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Research article

Characterization of frame vectors for $\mathcal{A}$-group-like unitary systems on Hilbert C^*-modules

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Abstract: In this paper, the notion of an $\mathcal{A}$-group-like unitary system on a Hilbert C^*-module is introduced and some basic properties are studied, where $\mathcal{A}$ is a unital C^*-algebra. Let $\mathcal{U}$ be such a unitary system. We prove that a complete Parseval frame vector for $\mathcal{U}$ can be dilated to a complete wandering vector. Also, it is shown that the set of all the complete Bessel vectors for $\mathcal{U}$ can be parameterized by the set of the adjointable operators in the double commutant of $\mathcal{U}$, and that the frame multiplicity of $\mathcal{U}$ is always finite.

Keywords: $\mathcal{A}$-group-like unitary systems; frame vectors; complete Bessel vectors; frame multiplicity; Hilbert C^*-modules
Mathematics Subject Classification: 42C15, 42C40, 46L08, 46H25

1. Introduction

Frames in Hilbert spaces were first introduced by Duffin and Schaeffer [7] to study the nonharmonic Fourier series. The frame theory has experienced tremendous development in the past decades. It was motivated by engineering applications and pure mathematics [1, 3–5]. Frank and Larson extended the concept of frame in Hilbert spaces to countably generated C^*-modules and investigated its main features [8, 9]. While a Hilbert C^*-module is a broader concept than a Hilbert space, there are several distinctions between the two. There are instances where adjoint operators are not present for certain bounded operators on Hilbert C^*-modules. It should be noted that although some definitions and results of modular frames may look similar to their Hilbert space frame counterparts, the complexity of the Hilbert C^*-module structure makes it difficult to simplify Hilbert space frames into generalizations. We refer to more discussion on essential differences between frames in Hilbert spaces and in C^*-modules [8, 9, 12]. Several generalizations of frames have been defined in Hilbert C^*-modules, as documented in [13, 15, 18, 21].

In recent years, there has been growing evidence that modular frames are closely related to other research areas, such as wavelet frame construction [16, 20]. Considering the theory and applications of structured frames in Hilbert spaces, such as Gabor and wavelet frames, have always been the main focus of Hilbert space frame theory, we believe the structured modular frames may be well suited for specific applications in terms of theoretical or application properties. Dai and Larson [6] and Han and Larson [11] have introduced a fresh perspective on analyzing structured frames by examining frame vectors for unitary systems. This approach has garnered significant interest in frame theory. Building on the concept of group-like unitary systems in Hilbert spaces proposed in [10], we have introduced the idea of $\mathcal{A}$-group-like unitary systems in Hilbert C^*-modules. The purpose of this paper is to explore the frames that are created by $\mathcal{A}$-group-like unitary systems.

The organizational structure of this paper is as follows. In Section 2, we state some notations and preliminaries, and introduce the notion of an $\mathcal{A}$-group-like unitary system $\mathcal{U}$ on Hilbert C^*-modules. Section 3 describes the dilation theory of frames induced by $\mathcal{U}$ on Hilbert C^*-modules. In Section 4, characterization of the complete Bessel vectors for $\mathcal{U}$ is obtained in terms of certain class of operators in $\mathcal{U}''$. Finally, we prove that the frame multiplicity of $\mathcal{U}$ is always finite in Section 5.

2. Preliminaries

In the following, we will review the fundamental definitions and characteristics of Hilbert C^*-modules and their frames. Suppose that $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{H}$ is a left $\mathcal{A}$-module. $\mathcal{H}$ is a *pre-Hilbert $\mathcal{A}$-module* if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H} \to \mathcal{A}$ such that the following properties hold:
(1) $\langle x, x\rangle \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.
(2) $\langle ax + y, z\rangle = a\langle x, z\rangle + \langle y, z\rangle$ for every $a \in \mathcal{A}$, every $x, y, z \in \mathcal{H}$.
(3) $\langle x, y\rangle = \langle y, x\rangle^*$ for every $x, y \in \mathcal{H}$.

For every $x \in \mathcal{H}$, we define

$$\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}.$$

If $\mathcal{H}$ is complete with $\|\cdot\|$, it is called a *Hilbert $\mathcal{A}$-module* (or a *Hilbert C^*-module* over $\mathcal{A}$). For every $a \in \mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$, and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x\rangle^{\frac{1}{2}}$.

Lemma 2.1 ([17]). *Assume $\mathcal{H}$, $\mathcal{K}$ are Hilbert $\mathcal{A}$-modules, and $T : \mathcal{H} \to \mathcal{K}$ is a linear map. Then the conditions listed below are equivalent:*
(1) *T is both bounded and $\mathcal{A}$-linear.*
(2) *There is $d \geq 0$ such that $\langle Tz, Tz\rangle \leq d\langle z, z\rangle, z \in \mathcal{H}$.*

A Hilbert $\mathcal{A}$-module $\mathcal{H}$ is called *finitely generated* if there is a finite set $\{z_1, \cdots z_n\}$ of $\mathcal{H}$ such that $z = \sum_{i=1}^{n} a_i z_i, a_i \in \mathcal{A}$ for each $z \in \mathcal{H}$, if the set of generators is countable, it is referred to as *countably generated*.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. We say that the operator $T : \mathcal{H} \to \mathcal{K}$ is *adjointable*, if there is another one $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tz, y\rangle = \langle z, T^*y\rangle$, $\forall z \in \mathcal{H}, y \in \mathcal{K}$. It's important to note that an adjointable operator is both $\mathcal{A}$-linear and bounded by default. Any families of adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ are referred to as $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$. If $\mathcal{K}$ equals $\mathcal{H}$, it is shortened to $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$. An operator $T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$ is classified as a unitary if $TT^* = I_{\mathcal{K}}$ and $T^*T = I_{\mathcal{H}}$. For further information on Hilbert C^*-modules, please refer to [14].

In this paper, $\mathcal{A}$ represents a unital C^*-algebra. $\mathcal{H}$ and $\mathcal{K}$ are Hilbert $\mathcal{A}$-modules that are either finitely or countably generated. Additionally, $\mathbb{I}$ is a countable index set. If T is a bounded $\mathcal{A}$-linear operator, then $\text{ran}T$, $\ker T$ and T^* refer to its range space, nullspace and adjoint, respectively.

In line with reference [9], we will define frames in C^*-modules.

Definition 2.2. A sequence $\{x_i\}_{i\in\mathbb{I}}$ of elements in $\mathcal{H}$ is said to be a *frame* if there exist two constants $A, B > 0$ such that

$$A\langle x, x\rangle \leq \sum_{i\in\mathbb{I}}\langle x, x_i\rangle\langle x_i, x\rangle \leq B\langle x, x\rangle \tag{2.1}$$

for every $x \in \mathcal{H}$. The constants A and B are called *frame bounds*. The frame $\{x_i\}_{i\in\mathbb{I}}$ is considered *tight* if $A = B$ and *Parseval* if $A = B = 1$. Likewise, if $\{x_i\}_{i\in\mathbb{I}}$ only satisfies the upper bound condition in (2.1), then $\{x_i\}_{i\in\mathbb{I}}$ is called a *Bessel sequence* with bound B. A frame is considered *standard* if the norm converges in the middle sum of (2.1).

We will consider only the standard (Parseval) frames and the standard Bessel sequences. This paper will focus on finitely or countably generated Hilbert C^*-modules, as the Kasparov Stabilization Theorem establishes a frame available for every such module. For a standard Bessel sequence $\{x_i\}_{i\in\mathbb{I}}$, the *analysis operator* T is an $\mathcal{A}$-linear bounded adjointable operator from $\mathcal{H}$ to $l^2(\mathcal{A})$ defined by

$$Tx = \{\langle x, x_i\rangle\}_{i\in\mathbb{I}}.$$

The adjoint operator $T^* : l^2(\mathcal{A}) \to \mathcal{H}$ is defined by

$$T^*\{a_i\}_{i\in\mathbb{I}} = \sum_{i\in\mathbb{I}} a_i x_i,$$

and it is called the *synthesis operator* of $\{x_i\}_{i\in\mathbb{I}}$. In fact, if $\{x_i\}_{i\in\mathbb{I}}$ is a standard frame for $\mathcal{H}$, the *frame operator*

$$S = T^*T : \mathcal{H} \to \mathcal{H}, \qquad Sx = \sum_{i\in\mathbb{I}}\langle x, x_i\rangle x_i$$

is a well-defined, positive, adjointable and invertible operator and the following *reconstruction formula*

$$x = \sum_{i\in\mathbb{I}}\langle x, S^{-1}x_i\rangle x_i \tag{2.2}$$

holds for all $x \in \mathcal{H}$. Moreover, $\{S^{-1}x_i\}_{i\in\mathbb{I}}$ is also a frame for $\mathcal{H}$ and is called the *canonical dual* of $\{x_i\}_{i\in\mathbb{I}}$.

An *orthonormal system* in $\mathcal{H}$, denoted by $\{f_i\}_{i\in\mathbb{I}}$, is a family of vectors such that $\langle f_i, f_j\rangle = 0, \forall i \neq j$ and $\langle f_i, f_i\rangle = 1, \forall i \in \mathbb{I}$. If an orthonormal system generates a dense submodule of $\mathcal{H}$, it is called an *orthonormal basis* for $\mathcal{H}$.

A *unitary system* $\mathcal{U}$ on $\mathcal{H}$ is a subset of all unitary operators in $\text{End}^*_{\mathcal{A}}(\mathcal{H})$ that includes the identity operator I. For a unitary system $\mathcal{U}$, a vector $x \in \mathcal{H}$ is called a *Parseval frame vector* (resp. *frame vector* or *Bessel vector*) for $\mathcal{U}$ if $\mathcal{U}x$ forms a Parseval frame (resp. frame or Bessel sequence) for $\overline{\text{span}}\{Ux : U \in \mathcal{U}\}$. When $\overline{\text{span}}\{Ux : U \in \mathcal{U}\} = \mathcal{H}$, the frame vector is said to be *complete*. If $\mathcal{U}x$ is an orthonormal basis for $\mathcal{H}$, x is a *complete wandering vector* for $\mathcal{U}$. The set of all complete wandering vectors for $\mathcal{U}$ is represented by $\mathcal{W}(\mathcal{U})$.

Suppose that $\mathcal{A}$ is a unital C^*-algebra with identity 1. We use $\mathcal{Z}(\mathcal{A}) = \{a \in \mathcal{A} : ab = ba, \ \forall\ b \in \mathcal{A}\}$ to represent the centre of $\mathcal{A}$. Inspired by the concept of group-like unitary systems in Hilbert spaces, which was proposed in [10], we introduce the notion of $\mathcal{A}$-group-like unitary systems in Hilbert C^*-modules. Let group($\mathcal{U}$) be the group generated by $\mathcal{U}$ and $\mathbb{A} = \{a \in \mathcal{Z}(\mathcal{A}) : aa^* = a^*a = 1\}$. We call $\mathcal{U}$ an *$\mathcal{A}$-group-like unitary system* if

$$\text{group}(\mathcal{U}) \subset \mathbb{A}\mathcal{U} = \{aU : a \in \mathbb{A}, U \in \mathcal{U}\} \tag{2.3}$$

and $\mathcal{U}$ is $\mathbb{A}$-linearly independent, i.e., $\mathbb{A}U \neq \mathbb{A}V$ as long as U and V are different elements of $\mathcal{U}$. Then there exists a function $f : \text{group}(\mathcal{U}) \to \mathbb{A}$ and a mapping $\tau : \text{group}(\mathcal{U}) \to \mathcal{U}$ such that $W = f(W)\tau(W)$ for all $W \in \text{group}(\mathcal{U})$. It is essential to provide clear definitions of f and τ. When W is equal to a_1U_1 and a_2U_2 with $U_1, U_2 \in \mathcal{U}, a_1, a_2 \in \mathbb{A}$, it can be determined that a_1 equals a_2 and U_1 equals U_2 since $\mathcal{U}$ is $\mathbb{A}$-linearly independent. When dealing with the mappings f and τ, there is a fundamental property that we can establish. The proof of this property is similar to the one presented in the literature's appendix [10].

Proposition 2.3. *Let $\mathcal{U}, f$ and τ be as above. Then*
(1) $\tau(U) = U, f(U) = 1, U \in \mathcal{U}$.
(2) $f(U\tau(VW))f(VW) = f(\tau(UV)W)f(UV), U, V, W \in \text{group}(\mathcal{U})$.
(3) $\tau(U\tau(VW)) = \tau(\tau(UV)W), U, V, W \in \text{group}(\mathcal{U})$.
(4) *If $V, W \in \text{group}(\mathcal{U})$, then*

$$\mathcal{U} = \{\tau(UV) : U \in \mathcal{U}\} = \{\tau(VU^{-1}) : U \in \mathcal{U}\} = \{\tau(VU^{-1}W) : U \in \mathcal{U}\} = \{\tau(V^{-1}U) : U \in \mathcal{U}\}.$$

(5) *Let $V, W \in \mathcal{U}$. Then the mappings from $\mathcal{U}$ to $\mathcal{U}$ listed below are all injective:*

$$U \to \tau(VU)(\text{resp}, \tau(UV), \tau(UV^{-1}), \tau(V^{-1}U), \tau(VU^{-1}), \tau(U^{-1}V), \tau(VU^{-1}W)).$$

Proof. The statement (1) is trivial. Statements (2) and (3) come from the equality

$$\begin{aligned} f(U\tau(VW))f(VW)\tau(U\tau(VW)) &= U(VW) = (UV)W \\ &= f(UV)f(\tau(UV)W)\tau(\tau(UV)W) \end{aligned}$$

and the assumption that $\mathcal{U}$ is $\mathbb{A}$-linear independent.

Since $\tau(\tau(SV^{-1})V) = \tau(S\tau(V^{-1}V)) = \tau(S) = S, \forall S \in \mathcal{U}$, which implies that the first equality in (4) holds. Notice

$$V\tau(S^{-1}V)^{-1} = f(S^{-1}V)V(S^{-1}V)^{-1} = f(S^{-1}V)S.$$

Hence $\tau(V\tau(S^{-1}V)^{-1}) = S$ since $S \in \mathcal{U}$. Similarly,

$$\tau(V\tau(WS^{-1}V)^{-1}W) = S.$$

Thus the rest of (4) holds.

For (5), suppose that $\tau(VU_1^{-1}W) = \tau(VU_2^{-1}W)$ for some $U_1, U_2 \in \mathcal{U}$. Then

$$f^*(VU_1^{-1}W)VU_1^{-1}W = f^*(VU_2^{-1}W)VU_2^{-1}W,$$

where $f^*(VU_1^{-1}W)$ denotes the adjoint of $f(VU_1^{-1}W)$. Hence

$$f(VU_1^{-1}W)U_1 = f(VU_2^{-1}W)U_2.$$

So $U_1 = U_2$, which implies that $U \to \tau(VU^{-1}W)$ is injective. Similarly, the rest of (5) hold. □

The *commutant* of a subset S in $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ is $S' = \{A \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}) : AS = SA, S \in \mathcal{S}\}$. The *local commutant* of a nonzero vector $\xi \in \mathcal{H}$ is denoted as

$$C_\xi(\mathcal{U}) = \{T \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}) : TU\xi = UT\xi, \forall U \in \mathcal{U}\}.$$

The Hilbert C^*-module $l^2_{\mathcal{U}}(\mathcal{A})$ is defined by

$$l^2_{\mathcal{U}}(\mathcal{A}) = \left\{\{a_U\}_{U\in\mathcal{U}} \subseteq \mathcal{A} : \sum_{U\in\mathcal{U}} a_U^* a_U \text{ converges in norm in } \mathcal{A}\right\}.$$

The standard orthonormal basis of $l^2_{\mathcal{U}}(\mathcal{A})$ is represented by $\{\chi_U\}_{U\in\mathcal{U}}$. Here, χ_U is 1 at U and 0 everywhere else. For each $U \in \mathcal{U}$, we define

$$L_U\chi_V = f(UV)\chi_{\tau(UV)}, \quad R_U\chi_V = f(VU^{-1})\chi_{\tau(VU^{-1})}, \quad V \in \mathcal{U}.$$

L and R here are the *left and right regular representations* of $\mathcal{U}$.

To prove our main results, we need the following:

Lemma 2.4. *Suppose that $\mathcal{A}$ is a commutative unital C^*-algebra. Then*

$$\mathcal{L} = \mathcal{R}' = \{R_U : U \in \mathcal{U}\}' \text{ and } \mathcal{R} = \mathcal{L}' = \{L_U : U \in \mathcal{U}\}',$$

where $\mathcal{L} = \{L_U : U \in \mathcal{U}\}''$ and $\mathcal{R} = \{R_U : U \in \mathcal{U}\}''$.

Proof. It is easy to check $L_UR_V = R_VL_U, \forall\, U, V \in \mathcal{U}$. Let $T \in \mathcal{L}'$ and $S \in \mathcal{R}'$. To prove this lemma, it is sufficient to prove $TS = ST$. Write

$$T\chi_I = \sum_{U\in\mathcal{U}} a_U\chi_U, \quad S\chi_I = \sum_{U\in\mathcal{U}} b_U\chi_U,\ a_U, b_U \in \mathcal{A}.$$

Then for any $V \in \mathcal{U}$, we have

$$\begin{aligned}
ST\chi_V &= STL_V\chi_I = SL_VT\chi_I = SL_V(\sum_{U\in\mathcal{U}} a_U\chi_U)\\
&= S(\sum_{U\in\mathcal{U}} a_Uf(VU)\chi_{\tau(VU)}) = S(\sum_{U\in\mathcal{U}} a_UR_{(VU)^{-1}}\chi_I)\\
&= \sum_{U\in\mathcal{U}} a_UR_{(VU)^{-1}}S\chi_I = \sum_{U\in\mathcal{U}} a_UR_{(VU)^{-1}}(\sum_{W\in\mathcal{U}} b_W\chi_W)\\
&= \sum_{U,W\in\mathcal{U}} a_Ub_Wf(WVU)\chi_{\tau(WVU)}.
\end{aligned}$$

The other side,

$$\begin{aligned}
TS\chi_V &= TSR_{V^{-1}}\chi_I = TR_{V^{-1}}S\chi_I = TR_{V^{-1}}(\sum_{W\in\mathcal{U}} b_W\chi_W)\\
&= T(\sum_{W\in\mathcal{U}} b_Wf(WV)\chi_{\tau(WV)}) = T(\sum_{W\in\mathcal{U}} b_WL_{(WV)}\chi_I)\\
&= \sum_{W\in\mathcal{U}} b_WL_{(WV)}T\chi_I = \sum_{W\in\mathcal{U}} b_WL_{(WV)}(\sum_{U\in\mathcal{U}} a_U\chi_U)\\
&= \sum_{U,W\in\mathcal{U}} b_Wa_Uf(WVU)\chi_{\tau(WVU)}.
\end{aligned}$$

Since $\mathcal{A}$ is a commutative C^*-algebra, which implies that $ST\chi_V = TS\chi_V$, and so $ST = TS$. $\square$

3. Dilation of complete Parseval frame vectors for $\mathcal{A}$-group-like unitary systems

This section aims to prove the dilation property of complete Parseval frame vectors for $\mathcal{U}$. A simple lemma below will be used to prove Theorem 3.2.

Lemma 3.1. *Suppose that $\mathcal{U}$ admits a complete Parseval frame vector η. Then $\mathcal{U}$ is unitarily equivalent to $\{L_U|_{\mathcal{K}} : U \in \mathcal{U}\}$, where $\mathcal{K} = T_\eta(\mathcal{H})$ and $T_\eta : \mathcal{H} \to l^2_{\mathcal{U}}(\mathcal{A})$ is the analysis operator defined by*

$$T_\eta x = \sum_{U\in\mathcal{U}} \langle x, U\eta\rangle \chi_U.$$

Proof. Because η is a complete Parseval frame vector for $\mathcal{U}$, it is easy to check that T_η is an adjointable isometry. By Theorems 15.3.5 and 15.3.8 in [19] we have

$$l^2_{\mathcal{U}}(\mathcal{A}) = T_\eta(\mathcal{H})^\perp \oplus T_\eta(\mathcal{H}).$$

Consider P as the orthogonal projection from $l^2_{\mathcal{U}}(\mathcal{A})$ onto $T_\eta(\mathcal{H})$. Then we have $T_\eta U\eta = P\chi_U$. In fact, let $V \in \mathcal{U}$,

$$\begin{aligned}
\langle T_\eta V\eta, P\chi_U\rangle &= \langle PT_\eta V\eta, \chi_U\rangle \\
&= \left\langle \sum_{W\in\mathcal{U}} \langle V\eta, W\eta\rangle \chi_W, \chi_U \right\rangle \\
&= \langle V\eta, U\eta\rangle = \langle T_\eta V\eta, T_\eta U\eta\rangle.
\end{aligned}$$

We first show that $T_\eta T_\eta^* = P$. In fact, for every $z \in l^2_{\mathcal{U}}(\mathcal{A})$, we can assume that $z = \sum_{U\in\mathcal{U}} a_U\chi_U$ for some $a_U \in \mathcal{A}$. Then we have

$$\begin{aligned}
T_\eta T_\eta^* z &= T_\eta T_\eta^* \sum_{U\in\mathcal{U}} a_U\chi_U = \sum_{U\in\mathcal{U}} a_U T_\eta T_\eta^* \chi_U \\
&= \sum_{U\in\mathcal{U}} a_U T_\eta U\eta = \sum_{U\in\mathcal{U}} a_U P\chi_U \\
&= P(\sum_{U\in\mathcal{U}} a_U\chi_U) = Pz.
\end{aligned}$$

Moreover, for each $V \in \mathcal{U}$, we have

$$\begin{aligned}
PL_V z &= PL_V \sum_{U\in\mathcal{U}} a_U\chi_U = \sum_{U\in\mathcal{U}} a_U PL_V\chi_U \\
&= \sum_{U\in\mathcal{U}} a_U Pf(VU)\chi_{\tau(VU)} = \sum_{U\in\mathcal{U}} a_U f(VU) T_\eta \tau(VU)\eta \\
&= \sum_{U\in\mathcal{U}} a_U T_\eta VU\eta = \sum_{U\in\mathcal{U}} a_U T_\eta V T_\eta^* \chi_U = T_\eta V T_\eta^* (\sum_{U\in\mathcal{U}} a_U\chi_U) \\
&= T_\eta V T_\eta^* z.
\end{aligned}$$

Therefore $PL_V = T_\eta V T_\eta^*$. For each $V \in \mathcal{U}$, we also have

$$L_U T_\eta V\eta = L_U \left(\sum_{W\in\mathcal{U}} \langle V\eta, W\eta\rangle \chi_W \right) = \sum_{W\in\mathcal{U}} \langle V\eta, W\eta\rangle f(UW)\chi_{\tau(UW)}$$

$$= \sum_{W \in \mathcal{U}} \langle UV\eta, UW\eta \rangle f(UW)\chi_{\tau(UW)}$$
$$= \sum_{W \in \mathcal{U}} \langle UV\eta, f(UW)\chi_{\tau(UW)}\eta \rangle f(UW)\chi_{\tau(UW)}$$
$$= \sum_{W \in \mathcal{U}} \langle UV\eta, \chi_{\tau(UW)}\eta \rangle \chi_{\tau(UW)} = T_\eta UV\eta.$$

Thus $L_U T_\eta = T_\eta U$. Finally, for any $U \in \mathcal{U}$,

$$PL_U = T_\eta U T_\eta^* = L_U T_\eta T_\eta^* = L_U P.$$

Hence $P \in \{L_U : U \in \mathcal{U}\}'$. Then

$$PT_\eta U T_\eta^* = PL_U T_\eta T_\eta^* = PL_U P,$$

i.e., $PT_\eta U = PL_U PT_\eta$. Since T_η is an isometry, we have

$$(PT_\eta)^* PT_\eta = T_\eta^* PPT_\eta = T_\eta^* PT_\eta = T_\eta^* T_\eta T_\eta^* T_\eta = I,$$

and PT_η is surjective, then PT_η is a unitary operator. Let $W = PT_\eta$, we have $WU = PL_U W$ for every $U \in \mathcal{U}$. Since $P(l^2_{\mathcal{U}}(\mathcal{A})) = T_\eta(\mathcal{H}) = \mathcal{K}$ and $P \in \{L_U : U \in \mathcal{U}\}'$, it is found that $\mathcal{U}$ is unitarily equivalent to $\{L_U|_{\mathcal{K}} : U \in \mathcal{U}\}$, where $\mathcal{K} = T_\eta(\mathcal{H})$. □

Next, we give the dilation theorem of frames induced by $\mathcal{A}$-group-like unitary systems on Hilbert C^*-modules.

Theorem 3.2. *Let $\mathcal{U}_1$ be an $\mathcal{A}$-group-like unitary system on a finitely or countable generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra $\mathcal{A}$, and let ξ be a complete Parseval frame vector for $\mathcal{U}_1$. Then there exists a finitely or countable generated Hilbert $\mathcal{A}$-module $\mathcal{M}$, an $\mathcal{A}$-group-like unitary system $\mathcal{U}_2$ on $\mathcal{M}$ and a complete Parseval frame vector η for $\mathcal{U}_2$ such that $\xi \oplus \eta$ is a complete wandering vector for $\mathcal{U}_1 \oplus \mathcal{U}_2$ on $\mathcal{H} \oplus \mathcal{M}$.*

Proof. Let $\mathcal{K} = l^2_{\mathcal{U}_1}(\mathcal{A})$. For each $U_1 \in \mathcal{U}_1$, let L_{U_1} be the left regular representation. Consider $\mathcal{U} = \{L_{U_1} : U_1 \in \mathcal{U}_1\}$. Now we consider the analysis operator associated with ξ which is denoted by W. Then W is an adjointable isometry and has closed range. Let P be the orthogonal projection onto ran$W$. We have $P\chi_{U_1} = WU_1\xi$. Let $U_1 = I$. We have $P\chi_I = W\xi$.

We first show that $P \in \mathcal{U}'$. For every $z \in \mathcal{H}, U_1, V_1 \in \mathcal{U}_1$,

$$L_{V_1} Wz = L_{V_1} \sum_{U_1 \in \mathcal{U}} \langle z, U_1\xi \rangle \chi_{U_1} = \sum_{U_1 \in \mathcal{U}} \langle z, U_1\xi \rangle f(V_1U_1)\chi_{\tau(V_1U_1)}$$
$$= \sum_{U_1 \in \mathcal{U}} \langle V_1 z, V_1U_1\xi \rangle f(V_1U_1)\chi_{\tau(V_1U_1)}$$
$$= \sum_{U_1 \in \mathcal{U}} \langle V_1 z, f(V_1U_1)\tau(V_1U_1)\xi \rangle f(V_1U_1)\chi_{\tau(V_1U_1)}$$
$$= \sum_{U_1 \in \mathcal{U}} \langle V_1 z, \tau(V_1U_1)\xi \rangle \chi_{\tau(V_1U_1)} = WV_1 z.$$

Then $L_{V_1}W = WV_1$ for all $V_1 \in \mathcal{U}_1$. Therefore, we have

$$\begin{aligned} PL_{V_1}\chi_{U_1} &= Pf(V_1U_1)\chi_{\tau(V_1U_1)} = f(V_1U_1)W\tau(V_1U_1)\xi \\ &= WV_1U_1\xi = L_{V_1}WU_1\xi = L_{V_1}P\chi_{U_1}. \end{aligned}$$

It follows that $P \in \mathcal{U}'$ and also $P^{\perp} \in \mathcal{U}'$.

Let $\mathcal{U}_2 = \{P^{\perp}L_{U_1}P^{\perp} : U_1 \in \mathcal{U}_1\}, \mathcal{M} = P^{\perp}\mathcal{H}, \eta = P^{\perp}\chi_I$. The test that $\mathcal{U}_2$ is an $\mathcal{A}$-group-like unitary system on $\mathcal{M}$ is easy.

Now, we show that η is a complete Parseval frame vector for $\mathcal{U}_2$. In fact, for every $z \in \mathcal{H}$,

$$\begin{aligned} &\sum_{U_2\in\mathcal{U}_2} \langle P^{\perp}z, U_2\eta\rangle\langle U_2\eta, P^{\perp}z\rangle \\ =&\sum_{U_1\in\mathcal{U}_1} \langle P^{\perp}z, P^{\perp}L_{U_1}P^{\perp}P^{\perp}\chi_I\rangle\langle P^{\perp}L_{U_1}P^{\perp}P^{\perp}\chi_I, P^{\perp}z\rangle \\ =&\sum_{U_1\in\mathcal{U}_1} \langle P^{\perp}z, P^{\perp}L_{U_1}\chi_I\rangle\langle P^{\perp}L_{U_1}\chi_I, P^{\perp}z\rangle \\ =&\sum_{U_1\in\mathcal{U}_1} \langle P^{\perp}z, P^{\perp}\chi_{U_1}\rangle\langle P^{\perp}\chi_{U_1}, P^{\perp}z\rangle \\ =&\sum_{U_1\in\mathcal{U}_1} \langle P^{\perp}z, P^{\perp}z\rangle = \langle P^{\perp}z, P^{\perp}z\rangle. \end{aligned}$$

The conclusion is proved by this. Finally, we show $\xi \oplus \eta$ is a complete wandering vector for $\mathcal{U}_1 \oplus \mathcal{U}_2$ on $\mathcal{H} \oplus \mathcal{M}$. Then for every $U_1, V_1 \in \mathcal{U}_1$, we have

$$\begin{aligned} &\langle U_1\xi \oplus U_2\eta, V_1\xi \oplus V_2\eta\rangle = \langle U_1\xi, V_1\xi\rangle + \langle U_2\eta, V_2\eta\rangle \\ =&\langle WU_1\xi, WV_1\xi\rangle + \langle P^{\perp}L_{U_1}P^{\perp}P^{\perp}\chi_I, P^{\perp}L_{V_1}P^{\perp}P^{\perp}\chi_I\rangle \\ =&\langle P\chi_{U_1}, P\chi_{V_1}\rangle + \langle P^{\perp}L_{U_1}\chi_I, P^{\perp}L_{V_1}\chi_I\rangle \\ =&\langle P\chi_{U_1}, \chi_{V_1}\rangle + \langle P^{\perp}\chi_{U_1}, \chi_{V_1}\rangle \\ =&\langle \chi_{U_1}, \chi_{V_1}\rangle = \begin{cases} 1, & U_1 = V_1, \\ 0, & U_1 \neq V_1. \end{cases} \end{aligned}$$

For $z_1 \in \mathcal{H}, z \in \mathcal{H} \oplus \mathcal{M}$, we also have

$$\begin{aligned} &\sum_{U_1\in\mathcal{U}_1, U_2\in\mathcal{U}_2} \langle z_1 \oplus P^{\perp}z, U_1\xi \oplus U_2\eta\rangle\langle U_1\xi \oplus U_2\eta, z_1 \oplus P^{\perp}z\rangle \\ =&\sum_{U_1\in\mathcal{U}_1, U_2\in\mathcal{U}_2} (\langle z_1, U_1\xi\rangle + \langle P^{\perp}z, U_2\eta\rangle)(\langle U_1\xi, z_1\rangle + \langle U_2\eta, P^{\perp}z\rangle) \\ =&\sum_{U_1\in\mathcal{U}_1} (\langle Wz_1, WU_1\xi\rangle + \langle P^{\perp}z, P^{\perp}L_{U_1}\chi_I\rangle)(\langle WU_1\xi, Wz_1\rangle + \langle P^{\perp}L_{U_1}\chi_I, P^{\perp}z\rangle) \\ =&\sum_{U_1\in\mathcal{U}_1} (\langle Wz_1, P\chi_{U_1}\rangle + \langle P^{\perp}z, P^{\perp}\chi_{U_1}\rangle)(\langle P\chi_{U_1}, Wz_1\rangle + \langle P^{\perp}\chi_{U_1}, P^{\perp}z\rangle) \\ =&\sum_{U_1\in\mathcal{U}_1} (\langle PWz_1, \chi_{U_1}\rangle + \langle P^{\perp}z, \chi_{U_1}\rangle)(\langle \chi_{U_1}, PWz_1\rangle + \langle \chi_{U_1}, P^{\perp}z\rangle) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{U_1 \in \mathcal{U}_1} (\langle PWz_1 + P^\perp z, \chi_{U_1}\rangle)(\langle \chi_{U_1}, PWz_1 + P^\perp z\rangle) \\
&= \langle PWz_1 + P^\perp z, PWz_1 + P^\perp z\rangle = \langle PWz_1, PWz_1\rangle + \langle P^\perp z, P^\perp z\rangle \\
&= \langle z_1, z_1\rangle + \langle P^\perp z, P^\perp z\rangle = \langle z_1 \oplus P^\perp z, z_1 \oplus P^\perp z\rangle.
\end{aligned}
$$

The conclusion is proved. □

Then we have

Corollary 3.3. *Let $\mathcal{U}$ be an $\mathcal{A}$-group-like unitary system on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ such that $\mathcal{W}(\mathcal{U}) \neq \emptyset$ and let ξ be a Parseval frame vector for $\mathcal{U}$. Then there exists a Parseval frame vector η for $\mathcal{U}$ such that $\xi \oplus \eta$ is a complete wandering vector for $\mathcal{U}$ on $\mathcal{H}$.*

4. The complete Bessel vectors

Let $\mathcal{U}$ be an $\mathcal{A}$-group-like unitary system on $\mathcal{H}$ over a unital commutative C^*-algebra $\mathcal{A}$, and let $\xi \in \mathcal{H}$ be a complete Bessel vector. The analysis operator with respect to ξ defined by

$$
T_\xi x = \sum_{U \in \mathcal{U}} \langle x, U\xi\rangle \chi_U, \quad x \in \mathcal{H}
$$

is adjointable and fulfils $T_\xi^* \chi_U = U\xi$ by [9]. It is well known $\{U\xi\}_{U\in\mathcal{U}}$ is a frame if and only if T_ξ^* is surjective [2]. In what follows, let $B_{\mathcal{U}}$ be the set of all complete Bessel vectors for $\mathcal{U}$.

Proposition 4.1. *Suppose that $\mathcal{U}$ is an $\mathcal{A}$-group-like unitary system on a Hilbert $\mathcal{A}$-module $\mathcal{H}$. If $\xi \in \mathcal{H}$ such that $\mathcal{U}\xi$ is a generator of $\mathcal{H}$, then $C_\xi(\mathcal{U}) = \mathcal{U}'$. Moreover, $T_\eta^* T_\zeta \in \mathcal{U}'$ for all $\eta, \zeta \in B_{\mathcal{U}}$.*

Proof. The inclusion " $\supseteq$ " is trivial. Let $T \in C_\xi(\mathcal{U})$ be arbitrary. Then

$$
\begin{aligned}
TUV\xi &= Tf(UV)\tau(UV)\xi = f(UV)\tau(UV)T\xi \\
&= UVT\xi = UTV\xi, \quad U, V \in \mathcal{U}.
\end{aligned}
$$

Since $\mathcal{U}\xi$ is the generator of $\mathcal{H}$, which implies that $TU = UT$. Thus $C_\xi(\mathcal{U}) \subseteq \mathcal{U}'$.

For the second part, for every $V \in \mathcal{U}$, we have

$$
\begin{aligned}
T_\eta^* T_\zeta V z &= \sum_{U\in\mathcal{U}} \langle Vz, U\zeta\rangle U\eta = V \sum_{U\in\mathcal{U}} \langle z, V^{-1}U\zeta\rangle V^{-1}U\eta \\
&= V \sum_{U\in\mathcal{U}} \langle z, f(V^{-1}U)\tau(V^{-1}U)\zeta\rangle f(V^{-1}U)\tau(V^{-1}U)\eta \\
&= V \sum_{U\in\mathcal{U}} \langle z, \tau(V^{-1}U)\zeta\rangle \tau(V^{-1}U)\eta = VT_\eta^* T_\zeta z,
\end{aligned}
$$

where $z \in \mathcal{H}, \eta, \zeta \in B_{\mathcal{U}}$. So we have $T_\eta^* T_\zeta \in \mathcal{U}'$ for all $\eta, \zeta \in B_{\mathcal{U}}$. □

The following proposition can be the analogue of the corresponding result for frames in Hilbert space [6].

Proposition 4.2. *Suppose that $\mathcal{H}$ has orthonormal bases and $\xi \in \mathcal{W}(\mathcal{U})$. Then $\eta \in B_{\mathcal{U}}$ if and only if there is an adjointable operator $T \in C_\xi(\mathcal{U})$ such that $\eta = T\xi$.*

Proof. Suppose that $\eta = T\xi$ for some adjointable operator $T \in C_\xi(\mathcal{U})$. Then for any $z \in \mathcal{H}$,

$$\begin{aligned}\sum_{U\in\mathcal{U}}\langle z, U\eta\rangle\langle U\eta, z\rangle &= \sum_{U\in\mathcal{U}}\langle z, UT\xi\rangle\langle UT\xi, z\rangle\\ &= \sum_{U\in\mathcal{U}}\langle z, TU\xi\rangle\langle TU\xi, z\rangle\\ &= \sum_{U\in\mathcal{U}}\langle T^*z, U\xi\rangle\langle U\xi, T^*z\rangle\\ &=\langle T^*z, T^*z\rangle \leq c\langle z, z\rangle,\end{aligned}$$

where the last inequality we use Lemma 2.1. Then $\eta \in B_{\mathcal{U}}$.

Now, suppose that $\eta \in B_{\mathcal{U}}$. The analysis operators with respect to ξ and η are represented by T_ξ and T_η, respectively. Let $T = T_\eta^* T_\xi$. Then T is adjointable. We now show that $\eta = T\xi$ and $T \in C_\xi(\mathcal{U})$. By Proposition 4.1, $T \in \mathcal{U}' \subseteq C_\xi(\mathcal{U})$. For any $U \in \mathcal{U}$,

$$TU\xi = \sum_{W\in\mathcal{U}}\langle U\xi, W\xi\rangle W\eta = U\eta.$$

Let $U = I$. Then $T\xi = \eta$. □

To parametrize the collection of all complete Bessel vectors for $\mathcal{U}$, we utilize Lemma 2.4 which introduces a natural $\mathcal{A}$-conjugate linear isomorphism π from $\mathcal{L}$ onto $\mathcal{L}'$. This is accomplished by defining

$$\pi(A)B\chi_I = BA^*\chi_I, \quad \forall A, B \in \mathcal{L}.$$

In particular, $\pi(A)\chi_I = A^*\chi_I, A \in \mathcal{L}$. The following is a parametrization of the set of all complete Bessel vectors for $\mathcal{U}$.

Theorem 4.3. *Let $\mathcal{U}$ be an $\mathcal{A}$-group-like unitary system on a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital commutative C^*-algebra $\mathcal{A}$. If $\eta \in \mathcal{H}$ is a complete Parseval frame vector, then*

$$B_{\mathcal{U}} = \{A\eta : A \in \mathcal{U}''\}.$$

Proof. According to Lemma 3.1, we can let $\mathcal{U} = \{L_U|_{\mathrm{ran}P} : U \in \mathcal{U}\}$ and $\eta = P\chi_I$, where P is an orthogonal projection in the commutant of $\{L_U : U \in \mathcal{U}\}$. Let $\mathcal{L} = \{L_U : U \in \mathcal{U}\}''$.

Let's assume $A \in \mathcal{U}''$. Then $A = PTP$ for some $T \in \mathcal{L}$. Thus

$$A\eta = PTP\eta = PT\eta = PT\chi_I.$$

Then we have

$$\begin{aligned}\sum_{U\in\mathcal{U}}\langle z, L_U A\eta\rangle\langle L_U A\eta, z\rangle &= \sum_{U\in\mathcal{U}}\langle z, L_U PT\chi_I\rangle\langle L_U PT\chi_I, z\rangle\\ &= \sum_{U\in\mathcal{U}}\langle z, PL_U\pi(T^*)\chi_I\rangle\langle PL_U\pi(T^*)\chi_I, z\rangle\\ &= \sum_{U\in\mathcal{U}}\langle z, P\pi(T^*)L_U\chi_I\rangle\langle P\pi(T^*)L_U\chi_I, z\rangle\end{aligned}$$

$$
\begin{aligned}
&= \sum_{U \in \mathcal{U}} \langle \pi(T^*)^* Pz, L_U \chi_I \rangle \langle L_U \chi_I, \pi(T^*)^* Pz \rangle \\
&= \left\langle \langle \sum_{U \in \mathcal{U}} \pi(T^*)^* Pz, \chi_U \rangle \chi_U, \pi(T^*)^* Pz \right\rangle \\
&= \langle \pi(T^*)^* Pz, \pi(T^*)^* Pz \rangle \leq c \langle z, z \rangle .
\end{aligned}
$$

Therefore, $A\eta \in B_{\mathcal{U}}$.

Conversely, let $\xi \in \operatorname{ran} P$ be a complete Bessel vector for $\mathcal{U}$. Then, $T\chi_U = L_U \xi$ defines an adjointable operator T. For each $V \in \mathcal{U}$, we have

$$
\begin{aligned}
T L_U \chi_V &= T f(UV) \chi_{\tau(UV)} = f(UV) T \chi_{\tau(UV)} \\
&= f(UV) L_{\tau(UV)} \xi = L_U L_V \xi = L_U T \chi_V .
\end{aligned}
$$

Therefore, $T \in \mathcal{L}'$. In particular, $T\chi_I = \xi$. Let $A = P\pi^{-1}(T^*)P \in \mathcal{U}''$. Note that $\pi^{-1}(T^*) \in \mathcal{L}$ and

$$
\pi^{-1}(T^*)\chi_I = \pi(\pi^{-1}(T))\chi_I = T\chi_I .
$$

Thus, we have

$$
A\eta = P\pi^{-1}(T^*)P\eta = P\pi^{-1}(T^*)P\chi_I = PT\chi_I = P\xi = \xi .
$$

Hence $B_{\mathcal{U}} = \{A\eta : A \in \mathcal{U}''\}$. □

Similar to the complete Bessel vectors, we have the following corollary.

Corollary 4.4. *Let $\mathcal{U}$ be an $\mathcal{A}$-group-like unitary system on a finitely or countably generated Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital commutative C^*-algebra $\mathcal{A}$. Suppose vector ξ is in $\mathcal{H}$, and vector $\eta \in \mathcal{H}$ is a complete Parseval frame vector for $\mathcal{U}$. If there's a unitary operator $A \in \mathcal{U}''$ that makes $\xi = A\eta$, then ξ is also a complete Parseval frame vector for $\mathcal{U}$.*

Proof. Through Lemma 3.1, we can assume that $\mathcal{U} = \{L_U|_{\operatorname{ran} P} : U \in \mathcal{U}\}$ and $\eta = P\chi_I$, where P is an orthogonal projection in $\{L_U : U \in \mathcal{U}\}'$. Let $\mathcal{L} = \{L_U : U \in \mathcal{U}\}''$.

First, we assume that there is a unitary operator $A \in \mathcal{U}''$ such that $\xi = A\eta$. Our research reveals that the vector $A\eta$ is a complete Parseval frame vector for $\mathcal{U}$. For $z \in \operatorname{ran} P$,

$$
\begin{aligned}
&\sum_{U \in \mathcal{U}} \langle z, UA\eta \rangle \langle UA\eta, z \rangle = \sum_{U \in \mathcal{U}} \langle z, L_U PA\eta \rangle \langle L_U PA\eta, z \rangle \\
&= \sum_{U \in \mathcal{U}} \langle z, L_U PAP\chi_I \rangle \langle L_U PAP\chi_I, z \rangle = \sum_{U \in \mathcal{U}} \langle z, L_U PA\chi_I \rangle \langle L_U PA\chi_I, z \rangle \\
&= \sum_{U \in \mathcal{U}} \langle z, PL_U A\chi_I \rangle \langle PL_U A\chi_I, z \rangle = \sum_{U \in \mathcal{U}} \langle Pz, L_U A\chi_I \rangle \langle L_U A\chi_I, Pz \rangle \\
&= \sum_{U \in \mathcal{U}} \langle z, L_U \pi(A^*)\chi_I \rangle \langle L_U \pi(A^*)\chi_I, z \rangle = \sum_{U \in \mathcal{U}} \langle z, \pi(A^*) L_U \chi_I \rangle \langle \pi(A^*) L_U \chi_I, z \rangle \\
&= \sum_{U \in \mathcal{U}} \langle \pi(A^*)^* z, \chi_U \rangle \langle \chi_U, \pi(A^*)^* z \rangle = \langle \pi(A^*)^* z, \pi(A^*)^* z \rangle = \langle z, z \rangle .
\end{aligned}
$$

The conclusion is obtained. □

5. Frame multiplicity of $\mathcal{A}$-group-like unitary systems

Let $\mathcal{U}$ be an $\mathcal{A}$-group-like unitary system. Two complete Parseval frame vectors ξ and η for $\mathcal{U}$ are said to be *equivalent* if $\{U\xi\}_{U\in\mathcal{U}}$ and $\{U\eta\}_{U\in\mathcal{U}}$ are unitarily equivalent frames, i.e. there is a unitary operator W such that $WU\xi = U\eta$ for any $U \in \mathcal{U}$. Well, we have the following equivalent condition.

Proposition 5.1. *Let $\mathcal{U}$ be an $\mathcal{A}$-group-like unitary system. Suppose that ξ_1 and ξ_2 are two complete Parseval frame vectors for $\mathcal{U}$, respectively. Then $\{U\xi_1\}_{U\in\mathcal{U}}$ and $\{U\xi_2\}_{U\in\mathcal{U}}$ are unitarily equivalent frames if and only if there is a unitary operator $\mathcal{W} \in \mathcal{U}'$ such that $\mathcal{W}\xi_1 = \xi_2$.*

Proof. Suppose that $\{U\xi_1\}_{U\in\mathcal{U}}$ and $\{U\xi_2\}_{U\in\mathcal{U}}$ are unitarily equivalent. Then there is a unitary operator $\mathcal{W}$ such that $\mathcal{W}U\xi_1 = U\xi_2$ for all $U \in \mathcal{U}$. In particular, $\mathcal{W}\xi_1 = \xi_2$. Then for any $U, V \in \mathcal{U}$, we have

$$\begin{aligned}\mathcal{W}UV\xi_1 &= \mathcal{W}f(UV)\tau(UV))\xi_1\\ &= f(UV)\tau(UV)\xi_2 = UV\xi_2\\ &= U\mathcal{W}V\xi_1.\end{aligned}$$

Then $\mathcal{W} \in \mathcal{U}'$.

Conversely, suppose that there is a unitary operator $\mathcal{W} \in \mathcal{U}'$ such that $\mathcal{W}\xi_1 = \xi_2$. Then $\mathcal{W}U = U\mathcal{W}$ and $\mathcal{W}U\xi_1 = U\mathcal{W}\xi_1 = U\xi_2$. Therefore, $\{U\xi_1\}_{U\in\mathcal{U}}$ and $\{U\xi_2\}_{U\in\mathcal{U}}$ are unitarily equivalent frames. □

We use $\mathcal{H}^{(n)}$ to denote $\mathcal{H}\oplus\mathcal{H}\oplus\cdots\oplus\mathcal{H}$ (n copies of $\mathcal{H}$) and $\mathcal{U}^{(n)}$ to denote the set $\{U\oplus U\oplus\cdots\oplus U : U \in \mathcal{U}\}$. It is easy to see $\mathcal{U}^{(n)}$ is an $\mathcal{A}$-group-like unitary system on $\mathcal{H}^{(n)}$. An $\mathcal{A}$-group-like unitary system $\mathcal{U}$ which satisfies $\mathcal{W}(\mathcal{U}) \neq \emptyset$ is said to have *frame multiplicity n* if n is the supremum of all the $k \in \mathbb{N}$ with the property that there exist complete Parseval frame vectors $\eta_i (i = 1, 2, \ldots k)$ such that $\eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_k$ is a complete Parseval frame vector for $\mathcal{U}^{(k)}$. We need the following to prove that the multiplicity of frame is finite.

Theorem 5.2. *Let $\mathcal{U}$ be an $\mathcal{A}$-group-like unitary system such that $\mathcal{W}(\mathcal{U}) \neq \emptyset$. Then the frame multiplicity of $\mathcal{U}$ is n if and only if there are isometry operators $V_i \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$ such that $V_i^* \in \mathcal{U}'$ and $V_i\mathcal{H}\perp V_j\mathcal{H}$ when $i \neq j, i, j = 1, 2, \cdots, n$.*

Proof. We only consider the case of $n = 2$. Assume that $\eta_1, \eta_2 \in \mathcal{H}$ are complete Parseval frame vectors for $\mathcal{U}$ such that $\{U\eta_1 \oplus U\eta_2 : U \in \mathcal{U}\}$ is a Parseval frame for $\mathcal{H}^{(2)}$. Let $\psi \in \mathcal{W}(\mathcal{U})$ and V_i be the analysis operator associated with $\eta_i, , i = 1, 2$, respectively. Then, V_i is an isometry, and

$$\begin{aligned}V_i^*UV\psi &= V_i^*f(UV)\tau(UV)\psi\\ &= f(UV)\tau(UV)\eta_i = UV\eta_i\\ &= UV_i^*V\psi,\end{aligned}$$

which implies that $V_i^* \in \mathcal{U}'$.

Since $\eta_1, \eta_2 \in \mathcal{H}$ are complete Parseval frame vectors for $\mathcal{U}$, respectively. Then for $y, z \in \mathcal{H}$,

$$y = \sum_{U\in\mathcal{U}}\langle y, U\eta_1\rangle U\eta_1, \qquad z = \sum_{U\in\mathcal{U}}\langle z, U\eta_2\rangle U\eta_2,$$

and

$$\begin{aligned}
y \oplus z &= \sum_{U\in\mathcal{U}} \langle y \oplus z, U\eta_1 \oplus U\eta_2\rangle U\eta_1 \oplus U\eta_2 \\
&= \left(\sum_{U\in\mathcal{U}} \langle y, U\eta_1\rangle U\eta_1 + \sum_{U\in\mathcal{U}} \langle z, U\eta_2\rangle U\eta_1\right) \\
&\oplus \left(\sum_{U\in\mathcal{U}} \langle y, U\eta_1\rangle U\eta_2 + \sum_{U\in\mathcal{U}} \langle z, U\eta_2\rangle U\eta_2\right) \\
&= \left(y + \sum_{U\in\mathcal{U}} \langle z, U\eta_2\rangle U\eta_1\right) \oplus \left(\sum_{U\in\mathcal{U}} \langle y, U\eta_1\rangle U\eta_2 + z\right),
\end{aligned}$$

which implies that $\sum_{U\in\mathcal{U}}\langle z, U\eta_2\rangle U\eta_1 = \sum_{U\in\mathcal{U}}\langle y, U\eta_1\rangle U\eta_2 = 0$. Then

$$\langle V_1 y, V_2 z\rangle = \left\langle \sum_{U\in\mathcal{U}} \langle y, U\eta_1\rangle U\eta_2, z\right\rangle = 0.$$

Since V_1, V_2 have closed ranges, we have $V_1\mathcal{H} \perp V_2\mathcal{H}$.

Conversely, let $\psi \in \mathcal{W}(\mathcal{U})$ and let $\eta_i = V_i^*\psi$. Then η_i is a complete Parseval frame vector for $\mathcal{U}$. We can identify V_i as an analysis operator associated with η_i. Since $V_1\mathcal{H} \perp V_2\mathcal{H}$ and V_1, V_2 are isometry operators. For any $y, z \in \mathcal{H}$, we have

$$\begin{aligned}
&\sum_{U\in\mathcal{U}} \langle y \oplus z, U\eta_1 \oplus U\eta_2\rangle\langle U\eta_1 \oplus U\eta_2, y \oplus z\rangle \\
=&\sum_{U\in\mathcal{U}} \langle y, U\eta_1\rangle\langle U\eta_1, y\rangle + \sum_{U\in\mathcal{U}} \langle y, U\eta_1\rangle\langle U\eta_2, z\rangle \\
&+ \sum_{U\in\mathcal{U}} \langle z, U\eta_2\rangle\langle U\eta_1, y\rangle + \sum_{U\in\mathcal{U}} \langle z, U\eta_2\rangle\langle U\eta_2, z\rangle \\
=&\langle V_1 y, V_1 y\rangle + \langle V_1 y, V_2 z\rangle + \langle V_2 z, V_1 y\rangle + \langle V_2 z, V_2 z\rangle \\
=&\langle y, y\rangle + \langle z, z\rangle = \langle y \oplus z, y \oplus z\rangle.
\end{aligned}$$

Therefore $\eta_1 \oplus \eta_2$ is a complete Parseval frame vector for $\mathcal{U}^{(2)}$. □

Below is an essential theorem in this section, which tells us that the frame multiplicity is always finite.

Theorem 5.3. *Let $\mathcal{U}$ be an $\mathcal{A}$-group-like unitary system such that $\mathcal{W}(\mathcal{U}) \neq \emptyset$. Then the frame multiplicity of $\mathcal{U}$ is finite.*

Proof. Instead, suppose the frame multiplicity of $\mathcal{U}$ is infinity. For any $k \in \mathbb{N}$, there exist complete Parseval frame vectors $\eta_i (i = 1, 2 \ldots k)$ for $\mathcal{U}$ such that $\eta_1 \oplus \eta_2 \oplus \cdots \oplus \eta_k$ is a complete Parseval frame vector for $\mathcal{U}^{(k)}$. Let T_{η_i} be the analysis operator associated with η_i, and P_i be the orthogonal projection from $\mathcal{H}$ onto $T_{\eta_i}(\mathcal{H})$. Just like the evidence presented in Theorems 3.2 and 5.2, we have $P_i \in \mathcal{U}'$ and $T_{\eta_i}(\mathcal{H}) \perp T_{\eta_j}(\mathcal{H})$ when $i \neq j$. Since $T_{\eta_i}(\mathcal{H}) = P_i(\mathcal{H})$, we have $P_i(\mathcal{H}) \perp P_j(\mathcal{H})$ when $i \neq j$. Then $P_iP_j = P_jP_i = 0$. Let $Q = \sum_i^k P_i$. It is an orthogonal projection and $Q \leq I$. Let $\psi \in \mathcal{W}(\mathcal{U})$. We have

$$\langle Q\psi, \psi\rangle = \sum_i^k \langle P_i\psi, P_i\psi\rangle$$

$$
\begin{aligned}
&= \sum_i^k \langle UP_i\psi, UP_i\psi\rangle = \sum_i^k \langle P_iU\psi, P_iU\psi\rangle \\
&= \sum_i^k \langle T_{\eta_i}U\eta_i, T_{\eta_i}U\eta_i\rangle = \sum_i^k \langle \eta_i, \eta_i\rangle \\
&\leq \langle \psi, \psi\rangle = \langle U\psi, U\psi\rangle = 1.
\end{aligned}
$$

Since $\langle \eta_i, \eta_i\rangle \in \mathcal{A}$ is positive element, which lead to a contradiction if we let $k \to \infty$. □

6. Conclusions

In this paper, we have introduced the concept of $\mathcal{A}$-group-like unitary system $\mathcal{U}$ and have proved that a complete Parseval frame vector for $\mathcal{U}$ on Hilbert C^*-module can be dilated to a complete wandering vector. Moreover, we have provided the parameterization of complete Bessel vector for $\mathcal{U}$. We also have proved that the frame multiplicity of $\mathcal{U}$ is always finite.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

We declare that there are no conflicts of interest.

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