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Research article

Dual ideal theory on L-algebras

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Abstract: This paper aims to study bounded algebras in another perspective-dual ideals of bounded L-algebras. As the dual concept of ideals in L-algebras, dual ideals are designed to characterize some significant properties of bounded L-algebras. We begin by providing a definition of dual ideals and discussing the relationships between ideals and dual ideals. Then, we prove that these dual ideals induce congruence relations and quotient L-algebras on bounded L-algebras. Naturally, in order to construct the first isomorphism theorem between bounded L-algebras, the relationship between dual ideals and morphisms between bounded L-algebras is investigated and that the kernels of any morphisms between bounded L-algebras are dual ideals is proven. Fortunately, although the first isomorphism theorem between arbitrary bounded L-algebras fails to be proven when using dual ideals, the theorem was proven when the range of morphism was good. Another main purpose of this study is to use dual ideals to characterize several kinds of bounded L-algebras. Therefore, first, the properties of dual ideals in some special bounded L-algebras are studied; then, some special bounded L-algebras are characterized by dual ideals. For example, a good L-algebra is a CL-algebra if and only if every dual ideal is C dual ideal is proven.

Keywords: L-algebra; dual ideal; isomorphism theorem; commutative dual ideal
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1. Introduction

Based on the L equation, the basic definition of L-algebras was introduced by German mathematician Wolfgang Rump in 2008 [1]. The purpose of this definition was to address the problem related to the set-theoretic solutions of the quantum Yang-Baxter equation. The L equation, which can be found in earlier papers [2, 3], has historically received little attention and has been virtually ignored. In [1], Rump studied the conditions under which an algebra can be a monoid and showed that every L-algebra has a self-similar closure. This allows us to use the properties of self-similar L-algebras to study L-algebras themselves. In recent years, L-algebras have attracted many scholars, who have conducted thorough research studies on L-algebras. In 2019, Wu et al. constructed lattice-ordered effect algebras in [4]. They proved that every lattice-ordered effect algebra can be generated from an L-algebra having the same ortho-complement. In 2020, Wu and Yang [5] proved that the monoid operation in the self-similar closure of OM-L-algebra is commutative and also provided the necessary and sufficient conditions for a KL-algebra to be a Boolean algebra. In 2021, Ciungu [6] discussed the relationships between some special L-algebras and normal L-algebras and claimed that L-algebras can be generated from other structures (such as BCK-algebras and pseudo MV-algebras). In 2022, Xin et al. [7] introduced the concept of pseudo L-algebras and studied their pseudo self-similar closures and structure groups.

When investigating algebraic structures, the study of ideals and filters is crucial. Mostly, they are used in inducing congruences relations and characterizing algebras. Therefore, studying various types of ideals and filters is crucial for the research of algebra itself. In [1], Rump provided the definition of ideals in L-algebras, and studied congruence relations and quotient algebras induced by ideals. In [8], Hua proposed state operators on L-algebras, as well as provided the concept of state L-algebras and state ideals and characterized a kind of state L-algebra by using state ideals. Ciungu defined a class of special ideals on L-algebras (commutative ideals) and studied the relationship between commutative ideals and CL-algebras in [6]. Iseki proposed the notion of prime ideals for commutative BCK-algebras in [9]. In [10], prime ideals in BCI-algebras were further developed, and various properties of prime ideals in BCI-algebras and BCK-algebras were verified, such as the relationship between prime ideals and maximal ideals. Ideals focus on fuzzy events with low truth values. The aforementioned ideals are the kinds of dual ideals in an L-algebra we are going to study. Thus, we can provide a new idea to study L-algebras by using dual ideals.

The main purpose of this article is to study L-algebras and its corresponding structure by using dual ideals. This article is organized as follows. In Section 2, we review some definitions and properties of L-algebras. In Section 3, the concept of dual ideals on bounded L-algebras is proposed. First, we give some properties of dual ideals. In addition, we discuss the connection between dual ideals and ideals in L-algebras. Furthermore, we prove that every dual ideal can induce both an equivalence relation and a congruence relation. In Section 4, we study the relationship between dual ideals and morphisms between L-algebras. We prove that for any morphism f from L-algebra L_1 to L-algebra L_2, then $\mathrm{Im}\, f$ is an L-algebra. Moreover, if L_1 and L_2 are both bounded L-algebras, then the inverse image of every dual ideal is a dual ideal as well. Using this fact, we find some connections between dual ideals and congruence relations that can induce quotient algebras. Furthermore, we prove that there is a one-to-one correspondence between dual ideals and congruence relations that can induce quotient algebras in involution L-algebras. Moreover, the isomorphism theorem is proven when L_2 is a good L-algebra. In Section 5, we introduce four special types of dual ideals on L-algebras. Then, we provide equivalent characterizations of special L-algebras (such as KL-algebras and CL-algebras) by using special dual ideals on good L-algebras.

2. Preliminaries

In this section, we recall some fundamental results regarding L-algebras.

Definition 2.1. *[1] An L-algebra is an algebra of type* $(2,0)$*, denoted by* $L = (L, \to, 1)$*, that satisfies the following conditions for any* $r, s, t \in L$*:*
$(L1)$ $r \to r = r \to 1 = 1$, $1 \to r = r$;
$(L2)$ $(r \to s) \to (r \to t) = (s \to r) \to (s \to t)$;
$(L3)$ $r \to s = s \to r = 1 \Rightarrow r = s$.
Condition $(L1)$ *asserts that* 1 *is a logical unit, which is unique. A partial ordering is defined by* $r \leq s \Leftrightarrow r \to s = 1$.

Definition 2.2. *[11] If an L-algebra L admits a smallest element* 0*, L is called a bounded L-algebra. If the mapping* $r \mapsto r'$ *is bijective, where* $r' = r \to 0$*, the bounded L-algebra L is called an L-algebra with negation. The inverse mapping is denoted by* $r \mapsto \widetilde{r}$.

Proposition 2.1. *[11] Let L be an L-algebra. If* $r \leq s$*, then* $t \to r \leq t \to s$ *for all* $r, s, t \in L$.

Proposition 2.2. *[11] Let L be an L-algebra. Then the following conditions are equivalent:*
(1) $s \leq r \to s$ *for all* $r, s \in L$;
(2) $r \leq t \Rightarrow t \to s \leq r \to s$ *for all* $r, s, t \in L$;
(3) $((r \to s) \to t) \to t \leq ((r \to s) \to t) \to ((s \to r) \to t)$ *for all* $r, s, t \in L$.

Definition 2.3. *[1] We call* $(L, \to, 1)$ *a semiregular L-algebra if it is an L-algebra and satisfies* $((r \to s) \to t) \to ((s \to r) \to t) = ((r \to s) \to t) \to t$ *for all* $r, s, t \in L$.

Proposition 2.3. *[11] Let L be a semiregular L-algebra with negation. Then* $r \leq s \Leftrightarrow s' \leq r'$ *holds for all* $r, s \in L$.

Proposition 2.4. *[11] Let L be a semiregular L-algebra. If* $r \to t \geq s \to t$*, then* $(r \to s) \to (r \to t) = r \to t$ *for all* $r, s, t \in L$.

Proposition 2.5. *[1] Let L be an L-algebra, and* $r, s \in L$*. Then* $r = s$ *if and only if* $r \to t = s \to t$ *for all* $t \in L$.

Definition 2.4. *[1] A monoid H equipped with a binary operation* $\to$ *is said to be a left hoop provided that the following conditions are satisfied for any* $r, s, t \in H$*:*
$(H1)$ $r \to r = 1$;
$(H2)$ $r \cdot s \to t = r \to (s \to t)$;
$(H3)$ $(r \to s) \cdot r = (s \to r) \cdot s$.

Definition 2.5. *[12] A self-similar L-algebra L can be described equationally as a monoid with a second operation* $\to$ *satisfying* $r, s, t \in L$*:*
(1) $s \to r \cdot s = r$;
(2) $r \cdot s \to t = r \to (s \to t)$;
(3) $(r \to s) \cdot r = (s \to r) \cdot s$.

Proposition 2.6. *[6] If L is a self-similar L-algebra, then the following statements hold for all* $r, s, t \in L$*:*
(1) $r \cdot s \leq t \Leftrightarrow r \leq s \to t$;
(2) $(r \to s) \cdot r \leq r, s$;
(3) $r \to s = r \cdot t \to s \cdot t$.

Definition 2.6. *[4] Let $(L, \rightarrow, 1)$ be an L-algebra. We call $F \subseteq L$ an ideal if the following conditions hold for all $r, s \in L$:*
$(F1)$ $1 \in F$;
$(F2)$ $r, r \rightarrow s \in F \Rightarrow s \in F$;
$(F3)$ $r \in F \Rightarrow (r \rightarrow s) \rightarrow s \in F$;
$(F4)$ $r \in F \Rightarrow s \rightarrow r, s \rightarrow (r \rightarrow s) \in F$.
If L satisfies the equation $r \rightarrow (s \rightarrow r) = 1$, then property $(F4)$ is no longer required.

Definition 2.7. *[6] Let $(L, \rightarrow, 1)$ be an L-algebra.*
(1) *If L satisfies condition K: $r \rightarrow (s \rightarrow r) = 1$, then L is called a KL-algebra.*
(2) *If L satisfies condition C: $(r \rightarrow (s \rightarrow t)) \rightarrow (s \rightarrow (r \rightarrow t)) = 1$, then L is called a CL-algebra.*

Proposition 2.7. *[6] Let $(L, \rightarrow, 1)$ be an L-algebra. If L satisfies condition C, then the following condition holds for all $x, y \in L$:*
(D) $r \rightarrow ((r \rightarrow s) \rightarrow s) = 1$.

3. Dual ideal of L-algebras

After preparing some preliminaries for our work in Section 2, we start our main work in this section. The main purpose of this section is to provide a definition of dual ideals that are within bounded *L*-algebras and to discuss the relationship between dual ideals and congruence relations.

Definition 3.1. *Let L be a bounded L-algebra. A subset I of L is called a dual ideal if it satisfies the following conditions for any elements r, s of L:*
$(I1)$ $0 \in I$;
$(I2)$ *if $r' \in I$ and $(r \rightarrow s)' \in I$, then $s' \in I$;*
$(I3)$ *if $r' \in I$, then $((r \rightarrow s) \rightarrow s)' \in I$;*
$(I4)$ *if $r' \in I$, then $(s \rightarrow r)' \in I$ and $(s \rightarrow (r \rightarrow s))' \in I$;*
$(I5)$ *if $r \leq s$ and $s \in I$, then $r \in I$.*
Let $Id(L)$ be the set of all dual ideals of L. A dual ideal I is said to be proper if $I \neq L$.

Remark 3.1. *If L satisfies condition (K), then $(I4)$ can be omitted. In fact, assuming that (K) holds and $s' \in I$, it follows from $(I2)$ that $(t \rightarrow s)' \in I$ since $(s \rightarrow (t \rightarrow s))' = 0 \in I$. By (K), we have $(t \rightarrow (s \rightarrow t))' = 0 \in I$. Therefore, $(I4)$ is also satisfied.*

Example 3.1. *Let $L = \{0, e, f, g, 1\}$. The operation $\rightarrow$ on L is defined by Table 1. Then, $(L, \rightarrow, 0, 1)$ is a bounded L-algebra and the partial order is determined by L is $0 \leq e \leq f \leq g \leq 1$. We can check that all dual ideals of L are $I_1 = \{0\}, I_2 = \{0, e\}$ and L.*

Table 1. Cayley table for the binary operation "→".

$\rightarrow$	0	e	f	g	1
0	1	1	1	1	1
e	f	1	1	1	1
f	0	e	1	1	1
g	0	e	f	1	1
1	0	e	f	g	1

Remark 3.2. *Although the identity $r' = r'''$ holds in residuated lattices, BE-algebras, and many other logic algebras, it may not generally hold in L-algebras as shown in Example 3.1 that $e' = f \neq 1 = e'''$. However, there is an interesting result stating that if $r' \in I$, then $r''' \in I$ for any dual ideal I of a bounded L-algebra. The proof directly follows the definition (I3) of dual ideals.*

Remark 3.3. *Suppose L is a bounded L-algebra. It is routine to prove that the intersection of any dual ideals is also a dual ideal. However, it is important to note that the union of dual ideals is not necessarily a dual ideal of L, as illustrated by the following example.*

Example 3.2. *[8] Let $L = \{0, m, n, 1\}$. The operation $\rightarrow$ on L is defined by Table 2. It can be checked that L is a bounded L-algebra and all dual ideals of L are $I_1 = \{0\}, I_2 = \{0, m\}, I_3 = \{0, n\}$, and L. We can see that the union of I_2 and I_3 is not a dual ideal anymore.*

Table 2. Cayley table for the binary operation "$\rightarrow$".

$\rightarrow$	0	m	n	1
0	1	1	1	1
m	n	1	n	1
n	m	m	1	1
1	0	m	n	1

We observe that $\{0\}$ is not necessarily a dual ideal in L-algebras, as demonstrated in the following example. However, we have discovered a sufficient condition, known as the "good condition" (i.e., $r' = 0$ if and only if $r = 1$), under which 0 becomes a dual ideal. Notably, the good condition is weaker than the involution. A bounded L-algebra that satisfies the good condition is referred to as a good L-algebra. Example 3.5 illustrates that the class of involutive L-algebras is a proper subclass of good L-algebras. Sections 4 and 5 will delve into further properties of good L-algebras.

Example 3.3. *Let $L = \{0, a, b, c, 1\}$. The operation $\rightarrow$ on L is defined by Table 3. Then, L is a bounded L-algebra and $\{0\}$ is not a dual ideal of L since $b' \in \{0\}$ and $(c \rightarrow b)' = b \notin \{0\}$.*

Table 3. Cayley table for the binary operation "$\rightarrow$".

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	a	1	a	1
c	0	a	a	1	1
1	0	a	b	c	1

Good L-algebras hold a significant position in our paper as a special type of bounded L-algebras. It is evident that any involutive L-algebra is also good, but the converse is not true, as demonstrated in Example 3.5. We observe that the bounded L-algebra in Example 3.2 satisfies the good condition. However, it is important to note that not all bounded L-algebras are necessarily good, as illustrated in Example 3.4.

Example 3.4. *Let $L = \{0, e, f, g, h, 1\}$. The binary operation $\to$ on L is shown in Table 4. Then, L is a bounded L-algebra and partial order induced by the operation is $0 \leq f \leq e \leq 1$ and $0 \leq h \leq g \leq e \leq 1$. L is not good since $e' = 0$.*

Table 4. Cayley table for the binary operation "$\to$".

$\to$	0	e	f	g	h	1
0	1	1	1	1	1	1
e	0	1	f	g	h	1
f	g	1	1	g	g	1
g	f	1	f	1	e	1
h	f	1	f	1	1	1
1	0	e	f	g	h	1

Example 3.5. *Let $L = \{0, e, f, g, 1\}$. The operation $\to$ on L is defined by Table 5. Then, L is a bounded L-algebra. Since $f'' = e$, L is good but not involutive.*

Table 5. Cayley table for the binary operation "$\to$".

$\to$	0	e	f	g	1
0	1	1	1	1	1
e	e	1	e	e	1
f	e	e	1	e	1
g	e	e	e	1	1
1	0	e	f	g	1

In the following two propositions, we will delve into the relationships between dual ideals and ideals in bounded L-algebras. To facilitate this discussion, we introduce the notation $N(A) = \{a \in L \mid a' \in A \subseteq L\}$ and refer to $N(A)$ as the complement of A in any bounded L-algebra. We will now employ this notation to establish the correspondence between these concepts.

Proposition 3.1. *Let L be a bounded L-algebra and $I \in Id(L)$. Then, $N(I)$ is an ideal of L.*

Proof. It is straightforward.

Proposition 3.2. *Let L be a bounded L-algebra satisfying that $(r \to s)'' = r'' \to s''$ and the operation$'$ is antitone. If F is an ideal of L, then $N(F) \in Id(L)$.*

Proof. Let F be an ideal of L. Since $0' = 1 \in F$, $0 \in N(F)$. Let $r', (r \to s)' \in N(F)$. Then $r'' \in F$ and $r'' \to s'' = (r \to s)'' \in F$. By $(F2)$, we get $s'' \in F$, i.e., $s' \in N(F)$. Let $r' \in N(F)$ (i.e., $r'' \in F$). It follows that $(r'' \to s'') \to s'' \in F$ by $(F3)$. Therefore, $((r \to s) \to s)'' \in F$, and $((r \to s) \to s)' \in N(F)$. Similarly, we have $(s \to r)', (s \to (r \to s))' \in N(F)$. Since F is an upset and $'$ is antitone, $N(F)$ is a down set. Thus, $N(F) \in Id(L)$.

Example 3.6. *Notice that in Example 3.3,* $0 = a'' = (c \to b)'' \neq c'' \to b'' = 1$ *and* $F = \{1\}$ *is an ideal of* L*, but* $N(F) = \{0\} \notin Id(L)$*. This shows that the condition* $(r \to s)'' = r'' \to s''$ *is necessary for Proposition 3.2.*

In the following section, we will demonstrate the significant implications of our concept of dual ideals in inducing congruence relations and quotient algebras in bounded L-algebras. This will facilitate the construction of the isomorphism theorem on bounded L-algebras, as presented in Section 4.

Subsequently, we will utilize dual ideals to induce binary relations on L and prove that these relations are indeed congruence relations. Moreover, we will prove that the quotient algebras induced by these congruence relations qualify as bounded L-algebras. The binary relation $\sim_I$, induced by a dual ideal I of L is defined as follows: $r \sim_I s$ if and only if $(r \to s)' \in I$ and $(s \to r)' \in I$.

Proposition 3.3. *The binary relation* $\sim_I$ *is an equivalence relation on the bounded L-algebra* L *for all* $I \in Id(L)$*.*

Proof. For any $r \in L$, since $(r \to r)' = 0 \in I$, $r \sim_I r$ and $\sim_I$ is reflexive. The symmetry of $\sim_I$ is apparent. To prove that $\sim_I$ is transitive, suppose $r, s, t \in L$ such that $r \sim_I s$ and $s \sim_I t$, which means $(r \to s)' \in I$, $(s \to r)' \in I$, $(s \to t)' \in I$, and $(t \to s)' \in I$. By $(I4)$, we have $((s \to r) \to (s \to t))' \in I$. Since $(s \to r) \to (s \to t) = (r \to s) \to (r \to t)$, we have $((r \to s) \to (r \to t))' \in I$. By $(I2)$, we get $(r \to t)' \in I$. Similarly, we have $(t \to r)' \in I$. Therefore, $r \sim_I t$ and $\sim_I$ is transitive.

Therefore, $\sim_I$ is an equivalence relation on L.

Theorem 3.1. *Let* L *be a bounded L-algebra and* $I \in Id(L)$*. Then* $\sim_I$ *is a congruence relation on* L*.*

Proof. To show that $\sim_I$ is a congruence relation on L, we need to prove that $\sim_I$ is compatible with $\to$ (i.e., $r \sim_I s$ implies $t \to r \sim_I t \to s$ and $r \to t \sim_I s \to t$ for all $r, s, t \in L$).

First, we prove that $t \to r \sim_I t \to s$ for any $r, s, t \in L$. Let $r \sim_I s$. Then, we have $(r \to s)' \in I$ and $(s \to r)' \in I$. By $(I4)$, $((s \to t) \to (s \to r))' \in I$. Since $(s \to t) \to (s \to r) = (t \to s) \to (t \to r)$, we have $((t \to s) \to (t \to r))' \in I$. Similarly, we have $((t \to r) \to (t \to s))' \in I$, hence $t \to r \sim_I t \to s$.

To prove $(r \to t) \sim_I (s \to t)$, we first prove that for any $u, v, w \in L$, if $u' \in I$ and $((u \to v) \to w)' \in I$, then $(v \to w)' \in I$. Since $((u \to v) \to w)' \in I$, by $(I4)$, we can conclude that $(((u \to v) \to v) \to ((u \to v) \to w))' \in I$. Since $((v \to (u \to v)) \to (v \to w))' = (((u \to v) \to v) \to ((u \to v) \to w))'$, so we have $((v \to (u \to v)) \to (v \to w))' \in I$. Since $u' \in I$, applying $(I4)$, we obtain $(v \to (u \to v))' \in I$. Therefore, by $(I2)$, we obtain $(v \to w)' \in I$.

Next, let $u = r \to s, v = r \to t, w = s \to t$ and $r \sim_I s$. Then $u' = (r \to s)' \in I$ and $(s \to r)' \in I$. By $(I3)$, we obtain $(((s \to r) \to (s \to t)) \to (s \to t))' \in I$. Since $((u \to v) \to w)' = (((r \to s) \to (r \to t)) \to (s \to t))' = (((s \to r) \to (s \to t)) \to (s \to t))' \in I$, it follows that $((r \to t) \to (s \to t))' \in I$. Similarly, we obtain $((s \to t) \to (r \to t))' \in I$. Hence, $r \to t \sim_I s \to t$.

We have shown that $\sim_I$ is compatible with $\to$. Therefore $\sim_I$ is a congruence relation on L.

Note: We denote $Con(L)$ as the set of all congruence relations of L. For any $\theta \in Con(L)$ and $r \in L$, we denote $[r]_\theta = \{s \in L \mid (r, s) \in \theta\}$.

Theorem 3.2. *Let* $(L, \to, 1)$ *be a bounded L-algebra and* $I \in Id(L)$*. We denote* $L/I = \{[r]_{\sim_I} \mid r \in L\}$ *and define a binary operation on* L/I*:* $[r]_{\sim_I} \to [s]_{\sim_I} = [r \to s]_{\sim_I}$ *for all* $[r]_{\sim_I}, [s]_{\sim_I} \in L/I$*. Then,* $(L/I, \to, [1]_{\sim_I})$ *is a bounded L-algebra and will be called the quotient L-algebra with respect to I.*

Proof. Let $[r]_{\sim_I} = [r_1]_{\sim_I}$ and $[s]_{\sim_I} = [s_1]_{\sim_I}$. Then $r \sim_I r_1$ and $s \sim_I s_1$. Since $\sim_I \in Con(L)$, we have $r \to s \sim_I r_1 \to s_1$. Therefore, $[r]_{\sim_I} \to [s]_{\sim_I} = [r \to s]_{\sim_I} = [r_1 \to s_1]_{\sim_I} = [r_1]_{\sim_I} \to [s_1]_{\sim_I}$. Therefore, $\to$ is well-defined in L/I.

Since the axioms of L-algebra defined by equations are naturally valid in quotient algebra, we only need to prove $L3$. Let $[r]_{\sim_I}, [s]_{\sim_I} \in L/I$ and $[r]_{\sim_I} \to [s]_{\sim_I} = [s]_{\sim_I} \to [r]_{\sim_I} = [1]_{\sim_I}$. It is easy to prove $[1]_{\sim_I} = \{t \in L \mid t' \in I\}$, since $[r]_{\sim_I} \to [s]_{\sim_I} = [r \to s]_{\sim_I} = [s]_{\sim_I} \to [r]_{\sim_I}$, we obtain $r \to s \in [1]_{\sim_I}$ and $s \to r \in [1]_{\sim_I}$ (i.e., $(r \to s)' \in I$ and $(s \to r)' \in I$). This implies $r \sim_I s$, i.e., $[r]_{\sim_I} = [s]_{\sim_I}$.

Let $[x]_{\sim_I} \in L/I$. Since $[0]_{\sim_I} \to [x]_{\sim_I} = [0 \to x]_{\sim_I} = [1]_{\sim_I}$, we have $[0]_{\sim_I} \le [x]_{\sim_I}$. Hence, L/I has a bottom element $[0]_{\sim_I}$.

Therefore, the triple $(L/I, \to, [1]_{\sim_I})$ forms a bounded L-algebra.

Remark 3.4. *Since the properties described by equations in an L-algebra can be naturally inherited by its quotient algebra, the quotient algebras of an involutive L-algebra and CL-algebra are involutive L-algebras and CL-algebras, respectively.*

4. Morphisms between L-algebras

The concept of a homomorphism between L-algebras, also known as a morphism, was introduced by Rump [1]. In this section, we primarily focus on the relationship between morphisms and dual ideals, and demonstrate that the concept of dual ideals has a significant impact on the study of morphisms. To begin, we will discuss certain properties of morphisms between L-algebras.

Definition 4.1. *[1] Let $E = (E, \to, 1)$ and $F = (F, \to, 1)$ be two L-algebras. A map $h : E \to F$ is called a morphism if it satisfies the following condition, for any $r, s \in E$:*

$$h(r \to s) = h(r) \to h(s).$$

Example 4.1. *Let $L_1 = \{0, m, n, x, y, 1\}$. Table 6 for the operation $\to$ is shown below.*

Table 6. Cayley table for the binary operation "→".

$\to$	0	m	n	x	y	1
0	1	1	1	1	1	1
m	y	1	y	1	y	1
n	m	m	1	1	1	1
x	0	m	y	1	y	1
y	m	m	x	x	1	1
1	0	m	n	x	y	1

Let L_2 be an arbitrary bounded L-algebra. We define a map $h : L_1 \to L_2$ as follows: $h(0) = h(n) = h(y) = 0$, and $h(m) = h(x) = h(1) = 1$. It can be easily verified that h is a morphism.

Example 4.2. *Let $L = \{0, l, m, n, 1\}$. Table 7 for the operation $\to$ is shown below.*

Table 7. Cayley table for the binary operation "$\to$".

$\to$	0	l	m	n	1
0	1	1	1	1	1
l	m	1	m	1	1
m	l	l	1	1	1
n	0	l	m	1	1
1	0	l	m	n	1

Then, $(L, \to, 1)$ is a bounded L-algebra. We define a map $h : L \to L$ as follows: $h(0) = 0, h(l) = m, h(m) = l, h(n) = n$, and $h(1) = 1$. We can check that h is an endomorphism.

Proposition 4.1. *[13] Let L_1 and L_2 be two L-algebras and $h : L_1 \to L_2$ a morphism. Then,*
(i) $h(1) = 1$;
(ii) *h is isotone.*

Remark 4.1. *Let L_1 and L_2 be two bounded L-algebras. A surjective morphism h from L_1 to L_2 is called an epimorphism. It is straightforward that if h is an epimorphism, then $h(0_1) = 0_2$ and $h(r') = h(r)'$ $(\forall r \in L_1)$.*

Lemma 4.1. *Let L_1 and L_2 be two L-algebras and $h : L_1 \to L_2$ a morphism. Then, $Im\, h = \{h(r) \mid r \in L_1\}$ is a subalgebra of L_2.*

Proof. Apparently, we only need to prove that $1 \in \mathrm{Im}\, h$ and $\mathrm{Im}\, h$ is closed under $\to$. By Proposition 4.1 (1), we have $1 \in \mathrm{Im}\, h$. Let $h(r_1) \in \mathrm{Im}\, h$ and $h(r_2) \in \mathrm{Im}\, h$. Then, $h(r_1) \to h(r_2) = h(r_1 \to r_2) \in \mathrm{Im}\, h$, hence $\mathrm{Im}\, h$ is closed under $\to$. Therefore, $\mathrm{Im}\, h$ is a subalgebra of L_2.

Remark 4.2. *In particular, if L_1 is an involutive L-algebra, then $Im\, h = \{h(r) \mid r \in L_1\}$ is also an involutive L-algebra. For any $s \in L_1$, we have $h(s) = h((s \to 0_1) \to 0_1) = (h(s) \to 0_2) \to 0_2 = h(s)''$; it follows that $Im\, h = \{h(r) \mid r \in L_1\}$ is involutive.*

Theorem 4.1. *Let L_1 and L_2 be two bounded L-algebras, $h : L_1 \to L_2$ be an epimorphism and $I \in Id(L_2)$. Then, $h^{-1}(I) = \{r \in L_1 \mid h(r) \in I\} \in Id(L_1)$.*

Proof. (1) Since $h : L_1 \to L_2$ is an epimorphism, $h(0_1) = 0_2 \in I$, thus $0_1 \in h^{-1}(I)$.

(2) If $r' \in h^{-1}(I)$ and $(r \to s)' \in h^{-1}(I)$, then $h(r)' = h(r') \in I$ and $(h(r) \to h(s))' = h((r \to s)') \in I$. By Definition 3.1, we obtain $h(s') = h(s)' \in I$, hence $s' \in h^{-1}(I)$.

(3) If $r' \in h^{-1}(I)$, then $h(r)' = h(r') \in I$. Hence, we obtain $h(((r \to s) \to s)') = ((h(r) \to h(s)) \to h(s))' \in I$, so $((r \to s) \to s)' \in h^{-1}(I)$.

(4) Similar to (3), let $r' \in h^{-1}(I)$ which is routine to prove that $(s \to r)' \in h^{-1}(I)$ and $(s \to (r \to s))' \in h^{-1}(I)$.

(5) Let $r \in h^{-1}(I)$ and $s \in L_1$ such that $s \le r$. Then, $h(r) \in I$ and $s \to r = 1$. By Definition 3.1, since $1 = h(1) = h(s \to r) = h(s) \to h(r)$, $h(s) \le h(r)$, we obtain $h(s) \in I$, which implies that $s \in h^{-1}(I)$.

Therefore, we have shown that $h^{-1}(I) \in Id(L_1)$.

Corollary 4.1. *Let L_1 and L_2 be two bounded L-algebras and $h : L_1 \to L_2$ an epimorphism. If $\{0\}$ is a dual ideal of L_2, $h^{-1}(0) = \{r \in L_1 \mid h(r) = 0\}$ is a dual ideal of L_1. Specially, if L_2 is good, $h^{-1}(0)$ is a dual ideal.*

Definition 4.2. *Let $(L, \to, 1)$ be an L-algebra. A mapping $h : L \to L$ is called idempotent if $h^2(r) = h(r)$ for all $r \in L$ where $h^2 = h \circ h$.*

Remark 4.3. *Let h be a mapping from L to L. The set of fixed points of h is denoted by $Fix_h = \{r \in L \mid h(r) = r\}$. If h is an endomorphism of L, then it is readily observed that Fix_h is a subalgebra of L.*

Example 4.3. *Let $L = \{0, a, b, 1\}$. The operations $\to_h$ and $\to_g$ on L are shown in Table 8.*

Table 8. Cayley table for the binary operations "$\to_h$" and "$\to_g$".

$\to_h$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1
$\to_g$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	0	a	1	1
1	0	a	b	1

We can verify that $(L, \to_h, 1)$ and $(L, \to_g, 1)$ are two L-algebras. Let $h : L \to L$ and $g : L \to L$ be defined as follows: $h(0) = h(a) = 0$, $h(b) = h(1) = 1$, and $g(0) = 0, g(a) = a$, $g(b) = b, g(1) = 1$. Since g is identified on $\{0, a, b, 1\}$, the claim $h \circ g = g \circ h$ is trivial.

Remark 4.4. *The following theorem states that if two idempotent mappings h and g on an L-algebra L which satisfies the condition $h \circ g = g \circ h$, then they are equal if and only if they have the same image or the same fixed point set.*

Theorem 4.2. *Let f and g be two idempotent mappings in L-algebra L such that $h \circ g = g \circ h$, then the following statements are equivalent:*
(1) *$h = g$;*
(2) *$Im\, h = Im\, g$;*
(3) *$Fix_h = Fix_g$.*

Proof. (1) $\Rightarrow$ (2) It is straightforward.

(2) $\Rightarrow$ (3) Clearly, we only need to prove that $\mathrm{Im}\, h = Fix_h$ for any h that is a idempotent. Let $r \in \mathrm{Im}\, h$. Then there exists $r_0 \in L$ such that $h(r_0) = r$, and it follows that $h(r) = h(h(r_0)) = h(r_0) = r$, which implies $r \in Fix_h$. Therefore, $\mathrm{Im}\, h \subseteq Fix_h$. Let $r \in Fix_h$; then $h(r) = r$, which implies $r \in \mathrm{Im}\, h$. Hence, $Fix_h \subseteq \mathrm{Im}\, h$. Therefore, $\mathrm{Im}\, h = Fix_h$. Since $\mathrm{Im}\, h = \mathrm{Im}\, g$, therefore we have $Fix_h = Fix_g$.

(3) $\Rightarrow$ (1) Assume that $Fix_h = Fix_g$. Let $r \in L$. Since $h(r) \in Fix_h = Fix_g$, we have $g(h(r)) = h(r)$. Similarly, we have $h(g(r)) = g(r)$; hence $h(r) = g(h(r)) = (g \circ h)(r) = (h \circ g)(r) = h(g(r)) = g(r)$. Therefore, $h = g$.

Definition 4.3. *Let $(L, \to, 1)$ be an L-algebra and $h : L \to L$ be an idempotent endomorphism on L. The pair (L, h) is referred to as an endomorphic L-algebra.*

Example 4.4. *As shown in Example 4.3, (L, f) is an endomorphic L-algebra.*

Example 4.5. *[14] Let L be an L-algebra and $h : L \to L$ be defined by $h(r) = r$ for all $r \in L$. Then, it is clear that (L, h) is an endomorphic L-algebra.*

Example 4.6. *[14] Let $L = \{l, m, n, 1\}$. Table 9 for the operation $\to$ is shown below.*

Table 9. Cayley table for the binary operation "→".

$\to$	l	m	n	1
l	1	m	n	1
m	1	1	n	1
n	1	1	1	1
1	l	m	n	1

Then, $(L, \to, 1)$ is an L-algebra. We can define a map $h : L \to L$ by setting $h(1) = h(l) = 1$ and $h(m) = h(n) = m$. However, h is not an endomorphism because $h(m \to n) = h(n) = m \neq 1 = m \to m = h(m) \to h(n)$.

Theorem 4.3. *Let $(L, \to, 1)$ be an L-algebra and $\psi : L \to L$. Define the self-map h_ψ on $L \times L$ as $h_\psi((r, s)) = (\psi(r), \psi(s))$. Then (L, ψ) is an endomorphic L-algebra if and only if $(L \times L, h_\psi)$ is an endomorphic L-algebra.*

Proof. If ψ is an idempotent endomorphism on L, let $(u_1, v_1), (u_2, v_2) \in L \times L$, then $h_\psi((u_1, v_1)) \to h_\psi((u_2, v_2)) = (\psi(u_1), \psi(v_1)) \to (\psi(u_2), \psi(v_2)) = (\psi(u_1) \to \psi(u_2), \psi(v_1) \to \psi(v_2)) = (\psi(u_1 \to u_2), \psi(v_1 \to v_2)) = h_\psi((u_1 \to u_2, v_1 \to v_2)) = h_\psi((u_1, v_1) \to (u_2, v_2))$. Hence h_ψ is a morphism on L^2. Since $h_\psi^2((u_1, v_1)) = h_\psi(h_\psi((u_1, v_1))) = h_\psi((\psi(u_1), \psi(v_1))) = (\psi^2(u_1), \psi^2(v_1)) = (\psi(u_1), \psi(v_1)) = h_\psi((u_1, v_1))$, thus h_ψ is an idempotent endomorphism on L^2.

Conversely, assume that h_ψ is an idempotent endomorphism on L^2. We obtain $(\psi(1), \psi(1)) = h_\psi((1, 1)) = (1, 1)$, and hence $\psi(1) = 1$. Since h_ψ is an idempotent endomorphism on L^2, we obtain $(1, \psi(u) \to \psi(v)) = (\psi(1) \to \psi(1), \psi(u) \to \psi(v)) = ((\psi(1), \psi(u)) \to (\psi(1), \psi(v))) = h_\psi(1, u) \to h_\psi(1, v) = h_\psi((1, u) \to (1, v)) = h_\psi((1 \to 1, u \to v)) = (\psi(1), \psi(u \to v)) = (1, \psi(u \to v))$, thus $\psi(u \to v) = \psi(u) \to \psi(v)$. Since h_ψ is idempotent, then $(\psi^2(u), \psi^2(u)) = h_\psi^2((u, u)) = h_\psi((u, u)) = (\psi(u), \psi(u))$, so $\psi^2(u) = \psi(u)$ for any $u \in L$. Therefore, ψ is an idempotent endomorphism on L.

Next, we quote the theorem stating that the kernel of a homomorphism between algebras is a congruence relation. By using this theorem, we successfully proved the isomorphism theorem on bounded L-algebras when the second one is good.

Lemma 4.2. *[15] Let R and S be two algebras of some type and $f : R \to S$ a morphism between them. Then, $Kerf = \{(r, s) \in R^2 : f(r) = f(s)\}$ is a congruence on R.*

Lemma 4.3. *Let L_1 be a bounded L-algebra, L_2 be a good L-algebra, and $f : L_1 \to L_2$ be an epimorphism. Then, $f^{-1}(0) = [0]_{Kerf}$ and $\sim_{[0]_{Kerf}} = Kerf$.*

Proof. Since f is an epimorphism, we have $[0]_{Kerf} = \{x \in L \mid f(x) = f(0)\} = \{x \in L \mid f(x) = 0\} = f^{-1}(0)$.

Now, let $(r, s) \in Kerf$, meaning $f(r) = f(s)$. It follows that $1 = f(r) \to f(s) = f(r \to s)$ and $1 = f(s) \to f(r) = f(s \to r)$. Thus, $f((r \to s)') = f(r \to s)' = 0$ and $f((s \to r)') = f(s \to r)' = 0$, which implies $(r \to s)' \in [0]_{Kerf}$ and $(s \to r)' \in [0]_{Kerf}$, indicating that $(r, s) \in \sim_{[0]_{Kerf}}$. Therefore, we have shown that $Kerf \subseteq \sim_{[0]_{Kerf}}$.

Conversely, let $(r, s) \in \sim_{[0]_{Kerf}}$, meaning $(r \to s)' \in [0]_{Kerf}$ and $(s \to r)' \in [0]_{Kerf}$. This implies that $f((r \to s)') = f(r \to s)' = 0$ and $f((s \to r)') = f(s \to r)' = 0$. Since L is a good L-algebra, we have $f(r \to s) = f(r) \to f(s) = 1$ and $f(s \to r) = f(s) \to f(r) = 1$, which implies $f(r) = f(s)$ and so $(r, s) \in Kerf$. Hence $\sim_{[0]_{Kerf}} \subseteq Kerf$.

Therefore, $\sim_{[0]_{Kerf}} = Kerf$.

Lemma 4.4. *[15] Suppose that R and S are two algebras of some type and $f : R \to S$ is a homomorphism from algebra R onto algebra S. Then, we can construct an isomorphism β from the quotient algebra $R/Kerf$ to S such that the homomorphism f can be expressed as the composition of the natural homomorphism υ from R to $R/Kerf$ and β, i.e., $f = \beta \circ \upsilon$.*

Theorem 4.4. *Let L_1 be a bounded L-algebra, L_2 be a good L-algebra and $f : L_1 \to L_2$ be an epimorphism, then*

$$L_1/\sim_{[0]_{Kerf}} \cong L_2.$$

Proof. This result follows directly from Lemmas 4.3 and 4.4.

In the following example, we will show that the mapping $\varphi : Id(L) \to Con(L)$, $I \mapsto \sim_I$, is neither injective nor surjective.

Example 4.7. *Let $L = \{0, m, n, 1\}$. Table 10 for the operation $\to$ is shown below.*

Table 10. Cayley table for the binary operation "→".

$\to$	0	m	n	1
0	1	1	1	1
m	n	1	1	1
n	0	m	1	1
1	0	m	n	1

We can verify that L is an L-algebra. By checking that $I = \{0\}$ is a dual ideal of L, we can compute the following:
(1) $[0]_{\sim_I} = \{0, m\}$ and $\{0, m\}$ is a dual ideal,
(2) $\sim_I = \{(0, 0), (m, m), (n, n), (1, 1), (0, m), (n, 1)\}$ and $\sim_{[0]_{\sim_I}} = \{(0, 0), (m, m), (n, n), (1, 1), (0, m), (n, 1)\}$.

We have shown $[0]_{\sim_I} \neq I$ and $\sim_I = \sim_{[0]_{\sim_I}}$, which implies that φ is not injective. Furthermore, it is evident that φ is order preserving. Considering that I is the smallest dual ideal of L, and the congruence induced by I is not the smallest congruence of L, it follows that the smallest congruence of L does not have an inverse image, so φ is not surjective.

The above example has shown that the congruence relations and dual ideals may not admit one-to-one correspondence. Following this, we will investigate some properties about congruences and dual ideals, and construct a bijection between dual ideals and congruences whose quotient algebras are L-algebras when L is involutive.

Lemma 4.5. *If L is an involutive L-algebra and $I \in Id(L)$, then $[0]_{\sim_I} = I$.*

Proof.

$$\begin{aligned}[0]_{\sim_I} &= \{r \in L | r \sim_I 0\} \\ &= \{r \in L | (r \to 0)' \in I\} \\ &= \{r \in L | r'' \in I\} \\ &= \{r \in L | r \in I\} \\ &= I.\end{aligned}$$

Note: We will use $QCon(L)$ to represent the set of all congruence relations whose induced quotient algebras are also L-algebras.

Lemma 4.6. *If L is an involutive L-algebra and $\theta \in QCon(L)$, then $[0]_\theta \in Id(L)$.*

Proof. Let $f : L \to L/\theta$ be the natural epimorphism. Since $(r, s) \in Kerf \Leftrightarrow f(r) = f(s) \Leftrightarrow [r]_\theta = [s]_\theta \Leftrightarrow (r, s) \in \theta$, we have $Kerf = \theta$. Since L/θ is involutive, $\{[0]_\theta\}$ is a dual ideal of L/θ. By Corollary 4.1, we obtain $[0]_\theta = [0]_{Kerf} = \{x \mid f(x) = f(0) = [0]_\theta\} = f^{-1}([0]_\theta) \in Id(L)$.

Lemma 4.7. *Let L be an involutive L-algebra and $\theta \in QCon(L)$, then $\sim_{[0]_\theta} = \theta$.*

Proof. By Lemma 4.6, $[0]_\theta \in Id(L)$; therefore, this lemma makes sense. Let $(r, s) \in \theta$. Since $\theta \in QCon(L)$, we have $(r \to s, s \to s) = (r \to s, 1) \in \theta$ and $(s \to r, s \to s) = (s \to r, 1) \in \theta$, it follows that $((r \to s)', 0) \in \theta$ and $((s \to r)', 0) \in \theta$. Hence $(r \to s)', (s \to r)' \in [0]_\theta$ (i.e., $r \sim_{[0]_\theta} s$, thus $\theta \subseteq \sim_{[0]_\theta}$).

Conversely, if $(r, s) \in \sim_{[0]_\theta}$, i.e., $(r \to s)', (s \to r)' \in [0]_\theta$. This implies that $((r \to s)', 0) \in \theta$, and so $(r \to s, 1) \in \theta$. Similarly, $(s \to r, 1) \in \theta$. Since $\theta \in QCon(L)$, we get $((r \to s) \to (r \to 0), r \to 0) \in \theta$ and $((s \to r) \to (s \to 0), s \to 0) \in \theta$. By Definition 2.1, we obtain $((s \to r) \to (s \to 0), s \to 0) = ((r \to s) \to (r \to 0), s \to 0) \in \theta$. Hence $(r \to 0, s \to 0) \in \theta$, and therefore $(r, s) \in \theta$. Thus $\sim_{[0]_\theta} \subseteq \theta$.

Therefore, $\sim_{[0]_\theta} = \theta$.

Theorem 4.5. *There exists a bijective mapping between $Id(L)$ and $QCon(L)$ for an involutive L-algebra L.*

Proof. It follows directly from Lemmas 4.5 and 4.7.

5. Special dual ideals in L-algebras

In this section, we will present several types of bounded L-algebras and utilize dual ideals to characterize them.

Definition 5.1. *Let L be a bounded L-algebra and $I \in Id(L)$. I is said to be a K dual ideal if it satisfies the following condition:* $(K1)$ $(r \to (s \to r))' \in I$ *for all* $r, s \in L$.

Example 5.1. *[4] Let* $L = \{0, e, f, g, 1\}$. *Table 11 for the operation → is shown below.*

Table 11. Cayley table for the binary operation "→".

$\to$	0	e	f	g	1
0	1	1	1	1	1
e	e	1	e	e	1
f	f	f	1	f	1
g	g	g	g	1	1
1	0	e	f	g	1

We can confirm that the only two dual ideals of L are $I_1 = \{0\}$ and L itself. It is evident that for any bounded L, the dual ideal L itself is a K dual ideal. In this example, another dual ideal of L $I_1 = \{0\}$ is not K dual ideal since $(e \to (f \to e))' = e \notin \{0\}$.

Theorem 5.1. *Let L be a good L-algebra. L is a KL-algebra if and only if every dual ideal of L is a K dual ideal.*

Proof. Assume that L is a KL-algebra and $I \in Id(L)$. For any $r, s \in L$, we have $(r \to (s \to r))' = 0 \in I$. Hence, I is a K dual ideal. This implies that every dual ideal of L is a K dual ideal.

Conversely, suppose that every dual ideal of L is a K dual ideal, and let $r, s \in L$. Since $\{0\}$ is a K dual ideal, we have $(r \to (s \to r))' = 0$. As L is a good L-algebra, we can conclude that $r \to (s \to r) = 1$. Therefore, L is a KL-algebra.

Definition 5.2. *Let L be a bounded L-algebra and $I \in Id(L)$. Then, I is said to be a C dual ideal if it satisfies the following condition:* $(C1)$ $((r \to (s \to t)) \to (s \to (r \to t)))' \in I$ *for all* $r, s, t \in L$.

Example 5.2. *Apparently, any bounded L-algebra as its dual ideal is C dual ideal. In Example 5.1,* $\{0\}$ *is not a C dual ideal since* $((e \to (f \to g)) \to (f \to (e \to g)))' = e \notin \{0\}$.

Theorem 5.2. *Let L be a good L-algebra. Then L is a CL-algebra if and only if every dual ideal of L is a C dual ideal.*

Proof. Assume that L is a CL-algebra and $I \in Id(L)$. For any $r, s, t \in L$, we have $(r \to (s \to t)) \to (s \to (r \to t)) = 1$, and $((r \to (s \to t)) \to (s \to (r \to t)))' = 0 \in I$; hence, I is a C dual ideal. Therefore, every dual ideal of L is a C dual ideal.

Conversely, suppose that every dual ideal of L is a C dual ideal, and let $r, s, t \in L$. Since $\{0\}$ is a C dual ideal, $((r \to (s \to t)) \to (s \to (r \to t)))' = 0$. As L is a good L-algebra, $(r \to (s \to t)) \to (s \to (r \to t)) = 1$. Therefore, L is a CL-algebra.

Recall that an L-algebra L is said to be commutative if it satisfies the condition K and $(r \to s) \to s = (s \to r) \to r$ for any $r, s \in L$.

Definition 5.3. *Let L be a bounded L-algebra and $I \in Id(L)$. Then, I is called a commutative dual ideal if it satisfies the following condition: $(I6)$ $(s \to r)' \in I$ implies $(((r \to s) \to s) \to r)' \in I$ for all $r, s \in L$.*

Note: Denote by $Id_c(L)$ the set of all commutative dual ideals of L.

Example 5.3. *In Example 3.4, it could be noted that $Id(L) = Id_c(L)$.*

Proposition 5.1. *Let L be a bounded CL-algebra, $M \in Id_c(L)$ and $N \in Id(L)$ such that $M \subseteq N$. Then $N \in Id_c(L)$.*

Proof. Let $r, s \in L$. Denote $t = s \to r$ and assume that $t' = (s \to r)' \in N$. By the fact that condition C implies D, we have $s \to (t \to r) = s \to ((s \to r) \to r) = 1$, therefore $(s \to (t \to r))' = 0 \in M$. By $(I6)$ and $M \subseteq N$, we obtain $((((t \to r) \to s) \to s) \to (t \to r))' \in M \subseteq N$. Due to condition C, we have $(t \to ((((t \to r) \to s) \to s) \to r))' \in N$. Since $t' \in N$, $((((t \to r) \to s) \to s) \to r)' \in N$. Since condition C implies K, we have $r \leq t \to r$, so $(t \to r) \to s \leq r \to s \Rightarrow (r \to s) \to s \leq ((t \to r) \to s) \to s \Rightarrow (((t \to r) \to s) \to s) \to r \leq ((r \to s) \to s) \to r \Rightarrow (((r \to s) \to s) \to r)' \leq ((((t \to r) \to s) \to s) \to r)' \in N$. Hence $(((r \to s) \to s) \to r)' \in N$, and we conclude that $N \in Id_c(L)$.

Corollary 5.1. *If L is a good CL-algebra, then $\{0\} \in Id_c(L)$ if and only if $Id(L) = Id_c(L)$.*

Lemma 5.1. *[6] Let $(L, \to, 1)$ be a CL-algebra. The following statements are equivalent:*
(1) *L is commutative;*
(2) *$r \to s = ((s \to r) \to r) \to s$ for any $r, s \in L$.*

Lemma 5.2. *[6] Let $(L, \to, 1)$ be a KL-algebra. The following statements are equivalent:*
(1) *L is commutative;*
(2) *$r \leq (r \to s) \to s$, for all $r, s \in L$ and $r = (r \to s) \to s$, whenever $s \leq r$.*

Theorem 5.3. *Let L be a good CL-algebra. The following statements are equivalent:*
(1) *L is commutative;*
(2) *$\{0\} \in Id_c(L)$;*
(3) *every dual ideal of L is a commutative dual ideal.*

Proof. (1) $\Rightarrow$ (2) Assume that L is commutative. By Lemma 5.1, we have $s \to r = ((r \to s) \to s) \to r$ for all $r, s \in L$. If $r, s \in L$ such that $(s \to r)' \in \{0\}$, then $(((r \to s) \to s) \to r)' \in \{0\}$; hence, $\{0\} \in Id_c(L)$.

(2) $\Rightarrow$ (1) Assume that $\{0\} \in Id_c(L)$ and $r, s \in L$ such that $s \leq r$, then $(s \to r)' = 0 \in \{0\}$. Since $\{0\}$ is commutative, we obtain $(((r \to s) \to s) \to r)' \in \{0\}$, so $(((r \to s) \to s) \to r)' = 0$. Since L is a good L-algebra, $((r \to s) \to s) \to r = 1$. As L is a CL-algebra, we obtain $r \leq (r \to s) \to s$, so $r = (r \to s) \to s$. By Lemma 5.2, we uncover that L is commutative.

(3) $\Rightarrow$ (2) This is straightforward.

(2) $\Rightarrow$ (3) Since $\{0\} \in Id_c(L)$ and $\{0\} \subseteq I$ for any $I \in Id(L)$, then $I \in Id_c(L)$ by Proposition 5.1.

Definition 5.4. *Suppose that $(L, \to, \cdot, 0, 1)$ is an algebra of type $(2, 1, 0, 0)$, where $(L, \to, 0, 1)$ is a bounded L-algebra and $(L, \cdot, 1)$ is a monoid. If, for all $r, s \in L - \{0\}$, $s \to r \cdot s = r$ and $r \cdot s \to s = 1$, then $(L, \to, \cdot, 0, 1)$ is called a partially self-similar L-algebra.*

Remark 5.1. *According to Definition 2.5, if a self-similar L-algebra L is bounded, then it contains only one element. Therefore, a new concept is introduced in Definition 5.4, namely, partial self-similar L-algebra. Its characteristic is the exclusion of the element 0 when defining partial self-similarity, while still preserving most of the properties of self-similar L-algebras.*

Proposition 5.2. *If L is a partially self-similar L-algebra, then the following statements hold for all $r, s \in L - \{0\}$:*
(1) $r \cdot s \to t = r \to (s \to t)$ *for all* $t \in L$;
(2) *if* $r \to s \in L - \{0\}$ *and* $s \to r \in L - \{0\}$, *then* $(r \to s) \cdot r = (s \to r) \cdot s$ *for all* $t \in L$.

Proof. (1) Let L be a partially self-similar L-algebra. We have $r \to (s \to t) = (s \to r \cdot s) \to (s \to t) = (r \cdot s \to s) \to (r \cdot s \to t)$. Since $r \cdot s \to s = 1$, we obtain $r \to (s \to t) = r \cdot s \to t$.
(2) By (1), we have $(r \to s) \cdot r \to t = (r \to s) \to (r \to t) = (s \to r) \to (s \to t) = (s \to r) \cdot s \to t$. By Proposition 2.5, we obtain $(r \to s) \cdot r = (s \to r) \cdot s$.

Proposition 5.3. *If L is a partially self-similar L-algebra and $r \in L - \{0\}$, then the following statements hold:*
(1) $s \cdot r \to t \cdot r = s \to t$ *for all* $s, t \in L - \{0\}$;
(2) *for all* $x \in L - \{0\}$, *there is some* $u \in L$ *such that* $x = r \to u$;
(3) *The map* $g : L - \{0\} \to L$, $s \mapsto s \cdot r$ *is injective;*
(4) *if* $s, t, r \to s \cdot t, ((t \to r) \to s) \cdot (r \to t) \in L - \{0\}$, *then* $r \to s \cdot t = ((t \to r) \to s) \cdot (r \to t)$.

Proof. (1) Since L is a partially self-similar L-algebra, according to Proposition 5.2 (1), we have $s \cdot r \to t \cdot r = s \to (r \to t \cdot r) = s \to t$.
(2) Since L is a partially self-similar L-algebra, assume that $x \in L - \{0\}$. By Definition 5.4, if you put $u = x \cdot r$, then we have $r \to u = x$.
(3) Suppose that $s \cdot r = t \cdot r$, $s, t \in L - \{0\}$, then $s \cdot r \to u = t \cdot r \to u$ for all $u \in L$. By Proposition 5.2(1), we obtain $s \to (r \to u) = t \to (r \to u)$. Since $s, t \in L - \{0\}$, by (2), there exist u_1 and u_2 in L such that $s = r \to u_1$ and $t = r \to u_2$. If you replace u by u_1 and u_2 in equation $s \to (r \to u) = t \to (r \to u)$, respectively, then we have $s \to s = t \to s$ and $s \to t = t \to t$; therefore, $s = t$ and g is injective.
(4) Using Proposition 5.2 (1), for all $u \in L$, we obtain $(r \to s \cdot t) \to (r \to u) = (s \cdot t \to r) \to (s \cdot t \to u) = (s \to (t \to r)) \to (s \to (t \to u)) = ((t \to r) \to s) \to ((t \to r) \to (t \to u)) = ((t \to r) \to s) \to ((r \to t) \to (r \to u)) = ((t \to r) \to s) \cdot (r \to t) \to (r \to u)$. Similar to (3), we have $r \to s \cdot t = ((t \to r) \to s) \cdot (r \to t)$.

Example 5.4. *[16] Let $G = (G, \vee, \wedge, +, -, 0)$ be an arbitrary ℓ-group and G^- be the negative cone of G, that is, $G^- = \{x \in G \mid x \leq 0\}$. For any $x, y \in G^-$, we define the following operations on G^-: $x \cdot y := x + y$, $x \to y := (y - x) \wedge 0$. First, add a symbol $-\infty$ in G^- and still denote G^- as $G^- \cup \{-\infty\}$, then define the following operation rules: $-\infty \to -\infty = 0, x \to -\infty = -\infty, -\infty \to x = 0, x \cdot -\infty = -\infty \cdot x = -\infty, -\infty \cdot -\infty = -\infty$. We can verify that $G^- = (G^-, \to, -\infty, 0)$ is a bounded L-algebra. Since for all $x, y \in G^- - \{-\infty\}$, $x \to y \cdot x = (x + y - x) \wedge 0 = y$ and $x \cdot y \to y = 0$, G^- is a partially self-similar L-algebra.*

A product for a L-algebra with negation: $r \cdot s := (r \to s')^{\sim}$, where $^{\sim}$ represents the inverse mapping of the bijection $'$ from Definition 2.2.

Definition 5.5. *Let L be an L-algebra with negation and $I \in Id(L)$. Then, I is called a self-similar dual ideal if it satisfies the condition:* $(S1)$ $((r \to s \cdot r) \to s)' \in I$, $(s \to (r \to s \cdot r))' \in I$ *and* $(r \cdot s \to s)' \in I$ *for all* $r, s \in L - \{0\}$.

Theorem 5.4. *If L is a L-algebra with negation, then L is a partially self-similar L-algebra if and only if every dual ideal of L is a self-similar dual ideal.*

Proof. Assume that L is a partially self-similar L-algebra and let $I \in Id(L)$. If $r, s \in L - \{0\}$, then $(r \to s \cdot r) \to s = 1$, $s \to (r \to s \cdot r) = 1$, and $r \cdot s \to s = 1$, so $((r \to s \cdot r) \to s)' = 0 \in I$, $(s \to (r \to s \cdot r))' = 0 \in I$, and $(r \cdot s \to s)' = 0 \in I$. Hence, I is a self-similar dual ideal. Thus, every dual ideal of L is a self-similar dual ideal.

Conversely, suppose that every dual ideal of L is a self-similar dual ideal, and let $r, s \in L - \{0\}$. Apparently, if L is a L-algebra with negation, then L is good. Therefore, we have $\{0\}$ as a self-similar dual ideal, $((r \to s \cdot r) \to s)' = 0$, $(s \to (r \to s \cdot r))' = 0$, and $(s \to (r \to s \cdot r))' = 0$. Hence $(r \to s \cdot r) \to s = 1$, $s \to (r \to s \cdot r) = 1$, and $r \cdot s \to s = 1$ (i.e., $r \to s \cdot r = s$ and $r \cdot s \leq s$). Therefore, L is a partially self-similar L-algebra.

6. Conclusions

This article introduces the concept of dual ideals and gives us a new tool to study bounded L-algebras. We prove that dual ideals induce congruence relations and quotient L-algebras. Consequently, by studying the relations between morphisms between bounded L-algebras and dual ideals, the isomorphism theorem is constructed by dual ideals. In Section 5, we use dual ideals to characterize special bounded L-algebras. Remarkably, the single point subset $\{0\}$ is not necessarily a dual ideal of a bounded L-algebra, which causes some difficulties in our paper. Therefore, we expect an improved definition which would designate $\{0\}$ as a dual ideal in any bounded L-algebra. This paper introduces the definition of dual ideals and conducts an initial study of their properties, and we look forward to a more in-depth study of bounded L-algebras and dual ideals.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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