina 45110, Greece

* **Correspondence:** Email: donny.p@sci.kmutnb.ac.th.

Abstract: In this paper, we initiate the study of existence and uniqueness of solutions for a coupled system involving Hilfer fractional quantum derivatives with nonlocal boundary value conditions containing q-Riemann-Liouville fractional derivatives and integrals. Our results are supported by some well-known fixed-point theories, including the Banach contraction mapping principle, Leray-Schauder alternative and the Krasnosel'skiĭ fixed-point theorem. Examples of these systems are also given in the end.

Keywords: quantum calculus; Hilfer derivative; systems; existence and uniqueness
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1. Introduction

An interesting area of research is the topic of coupled systems of fractional differential equations, as subjected to various types of nonlocal boundary conditions, since many real word problems can be modeled by these systems; see, for example, [1–4] and the references therein. The use of nonlocal boundary conditions is also favorable for numerous problems in physics and other areas of applied mathematics. In the literature, one can find many fractional derivative operators, the most known of them are those of Riemann-Liouville, Caputo, Katugampola, Hadamard, Hilfer, etc. See [5–10] for the treatments on these operators. Also, see [11] for a variety of results of boundary value problems that use different kinds of fractional-order derivative operators with nonlocal boundary conditions. Aside from fractional calculus, there is also a notion of fractal calculus which links the idea of fractional calculus with fractional dimensions. See [12] for details and applications to the dynamics of porous media and hierarchical structures.

In [13], Hilfer gives a new fractional derivative operator, which generalizes the Riemann-Liouville and Caputo fractional derivatives by adjusting the parameters based on its definition. See [14] for several benefits of using the Hilfer derivative. We refer the interested reader to [15–18] and the references therein for problems including the Hilfer fractional derivative.

The study of calculus without the notion of limits is typically known as quantum calculus or q-calculus. The first person to establish what is known as the q-derivative and q-integral is Jackson [19]. There are many applications of quantum calculus, such as physics, number theory, integer partitions, orthogonal polynomials and hypergeometric functions; see [20, 21]. There has also been a generalization of q-derivatives and q-integrals into orders other than integers by Al-Salam [22] and Agarwal [23], which is used in the development of the q-difference calculus. For details on q-fractional calculus and equations, see the monograph of [24]. For new results on the topic, see [25–33] and the references therein. Tariboon and Ntouyas [34] also introduced quantum calculus on finite intervals. For details on quantum calculus and recent results, we refer the interested reader to the monograph of [35].

Recently, in [36], the generalization of the Hilfer fractional derivative, which was developed by R. Hilfer in [13], was introduced, called the Hilfer fractional quantum derivative. Also, in [36], studies on such initial and boundary value problems were done via fixed-point theory, with a new class of boundary value problems using the Hilfer quantum fractional derivative of the following form:

$$\begin{cases} {}_a^H D_q^{\alpha,\beta} x(t) = f(t, x(t)), & a < t < b, \\ x(a) = 0, \qquad x(b) = \sum_{i=1}^{m} \lambda_i \left({}_a I_{q_i}^{\kappa_i} x \right)(\eta_i), \end{cases} \tag{1.1}$$

where ${}_a^H D_q^{\alpha,\beta}$ is the Hilfer quantum fractional derivative, whose order is $\alpha \in (1, 2)$ with $\beta \in (0, 1)$ and $q \in (0, 1)$, f is a function defined from $[a, b] \times \mathbb{R}$ to $\mathbb{R}$, ${}_a I_{q_i}^{\kappa_i}$ is the quantum fractional integral, whose order is $\kappa_i > 0$, $\lambda_i \in \mathbb{R}$, $q_i \in (0, 1)$ and $\eta_i \in [a, b]$ for each $1 \le i \le m$. The existence and uniqueness of the solutions for such a system were established via Banach's fixed-point theorem.

In the present paper, our aim is to enrich the literature on the Hilfer quantum fractional derivative by combining the Hilfer quantum and Riemann-Liouville fractional derivative operators with q-Riemann-Liouville integral operators. Such settings for this combination, as far as we know, are new in the literature. Thus, in the present paper, we investigate the existence and uniqueness of the solutions to the following coupled system involving Hilfer fractional quantum derivatives with nonlocal boundary value conditions containing q-Riemann-Liouville fractional derivatives and integrals of the following form:

$$\begin{cases} ({}_a^H D_{q_1}^{\alpha_1,\beta_1} y)(t) = g\left(t, y(t), z(t)\right), & a < t < b, \\ ({}_a^H D_{q_2}^{\alpha_2,\beta_2} z)(t) = h\left(t, y(t), z(t)\right), & a < t < b, \\ y(a) = 0, \;\; y(b) = \mu_2 \left({}_a^{RL} D_{q_2}^{\lambda_2} z \right)(\eta_2) + \delta_2 \left({}_a I_{q_2}^{\epsilon_2} z \right)(\xi_2), \\ z(a) = 0, \;\; z(b) = \mu_1 \left({}_a^{RL} D_{q_1}^{\lambda_1} y \right)(\eta_1) + \delta_1 \left({}_a I_{q_1}^{\epsilon_1} y \right)(\xi_1), \end{cases} \tag{1.2}$$

where $\alpha_i \in (1, 2)$, $\beta_i, q_i, \lambda_i \in (0, 1)$, $i = 1, 2$, $a \ge 0$, $\epsilon_1, \epsilon_2 > 0$, $\eta_j, \xi_j \in [a, b]$, $\mu_j, \delta_j \in \mathbb{R}$, $j = 1, 2$, the operators ${}_a^H D_{q_r}^{\alpha_r,\beta_r}$ and ${}_a^{RL} D_{q_r}^{\lambda_r}$ are the Hilfer and Riemann-Liouville quantum fractional derivatives, respectively, with orders α_r, λ_r, quantum number q_r and Hilfer parameter β_r, $r = 1, 2$;

$g, h : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are the given functions, ${}_aI_{q_k}^{\epsilon_k}$ is the q_k-Riemann-Liouville integral operator with orders $\epsilon_k > 0$, $k = 1, 2$ and all fractional operators are initiated at a point a.

Existence and uniqueness results for the system (1.2) are obtained by using classical fixed-point theorems. Thus, via Banach's contraction mapping principle, the existence and uniqueness of the solution is guaranteed, while the Leray-Schauder alternative and Krasnosel'skiĭ fixed-point theorem are used to show the solution's existence. These obtained results are new and will enrich the literature on this new topic of research, for which the existing results are very limited.

We emphasize that, in this paper, we initiate the study of a coupled system in which we combine the following:

- Hilfer fractional quantum derivatives,
- q-Riemann-Liouville integral operators,
- q-Riemann-Liouville differential operators,
- mixed nonlocal boundary conditions including the Riemann-Liouville derivative and integral operators.

The used method is standard, but its configuration in the coupled quantum system (1.2) is new.

The organization of this paper is as follows. Section 2 covers some preliminaries and lemmas, with some basic results from the topics of q-calculus up to the Hilfer fractional quantum derivative. Also, we prove an auxiliary lemma in order to transform the given nonlinear system into a fixed-point problem. In Section 3, we prove the results on existence and uniqueness of the solution for the Hilfer coupled quantum system (1.2). Finally, illustrative examples are constructed in Section 4.

2. Preliminaries

In this section, we recall some definitions and basic properties from quantum calculus, fractional quantum calculus and the Hilfer fractional quantum derivative.

2.1. Quantum calculus

Let $y : [a, b] \to \mathbb{R}$ be a given function. The quantum derivative on $[a, b]$ (which was introduced by Tariboon and Ntouyas in 2013 [34]) is defined by

$$ {}_aD_q y(t) = \frac{y(t) - y(qt + (1-q)a)}{(1-q)(t-a)}, \quad t \in (a, b], \tag{2.1}$$

and ${}_aD_q y(a) = \lim_{t \to a} \left({}_aD_q y \right)(t)$. If $a = 0$, for $t \in (0, b]$, it is reduced to

$$ D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \tag{2.2}$$

which is the Jackson q-derivative (first defined by Jackson in 1910 [19]). As is customary, the quantum integral (q-integral) of a function y is defined as

$$ {}_aI_q y(t) = \int_a^t y(s)\, {}_ad_q s = (1-q)(t-a) \sum_{i=0}^{\infty} q^i y(q^i t + (1-q^i)a). \tag{2.3}$$

For the setting $a = 0$, we will obtain the Jackson q-integral as

$$I_q y(t) = \int_0^t y(s) d_q s = (1-q)t \sum_{i=0}^{\infty} q^i y(q^i t). \tag{2.4}$$

It is not difficult to define the q-derivative and q-integral of higher order, e.g., ${}_aD_q^n$ and ${}_aI_q^n$, $n \in \mathbb{Z}$, i.e., the set of integers. The story of fractional quantum calculus was begun by generalizing the integer-order n by $\omega \in \mathbb{R}$ based on definitions (2.2) and (2.4) by Al-Salam in 1966 [22] and Agarwal in 1969 [23]. Tariboon et al. replaced n by ω by using definitions (2.1) and (2.3) in 2015 [37]. The key tool of success is the q-power function, which is contained inside the integration. The q-power function defined on $[a, b]$ is as follows:

$${}_a(t-s)_q^{(\omega)} = \prod_{i=0}^{\infty} \frac{(t -_a \Psi_q^i(s))}{(t -_a \Psi_q^{i+\omega}(s))},$$

where ${}_a\Psi_q^i(t) = q^i t + (1-q^i)a$ and $\omega \in \mathbb{R}$. If $a = 0$, it is reduced to ${}_0\Psi_q^i(t) = q^i t$, which gives

$$(t-s)_q^{(\omega)} = \prod_{i=0}^{\infty} \frac{(t - q^i s)}{(t - q^{i+\omega} s)}, \tag{2.5}$$

and it appears in the kernel of fractional quantum calculus in [22, 23]. Note that, if $\omega = k$ is an integer, we can rewrite (2.5) as

$$(t-s)_q^{(k)} = \prod_{i=0}^{k-1} (t - q^i s), \quad k \in \mathbb{N} \cup \{\infty\}.$$

Definition 2.1. *[37] The fractional quantum derivative of Riemann-Liouville type of order $\omega \geq 0$ on the interval $[a, b]$ is*

$$\begin{aligned} ({}_a^{RL}D_q^{\omega} y)(t) &= ({}_aD_q^n {}_aI_q^{n-\omega} y)(t) \\ &= \frac{1}{\Gamma_q(n-\omega)} \left({}_aD_q^n\right) \int_a^t {}_a(t -_a \Phi_q(s))_q^{(n-\omega-1)} y(s) \, {}_ad_q s, \quad \omega > 0, \end{aligned}$$

and $({}_a^{RL}D_q^0 f)(t) = f(t)$. Here, n is the smallest integer such that $\omega \leq n$, and $\Gamma_q(v)$ is defined as

$$\Gamma_q(v) = \frac{(1-q)_q^{(v-1)}}{(1-q)^{v-1}}, \quad v \in \mathbb{R} \backslash \{0, -1, -2, \ldots\}.$$

Also, ${}_a\Phi_q$ is the q-shifting operator

$${}_a\Phi_q(m) = qm + (1-q)a.$$

Definition 2.2. *[37] Let $\omega \geq 0$ and $y : [a, b] \to \mathbb{R}$. The Riemann-Liouville-type fractional q-integral of y is defined as*

$$({}_aI_q^{\omega} y)(t) = \frac{1}{\Gamma_q(\omega)} \int_a^t {}_a(t - {}_a\Phi_q(s))_q^{(\omega-1)} y(s) {}_ad_q s, \quad \omega > 0, \quad t \in [a, b];$$

and, $({}_aI_q^0 y)(t) = y(t)$, provided that the right-hand side exists.

Definition 2.3. *[36] The Hilfer fractional quantum derivative of order $\omega > 0$, with the parameter $\zeta \in [0,1]$, of a function y defined on $[a,b]$, is defined as*

$$ {}_a^H D_q^{\omega,\zeta} y(t) = {}_a I_q^{\zeta(n-\omega)} \, {}_a D_q^n \, {}_a I_q^{(1-\zeta)(n-\omega)} y(t), $$

where $\omega \in (n-1,n)$, with $q \in (0,1)$ and $t > a$.

We remark that the Hilfer fractional quantum derivative is an interpolation between two types of fractional derivatives, that is, if $\zeta = 0$, we obtain the fractional quantum Riemann-Liouville derivative as

$$ {}_a^H D_q^{\omega,0} y(t) = {}_a^{RL} D_q^{\omega} y(t), $$

and if $\zeta = 1$, we obtain the fractional quantum Caputo derivative:

$$ {}_a^H D_q^{\omega,1} y(t) = {}_a^C D_q^{\omega} y(t), $$

which is defined by

$$ \begin{aligned} ({}_a^C D_q^{\omega} y)(t) &= {}_a I_q^{n-\omega} \left({}_a D_q^n y \right)(t), \\ &= \frac{1}{\Gamma_q(n-\omega)} \int_a^t {}_a(t - {}_a \Phi_q(s))_q^{(n-\omega-1)} \left({}_a D_q^n y \right)(s) \, {}_a d_q s, \quad \omega > 0. \end{aligned} $$

Lemma 2.1. *[36] Assume that $y \in C^n([a,b],\mathbb{R})$, $\omega \in (n-1,n)$, $\zeta \in (0,1)$ and $q \in (0,1)$. Then,*

(i) ${}_a I_q^{\omega} \left({}_a^H D_q^{\omega,\zeta} y \right)(t) = y(t) - \sum_{j=1}^{n} \frac{(t-a)^{\gamma-j}}{\Gamma_q(\gamma-j+1)} \left({}_a^{RL} D_q^{\gamma-j} y \right)(a),$

(ii) ${}_a^H D_q^{\omega,\zeta} \left({}_a I_q^{\omega} y \right)(t) = y(t),$

where $\gamma = \omega + \zeta(n-\omega)$.

Lemma 2.2. *[36] Let $\omega \in (0,\delta)$ and $q \in (0,1)$. Then,*

(a) ${}_a^{RL} D_q^{\omega} (t-a)^{\delta} = \frac{\Gamma_q(\delta+1)}{\Gamma_q(\delta-\omega+1)} (t-a)^{\delta-\omega},$

(b) ${}_a I_q^{\omega} (t-a)^{\delta} = \frac{\Gamma_q(\delta+1)}{\Gamma_q(\delta+\omega+1)} (t-a)^{\delta+\omega}.$

2.2. An auxiliary result

In the next lemma, an auxiliary result is proved, which is the basic tool in transforming the nonlinear problem (1.2) into a fixed-point problem, and we deal with a linear variant of the problem (1.2). For convenience, we set the following:

$$ \begin{aligned} \Phi_1 &= \frac{\mu_1 \Gamma_{q_1}(\gamma_1)}{\Gamma_{q_1}(\gamma_1 - \lambda_1)} (\eta_1 - a)^{\gamma_1 - \lambda_1 - 1} + \frac{\delta_1 \Gamma_{q_1}(\gamma_1)}{\Gamma_{q_1}(\gamma_1 + \epsilon_1)} (\xi_1 - a)^{\gamma_1 + \epsilon_1 - 1}, \\ \Phi_2 &= \frac{\mu_2 \Gamma_{q_2}(\gamma_2)}{\Gamma_{q_2}(\gamma_2 - \lambda_2)} (\eta_2 - a)^{\gamma_2 - \lambda_2 - 1} + \frac{\delta_2 \Gamma_{q_2}(\gamma_2)}{\Gamma_{q_2}(\gamma_2 + \epsilon_2)} (\xi_2 - a)^{\gamma_2 + \epsilon_2 - 1}, \\ \Phi_3 &= (b-a)^{\gamma_1 + \gamma_2 - 2} - \Phi_1 \Phi_2. \end{aligned} \tag{2.6} $$

Lemma 2.3. *Let $g_1, h_1 \in C^2([a,b],\mathbb{R})$ and $\Phi_3 \neq 0$. Then, the solution of the linear system*

$$\begin{cases} ({}^{H}_{a}D^{\alpha_1,\beta_1}_{q_1}y)(t) = g_1(t), \qquad a < t < b, \\ ({}^{H}_{a}D^{\alpha_2,\beta_2}_{q_2}z)(t) = h_1(t), \qquad a < t < b, \\ y(a) = 0, \;\; y(b) = \mu_2 \left({}^{RL}_{a}D^{\lambda_2}_{q_2}z\right)(\eta_2) + \delta_2 \left({}_{a}I^{\epsilon_2}_{q_2}z\right)(\xi_2), \\ z(a) = 0, \;\; z(b) = \mu_1 \left({}^{RL}_{a}D^{\lambda_1}_{q_1}y\right)(\eta_1) + \delta_1 \left({}_{a}I^{\epsilon_1}_{q_1}y\right)(\xi_1), \end{cases} \tag{2.7}$$

is uniquely given by

$$\begin{aligned} y(t) \;\; = \;\; & (t-a)^{\gamma_1-1}\Big\{ -\frac{(b-a)^{\gamma_2-1}}{\Phi_3}{}_{a}I^{\alpha_1}_{q_1}g_1(b) + \mu_2\frac{(b-a)^{\gamma_2-1}}{\Phi_3}{}_{a}I^{\alpha_2-\lambda_2}_{q_2}h_1(\eta) \\ & +\delta_2\frac{(b-a)^{\gamma_2-1}}{\Phi_3}{}_{a}I^{\alpha_2+\epsilon_2}_{q_2}h_1(\xi_2) - \frac{\Phi_2}{\Phi_3}{}_{a}I^{\alpha_2}_{q_2}h_1(b) + \mu_1\frac{\Phi_2}{\Phi_3}{}_{a}I^{\alpha_1-\lambda_1}_{q_1}g_1(\eta_1) \\ & + \delta_1\frac{\Phi_2}{\Phi_3}{}_{a}I^{\alpha_1+\epsilon_1}_{q_1}g_1(\xi_1)\Big\} + {}_{a}I^{\alpha_1}_{q_1}g_1(t) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} z(t) \;\; = \;\; & (t-a)^{\gamma_2-1}\Big\{ -\frac{(b-a)^{\gamma_1-1}}{\Phi_3}{}_{a}I^{\alpha_2}_{q_2}h_1(b) + \mu_1\frac{(b-a)^{\gamma_1-1}}{\Phi_3}{}_{a}I^{\alpha_1-\lambda_1}_{q_1}g_1(\eta_1) \\ & +\delta_1\frac{(b-a)^{\gamma_1-1}}{\Phi_3}{}_{a}I^{\alpha_1+\epsilon_1}_{q_1}g_1(\xi_1) - \frac{\Phi_1}{\Phi_3}{}_{a}I^{\alpha_1}_{q_1}g_1(b) + \mu_2\frac{\Phi_1}{\Phi_3}{}_{a}I^{\alpha_2-\lambda_2}_{q_2}h_1(\eta_2) \\ & + \delta_2\frac{\Phi_1}{\Phi_3}{}_{a}I^{\alpha_2+\epsilon_2}_{q_2}h_1(\xi_2)\Big\} + {}_{a}I^{\alpha_2}_{q_2}h_1(t), \end{aligned} \tag{2.9}$$

where $\Phi_i, i = 1,2,3$ are given in (2.6).

Proof. For the first equation in (2.7), taking the Riemann-Liouville fractional integral of order α_1 and quantum number q_1 from a to t on both sides and applying Lemma 2.1, we have

$$\begin{aligned} y(t) \;\; = \;\; & \frac{(t-a)^{\gamma_1-1}}{\Gamma_{q_1}(\gamma_1)}{}^{RL}_{a}D^{\gamma_1-1}_{q_1}y(a) + \frac{(t-a)^{\gamma_1-2}}{\Gamma_{q_1}(\gamma_1-1)}{}^{RL}_{a}D^{\gamma_1-2}_{q_1}y(a) + {}_{a}I^{\alpha_1}_{q_1}g_1(t) \\ := \;\; & (t-a)^{\gamma_1-1}K_1 + (t-a)^{\gamma_1-2}K_2 + {}_{a}I^{\alpha_1}_{q_1}g_1(t), \end{aligned} \tag{2.10}$$

where $\gamma_1 = \alpha_1 + \beta_1(2-\alpha_1)$ and $K_1, K_2 \in \mathbb{R}$. Since $1 < \alpha_1 < 2$ and $\gamma_1 \in (\alpha_1, 2)$, we obtain that $K_2 \equiv 0$ by that condition $y(a) = 0$ and (2.10) is presented as

$$y(t) = (t-a)^{\gamma_1-1}K_1 + {}_{a}I^{\alpha_1}_{q_1}g_1(t). \tag{2.11}$$

Lemma 2.2 with (2.11) leads to

$${}^{RL}_{a}D^{\lambda_1}_{q_1}y(\eta_1) = K_1\frac{\Gamma_{q_1}(\gamma_1)}{\Gamma_{q_1}(\gamma_1-\lambda_1)}(\eta_1-a)^{\gamma_1-\lambda_1-1} + {}_{a}I^{\alpha_1-\lambda_1}_{q_1}g_1(\eta_1) \tag{2.12}$$

and

$${}_{a}I^{\epsilon_1}_{q_1}y(\xi_1) = K_1\frac{\Gamma_{q_1}(\gamma_1)}{\Gamma_{q_1}(\gamma_1+\epsilon_1)}(\xi_1-a)^{\gamma_1+\epsilon_1-1} + {}_{a}I^{\alpha_1+\epsilon_1}_{q_1}g_1(\xi_1). \tag{2.13}$$

In the same way as with the second equation of (2.7), taking the fractional integral of Riemann-Liouville type of order α_2 and quantum number q_2, combined with the condition $z(a) = 0$, we obtain

$$z(t) = (t-a)^{\gamma_2-1}C_1 + {}_aI_{q_2}^{\alpha_2}h_1(t), \tag{2.14}$$

where $\gamma_2 = \alpha_2 + \beta_2(2-\alpha_2)$ and $C_1 \in \mathbb{R}$. Applying Lemma 2.2 with quantum number q_2 in (2.14), we get

$${}_a^{RL}D_{q_2}^{\lambda_2}z(\eta_2) = C_1\frac{\Gamma_{q_2}(\gamma_2)}{\Gamma_{q_2}(\gamma_2-\lambda_2)}(\eta_2-a)^{\gamma_2-\lambda_2-1} + {}_aI_{q_2}^{\alpha_2-\lambda_2}h_1(\eta_2) \tag{2.15}$$

and

$${}_aI_{q_2}^{\epsilon_2}z(\xi_2) = C_1\frac{\Gamma_{q_2}(\gamma_2)}{\Gamma_{q_2}(\gamma_2+\epsilon_2)}(\xi_2-a)^{\gamma_2+\epsilon_2-1} + {}_aI_{q_2}^{\alpha_2+\epsilon_2}h_1(\xi_2). \tag{2.16}$$

Substituting (2.12), (2.13), (2.15) and (2.16) in the second condition of the third and fourth equalities of (2.7), we obtain the system below.

$$\begin{aligned}
(b-a)^{\gamma_1-1}K_1 - \Phi_2C_1 &= -{}_aI_{q_1}^{\alpha_1}g_1(b) + \mu_{2a}I_{q_2}^{\alpha_2-\lambda_2}h_1(\eta_2) + \delta_{2a}I_{q_2}^{\alpha_2+\epsilon_2}h_1(\xi_2),\\
-\Phi_1K_1 + (b-a)^{\gamma_2-1}C_1 &= -{}_aI_{q_2}^{\alpha_2}h_1(b) + \mu_{1a}I_{q_1}^{\alpha_1-\lambda_1}g_1(\eta_1) + \delta_{1a}I_{q_1}^{\alpha_1+\epsilon_1}g_1(\xi_1).
\end{aligned}$$

Solving the above system for K_1 and C_1, we have

$$\begin{aligned}
K_1 &= \frac{(b-a)^{\gamma_2-1}}{\Phi_3}\left(-{}_aI_{q_1}^{\alpha_1}g_1(b) + \mu_{2a}I_{q_2}^{\alpha_2-\lambda_2}h_1(\eta_2) + \delta_{2a}I_{q_2}^{\alpha_2+\epsilon_2}h_1(\xi_2)\right)\\
&\quad+\frac{\Phi_2}{\Phi_3}\left(-{}_aI_{q_2}^{\alpha_2}h_1(b) + \mu_{1a}I_{q_1}^{\alpha_1-\lambda_1}g_1(\eta_1) + \delta_{1a}I_{q_1}^{\alpha_1+\epsilon_1}g_1(\xi_1)\right)
\end{aligned}$$

and

$$\begin{aligned}
C_1 &= \frac{(b-a)^{\gamma_1-1}}{\Phi_3}\left(-{}_aI_{q_2}^{\alpha_2}h_1(b) + \mu_{1a}I_{q_1}^{\alpha_1-\lambda_1}g_1(\eta_1) + \delta_{1a}I_{q_1}^{\alpha_1+\epsilon_1}g_1(\xi_1)\right)\\
&\quad+\frac{\Phi_1}{\Phi_3}\left(-{}_aI_{q_1}^{\alpha_1}g_1(b) + \mu_{2a}I_{q_2}^{\alpha_2-\lambda_2}h_1(\eta_2) + \delta_{2a}I_{q_2}^{\alpha_2+\epsilon_2}h_1(\xi_2)\right).
\end{aligned}$$

Putting constants K_1 and C_1 into (2.11) and (2.14), (2.8) and (2.9) are then established. The converse can be verified by direct computation. The proof is finished. □

2.3. Fixed-point theorems

The following fixed-point theorems are used in the proofs of our main results.

Lemma 2.4. *[38] (Banach fixed-point theorem) Let X be a Banach space, $D \subset X$ be closed and $F : D \to D$ be a strict contraction, i.e., $|Fx - Fy| \le k|x - y|$ for some $k \in (0, 1)$ and all $x, y \in D$. Then, F has a fixed point in D.*

Lemma 2.5. *[39] (Leray-Schauder alternative) Let $T : E \to E$ be an operator on a Banach space E such that T is completely continuous. Let*

$$\xi_T = \{x \in E | x = \lambda T(x) \quad \textit{for some} \quad 0 < \lambda < 1\};$$

then, either ξ_T is unbounded or the operator T has at least one fixed point.

Lemma 2.6. *[40] (Krasnosel'skiĭ's fixed-point theorem) For a Banach space X, let $\emptyset \neq M \subset X$ be a closed, bounded and convex subset. Let A and B be operators on M such that*

(*a*) *$Ax + By \in M$ for any $x, y \in M$,*
(*b*) *A is continuous and compact,*
(*c*) *B is a contraction mapping.*

Then, $A + B$ has a fixed point, i.e., $Az + Bz = z$ for some $z \in M$.

3. Existence results

Let $X = C([a,b],\mathbb{R})$ be the Banach space of all continuous functions from $[a,b]$ to $\mathbb{R}$, as endowed with the supremum norm $\|y\| = \sup\{|y(t)| : t \in [a,b]\}$. The product space $(X\times X, \|(y,z)\|)$ with the norm $\|(y,z)\| = \|y\| + \|z\|$ is also a Banach space.

In view of Lemma 2.3, define an operator $\mathcal{A} : X \times X \to X \times X$, where

$$\mathcal{A}(y,z) = (\mathcal{A}_1(y,z),\ \mathcal{A}_2(y,z)), \tag{3.1}$$

$$\begin{aligned}
&\mathcal{A}_1(y,z)(t)\\
= \;& (t-a)^{\gamma_1-1}\Big\{ -\frac{(b-a)^{\gamma_2-1}}{\Phi_3}\, {}_aI_{q_1}^{\alpha_1} g\,(b, y(b), z(b))\\
&+\mu_2\frac{(b-a)^{\gamma_2-1}}{\Phi_3}\, {}_aI_{q_2}^{\alpha_2-\lambda_2} h\,(\eta_2, y(\eta_2), z(\eta_2)) + \delta_2\frac{(b-a)^{\gamma_2-1}}{\Phi_3}\, {}_aI_{q_2}^{\alpha_2+\epsilon_2} h\,(\xi_2, y(\xi_2), z(\xi_2))\\
&-\frac{\Phi_2}{\Phi_3}\, {}_aI_{q_2}^{\alpha_2} h\,(b, y(b), z(b)) + \mu_1\frac{\Phi_2}{\Phi_3}\, {}_aI_{q_1}^{\alpha_1-\lambda_1} g\,(\eta_1, y(\eta_1), z(\eta_1))\\
&+\delta_1\frac{\Phi_2}{\Phi_3}\, {}_aI_{q_1}^{\alpha_1+\epsilon_1} g\,(\xi_1, y(\xi_1), z(\xi_1))\Big\} + {}_aI_{q_1}^{\alpha_1} g\,(t, y(t), z(t))
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\mathcal{A}_2(y,z)(t)\\
= \;& (t-a)^{\gamma_2-1}\Big\{ -\frac{(b-a)^{\gamma_1-1}}{\Phi_3}\, {}_aI_{q_2}^{\alpha_2} h\,(b, y(b), z(b))\\
&+\mu_1\frac{(b-a)^{\gamma_1-1}}{\Phi_3}\, {}_aI_{q_1}^{\alpha_1-\lambda_1} g\,(\eta_1, y(\eta_1), z(\eta_1)) + \delta_1\frac{(b-a)^{\gamma_1-1}}{\Phi_3}\, {}_aI_{q_1}^{\alpha_1+\epsilon_1} g\,(\xi_1, y(\xi_1), z(\xi_1))\\
&-\frac{\Phi_1}{\Phi_3}\, {}_aI_{q_1}^{\alpha_1} g\,(b, y(b), z(b)) + \mu_2\frac{\Phi_1}{\Phi_3}\, {}_aI_{q_2}^{\alpha_2-\lambda_2} h\,(\eta_2, y(\eta_2), z(\eta_2))\\
&+\delta_2\frac{\Phi_1}{\Phi_3}\, {}_aI_{q_2}^{\alpha_2+\epsilon_2} h\,(\xi_2, y(\xi_2), z(\xi_2))\Big\} + {}_aI_{q_2}^{\alpha_2} h\,(t, y(t), z(t)).
\end{aligned} \tag{3.3}$$

For convenience, we set the following:

$$\begin{aligned}
\mathbb{M}_1 \;=\; & \frac{(b-a)^{\gamma_1+\gamma_2+\alpha_1-2}}{|\Phi_3|\Gamma_{q_1}(\alpha_1+1)} + \frac{|\Phi_2|}{|\Phi_3|}\left(\frac{|\mu_1|\,(b-a)^{\gamma_1+\alpha_1-\lambda_1-1}}{\Gamma_{q_1}(\alpha_1-\lambda_1+1)} + \frac{|\delta_1|\,(b-a)^{\gamma_1+\alpha_1+\epsilon_1-1}}{\Gamma_{q_1}(\alpha_1+\epsilon_1+1)}\right)\\
&+\frac{(b-a)^{\alpha_1}}{\Gamma_{q_1}(\alpha_1+1)},
\end{aligned}$$

$$\mathbb{M}_2 = \frac{|\mu_2|\,(b-a)^{\gamma_1+\gamma_2+\alpha_2-\lambda_2-2}}{|\Phi_3|\Gamma_{q_2}(\alpha_2-\lambda_2+1)} + \frac{|\delta_2|\,(b-a)^{\gamma_1+\gamma_2+\alpha_2+\epsilon_2-2}}{|\Phi_3|\Gamma_{q_2}(\alpha_2+\epsilon_2+1)} + \frac{|\Phi_2|}{|\Phi_3|}\cdot\frac{(b-a)^{\gamma_1+\alpha_2-1}}{\Gamma_{q_2}(\alpha_2+1)},$$
$$\mathbb{M}_3 = \frac{|\mu_1|\,(b-a)^{\gamma_1+\gamma_2+\alpha_1-\lambda_1-2}}{|\Phi_3|\Gamma_{q_1}(\alpha_1-\lambda_1+1)} + \frac{|\delta_1|\,(b-a)^{\gamma_1+\gamma_2+\alpha_1+\epsilon_1-2}}{|\Phi_3|\Gamma_{q_1}(\alpha_1+\epsilon_1+1)} + \frac{|\Phi_1|}{|\Phi_3|}\cdot\frac{(b-a)^{\gamma_2+\alpha_1-1}}{\Gamma_{q_1}(\alpha_1+1)},$$
$$\mathbb{M}_4 = \frac{(b-a)^{\gamma_1+\gamma_2+\alpha_2-2}}{|\Phi_3|\Gamma_{q_2}(\alpha_2+1)} + \frac{|\Phi_1|}{|\Phi_3|}\left(\frac{|\mu_2|\,(b-a)^{\gamma_2+\alpha_2-\lambda_2-1}}{\Gamma_{q_2}(\alpha_2-\lambda_2+1)} + \frac{|\delta_2|\,(b-a)^{\gamma_2+\alpha_2+\epsilon_2-1}}{\Gamma_{q_2}(\alpha_2+\epsilon_2+1)}\right) + \frac{(b-a)^{\alpha_2}}{\Gamma_{q_2}(\alpha_2+1)}. \tag{3.4}$$

We now prove an existence and uniqueness result for the Hilfer fractional quantum system (1.2) by using Banach's contraction mapping principle (Lemma 2.4).

Theorem 3.1. *Let $g, h : [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be such that the following holds:*

(H_1) *There exist real numbers $\mathfrak{m}_i, \mathfrak{n}_i \geq 0$, $i = 1,2$ such that, for every $t \in [a,b]$ and $y_i, z_i \in \mathbb{R}$, $i = 1,2$,*

$$|g(t,y_1,z_1) - g(t,y_2,z_2)| \leq \mathfrak{m}_1|y_1-y_2| + \mathfrak{m}_2|z_1-z_2|$$

and

$$|h(t,y_1,z_1) - h(t,y_2,z_2)| \leq \mathfrak{n}_1|y_1-y_2| + \mathfrak{n}_2|z_1-z_2|.$$

Then, if

$$(\mathbb{M}_1+\mathbb{M}_3)(\mathfrak{m}_1+\mathfrak{m}_2) + (\mathbb{M}_2+\mathbb{M}_4)(\mathfrak{n}_1+\mathfrak{n}_2) < 1, \tag{3.5}$$

where $\mathbb{M}_i, i = 1,2,3,4$ are defined in (3.4), the Hilfer fractional quantum system (1.2) has a unique solution on $[a,b]$.

Proof. Let us define $B_r = \{(y,z)\in X\times X : \|(y,z)\| \leq r\}$, with

$$r > \frac{(\mathbb{M}_1+\mathbb{M}_3)N_1 + (\mathbb{M}_2+\mathbb{M}_4)N_2}{1-[(\mathbb{M}_1+\mathbb{M}_3)(\mathfrak{m}_1+\mathfrak{m}_2) + (\mathbb{M}_2+\mathbb{M}_4)(\mathfrak{n}_1+\mathfrak{n}_2)]},$$

and we have that $\sup_{t\in[a,b]}|g(t,0,0)| = N_1$ and $\sup_{t\in[a,b]}|h(t,0,0)| = N_2$.

We first show that $\mathcal{A}B_r \subset B_r$. Note that

$$\begin{aligned} |g(t,y(t),z(t))| &\leq |g(t,y(t),z(t)) - g(t,0,0)| + |g(t,0,0)| \leq \mathfrak{m}_1|y(t)| + \mathfrak{m}_2|z(t)| + N_1 \\ &\leq \mathfrak{m}_1\|y\| + \mathfrak{m}_2\|z\| + N_1, \end{aligned}$$

and, similarly,

$$|h(t,y(t),z(t))| \leq \mathfrak{n}_1\|y\| + \mathfrak{n}_2\|z\| + N_2.$$

Then, for $(y,z)\in B_r$, we have

$$\begin{aligned} &|\mathcal{A}_1(y,z)(t)| \\ \leq\ & (b-a)^{\gamma_1-1}\Big[\frac{(b-a)^{\gamma_2-1}}{|\Phi_3|}\Big({}_aI_{q_1}^{\alpha_1}|g(b,y(b),z(b))| + |\mu_2|\,{}_aI_{q_2}^{\alpha_2-\lambda_2}|h(\eta_2,y(\eta_2),z(\eta_2))| \\ &+|\delta_2|\,{}_aI_{q_2}^{\alpha_2+\epsilon_2}|h(\xi_2,y(\xi_2),z(\xi_2))|\Big) \end{aligned}$$

$$
\begin{aligned}
&+\frac{|\Phi_2|}{|\Phi_3|}\Big({}_aI_{q_2}^{\alpha_2}|h(b,y(b),z(b))|+|\mu_1|\,{}_aI_{q_1}^{\alpha_1-\lambda_1}|g(\eta_1,y(\eta_1),z(\eta_1))| \\
&+|\delta_1|\,{}_aI_{q_1}^{\alpha_1+\epsilon_1}|g(\xi_1,y(\xi_1),z(\xi_1))|\Big)\Big]+{}_aI_{q_1}^{\alpha_1}|g(t,y(t),z(t))| \\
\leq\ & \Bigg\{(b-a)^{\gamma_1-1}\left[\frac{(b-a)^{\gamma_2-1}}{|\Phi_3|}{}_aI_{q_1}^{\alpha_1}(1)(b)+\frac{|\Phi_2|}{|\Phi_3|}\Big(|\mu_1|\,{}_aI_{q_1}^{\alpha_1-\lambda_1}(1)(b)+|\delta_1|\,{}_aI_{q_1}^{\alpha_1+\epsilon_1}(1)(b)\Big)\right] \\
&+{}_aI_{q_1}^{\alpha_1}(1)(b)\Bigg\}(\mathfrak{m}_1\|y\|+\mathfrak{m}_2\|z\|+N_1) \\
&+(b-a)^{\gamma_1-1}\left[\frac{(b-a)^{\gamma_2-1}}{|\Phi_3|}\Big(|\mu_2|\,{}_aI_{q_2}^{\alpha_2-\lambda_2}(1)(b)+|\delta_2|\,{}_aI_{q_2}^{\alpha_2+\epsilon_2}(1)(b)\Big)+\frac{|\Phi_2|}{|\Phi_3|}{}_aI_{q_2}^{\alpha_2}(1)(b)\right] \\
&\times(\mathfrak{n}_1\|y\|+\mathfrak{n}_2\|z\|+N_2) \\
=\ & \mathbb{M}_1(\mathfrak{m}_1\|y\|+\mathfrak{m}_2\|z\|+N_1)+\mathbb{M}_2(\mathfrak{n}_1\|y\|+\mathfrak{n}_2\|z\|+N_2) \\
=\ & (\mathbb{M}_1\mathfrak{m}_1+\mathbb{M}_2\mathfrak{n}_1)\|y\|+(\mathbb{M}_1\mathfrak{m}_2+\mathbb{M}_2\mathfrak{n}_2)\|z\|+\mathbb{M}_1N_1+\mathbb{M}_2N_2 \\
\leq\ & [\mathbb{M}_1(\mathfrak{m}_1+\mathfrak{m}_2)+\mathbb{M}_2(\mathfrak{n}_1+\mathfrak{n}_2)]r+\mathbb{M}_1N_1+\mathbb{M}_2N_2.
\end{aligned}
$$

Similarly, we also obtain

$$
\begin{aligned}
&|\mathcal{A}_2(y,z)(t)| \\
\leq\ & (b-a)^{\gamma_2-1}\left[\frac{(b-a)^{\gamma_1-1}}{|\Phi_3|}\Big(|\mu_1|\,{}_aI_{q_1}^{\alpha_1-\lambda_1}(1)(b)+|\delta_1|\,{}_aI_{q_1}^{\alpha_1+\epsilon_1}(1)(b)\Big)+\frac{|\Phi_1|}{|\Phi_3|}{}_aI_{q_1}^{\alpha_1}(1)(b)\right] \\
&\times(\mathfrak{m}_1\|y\|+\mathfrak{m}_2\|z\|+N_1) \\
&+\Bigg\{(b-a)^{\gamma_2-1}\left[\frac{(b-a)^{\gamma_1-1}}{|\Phi_3|}{}_aI_{q_2}^{\alpha_2}(1)(b)+\frac{|\Phi_1|}{|\Phi_3|}\Big(|\mu_2|\,{}_aI_{q_2}^{\alpha_2-\lambda_2}(1)(b)+|\delta_2|\,{}_aI_{q_2}^{\alpha_2+\epsilon_2}(1)(b)\Big)\right] \\
&+{}_aI_{q_2}^{\alpha_2}(1)(b)\Bigg\}(\mathfrak{n}_1\|y\|+\mathfrak{n}_2\|z\|+N_2) \\
=\ & \mathbb{M}_3(\mathfrak{m}_1\|y\|+\mathfrak{m}_2\|z\|+N_1)+\mathbb{M}_4(\mathfrak{n}_1\|y\|+\mathfrak{n}_2\|z\|+N_2) \\
=\ & (\mathbb{M}_3\mathfrak{m}_1+\mathbb{M}_4\mathfrak{n}_1)\|y\|+(\mathbb{M}_3\mathfrak{m}_2+\mathbb{M}_4\mathfrak{n}_2)\|z\|+\mathbb{M}_3N_1+\mathbb{M}_4N_2 \\
\leq\ & [\mathbb{M}_3(\mathfrak{m}_1+\mathfrak{m}_2)+\mathbb{M}_4(\mathfrak{n}_1+\mathfrak{n}_2)]r+\mathbb{M}_3N_1+\mathbb{M}_4N_2.
\end{aligned}
$$

From the foregoing inequalities, we then conclude that

$$
\begin{aligned}
\|\mathcal{A}(y,z)\| &= \|\mathcal{A}_1(y,z)\|+\|\mathcal{A}_2(y,z)\| \\
&\leq [\mathbb{M}_1(\mathfrak{m}_1+\mathfrak{m}_2)+\mathbb{M}_2(\mathfrak{n}_1+\mathfrak{n}_2)]r+\mathbb{M}_1N_1+\mathbb{M}_2N_2 \\
&\quad +[\mathbb{M}_3(\mathfrak{m}_1+\mathfrak{m}_2)+\mathbb{M}_4(\mathfrak{n}_1+\mathfrak{n}_2)]r+\mathbb{M}_3N_1+\mathbb{M}_4N_2 \\
&\leq r;
\end{aligned}
$$

thus, $\mathcal{A}B_r \subset B_r$. The remaining part is to show that $\mathcal{A}$ is a contraction mapping. For $(y_1,z_1),(y_2,z_2) \in X\times X$, we have

$$
\begin{aligned}
&|\mathcal{A}_1(y_2,z_2)(t)-\mathcal{A}_1(y_1,z_1)(t)| \\
\leq\ & (b-a)^{\gamma_1-1}\Big[\frac{(b-a)^{\gamma_2-1}}{|\Phi_3|}\Big({}_aI_{q_1}^{\alpha_1}|g(b,y_2(b),z_2(b))-g(b,y_1(b),z_1(b))| \\
&+|\mu_2|\,{}_aI_{q_2}^{\alpha_2-\lambda_2}|h(\eta_2,y_2(\eta_2),z_2(\eta_2))-h(\eta_2,y_1(\eta_2),z_1(\eta_2))|
\end{aligned}
$$

$$
\begin{aligned}
& + |\delta_2| \, {}_aI_{q_2}^{\alpha_2+\epsilon_2} |h(\xi_2, y_2(\xi_2), z_2(\xi_2)) - h(\xi_2, y_1(\xi_2), z_1(\xi_2))|\Big) \\
& + \frac{|\Phi_2|}{|\Phi_3|} \Big({}_aI_{q_2}^{\alpha_2} |h(b, y_2(b), z_2(b)) - h(b, y_1(b), z_1(b))| \\
& + |\mu_1| \, {}_aI_{q_1}^{\alpha_1-\lambda_1} |g(\eta_1, y_2(\eta_1), z_2(\eta_1)) - g(\eta_1, y_1(\eta_1), z_1(\eta_1))| \\
& + |\delta_1| \, {}_aI_{q_1}^{\alpha_1+\epsilon_1} |g(\xi_1, y_2(\xi_1), z_2(\xi_1)) - g(\xi_1, y_1(\xi_1), z_1(\xi_1))|\Big)\Big] \\
& + {}_aI_{q_1}^{\alpha_1} |g(t, y_2(t), z_2(t)) - g(t, y_1(t), z_1(t))| \\
\leq \;& (b-a)^{\gamma_1-1} \Bigg[\frac{(b-a)^{\gamma_2-1}}{|\Phi_3|} \Big\{ {}_aI_{q_1}^{\alpha_1} (\mathfrak{m}_1\|y_1 - y_2\| + \mathfrak{m}_2\|z_1 - z_2\|)(b) \\
& + |\mu_2| \, {}_aI_{q_2}^{\alpha_2-\lambda_2} (\mathfrak{n}_1\|y_1 - y_2\| + \mathfrak{n}_2\|z_1 - z_2\|)(b) \\
& + |\delta_2| \, {}_aI_{q_2}^{\alpha_2+\epsilon_2} (\mathfrak{n}_1\|y_1 - y_2\| + \mathfrak{n}_2\|z_1 - z_2\|)(b) \Big\} \\
& + \frac{|\Phi_2|}{|\Phi_3|} \Big\{ {}_aI_{q_2}^{\alpha_2} (\mathfrak{n}_1\|y_1 - y_2\| + \mathfrak{n}_2\|z_1 - z_2\|)(b) \\
& + |\mu_1| \, {}_aI_{q_1}^{\alpha_1-\lambda_1} (\mathfrak{m}_1\|y_1 - y_2\| + \mathfrak{m}_2\|z_1 - z_2\|)(b) \\
& + \; |\delta_1| \, {}_aI_{q_1}^{\alpha_1+\epsilon_1} (\mathfrak{m}_1\|y_1 - y_2\| + \mathfrak{m}_2\|z_1 - z_2\|)(b) \Big\} \Bigg] \\
& + {}_aI_{q_1}^{\alpha_1} (\mathfrak{m}_1\|y_1 - y_2\| + \mathfrak{m}_2\|z_1 - z_2\|)(b) \\
= \;& \mathbb{M}_1(\mathfrak{m}_1\|y_1 - y_2\| + \mathfrak{m}_2\|z_1 - z_2\|) + \mathbb{M}_2(\mathfrak{n}_1\|y_1 - y_2\| + \mathfrak{n}_2\|z_1 - z_2\|) \\
= \;& (\mathbb{M}_1\mathfrak{m}_1 + \mathbb{M}_2\mathfrak{n}_1)\|y_1 - y_2\| + (\mathbb{M}_1\mathfrak{m}_2 + \mathbb{M}_2\mathfrak{n}_2)\|z_1 - z_2\| \\
\leq \;& [\mathbb{M}_1(\mathfrak{m}_1 + \mathfrak{m}_2) + \mathbb{M}_2(\mathfrak{n}_1 + \mathfrak{n}_2)](\|y_1 - y_2\| + \|z_1 - z_2\|),
\end{aligned}
$$

and, similarly,

$$
|\mathcal{A}_2(y_2, z_2)(t) - \mathcal{A}_2(y_1, z_1)(t)| \leq [\mathbb{M}_3(\mathfrak{m}_1 + \mathfrak{m}_2) + \mathbb{M}_4(\mathfrak{n}_1 + \mathfrak{n}_2)](\|y_1 - y_2\| + \|z_1 - z_2\|).
$$

Then, for the operator $\mathcal{A}$, we have

$$
\begin{aligned}
\|\mathcal{A}(y_2, z_2) - \mathcal{A}(y_1, z_1)\| &\leq \|\mathcal{A}(y_1, z_1)\| + \|\mathcal{A}(y_2, z_2)\| \\
&\leq [(\mathbb{M}_1 + \mathbb{M}_3)(\mathfrak{m}_1 + \mathfrak{m}_2) + (\mathbb{M}_2 + \mathbb{M}_4)(\mathfrak{n}_1 + \mathfrak{n}_2)] \\
&\quad \times(\|y_1 - y_2\| + \|z_1 - z_2\|).
\end{aligned}
$$

From the assumption (3.5), the operator $\mathcal{A}$ is a contraction mapping. By using Banach's fixed-point theorem, we have that $\mathcal{A}$ has a unique fixed point, which in turn creates the unique solution of the Hilfer fractional quantum system (1.2) on the interval $[a, b]$, which completes the proof. □

The next existence result relies on the Leray-Schauder alternative (Lemma 2.5).

Theorem 3.2. *Assume that $g, h \in C([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. In addition, we suppose the following:*

(H_2) *There exist real numbers $u_0, v_0 > 0$ and $u_i, v_i \geq 0$ for $i = 1, 2$ such that, for every $t, y_i \in \mathbb{R}$, $i = 1, 2$,*

$$
\begin{aligned}
|g(t, y_1, y_2)| &\leq u_0 + u_1|y_1| + u_2|y_2|, \\
|h(t, y_1, y_2)| &\leq v_0 + v_1|y_1| + v_2|y_2|.
\end{aligned}
$$

If

$$(\mathbb{M}_1+\mathbb{M}_3)u_1+(\mathbb{M}_2+\mathbb{M}_4)v_1<1 \quad \textit{and} \quad (\mathbb{M}_1+\mathbb{M}_3)u_2+(\mathbb{M}_2+\mathbb{M}_4)v_2<1,$$

where $\mathbb{M}_i$, $i=1,2,3,4$ *is as defined in* (3.4)*, then there exists at least one solution for the Hilfer fractional quantum system (1.2) on* $[a,b]$.

Proof. The operator $\mathcal{A}: X\times X\to X\times X$ is continuous since the functions $g,h:[a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous. Next, consider a bounded subset $B_\rho=\{(y,z)\in X\times X:\|(y,z)\|\le\rho\}$ of $X\times X$. Note that, for $(y,z)\in B_\rho$,

$$\begin{aligned}|g(t,y(t),z(t))| &\le u_0+u_1|y|+u_2|z|\le u_0+(u_1+u_2)\rho=L_1,\\ |h(t,y(t),z(t))| &\le v_0+v_1|y|+v_2|z|\le v_0+(v_1+v_2)\rho=L_2.\end{aligned}$$

We will show that $\mathcal{A}B_\rho$ is uniformly bounded. For $(y,z)\in B_\rho$, we obtain

$$\begin{aligned} & |\mathcal{A}_1(y,z)(t)| \\ \le\ & (b-a)^{\gamma_1-1}\Big[\frac{(b-a)^{\gamma_2-1}}{|\Phi_3|}\Big({}_aI_{q_1}^{\alpha_1}|g(b,y(b),z(b))|+|\mu_2|\,{}_aI_{q_2}^{\alpha_2-\lambda_2}|h(\eta_2,y(\eta_2),z(\eta_2))| \\ & +|\delta_2|\,{}_aI_{q_2}^{\alpha_2+\epsilon_2}|h(\xi_2,y(\xi_2),z(\xi_2))|\Big) \\ & +\frac{|\Phi_2|}{|\Phi_3|}\Big({}_aI_{q_2}^{\alpha_2}|h(b,y(b),z(b))|+|\mu_1|\,{}_aI_{q_1}^{\alpha_1-\lambda_1}|g(\eta_1,y(\eta_1),z(\eta_1))| \\ & +|\delta_1|\,{}_aI_{q_1}^{\alpha_1+\epsilon_1}|g(\xi_1,y(\xi_1,z(\xi_1)|)\Big)\Big]+{}_aI_{q_1}^{\alpha_1}|g(t,y(t),z(t))| \\ \le\ & \left\{(b-a)^{\gamma_1-1}\left[\frac{(b-a)^{\gamma_2-1}}{|\Phi_3|}{}_aI_{q_1}^{\alpha_1}(1)(b)+\frac{|\Phi_2|}{|\Phi_3|}\Big(|\mu_1|\,{}_aI_{q_1}^{\alpha_1-\lambda_1}(1)(b)+|\delta_1|\,{}_aI_{q_1}^{\alpha_1+\epsilon_1}(1)(b)\Big)\right]\right. \\ & \left.+\,{}_aI_{q_1}^{\alpha_1}(1)(b)\right\}L_1 \\ & +(b-a)^{\gamma_1-1}\left[\frac{(b-a)^{\gamma_2-1}}{|\Phi_3|}\Big(|\mu_2|\,{}_aI_{q_2}^{\alpha_2-\lambda_2}(1)(b)+|\delta_2|\,{}_aI_{q_2}^{\alpha_2+\epsilon_2}(1)(b)\Big)+\frac{|\Phi_2|}{|\Phi_3|}{}_aI_{q_2}^{\alpha_2}(1)(b)\right]L_2 \\ =\ & \mathbb{M}_1L_1+\mathbb{M}_2L_2, \end{aligned}$$

which implies that

$$\|\mathcal{A}_1(y,z)\|\le\mathbb{M}_1L_1+\mathbb{M}_2L_2.$$

Similarly, we can show that

$$\|\mathcal{A}_2(y,z)\|\le\mathbb{M}_3L_1+\mathbb{M}_4L_2.$$

Therefore, $\mathcal{A}$ is uniformly bounded, as $\|\mathcal{A}(y,z)\|\le(\mathbb{M}_1+\mathbb{M}_3)L_1+(\mathbb{M}_2+\mathbb{M}_4)L_2$. Next, we will prove that $\mathcal{A}$ is equicontinuous. For $\tau_1,\tau_2\in[a,b]$ with $\tau_2>\tau_1$, we have that

$$\begin{aligned} & |\mathcal{A}_1(y,z)(\tau_2)-\mathcal{A}_1(y,z)(\tau_1)| \\ \le\ & \Big|(\tau_2-a)^{\gamma_1-1}-(\tau_1-a)^{\gamma_1-1}\Big|\Big[\frac{(b-a)^{\gamma_2-1}}{|\Phi_3|}\Big({}_aI_{q_1}^{\alpha_1}|g(b,y(b),z(b))| \\ & +|\mu_2|\,{}_aI_{q_2}^{\alpha_2-\lambda_2}|h(\eta_2,y(\eta_2),z(\eta_2))|+|\delta_2|\,{}_aI_{q_2}^{\alpha_2+\epsilon_2}|h(\xi_2,y(\xi_2),z(\xi_2))\Big) \\ & +\frac{|\Phi_2|}{|\Phi_3|}\Big({}_aI_{q_2}^{\alpha_2}|h(b,y(b),z(b))|+|\mu_1|\,{}_aI_{q_1}^{\alpha_1-\lambda_1}|g(\eta_1,y(\eta_1),z(\eta_1))| \end{aligned}$$

$$
\begin{aligned}
& +|\delta_1| \, {}_aI_{q_1}^{\alpha_1+\epsilon_1}|g(\xi_1, y(\xi_1), z(\xi_1)|)\Big)\Big] + \Big|{}_aI_{q_1}^{\alpha_1} g(\tau_2, y(\tau_2), z(\tau_2)) - {}_aI_{q_1}^{\alpha_1} g(\tau_1, y(\tau_1), z(\tau_1))\Big| \\
\leq \;& \Big|(\tau_2 - a)^{\gamma_1-1} - (\tau_1 - a)^{\gamma_1-1}\Big| \Bigg[\frac{(b-a)^{\gamma_2-1}}{|\Phi_3|}\Big(L_1 \, {}_aI_{q_1}^{\alpha_1}(b) + L_2|\mu_2| \, {}_aI_{q_2}^{\alpha_2-\lambda_2}(\eta_2) \\
& +L_2|\delta_2| \, {}_aI_{q_2}^{\alpha_2+\epsilon_2}(\xi_2)\Big) + L_2\frac{|\Phi_2|}{|\Phi_3|}\Big({}_aI_{q_2}^{\alpha_2}(b) + L_1|\mu_1| \, {}_aI_{q_1}^{\alpha_1-\lambda_1}(\eta_1) \\
& +L_1|\delta_1| \, {}_aI_{q_1}^{\alpha_1+\epsilon_1}(\xi_1)\Big)\Bigg] \\
& +\frac{L_1}{\Gamma_q(\alpha_1)}\int_a^{\tau_1}\Big|{}_a(\tau_2 - {}_a\Phi_q(s))_q^{(\alpha_1-1)} - {}_a(\tau_1 - {}_a\Phi_q(s))_q^{(\alpha_1-1)}\Big| \, {}_ad_qs \\
& +\frac{L_1}{\Gamma_q(\alpha_1)}\int_{\tau_1}^{\tau_2}|{}_a(\tau_2 - {}_a\Phi_q(s))_q^{(\alpha_1-1)}| \, {}_ad_qs,
\end{aligned}
$$

which converges to zero as $\tau_1 \to \tau_2$ independently of (y, z).

Similar analysis also yields

$$|\mathcal{A}_2(y, z)(\tau_2) - \mathcal{A}_2(y, z)(\tau_1)| \to 0 \quad \text{as} \quad \tau_1 \to \tau_2.$$

Hence, $\mathcal{A}(y, z)$ is equicontinuous. From the Arzelá-Ascoli theorem, the set $\mathcal{A}$ is relatively compact; thus, the operator $\mathcal{A}$ is completely continuous.

For the final part, we will show that the set

$$\xi_{\mathcal{A}} = \{(y, z) \in X \times X | (y, z) = \lambda\mathcal{A}(y, z) \quad \text{for some} \quad 0 \leq \lambda \leq 1\}$$

is bounded. Consider $(y, z) \in \xi_{\mathcal{A}}$ so that $(y, z) = \lambda\mathcal{A}(y, z)$ for some $\lambda \in [0, 1]$. Then,

$$y(t) = \lambda\mathcal{A}_1(y, z)(t), \quad z(t) = \lambda\mathcal{A}_2(y, z)(t) \quad \text{for all } t \in [a, b].$$

Following the steps of proving the uniform boundedness, and by using (H_2), we can easily derive that

$$
\begin{aligned}
\|y\| &\leq \mathbb{M}_1(u_0 + u_1\|y\| + u_2\|z\|) + \mathbb{M}_2(v_0 + v_1\|y\| + v_2\|z\|), \\
\|z\| &\leq \mathbb{M}_3(u_0 + u_1\|y\| + u_2\|z\|) + \mathbb{M}_4(v_0 + v_1\|y\| + v_2\|z\|),
\end{aligned}
$$

from which we get

$$
\begin{aligned}
\|y\| + \|z\| \leq \;& (\mathbb{M}_1 + \mathbb{M}_3)u_0 + (\mathbb{M}_2 + \mathbb{M}_4)v_0 + [(\mathbb{M}_1 + \mathbb{M}_3)u_1 + (\mathbb{M}_2 + \mathbb{M}_4)v_1]\|y\| \\
& +[(\mathbb{M}_1 + \mathbb{M}_3)u_2 + (\mathbb{M}_2 + \mathbb{M}_4)v_2]\|z\|.
\end{aligned}
$$

By selecting

$$\mathbb{M}_0 = \min\{1 - [(\mathbb{M}_1 + \mathbb{M}_3)u_1 + (\mathbb{M}_2 + \mathbb{M}_4)v_1], 1 - [(\mathbb{M}_1 + \mathbb{M}_3)u_2 + (\mathbb{M}_2 + \mathbb{M}_4)v_2]\},$$

we then obtain the inequality

$$\|(y, z)\| \leq \frac{(\mathbb{M}_1 + \mathbb{M}_3)u_0 + (\mathbb{M}_2 + \mathbb{M}_4)v_0}{\mathbb{M}_0}. \tag{3.6}$$

Hence, the set $\xi_{\mathcal{A}}$ is bounded. Using the Leray-Schauder alternative, we conclude that the operator $\mathcal{A}$ has at least one fixed point, which creates a solution for our Hilfer fractional quantum system (1.2) on $[a, b]$. This completes the proof. □

For the final existence result, we apply Krasnosel'skiĭ's fixed-point theorem (Lemma 2.6).

Theorem 3.3. *Assume that $g, h \in C([a,b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ satisfies (H_1). In addition, we suppose the following:*

(H_3) *There exists nonnegative functions $\mathbb{P}, \mathbb{Q} \in C([a,b], \mathbb{R})$ such that*

$$|g(t,y,z)| \leq \mathbb{P}(t) \quad and \quad |h(t,y,z)| \leq \mathbb{Q}(t) \, for \, (t,x,y) \in [a,b] \times \mathbb{R} \times \mathbb{R}.$$

If

$$(\mathfrak{m}_1 + \mathfrak{m}_2) \frac{(b-a)^{\alpha_1}}{\Gamma_{q_1}(\alpha_1 + 1)} + (\mathfrak{n}_1 + \mathfrak{n}_2) \frac{(b-a)^{\alpha_2}}{\Gamma_{q_2}(\alpha_2 + 1)} < 1, \tag{3.7}$$

then there exists at least one solution for the Hilfer fractional quantum system (1.2) on $[a,b]$.

Proof. We first decompose the operator $\mathcal{A}$ into $\mathcal{A}_1 = \mathcal{A}_{1,1} + \mathcal{A}_{1,2}$ and $\mathcal{A}_2 = \mathcal{A}_{2,1} + \mathcal{A}_{2,2}$ as follows:

$$\begin{aligned}
\mathcal{A}_{1,1}(y,z)(t) &= (t-a)^{\gamma_1 - 1} \Bigg[\frac{(b-a)^{\gamma_2 - 1}}{\Phi_3} \Big(-{}_aI_{q_1}^{\alpha_1} g(b, y(b), z(b)) + \mu_2 \, {}_aI_{q_2}^{\alpha_2 - \lambda_2} h(\eta_2, y(\eta_2), z(\eta_2)) \\
&\quad + \delta_2 \, {}_aI_{q_2}^{\alpha_2 + \epsilon_2} h(\xi_2, y(\xi_2), z(\xi_2)) \Big) + \frac{\Phi_2}{\Phi_3} \Big(-{}_aI_{q_2}^{\alpha_2} h(b, y(b), z(b)) \\
&\quad + \mu_1 \, {}_aI_{q_1}^{\alpha_1 - \lambda_1} g(\eta_1, y(\eta_1), z(\eta_1)) + \delta_1 \, {}_aI_{q_1}^{\alpha_1 + \epsilon_1} g(\xi_1, y(\xi_1), z(\xi_1)) \Big) \Bigg], \quad t \in [a,b], \\
\mathcal{A}_{1,2}(y,z)(t) &= {}_aI_{q_1}^{\alpha_1} g(t, y(t), z(t)), \quad t \in [a,b], \\
\mathcal{A}_{2,1}(y,z)(t) &= (t-a)^{\gamma_2 - 1} \Bigg[\frac{(b-a)^{\gamma_1 - 1}}{\Phi_3} \Big(-{}_aI_{q_2}^{\alpha_2} h(b, y(b), z(b)) + \mu_1 \, {}_aI_{q_1}^{\alpha_1 - \lambda_1} g(\eta_1, y(\eta_1), z(\eta_1)) \\
&\quad + \delta_1 \, {}_aI_{q_1}^{\alpha_1 + \epsilon_1} g(\xi_1, y(\xi_1), z(\xi_1)) \Big) + \frac{\Phi_1}{\Phi_3} \Big(-{}_aI_{q_1}^{\alpha_1} g(b, y(b), z(b)) \\
&\quad + \mu_2 \, {}_aI_{q_2}^{\alpha_2 - \lambda_2} h(\eta_2, y(\eta_2), z(\eta_2)) + \delta_2 \, {}_aI_{q_2}^{\alpha_2 + \epsilon_2} h(\xi_2, y(\xi_2), z(\xi_2)) \Big) \Bigg], \quad t \in [a,b], \\
\mathcal{A}_{2,2}(y,z)(t) &= {}_aI_{q_2}^{\alpha_2} h(t, y(t), z(t)), \quad t \in [a,b].
\end{aligned}$$

Let $B_\delta = \{(y,z) \in X \times X \| \|(y,z)\| \leq \delta\}$ be a closed and bounded ball with

$$\delta \geq (\mathbb{M}_1 + \mathbb{M}_3)\|P\| + (\mathbb{M}_2 + \mathbb{M}_4)\|Q\|.$$

For $(y_1, z_1), (y_2, z_2) \in B_\delta$, as in Theorem 3.2, we have

$$|\mathcal{A}_{1,1}(y_1, z_1)(t) + \mathcal{A}_{1,2}(y_2, z_2)(t)| \leq \mathbb{M}_1\|\mathbb{P}\| + \mathbb{M}_2\|\mathbb{Q}\|$$

and

$$|\mathcal{A}_{2,1}(y_1, z_1)(t) + \mathcal{A}_{2,2}(y_2, z_2)(t)| \leq \mathbb{M}_3\|\mathbb{P}\| + \mathbb{M}_4\|\mathbb{Q}\|.$$

Consequently,

$$\|(\mathcal{A}_{1,1} + \mathcal{A}_{2,1})(y_1, z_1) + (\mathcal{A}_{1,2} + \mathcal{A}_{2,2})(y_2, z_2)\| \leq (\mathbb{M}_1 + \mathbb{M}_3)\|\mathbb{P}\| + (\mathbb{M}_2 + \mathbb{M}_4)\|\mathbb{Q}\| < \delta,$$

which means that $(\mathcal{A}_{1,1} + \mathcal{A}_{2,1})(y_1, z_1) + (\mathcal{A}_{1,2} + \mathcal{A}_{2,2})(y_2, z_2) \in B_\delta$.

Next, we will show that the operator $(\mathcal{A}_{1,1}, \mathcal{A}_{2,1})$ is both compact and continuous. The continuity of $(\mathcal{A}_{1,1}, \mathcal{A}_{2,1})$ follows directly from the fact that the functions g and h are continuous on $[a, b] \times \mathbb{R} \times \mathbb{R}$. Also, for each $(y, z) \in B_\delta$,

$$
\begin{aligned}
|\mathcal{A}_{1,1}(y,z)(t)| &\leq (b-a)^{\gamma_1-1}\Bigg[\frac{(b-a)^{\gamma_2-1}}{|\Phi_3|}\Big(\|\mathbb{P}\|\,{}_aI_{q_1}^{\alpha_1}(1)(b) + \|\mathbb{Q}\||\mu_2|\,{}_aI_{q_2}^{\alpha_2-\lambda_2}(1)(b)\Big) \\
&\quad + \|\mathbb{Q}\||\delta_2|\,{}_aI_{q_2}^{\alpha_2+\epsilon_2}(1)(b) + \frac{|\Phi_2|}{|\Phi_3|}\Big(\|\mathbb{Q}\|\,{}_aI_{q_2}^{\alpha_2}(1)(b) + \|\mathbb{P}\||\mu_1|\,{}_aI_{q_1}^{\alpha_1-\lambda_1}(1)(b)\Big)\Bigg] \\
&\quad + \|\mathbb{P}\||\delta_1|\,{}_aI_{q_1}^{\alpha_1+\epsilon_1}(1)(b) \\
&= \Big(\mathbb{M}_1 - \frac{(b-a)^{\alpha_1}}{\Gamma_{q_1}(\alpha_1+1)}\Big)\|\mathbb{P}\| + \mathbb{M}_2\|\mathbb{Q}\|,
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathcal{A}_{2,1}(y,z)(t)| &\leq (b-a)^{\gamma_2-1}\Bigg[\frac{(b-a)^{\gamma_1-1}}{|\Phi_3|}\Big(\|\mathbb{Q}\|\,{}_aI_{q_2}^{\alpha_2}(1)(b) + \|\mathbb{P}\||\mu_1|{}_aI_{q_1}^{\alpha_1-\lambda_1}(1)(b)\Big) \\
&\quad + \|\mathbb{P}\||\delta_1|\,{}_aI_{q_1}^{\alpha_1+\epsilon_1}(1)(b) + \frac{|\Phi_1|}{|\Phi_3|}\Big(\|\mathbb{P}\|\,{}_aI_{q_1}^{\alpha_1}(1)(b) + \|\mathbb{Q}\||\mu_2|{}_aI_{q_2}^{\alpha_2-\lambda_2}(1)(b)\Big)\Bigg] \\
&\quad + \|\mathbb{Q}\||\delta_2|\,{}_aI_{q_2}^{\alpha_2+\epsilon_2}(1)(b) \\
&= \mathbb{M}_3\|\mathbb{P}\| + \Big(\mathbb{M}_4 - \frac{(b-a)^{\alpha_2}}{\Gamma_{q_2}(\alpha_2+1)}\Big)\|\mathbb{Q}\|.
\end{aligned}
$$

Hence,

$$
\|(\mathcal{A}_{1,1}(y,z) + \mathcal{A}_{2,1}(y,z))\| \leq \Big(\mathbb{M}_1 + \mathbb{M}_3 - \frac{(b-a)^{\alpha_1}}{\Gamma_{q_1}(\alpha_1+1)}\Big)\|\mathbb{P}\| + \Big(\mathbb{M}_2 + \mathbb{M}_4 - \frac{(b-a)^{\alpha_2}}{\Gamma_{q_2}(\alpha_2+1)}\Big)\|\mathbb{Q}\|;
$$

thus, the set $(\mathcal{A}_{1,1}, \mathcal{A}_{2,1})B_\delta$ is uniformly bounded. Furthermore, for any $\tau_1, \tau_2 \in [a, b]$ such that $\tau_1 < \tau_2$, and for any $(y, z) \in B_\delta$, we have

$$
\begin{aligned}
& |\mathcal{A}_{1,1}(y,z)(\tau_2) - \mathcal{A}_{1,1}(y,z)(\tau_1)| \\
\leq\ & \Big|(\tau_2-a)^{\gamma_1-1} - (\tau_1-a)^{\gamma_1-1}\Big|\Bigg[\frac{(b-a)^{\gamma_2-1}}{|\Phi_3|}\Big({}_aI_{q_1}^{\alpha_1}|g(b, y(b), z(b))| \\
& + |\mu_2|{}_aI_{q_2}^{\alpha_2-\lambda_2}|h(\eta_2, y(\eta_2), z(\eta_2))| + |\delta_2|{}_aI_{q_2}^{\alpha_2+\epsilon_2}|h(\xi_2, y(\xi_2), z(\xi_2))|\Big) \\
& + \frac{|\Phi_2|}{|\Phi_3|}\Big({}_aI_{q_2}^{\alpha_2}|h(b, y(b), z(b))| + |\mu_1|{}_aI_{q_1}^{\alpha_1-\lambda_1}|g(\eta_1, y(\eta_1), z(\eta_1))| \\
& + |\delta_1|{}_aI_{q_1}^{\alpha_1+\epsilon_1}|g(\xi_1, y(\xi_1), z(\xi_1))|\Big)\Bigg] \\
\leq\ & \Big|(\tau_2-a)^{\gamma_1-1} - (\tau_1-a)^{\gamma_1-1}\Big|\Bigg[\frac{(b-a)^{\gamma_2-1}}{|\Phi_3|}\Big(\|\mathbb{P}\|\,{}_aI_{q_1}^{\alpha_1}(1)(b) + \|\mathbb{Q}\||\mu_2|\,{}_aI_{q_2}^{\alpha_2-\lambda_2}(1)(\eta_2) \\
& + \|\mathbb{Q}\||\delta_2|\,{}_aI_{q_2}^{\alpha_2+\epsilon_2}(1)(\xi_2)\Big) + \frac{|\Phi_2|}{|\Phi_3|}\Big(\|\mathbb{Q}\|\,{}_aI_{q_2}^{\alpha_2}(1)(b) + \|\mathbb{P}\||\mu_1|\,{}_aI_{q_1}^{\alpha_1-\lambda_1}(1)(\eta_1) \\
& + \|\mathbb{P}\||\delta_1|\,{}_aI_{q_1}^{\alpha_1+\epsilon_1}(1)(\xi_1)\Big)\Bigg],
\end{aligned}
$$

which converges to zero as $\tau_1 \to \tau_2$ independently of $(y,z) \in B_\delta$. Similarly, we can prove that $|\mathcal{A}_{2,1}(y,z)(\tau_2) - \mathcal{A}_{2,1}(y,z)(\tau_1)| \to 0$ as $\tau_1 \to \tau_2$ independently of $(y,z) \in B_\delta$. Hence,

$$|(\mathcal{A}_{1,1},\mathcal{A}_{2,1})(y,z)(\tau_2) - (\mathcal{A}_{1,1},\mathcal{A}_{2,1})(y,z)(\tau_1)| \to 0 \quad \text{as} \quad \tau_1 \to \tau_2,$$

which implies that the set $(\mathcal{A}_{1,1},\mathcal{A}_{2,1})B_\delta$ is equicontinuous. By the Arzelá-Ascoli theorem, we deduce that the operator $(\mathcal{A}_{1,1},\mathcal{A}_{2,1})$ is compact.

For the final step, we will show that the operator $(\mathcal{A}_{1,2},\mathcal{A}_{2,2})$ is a contraction mapping. Let us consider $(y_1,z_1),(y_2,z_2) \in B_\delta$. From the hypothesis (H_1), we obtain

$$\begin{aligned}
|\mathcal{A}_{1,2}(y_1,z_1)(t) - \mathcal{A}_{1,2}(y_2,z_2)(t)| &\leq {}_aI_{q_1}^{\alpha_1}|g(t,y_2(t),z_2(t)) - g(t,y_1(t),z_1(t))| \\
&\leq (\mathfrak{m}_1\|y_1 - y_2\| + \mathfrak{m}_2\|z_1 - z_2\|)\, {}_aI_{q_1}^{\alpha_1}(1)(b) \\
&\leq (\mathfrak{m}_1 + \mathfrak{m}_2)\frac{(b-a)^{\alpha_1}}{\Gamma_{q_1}(\alpha_1+1)}(\|y_1 - y_2\| + \|z_1 - z_2\|).
\end{aligned}$$

Also,

$$\begin{aligned}
|\mathcal{A}_{2,2}(y_1,z_1)(t) - \mathcal{A}_{2,2}(y_2,z_2)(t)| &\leq {}_aI_{q_2}^{\alpha_1}|h(t,y_2(t),z_2(t)) - h(t,y_1(t),z_1(t))| \\
&\leq (\mathfrak{n}_1 + \mathfrak{n}_2)\frac{(b-a)^{\alpha_2}}{\Gamma_{q_2}(\alpha_2+1)}(\|y_1 - y_2\| + \|z_1 - z_2\|).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|(\mathcal{A}_{1,2},\mathcal{A}_{2,2})(y_1,z_1) - (\mathcal{A}_{1,2},\mathcal{A}_{2,2})(y_2,z_2)\| \\
\leq\ &\Big[(\mathfrak{m}_1 + \mathfrak{m}_2)\frac{(b-a)^{\alpha_1}}{\Gamma_{q_1}(\alpha_1+1)} + (\mathfrak{n}_1 + \mathfrak{n}_2)\frac{(b-a)^{\alpha_2}}{\Gamma_{q_2}(\alpha_2+1)})\Big](\|y_1 - y_2\| + \|z_1 - z_2\|).
\end{aligned}$$

By the inequality in (3.7), the operator $(\mathcal{A}_{1,2},\mathcal{A}_{2,2})$ is a contraction mapping. Using Krasnosel'skiĭ's fixed-point theorem, we conclude that there exists at least one solution for the Hilfer fractional quantum system (1.2) on $[a,b]$. The proof is now completed. □

Remark 3.1. *By interchanging the roles of the operators $\mathcal{A}_{1,1},\mathcal{A}_{1,2}$ and $\mathcal{A}_{2,1},\mathcal{A}_{2,2}$ in the foregoing result, we can obtain a second existence result by replacing condition (3.7) with the following:*

$$\Big(\mathbb{M}_1 - \frac{(b-a)^{\alpha_1}}{\Gamma_{q_1}(\alpha_1+1)} + \mathbb{M}_3\Big)(\mathfrak{m}_1 + \mathfrak{m}_2) + \Big(\mathbb{M}_2 + \mathbb{M}_4 - \frac{(b-a)^{\alpha_2}}{\Gamma_{q_2}(\alpha_2+1)}\Big)(\mathfrak{n}_1 + \mathfrak{n}_2) < 1.$$

4. Examples

In this section, we will show some applications of our results to the nonlinear Hilfer fractional quantum system with Riemann-Liouville fractional derivatives and integral boundary conditions of the following form:

$$
\begin{cases}
\left({}^{H}_{\frac{1}{8}}D^{\frac{5}{4},\frac{1}{3}}_{\frac{2}{5}}y\right)(t) = g\left(t, y(t), z(t)\right), & \dfrac{1}{8} < t < \dfrac{11}{8}, \\
\left({}^{H}_{\frac{1}{8}}D^{\frac{7}{4},\frac{2}{3}}_{\frac{4}{5}}z\right)(t) = h\left(t, y(t), z(t)\right), & \dfrac{1}{8} < t < \dfrac{11}{8}, \\
y\left(\dfrac{1}{8}\right) = 0, \quad y\left(\dfrac{11}{8}\right) = \dfrac{1}{\pi^2}\left({}^{RL}_{\frac{1}{8}}D^{\frac{3}{5}}_{\frac{4}{5}}z\right)\left(\dfrac{7}{8}\right) + \dfrac{2}{33}\left({}_{\frac{1}{8}}I^{\frac{5}{6}}_{\frac{4}{5}}z\right)\left(\dfrac{9}{8}\right), \\
z\left(\dfrac{1}{8}\right) = 0, \quad z\left(\dfrac{11}{8}\right) = \dfrac{3}{44}\left({}^{RL}_{\frac{1}{8}}D^{\frac{1}{5}}_{\frac{2}{5}}y\right)\left(\dfrac{3}{8}\right) + \dfrac{1}{e^2}\left({}_{\frac{1}{8}}I^{\frac{7}{6}}_{\frac{2}{5}}y\right)\left(\dfrac{5}{8}\right).
\end{cases} \tag{4.1}
$$

Here, we set $\alpha_1 = 5/4$, $\alpha_2 = 7/4$, $\beta_1 = 1/3$, $\beta_2 = 2/3$, $q_1 = 2/5$, $q_2 = 4/5$, $a = 1/8$, $b = 11/8$, $\eta_1 = 3/8$, $\eta_2 = 7/8$, $\xi_1 = 5/8$, $\xi_2 = 9/8$, $\lambda_1 = 1/5$, $\lambda_2 = 3/5$, $\epsilon_1 = 7/6$, $\epsilon_2 = 5/6$, $\mu_1 = 3/44$, $\mu_2 = 1/\pi^2$, $\delta_1 = 1/e^2$ and $\delta_2 = 2/33$. From these values, we can compute that $\gamma_1 = 3/2$ and $\gamma_2 = 23/12$; using WolframAlpha, we obtain the following: $\Gamma_{q_1}(\gamma_1) \approx 0.9303873679$, $\Gamma_{q_2}(\gamma_2) \approx 0.9715412324$, $\Gamma_{q_1}(\gamma_1 - \lambda_1) \approx 0.9353312130$, $\Gamma_{q_2}(\gamma_2 - \lambda_2) \approx 0.9055725943$, $\Gamma_{q_1}(\gamma_1 + \epsilon_1) \approx 1.2293719126$, $\Gamma_{q_2}(\gamma_2 + \epsilon_2) \approx 0.9622467967$, $\Gamma_{q_1}(\alpha_1 + 1) \approx 1.0689410188$, $\Gamma_{q_1}(\alpha_1 - \lambda_1 + 1) \approx 1.0121177344$, $\Gamma_{q_1}(\alpha_1 + \epsilon_1 + 1) \approx 1.6721250919$, $\Gamma_{q_2}(\alpha_2 + 1) \approx 1.5005722384$, $\Gamma_{q_2}(\alpha_2 - \lambda_2 + 1) \approx 1.0631833692$, $\Gamma_{q_2}(\alpha_2 + \epsilon_2 + 1) \approx 2.9452632392$; using Maple, we obtain the following: $\Phi_1 \approx 0.07700624048$, $\Phi_2 \approx 0.1604286114$, $\Phi_3 \approx 1.359440900$, $\mathbb{M}_1 \approx 2.513723742$, $\mathbb{M}_2 \approx 0.2911846155$, $\mathbb{M}_3 \approx 0.3119090080$, $\mathbb{M}_4 \approx 1.989599422$.

(i) Let the functions g and h that appear in problem (4.1) be the nonlinear unbounded functions on $[1/8, 11/8]$ that are respectively defined by

$$
\begin{cases}
g(t, y, z) = \dfrac{1}{4(4t+5)}\left(\dfrac{y^2 + 2|y|}{1+|y|}\right) + \dfrac{e^{-(8t-1)^2}}{12}\sin|z| + \dfrac{1}{2}, \\
h(t, y, z) = \dfrac{1}{9}\cos^2 \pi t \tan^{-1} y + \dfrac{1}{3(8t+9)}\left(\dfrac{2z^2 + 3|z|}{1+|z|}\right) + \dfrac{1}{3}.
\end{cases} \tag{4.2}
$$

Then, g and h satisfy the Lipschitz condition, (H_1), as follows:

$$
|g(t, y_1, z_1) - g(t, y_2, z_2)| \le \frac{1}{11}|y_1 - y_2| + \frac{1}{12}|z_1 - z_2|
$$

and

$$
|h(t, y_1, z_1) - h(t, y_2, z_2)| \le \frac{1}{9}|y_1 - y_2| + \frac{1}{10}|z_1 - z_2|,
$$

with $m_1 = 1/11$, $m_2 = 1/12$, $n_1 = 1/9$, $n_2 = 1/10$. Therefore, the following inequality holds:

$$
(\mathbb{M}_1 + \mathbb{M}_3)(\mathfrak{m}_1 + \mathfrak{m}_2) + (\mathbb{M}_2 + \mathbb{M}_4)(\mathfrak{n}_1 + \mathfrak{n}_2) \approx 0.9738439529 < 1.
$$

Thus, (3.5) is true. By Theorem 3.1, this system (4.1), with g and h defined in (4.2), has a unique solution on $[1/8, 11/8]$.

(ii) Now, assume that the functions g and h in (4.1) are respectively given by

$$
\begin{cases}
g(t, y, z) = \dfrac{2}{16t+1} + \dfrac{|y|^{2023}}{5(1+y^{2022})}e^{-z^2} + \dfrac{1}{7}\left(\dfrac{y^4\sin^8 y}{1+y^4}\right)z, \\
h(t, y, z) = \dfrac{2}{11}t + \dfrac{1}{3\pi}\left(\dfrac{|z|^5\tan^{-1} z}{1+|z|^5}\right)y + \dfrac{1}{4}\left(\dfrac{z^{2024}}{1+|z|^{2023}}\right)\cos^{12} y.
\end{cases} \tag{4.3}
$$

We can see that both functions do not satisfy the Lipschitz condition. However, we can find the bounded plane of each of them to be as follows:

$$|g(t, y_1, y_2)| \leq \frac{2}{3} + \frac{1}{5}|y_1| + \frac{1}{7}|y_2|$$

and

$$|h(t, y_1, y_2)| \leq \frac{1}{4} + \frac{1}{6}|y_1| + \frac{1}{4}|y_2|,$$

which satisfy condition (H_2) with $u_0 = 2/3$, $u_1 = 1/5$, $u_2 = 1/7$, $v_0 = 1/4$, $v_1 = 1/6$ and $v_2 = 1/4$. Since

$$(\mathbb{M}_1 + \mathbb{M}_3)u_1 + (\mathbb{M}_2 + \mathbb{M}_4)v_1 \approx 0.9452572230 < 1$$

and

$$(\mathbb{M}_1 + \mathbb{M}_3)u_2 + (\mathbb{M}_2 + \mathbb{M}_4)v_2 \approx 0.9738578309 < 1,$$

by Theorem 3.2, this system (4.1), with g and h given in (4.3), has at least one solution on $[1/8, 11/8]$.

(*iii*) Finally, let g and h in (4.1) be the nonlinear functions respectively given by

$$\begin{cases} g(t, y, z) = \dfrac{1}{2(8t+1)}\left(\dfrac{|y|}{1+|y|}\right) + \dfrac{e^{-(8t-1)^2}}{5}\sin|z| + \dfrac{1}{2}, \\ h(t, y, z) = \dfrac{1}{5}\cos^2 \pi t \tan^{-1} y + \dfrac{1}{3(8t+1)}\left(\dfrac{|z|}{1+|z|}\right) + \dfrac{1}{3}. \end{cases} \tag{4.4}$$

Then, we have

$$|g(t, y_1, y_2)| \leq \frac{1}{2(8t+1)} + \frac{e^{-(8t-1)^2}}{5} + \frac{1}{2} := P(t)$$

and

$$|h(t, y_1, y_2)| \leq \frac{\pi}{10}\cos^2 \pi t + \frac{1}{3(8t+1)} + \frac{1}{3} := Q(t).$$

Thus, the condition (H_3) in Theorem 3.3 is satisfied. In addition, the Lipschitz condition is satisfied since

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq \frac{1}{4}|y_1 - y_2| + \frac{1}{5}|z_1 - z_2|$$

and

$$|h(t, y_1, z_1) - h(t, y_2, z_2)| \leq \frac{1}{5}|y_1 - y_2| + \frac{1}{6}|z_1 - z_2|,$$

with $m_1 = 1/4$, $m_2 = 1/5$, $n_1 = 1/5$, $n_2 = 1/6$. Since

$$(\mathfrak{m}_1 + \mathfrak{m}_2)\frac{(b-a)^{\alpha_1}}{\Gamma_{q_1}(\alpha_1 + 1)} + (\mathfrak{n}_1 + \mathfrak{n}_2)\frac{(b-a)^{\alpha_2}}{\Gamma_{q_2}(\alpha_2 + 1)} \approx 0.9174947461 < 1,$$

by Theorem 3.3, this system (4.1), with g and h given in (4.4), has at least one solution on $[1/8, 11/8]$.

It is important to notice that the functions g and h given in (4.4), although they satisfy the Lipschitz condition, do not guarantee uniqueness because

$$(\mathbb{M}_1 + \mathbb{M}_3)(\mathfrak{m}_1 + \mathfrak{m}_2) + (\mathbb{M}_2 + \mathbb{M}_4)(\mathfrak{n}_1 + \mathfrak{n}_2) \approx 2.107822219 > 1.$$

5. Conclusions

We investigated a new problem that appeared for the first time in the literature by combining Hilfer quantum and Riemann-Liouville fractional derivative operators with q-Riemann-Liouville inregral operators. We established the existence and uniqueness of solutions to a coupled system involving Hilfer fractional quantum derivatives supplemented by nonlocal boundary value conditions containing both q-Riemann-Liouville fractional derivatives and integrals. We first transformed the given system into a fixed-point problem by using its linear variant. Then, we established the existence and uniqueness of a solution via Banach's contraction mapping principle; we also obtained two existence results by using the Leray-Schauder alternative and Krasnosel'skiĭ's fixed-point theorem. The applicability of these theoretical results are demonstrated through constructed numerical examples. These new results will enrich the literature on this new topic of research. Also, by appropriately fixing the parameters involved in the problem, our results imply several results on the existence and uniqueness for other coupled systems. Consider the following examples: (i) a coupled system of Hilfer fractional quantum differential equations with q-Riemann-Liouville fractional derivatives is obtained when $\delta_1 = \delta_2 = 0$; (ii) a coupled system of Hilfer fractional quantum differential equations with q-Riemann-Liouville fractional integrals is obtained when $\mu_1 = \mu_2 = 0$; (iii) a coupled system of Hilfer fractional quantum differential equations with mixed q-Riemann-Liouville fractional derivatives and integrals is obtained when either δ_1, δ_2 or μ_1, μ_2 is zero.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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