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Research article

Coupled systems of ψ-Hilfer generalized proportional fractional nonlocal mixed boundary value problems

Sunisa Theswan[1], **Sotiris K. Ntouyas**[2] **and Jessada Tariboon**[1,*]

[1] Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand

[2] Department of Mathematics, University of Ioannina, Ioannina 45110, Greece

* **Correspondence:** Email: jessada.t@sci.kmutnb.ac.th.

Abstract: In this paper, we investigate a coupled system of Hilfer-type nonlinear proportional fractional differential equations supplemented with mixed multi-point and integro-multi-point boundary conditions. We used standard methods from functional analysis and especially fixed point theory. Two existence results are established using the Leray-Schauder's alternative and the Krasnosel'skii's fixed point theorem, while the existence of a unique solution is achieved via the Banach's contraction mapping principle. Finally, numerical examples are constructed to illustrate the main theoretical results. Our results are novel, wider in scope, produce a variety of new results as special cases and contribute to the existing literature on nonlocal systems of nonlinear ψ-Hilfer generalized fractional proportional differential equations.

Keywords: coupled system; Hilfer fractional proportional derivative; nonlocal boundary conditions; multi-point boundary conditions; integral boundary conditions; fixed point theorems
Mathematics Subject Classification: 26A33, 34A08, 34B10

1. Introduction

In the last years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives. These kind of equations have gained considerable importance due to their application in various sciences, such as physics, biology, economics, mechanics, chemistry, control theory, engineering, signal and image processing, etc [1–7].

Nonlinear coupled systems of fractional order differential equations appear often in investigations connected with disease models [8], anomalous diffusion [9] and ecological models [10]. Unlike the classical derivative operator, one can find a variety of its fractional counterparts such as Riemann-

Liouville, Caputo, Hadamard, Erdelyi-Kober, Hilfer, Caputo-Hadamard, etc. Recently, a new class of fractional proportional derivative operators was introduced and discussed in [11–13]. Then, the concept of Hilfer type generalized proportional fractional derivative operators was proposed in [14]. For the detailed advantages of the Hilfer derivative, see [15] and a recent application in calcium diffusion in [16].

Many researchers studied initial and boundary value problems for differential equations and inclusions including different kinds of fractional derivative operators, for instance, see [17–21]. In [22], the authors studied a nonlocal-initial value problem of order in $(0,1)$ involving a $\overline{\psi}_*$-Hilfer generalized proportional fractional derivative of a function with respect to another function. Recently, in [23], the authors investigated the existence and uniqueness of solutions for a nonlocal mixed boundary value problem for Hilfer fractional $\overline{\psi}_*$-proportional type differential equations and inclusions of order in $(1,2]$ of the form

$$\begin{cases} \mathbb{D}^{\alpha,\beta,\sigma,\psi}_{\mathtt{w}_1^+}u(t) = \Upsilon(t,u(t)), \quad t\in[\mathtt{w}_1,\mathtt{w}_2],\ 0\le \mathtt{w}_1<\mathtt{w}_2, \\ u(\mathtt{w}_1)=0, \\ u(\mathtt{w}_2) = \sum\limits_{j=1}^{m}\eta_j u(\xi_j) + \sum\limits_{i=1}^{n}\zeta_i\, I^{\phi_i,\sigma,\psi}_{c^+}u(\theta_i) + \sum\limits_{k=1}^{r}\lambda_k\, D^{\delta_k,\beta,\sigma,\psi}_{c^+}u(\mu_k), \end{cases} \tag{1.1}$$

where $\mathbb{D}^{\chi,\beta,\sigma,\psi}_{c+}$, denotes the ψ-Hilfer generalized proportional fractional derivative operator of order $\chi\in\{\alpha,\delta_k\}$, $\alpha,\delta_k\in(1,2]$ and type $\beta\in[0,1]$, respectively, $\sigma\in(0,1]$, $\eta_j,\zeta_i,\lambda_k\in\mathbb{R}$ are given constants, Υ: $[\mathtt{w}_1,\mathtt{w}_2]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function, $I^{\phi_i;\sigma,\psi}_{\mathtt{w}_1+}$ is the generalized proportional fractional integral operator of order $\phi_i>0$ and $\xi_j,\theta_i,\mu_k\in(\mathtt{w}_1,\mathtt{w}_2)$, $j=1,2,\cdots,m$, $i=1,2,\cdots,n$, $k=1,2,\cdots,r$, are given points.

In [24], the authors discussed the existence of solutions for a nonlinear coupled system of (k,ψ)-Hilfer fractional differential equations of different orders in $(1,2]$, complemented with coupled (k,ψ)-Riemann-Liouville fractional integral boundary conditions. In [25] a coupled system of Hilfer type generalized proportional fractional differential equations with nonlocal multi-point boundary conditions of the form

$$\begin{cases} \left(\mathbb{D}^{\delta_1,\eta,\sigma}_{\mathtt{w}_1^+} + k\mathbb{D}^{\delta_1-1,\eta,\sigma}_{\mathtt{w}_1^+}\right)\tau_1(t) = \Upsilon_1(t,\tau_1(t),\tau_2(t)), \quad t\in[\mathtt{w}_1,\mathtt{w}_2], \\ \left(\mathbb{D}^{\delta_2,\eta,\sigma}_{\mathtt{w}_1^+} + k_1\mathbb{D}^{\delta_2-1,\eta,\sigma}_{\mathtt{w}_1^+}\right)\tau_2(t) = \Upsilon_2(t,\tau_1(t),\tau_2(t)), \quad t\in[\mathtt{w}_1,\mathtt{w}_2], \\ \tau_1(\mathtt{w}_1)=0, \qquad \tau_1(\mathtt{w}_2) = \sum\limits_{j=1}^{m}\theta_j\tau_2(\xi_j), \\ \tau_2(\mathtt{w}_1)=0, \qquad \tau_2(\mathtt{w}_2) = \sum\limits_{i=1}^{n}\varepsilon_i\tau_1(\lambda_i), \end{cases} \tag{1.2}$$

is investigated, in which $\mathbb{D}^{\delta_1,\eta,\sigma}_{\mathtt{w}_1^+}$ and $\mathbb{D}^{\delta_2,\eta,\sigma}_{\mathtt{w}_1^+}$ are the fractional derivatives of Hilfer generalized proportional type of order $1<\delta_1,\delta_2<2$, the Hilfer parameter $0\le\eta\le1$, $\sigma\in(0,1]$, $k,k_1\in\mathbb{R}$, Υ_1,Υ_2: $[\mathtt{w}_1,\mathtt{w}_2]\times\mathbb{R}\to\mathbb{R}$ are continuous functions, $\mathtt{w}_1\ge0$, $\theta_j,\varepsilon_i\in\mathbb{R}$, $\xi_j,\lambda_i\in(\mathtt{w}_1,\mathtt{w}_2)$ for $i=1,2,3,\cdots,n$ and $j=1,2,\cdots,m$. Existence and uniqueness results are proved by applying classical Banach and Krasnosel'skii fixed-point theorems, and the Leray-Schauder alternative.

Very recently in [26] the authors established existence and uniqueness results for a class of coupled systems of nonlinear Hilfer-type fractional $\overline{\psi}_*$-proportional differential equations equipped with nonlocal multi-point and integro-multi-strip coupled boundary conditions of the form:

$$\begin{cases} \mathbb{D}_{\mathtt{w}_1+}^{\rho_1,\varphi_1,\vartheta_*,\overline{\psi}_*}\sigma(z) = \Psi_1(z,\sigma(z),\tau(z)), \quad z\in[\mathtt{w}_1,\mathtt{w}_2], \\ \mathbb{D}_{\mathtt{w}_1+}^{\rho_2,\varphi_2,\vartheta_*,\overline{\psi}_*}\tau(z) \ = \Psi_2(z,\sigma(z),\tau(z)), \quad z\in[\mathtt{w}_1,\mathtt{w}_2], \\ \sigma(\mathtt{w}_1)=0, \quad \displaystyle\int_{\mathtt{w}_1}^{\mathtt{w}_2}\overline{\psi}'_*(s)\sigma(s)ds = \sum_{i=1}^{n}\kappa_i\int_{\xi_i}^{\eta_i}\overline{\psi}'_*(s)\tau(s)ds + \sum_{j=1}^{m}\theta_j\tau(\zeta_j), \\ \tau(\mathtt{w}_1) \ = 0, \quad \displaystyle\int_{\mathtt{w}_1}^{\mathtt{w}_2}\overline{\psi}'_*(s)\tau(s)ds = \sum_{i=1}^{n}\phi_i\int_{\delta_i}^{\epsilon_i}\overline{\psi}'_*(s)\sigma(s)ds + \sum_{j=1}^{m}\vartheta_j\sigma(z_j), \end{cases} \tag{1.3}$$

where $\mathbb{D}_{\mathtt{w}_1+}^{\rho_\kappa,\varphi_a,\vartheta_*,\overline{\psi}_*}$ and $\kappa = 1,2$ denote the Hilfer fractional $\overline{\psi}_*$-proportional derivative operator of the order $\rho_\kappa \in (1,2]$ and type $\varphi_i \in [0,1]$, $\vartheta_* \in (0,1]$, $\mathtt{w}_1 < \zeta_j < \xi_i < \eta_i < \mathtt{w}_2$, $\mathtt{w}_1 < \delta_j < z_i < \epsilon_i < \mathtt{w}_2$, $\kappa_i,\theta_j,\phi_i,\vartheta_j \in \mathbb{R}$, $i = 1,2,\cdots,n$, $j = 1,2,\cdots,m$, $\overline{\psi}_*$: $[\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ is an increasing function with $\overline{\psi}'_*(z) \neq 0$ for all $z \in [\mathtt{w}_1,\mathtt{w}_2]$ and Ψ_1,Ψ_2: $[\mathtt{w}_1,\mathtt{w}_2]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions.

In this work, motivated by the above mentioned papers, we study a coupled system of ψ-Hilfer generalized proportional sequential fractional differential equations with mixed nonlocal and integro-multi-point boundary conditions, of the form

$$\begin{cases} ({}^HD_{\mathtt{w}_1}^{\alpha_1,\beta_1,\rho,\psi}k_1)(t) + \lambda_1({}^HD_{\mathtt{w}_1}^{\alpha_1-1,\beta_1,\rho,\psi}k_1)(t) = \Upsilon_1(t,k_1(t),k_2(t)), \ \ t\in[\mathtt{w}_1,\mathtt{w}_2], \\ ({}^HD_{\mathtt{w}_1}^{\alpha_2,\beta_2,\rho,\psi}k_2)(t) + \lambda_2({}^HD_{\mathtt{w}_1}^{\alpha_2-1,\beta_2,\rho,\psi}k_2)(t) = \Upsilon_2(t,k_2(t),k_1(t)), \ \ t\in[\mathtt{w}_1,\mathtt{w}_2], \\ k_1(\mathtt{w}_1)=0, \qquad k_1(\mathtt{w}_2) = \displaystyle\sum_{i=1}^{n}\eta_i\,k_2(\xi_i) + \sum_{j=1}^{m}\zeta_j\,{}^pI^{\Phi_j,\rho,\psi}k_2(\theta_j), \\ k_2(\mathtt{w}_1)=0, \qquad k_2(\mathtt{w}_2) = \displaystyle\sum_{k=1}^{r}\aleph_k\,k_1(\varrho_k) + \sum_{l=1}^{q}\Theta_l\,{}^pI^{\upsilon_l,\rho,\psi}k_1(\vartheta_l), \end{cases} \tag{1.4}$$

where ${}^HD_{\mathtt{w}_1}^{\mathcal{X},\beta_\iota,\rho,\psi}$ denotes the ψ-Hilfer generalized proportional fractional derivative operator of order $\mathcal{X} \in \{\alpha_1,\alpha_2\}$ with the parameters $\beta_\iota, \iota \in \{1,2\}$, $1 < \mathcal{X} \leq 2$, $0 \leq \beta_\iota \leq 1$, ${}^pI_{\mathtt{w}_1}^{\mathcal{Y},\rho,\psi}$ is a generalized proportional fractional integral operator of order $\mathcal{Y} > 0$, $\mathcal{Y} \in \{\Phi_j,\upsilon_l\}$, $\lambda_1,\lambda_2,\eta_i,\zeta_j,\aleph_k,\Theta_l \in \mathbb{R}\backslash\{0\}$, $\xi_i,\theta_j,\varrho_k,\vartheta_l \in (a,b)$, $i = 1,2,\cdots,n$, $j = 1,2,\cdots,m$, $k = 1,2,\cdots,r$, $l = 1,2,\cdots,q$ and Υ_1,Υ_2: $[\mathtt{w}_1,\mathtt{w}_2]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are nonlinear continuous functions.

Here, we emphasize that problem (1.4) is novel, and its investigation will enhance the scope of the literature on nonlocal Hilfer-type fractional ψ-proportional systems. Note that, when $\psi(t) = t$, problem (1.4) reduces to a coupled system of Hilfer generalized proportional fractional differential equations with mixed nonlocal multi-point and integro-multi-point boundary conditions; while if $\rho = 1$, reduces to a coupled system of ψ-Hilfer fractional differential equations with mixed nonlocal multi-point and integro-multi-point boundary conditions. If $\psi(t) = t$, $\zeta_j = 0$, $\Theta_l = 0$, problem (1.4) is reduced to problem (1.2).

In solving (1.4), we first convert it into an equivalent fixed point problem, with the help of an auxiliary result based on a linear variant (1.4). Afterward, under different assumptions, we apply different fixed point theorems to establish our results on existence and uniqueness of solutions. For the first result (Theorem 3.1), we apply the Leray-Schauder's alternative to show that there exists at least one solution for the problem (1.4). The second result (Theorem 3.2), relying on Krasnosel'skii's fixed point theorem, shows that the problem (1.4) has at least one solution under different assumptions, and the last result (Theorem 3.3), shows the existence of a unique solution to the problem (1.4) by means of Banach's contraction mapping principle. In Section 4, we illustrate all the obtained theoretical results with the aid of constructed numerical examples. We emphasize that the problem (1.4) is novel and its investigation will enhance the scope of the literature on coupled systems of ψ-Hilfer generalized proportional fractional differential equations with mixed nonlocal and integro-multi-point boundary conditions. The used method is standard, but its configuration in the problem (1.4) is new.

The structure of the rest of the paper is organized as follows: In Section 2, some necessary definitions and preliminary results related to our problem are presented. Section 3 contains the main results for the problem (1.4), while numerical examples illustrating these results are constructed in Section 4. A brief conclusion closes the paper.

2. Preliminaries

In this section, we introduce some necessary definitions and preliminary results needed in main results later.

Definition 2.1. *[11, 12] Let the functions ϑ_0, ϑ_1: $[0,1]\times\mathbb{R}\to[0,\infty)$ be continuous such that for all $t\in\mathbb{R}$ and for $\rho\in[0,1]$, we get*

$$\lim_{\rho\to 0^+}\vartheta_0(\rho,t)=0,\ \lim_{\rho\to 0^+}\vartheta_1(\rho,t)=1,\ \lim_{\rho\to 1^-}\vartheta_0(\rho,t)=1,\ \lim_{\rho\to 1^-}\vartheta_1(\rho,t)=0$$

and

$$\vartheta_0(\rho,t)\neq 0,\ 0<\rho\leq 1,\ \vartheta_1(\rho,t)\neq 0,\ 0\leq\rho<1.$$

Let also $\psi(t)$ be a strictly positive increasing continuous function. So, the proportional differential operator of order ρ of function $\Upsilon(t)$ with respect to function $\psi(t)$ is defined by

$${}^{p}D^{\rho,\psi}\Upsilon_1(t)=\vartheta_1(\rho,t)\Upsilon(t)+\vartheta_0(\rho,t)\frac{\Upsilon'(t)}{\psi'(t)}.$$

Moreover, if $\vartheta_0(\rho,t)=\rho$ and $\vartheta_1(\rho,t)=1-\rho$, then operator ${}^{p}D^{\rho,\psi}$ becomes

$${}^{p}D^{\rho,\psi}\Upsilon(t)=(1-\rho)\Upsilon_1(t)+\rho\frac{\Upsilon'(t)}{\psi'(t)}.$$

The integral corresponding to the above proportional derivative is defined as

$${}^{p}I^{1,\rho,\psi}_{w_1}\Upsilon_1(t)=\frac{1}{\rho}\int_{w_1}^{t}e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(s))}\Upsilon(s)\psi'(s)ds,$$

where

$${}^{p}I^{0,\rho,\psi}_{w_1}\Upsilon_1(t)=\Upsilon_1(t).$$

The generalized proportional integral of order n corresponding to proportional derivative ${}^{p}D^{n,\rho,\psi}\Upsilon_1(t)$, is given by

$${}^{p}I^{n,\rho,\psi}_{w_1}\Upsilon_1(t)=\frac{1}{\rho^n\Gamma(n)}\int_{w_1}^{t}e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{n-1}\Upsilon(s)\psi'(s)ds.$$

Based on the generalized proportional integral of order n, we can obtain the following general proportional fractional integral and derivative.

Definition 2.2. *[11, 12] Let $\rho \in (0,1]$ and $\alpha > 0$. The fractional proportional integral of order α of the function f with respect to function ψ is defined by*

$$({}^{p}I_{\mathtt{w}_1}^{\alpha,\rho,\psi}\Upsilon)(t) = \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_{\mathtt{w}_1}^{t} e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\alpha-1}\Upsilon(s)\psi'(s)ds.$$

Definition 2.3. *[11, 12] Let $\rho \in (0,1]$ $\alpha > 0$ and $\psi(t)$ is a continuous function on $[\mathtt{w}_1, \mathtt{w}_2]$, $\psi'(t) > 0$. The generalized proportional fractional derivative of order α of the function Υ with respect to function ψ is defined by*

$$({}^{p}D_{\mathtt{w}_1}^{\alpha,\rho,\psi}\Upsilon)(t) = \frac{{}^{p}D^{n,\rho,\psi}}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_{\mathtt{w}_1}^{t} e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{n-\alpha-1}\Upsilon(s)\psi'(s)ds,$$

where

$${}^{p}D^{n,\rho,\psi} = \underbrace{{}^{p}D^{\rho,\psi}\cdot{}^{p}D^{\rho,\psi}\cdot{}^{p}D^{\rho,\psi}\ldots{}^{p}D^{\rho,\psi}}_{n\ times}.$$

Below we present the generalized proportional fractional derivatives of order α of function Υ with respect to another function ψ in Hilfer sense.

Definition 2.4. *[27] For $\rho \in (0,1]$. Let functions $\Upsilon, \psi \in C^{m}([\mathtt{w}_1, \mathtt{w}_2], \mathbb{R})$ and ψ be positive and strictly increasing with $\psi'(t) \neq 0$, for all $t \in [\mathtt{w}_1, \mathtt{w}_2]$. The ψ-Hilfer generalized propotional fractional derivative of order α and type β for Υ with respect to another function ψ is defined by*

$$({}^{H}D_{\mathtt{w}_1}^{\alpha,\beta,\rho,\psi}\Upsilon)(t) = {}^{p}I_{\mathtt{w}_1}^{\beta(n-\alpha),\rho,\psi}[D^{n,\rho,\psi}({}^{p}I_{\mathtt{w}_1}^{(1-\beta)(n-\alpha),\rho,\psi}\Upsilon)](t),$$

where order $n-1 < \alpha < n$ and $0 \le \beta \le 1$.

If $\gamma = \alpha + \beta(n-\alpha)$, then the ψ-Hilfer generalized proportional derivative ${}^{H}D_{\mathtt{w}_1}^{\alpha,\beta,\rho,\psi}$ is equivalent to

$$({}^{H}D_{\mathtt{w}_1}^{\alpha,\beta,\rho,\psi}\Upsilon)(t) = {}^{p}I_{\mathtt{w}_1}^{\beta(n-\alpha),\rho,\psi}[D^{n,\rho,\psi}({}^{p}I_{\mathtt{w}_1}^{(1-\beta)(n-\alpha),\rho,\psi}\Upsilon)](t) = ({}^{p}I_{\mathtt{w}_1}^{\beta(n-\alpha),\rho,\psi}D_{\mathtt{w}_1}^{\gamma,\rho,\psi}\Upsilon)(t).$$

Lemma 2.1. *[27] Let $n-1 < \alpha < n \in \mathbb{N}$, $0 < \rho \le 1$, $0 \le \beta \le 1$ and $n-1 < \gamma < n$ such that $\gamma = \alpha + n\beta - \alpha\beta$. If $\Upsilon \in C([\mathtt{w}_1, \mathtt{w}_2], \mathbb{R})$ and $I_{\mathtt{w}_1}^{(n-\gamma,\rho,\psi)} \in C^{n}([\mathtt{w}_1, \mathtt{w}_2], \mathbb{R})$, then*

$$\left({}^{p}I_{\mathtt{w}_1}^{\alpha,\rho,\psi}\ {}^{H}D_{\mathtt{w}_1}^{\alpha,\beta,\rho,\psi}\Upsilon\right)(t) = \Upsilon(t) - \sum_{j=1}^{n}\frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma-j}}{\rho^{\gamma-j}\Gamma(\gamma-j+1)}\left({}^{p}I_{\mathtt{w}_1}^{j-\gamma,\rho,\psi}f\right)(\mathtt{w}_1).$$

Lemma 2.2. *Let $1 < \alpha_1, \alpha_2 < 2$, $0 \le \beta_1, \beta_2 \le 1$, $\gamma_i = \alpha_i + \beta_i(2-\alpha_i)$, $i = 1,2$, $\ddot{\Lambda} \neq 0$ and $z, w \in C([\mathtt{w}_1, \mathtt{w}_2], \mathbb{R})$. Then the pair (k_1, k_2) is the solution of the coupled system*

$$\begin{cases} ({}^{H}D_{\mathtt{w}_1}^{\alpha_1,\beta_1,\rho,\psi}k_1)(t) + \lambda_1({}^{H}D_{\mathtt{w}_1}^{\alpha_1-1,\beta_1,\rho,\psi}k_1)(t) = z(t), \\ ({}^{H}D_{\mathtt{w}_1}^{\alpha_2,\beta_2,\rho,\psi}k_2)(t) + \lambda_2({}^{H}D_{\mathtt{w}_1}^{\alpha_2-1,\beta_2,\rho,\psi}k_2)(t) = w(t), \\ k_1(\mathtt{w}_1) = 0, \qquad k_1(\mathtt{w}_2) = \sum_{i=1}^{n}\eta_i\, k_2(\xi_i) + \sum_{j=1}^{m}\zeta_j\ {}^{p}I_{\mathtt{w}_1}^{\Phi_j,\rho,\psi}k_2(\theta_j), \\ k_2(\mathtt{w}_1) = 0, \qquad k_2(\mathtt{w}_2) = \sum_{k=1}^{r}\aleph_k\, k_1(\varrho_k) + \sum_{l=1}^{q}\Theta_l\ {}^{p}I_{\mathtt{w}_1}^{\upsilon_l,\rho,\psi}k_1(\vartheta_l), \end{cases} \tag{2.1}$$

if and only if

$$
\begin{aligned}
k_1(t) &= \frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\ddot{\Lambda}\rho^{\gamma_1-1}\Gamma(\gamma_1)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_1-1}\Bigg[Y_1\Big(\sum_{i=1}^{n}\eta_i\,{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}w(\xi_i)-\lambda_2\sum_{i=1}^{n}\eta_i\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(\xi_i) \\
&\quad+\sum_{j=1}^{m}\zeta_j\,{}^{p}I_{\mathtt{w}_1}^{\alpha_2+\Phi_j,\rho,\psi}w(\theta_j)-\lambda_2\sum_{j=1}^{m}\zeta_j\,{}^{p}I_{\mathtt{w}_1}^{1+\Phi_j,\rho,\psi}k_2(\theta_j)-{}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}z(\mathtt{w}_2)+\lambda_1\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(\mathtt{w}_2)\Big) \\
&\quad+X_2\Big(\sum_{k=1}^{r}\aleph_k\,{}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}z(\varrho_k)-\lambda_1\sum_{k=1}^{r}\aleph_k\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(\varrho_k)+\sum_{l=1}^{q}\Theta_l\,{}^{p}I_{\mathtt{w}_1}^{\alpha_1+\upsilon_l,\rho,\psi}z(\vartheta_l) \\
&\quad-\lambda_1\sum_{l=1}^{q}\Theta_l\,{}^{p}I_{\mathtt{w}_1}^{1+\upsilon_l,\rho,\psi}k_1(\vartheta_l)-{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}w(\mathtt{w}_2)+\lambda_2\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(\mathtt{w}_2)\Big)\Bigg] \\
&\quad+{}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}z(t)-\lambda_1\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(t) \qquad (2.2)
\end{aligned}
$$

and

$$
\begin{aligned}
k_2(t) &= \frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\ddot{\Lambda}\rho^{\gamma_2-1}\Gamma(\gamma_2)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_2-1}\Bigg[Y_2\Big(\sum_{i=1}^{n}\eta_i\,{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}w(\xi_i)-\lambda_2\sum_{i=1}^{n}\eta_i\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(\xi_i) \\
&\quad+\sum_{j=1}^{m}\zeta_j\,{}^{p}I_{\mathtt{w}_1}^{\alpha_2+\Phi_j,\rho,\psi}w(\theta_j)-\lambda_2\sum_{j=1}^{m}\zeta_j\,{}^{p}I_{\mathtt{w}_1}^{1+\Phi_j,\rho,\psi}k_2(\theta_j)-{}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}z(\mathtt{w}_2)+\lambda_1\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(\mathtt{w}_2)\Big) \\
&\quad+X_1\Big(\sum_{k=1}^{r}\aleph_k\,{}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}z(\varrho_k)-\lambda_1\sum_{k=1}^{r}\aleph_k\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(\varrho_k)+\sum_{l=1}^{q}\Theta_l\,{}^{p}I_{\mathtt{w}_1}^{\alpha_1+\upsilon_l,\rho,\psi}z(\vartheta_l) \\
&\quad-\lambda_1\sum_{l=1}^{q}\Theta_l\,{}^{p}I_{\mathtt{w}_1}^{1+\upsilon_l,\rho,\psi}k_1(\vartheta_l)-{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}w(\mathtt{w}_2)+\lambda_2\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(\mathtt{w}_2)\Big)\Bigg] \\
&\quad+{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}w(t)-\lambda_2\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(t), \qquad (2.3)
\end{aligned}
$$

where $\ddot{\Lambda}=X_1Y_1-X_2Y_2$,

$$
\begin{aligned}
X_1 &= \frac{e^{\frac{\rho-1}{\rho}(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_1-1}\Gamma(\gamma_1)}(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\gamma_1-1}, \\
X_2 &= \sum_{i=1}^{n}\frac{\eta_i\,e^{\frac{\rho-1}{\rho}(\psi(\xi_i)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)}(\psi(\xi_i)-\psi(\mathtt{w}_1))^{\gamma_2-1} \\
&\quad+\sum_{j=1}^{m}\frac{\zeta_j\,e^{\frac{\rho-1}{\rho}(\psi(\theta_j)-\psi(\mathtt{w}_1))}}{\rho^{\Phi_j+\gamma_2-1}\Gamma(\Phi_j+\gamma_2)}(\psi(\theta_j)-\psi(\mathtt{w}_1))^{\Phi_j+\gamma_2-1}, \qquad (2.4) \\
Y_1 &= \sum_{k=1}^{r}\aleph_k\,\frac{e^{\frac{\rho-1}{\rho}(\psi(\varrho_k)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_1-1}\Gamma(\gamma_1)}(\psi(\varrho_k)-\psi(\mathtt{w}_1))^{\gamma_1-1} \\
&\quad+\sum_{l=1}^{q}\frac{\Theta_l\,e^{\frac{\rho-1}{\rho}(\psi(\vartheta_l)-\psi(\mathtt{w}_1))}}{\rho^{\upsilon_l+\gamma_1-1}\Gamma(\upsilon_l+\gamma_1)}(\psi(\vartheta_l)-\psi(\mathtt{w}_1))^{\upsilon_l+\gamma_1-1}, \\
Y_2 &= \frac{e^{\frac{\rho-1}{\rho}(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)}(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\gamma_2-1}.
\end{aligned}
$$

Proof. Let the pair (k_1, k_2) be the solution of the system (2.1). We take the Riemann-Liouville integrals to Eq (2.1),

$$\begin{cases} {}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}[({}^{H}D_{\mathtt{w}_1}^{\alpha_1,\beta_1,\rho,\psi}k_1)(t) + \lambda_1({}^{H}D_{\mathtt{w}_1}^{\alpha_1-1,\beta_1,\rho,\psi}k_1)(t)] = {}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}z(t), \\ {}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}[({}^{H}D_{\mathtt{w}_1}^{\alpha_2,\beta_2,\rho,\psi}k_2)(t) + \lambda_2({}^{H}D_{\mathtt{w}_1}^{\alpha_2-1,\beta_2,\rho,\psi}k_2)(t)] = {}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}w(t). \end{cases} \tag{2.5}$$

Then, applying Lemma 2.1 with $n = 2$ to Eq (2.5), we get

$$\begin{aligned} k_1(t) &= {}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}z(t) - \lambda_1\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(t) + \frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_1-1}\Gamma(\gamma_1)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_1-1}({}^{p}I_{\mathtt{w}_1}^{1-\gamma_1,\rho,\psi}x)(\mathtt{w}_1) \\ &\quad + \frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_1-2}\Gamma(\gamma_1-1)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_1-2}({}^{p}I_{\mathtt{w}_1}^{2-\gamma_1,\rho,\psi}x)(\mathtt{w}_1) \\ &= {}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}z(t) - \lambda_1\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(t) + c_0\frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_1-1}\Gamma(\gamma_1)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_1-1} \\ &\quad + c_1\frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_1-2}\Gamma(\gamma_1-1)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_1-2} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} k_2(t) &= {}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}w(t) - \lambda_2\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(t) + \frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_2-1}({}^{p}I_{\mathtt{w}_1}^{1-\gamma_2,\rho,\psi}y)(\mathtt{w}_1) \\ &\quad + \frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_2-2}\Gamma(\gamma_2-1)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_2-2}({}^{p}I_{\mathtt{w}_1}^{2-\gamma_2,\rho,\psi}y)(\mathtt{w}_1) \\ &= {}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}w(t) - \lambda_2\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(t) + c_2\frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_2-1} \\ &\quad + c_3\frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_2-2}\Gamma(\gamma_2-1)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_2-2}, \end{aligned} \tag{2.7}$$

where

$$c_0 = ({}^{p}I_{\mathtt{w}_1}^{1-\gamma_1,\rho,\psi}x)(\mathtt{w}_1),\ c_1 = ({}^{p}I_{\mathtt{w}_1}^{2-\gamma_1,\rho,\psi}x)(\mathtt{w}_1),\ c_2 = ({}^{p}I_{\mathtt{w}_1}^{1-\gamma_2,\rho,\psi}y)(\mathtt{w}_1) \text{ and } c_3 = ({}^{p}I_{\mathtt{w}_1}^{2-\gamma_2,\rho,\psi}y)(\mathtt{w}_1).$$

From the conditions $k_1(\mathtt{w}_1) = 0$ and $k_2(\mathtt{w}_1) = 0$ we get $c_1 = 0$ and $c_3 = 0$, since $\gamma_1 \in [\alpha_1, 2]$ and $\gamma_2 \in [\alpha_2, 2]$ (see [27]), and Eqs (2.6) and (2.7) are reduced to

$$k_1(t) = {}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}z(t) - \lambda_1\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(t) + c_0\frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_1-1}\Gamma(\gamma_1)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_1-1}, \tag{2.8}$$

$$k_2(t) = {}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}w(t) - \lambda_2\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(t) + c_2\frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_2-1}. \tag{2.9}$$

From the boundary conditions

$$k_1(\mathtt{w}_2) = \sum_{i=1}^{n}\eta_i\, k_2(\xi_i) + \sum_{j=1}^{m}\zeta_j\, {}^{p}I_{\mathtt{w}_1}^{\Phi_j,\rho,\psi}k_2(\theta_j)$$

and

$$k_2(\mathtt{w}_2) = \sum_{k=1}^{r} \aleph_k \, k_1(\varrho_k) + \sum_{l=1}^{q} \Theta_l {}^pI_{\mathtt{w}_1}^{\upsilon_l,\rho,\psi} k_1(\vartheta_l),$$

we get

$$\begin{aligned}
&{}^pI_{\mathtt{w}_1}^{\alpha_1,\rho,\psi} z(\mathtt{w}_2) - \lambda_1 \, {}^pI_{\mathtt{w}_1}^{1,\rho,\psi} k_1(\mathtt{w}_2) + c_0 \frac{e^{\frac{\rho-1}{\rho}(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_1-1}\Gamma(\gamma_1)}(\psi(\mathtt{w}_2) - \psi(\mathtt{w}_1))^{\gamma_1-1} \\
&= \sum_{i=1}^{n} \eta_i \, {}^pI_{\mathtt{w}_1}^{\alpha_2,\rho,\psi} w(\xi_i) - \lambda_2 \sum_{i=1}^{n} \eta_i \, {}^pI_{\mathtt{w}_1}^{1,\rho,\psi} k_2(\xi_i) + c_2 \sum_{i=1}^{n} \frac{\eta_i \, e^{\frac{\rho-1}{\rho}(\psi(\xi_i)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)}(\psi(\xi_i) - \psi(\mathtt{w}_1))^{\gamma_2-1} \\
&\quad + \sum_{j=1}^{m} \zeta_j \, {}^pI_{\mathtt{w}_1}^{\alpha_2+\Phi_j,\rho,\psi} w(\theta_j) - \lambda_2 \sum_{j=1}^{m} \zeta_j \, {}^pI_{\mathtt{w}_1}^{1+\Phi_j,\rho,\psi} k_2(\theta_j) \\
&\quad + c_2 \sum_{j=1}^{m} \frac{\zeta_j \, e^{\frac{\rho-1}{\rho}(\psi(\theta_j)-\psi(\mathtt{w}_1))}}{\rho^{\Phi_j+\gamma_2-1}\Gamma(\Phi_j+\gamma_2)}(\psi(\theta_j) - \psi(\mathtt{w}_1))^{\Phi_j+\gamma_2-1}
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&{}^pI_{\mathtt{w}_1}^{\alpha_2,\rho,\psi} w(\mathtt{w}_2) - \lambda_2 \, {}^pI_{\mathtt{w}_1}^{1,\rho,\psi} k_2(\mathtt{w}_2) + c_2 \frac{e^{\frac{\rho-1}{\rho}(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_2-1}\Gamma(\gamma_2)}(\psi(\mathtt{w}_2) - \psi(\mathtt{w}_1))^{\gamma_2-1} \\
&= \sum_{k=1}^{r} \aleph_k \, {}^pI_{\mathtt{w}_1}^{\alpha_1,\rho,\psi} z(\varrho_k) - \lambda_1 \sum_{k=1}^{r} \aleph_k \, {}^pI_{\mathtt{w}_1}^{1,\rho,\psi} k_1(\varrho_k) + c_0 \sum_{k=1}^{r} \aleph_k \frac{e^{\frac{\rho-1}{\rho}(\psi(\varrho_k)-\psi(\mathtt{w}_1))}}{\rho^{\gamma_1-1}\Gamma(\gamma_1)}(\psi(\varrho_k) - \psi(\mathtt{w}_1))^{\gamma_1-1} \\
&\quad + \sum_{l=1}^{q} \Theta_l \, {}^pI_{\mathtt{w}_1}^{\alpha_1+\upsilon_l,\rho,\psi} z(\vartheta_l) - \lambda_1 \sum_{l=1}^{q} \Theta_l \, {}^pI_{\mathtt{w}_1}^{1+\upsilon_l,\rho,\psi} k_1(\vartheta_l) \\
&\quad + c_0 \sum_{l=1}^{q} \frac{\Theta_l \, e^{\frac{\rho-1}{\rho}(\psi(\vartheta_l)-\psi(\mathtt{w}_1))}}{\rho^{\upsilon_l+\gamma_1-1}\Gamma(\upsilon_l+\gamma_1)}(\psi(\vartheta_l) - \psi(\mathtt{w}_1))^{\upsilon_l+\gamma_1-1}.
\end{aligned} \tag{2.11}$$

From Eqs (2.10) and (2.11), by using the notations (2.4) we get the system

$$X_1 c_0 - X_2 c_2 = M, \tag{2.12}$$

$$Y_2 k_2 c_0 + Y_1 c_2 = N, \tag{2.13}$$

where

$$\begin{aligned}
M &= \sum_{i=1}^{n} \eta_i \, {}^pI_{\mathtt{w}_1}^{\alpha_2,\rho,\psi} w(\xi_i) - \lambda_2 \sum_{i=1}^{n} \eta_i \, {}^pI_{\mathtt{w}_1}^{1} k_2(\xi_i) + \sum_{j=1}^{m} \zeta_j \, {}^pI_{\mathtt{w}_1}^{\alpha_2+\Phi_j,\rho,\psi} w(\theta_j) \\
&\quad -\lambda_2 \sum_{j=1}^{m} \zeta_j \, {}^pI_{\mathtt{w}_1}^{1+\Phi_j} k_2(\theta_j) - {}^pI_{\mathtt{w}_1}^{\alpha_1,\rho,\psi} v(\mathtt{w}_2) + \lambda_1 {}^pI_{\mathtt{w}_1}^{1} k_1(\mathtt{w}_2), \\
N &= \sum_{k=1}^{r} \aleph_k \, {}^pI_{\mathtt{w}_1}^{\alpha_1,\rho,\psi} v(\varrho_k) - \lambda_1 \sum_{k=1}^{r} \aleph_k \, {}^pI_{\mathtt{w}_1}^{1} k_1(\varrho_k) + \sum_{l=1}^{q} \Theta_l \, {}^pI_{\mathtt{w}_1}^{\alpha_1+\upsilon_l,\rho,\psi} v(\vartheta_l) \\
&\quad -\lambda_1 \sum_{l=1}^{q} \Theta_l \, {}^pI_{\mathtt{w}_1}^{1+\upsilon_l} k_1(\vartheta_l) - {}^pI_{\mathtt{w}_1}^{\alpha_2,\rho,\psi} w(\mathtt{w}_2) + \lambda_2 {}^pI_{\mathtt{w}_1}^{1} k_2(\mathtt{w}_2).
\end{aligned}$$

By solving the above system, we obtain the constants c_0 and c_2 as

$$c_0 = \frac{Y_1M + X_2N}{X_1Y_1 - X_2Y_2} \text{ and } c_2 = \frac{Y_2M + X_1N}{X_1Y_1 - X_2Y_2}.$$

Now substitute the values of c_0 and c_2 into Eqs (2.8) and (2.9) and yield Eqs (2.2) and (2.3), as desired. We can prove the converse of the lemma by direct computation. □

3. Existence results

In this section, we prove the existence and uniqueness results for the problem (1.4) by using three fixed ponit theorems.

First, we defined the spaces

$$X = \{k_1 \,|\, k_1(t) \in C([\mathtt{w}_1, \mathtt{w}_2], \mathbb{R})\}$$

with the norm

$$\|k_1\| = \sup\{|k_1(t)|,\ t \in [\mathtt{w}_1, \mathtt{w}_2]\},$$

and

$$Y = \{k_2 \,|\, k_2(t) \in C([\mathtt{w}_1, \mathtt{w}_2], \mathbb{R})\}$$

with the norm

$$\|k_2\| = \sup\{|k_2(t)|, t \in [\mathtt{w}_1, \mathtt{w}_2]\}.$$

Then it is well known that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are Banach apaces. Obviously, the product space of $X \times Y$ endowed with norm $\|(k_1, k_2)\| = \|k_1\| + \|k_2\|$ for $(k_1, k_2) \in X \times Y$ is a Banach space.

In view of the Lemma 2.2, we define an operator $T\colon X \times Y \to X \times Y$ by

$$T(k_1, k_2)(t) = (T_1(k_1, k_2)(t), T_2(k_1, k_2)(t)),$$

where

$$\begin{aligned}
T_1(k_1,k_2)(t) \;\; = \;\; & \frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\ddot{\Lambda}\rho^{\gamma_1-1}\Gamma(\gamma_1)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_1-1}\Bigg[Y_1\Big(\sum_{i=1}^{n}\eta_i\, {}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}\Upsilon_2(\xi_i,k_1(\xi_i),k_2(\xi_i)) \\
& -\lambda_2\sum_{i=1}^{n}\eta_i\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(\xi_i) + \sum_{j=1}^{m}\zeta_j\, {}^{p}I_{\mathtt{w}_1}^{\alpha_2+\Phi_j,\rho,\psi}\Upsilon_2(\theta_j,k_1(\theta_j),k_2(\theta_j)) \\
& -\lambda_2\sum_{j=1}^{m}\zeta_j\, {}^{p}I_{\mathtt{w}_1}^{1+\Phi_j,\rho,\psi}k_2(\theta_j) - {}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}\Upsilon_1(\mathtt{w}_2,k_1(\mathtt{w}_2),k_2(\mathtt{w}_2)) + \lambda_1\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(\mathtt{w}_2)\Big) \\
& +X_2\Big(\sum_{k=1}^{r}\aleph_k\, {}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}\Upsilon_1(\varrho_k,k_1(\varrho_k),k_2(\varrho_k)) - \lambda_1\sum_{k=1}^{r}\aleph_k\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(\varrho_k) \\
& +\sum_{l=1}^{q}\Theta_l\, {}^{p}I_{\mathtt{w}_1}^{\alpha_1+\upsilon_l,\rho,\psi}\Upsilon_1(\vartheta_l,k_1(\vartheta_l),k_2(\vartheta_l)) - \lambda_1\sum_{l=1}^{q}\Theta_l\, {}^{p}I_{\mathtt{w}_1}^{1+\upsilon_l,\rho,\psi}k_1(\vartheta_l) \\
& -{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}\Upsilon_2(\mathtt{w}_2,k_1(\mathtt{w}_2),k_2(\mathtt{w}_2)) + \lambda_2\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(\mathtt{w}_2)\Big)\Bigg] + {}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}\Upsilon_1(t,k_1(t),k_2(t)) \\
& -\lambda_1\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(t)
\end{aligned}$$

and

$$
\begin{aligned}
T_2(k_1,k_2)(t) &= \frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\ddot{\Lambda}\rho^{\gamma_2-1}\Gamma(\gamma_2)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_2-1}\Bigg[Y_2\Big(\sum_{i=1}^{n}\eta_i\, {}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}\Upsilon_2(\xi_i,k_1(\xi_i),k_2(\xi_i))\\
&-\lambda_2\sum_{i=1}^{n}\eta_i\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(\xi_i)+\sum_{j=1}^{m}\zeta_j\, {}^{p}I_{\mathtt{w}_1}^{\alpha_2+\Phi_j,\rho,\psi}\Upsilon_2(\theta_j,k_1(\theta_j),k_2(\theta_j))\\
&-\lambda_2\sum_{j=1}^{m}\zeta_j\, {}^{p}I_{\mathtt{w}_1}^{1+\Phi_j,\rho,\psi}k_2(\theta_j)-{}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}\Upsilon_1(\mathtt{w}_2,k_1(\mathtt{w}_2),k_2(\mathtt{w}_2))+\lambda_1\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(\mathtt{w}_2)\Big)\\
&+X_1\Big(\sum_{k=1}^{r}\aleph_k\, {}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}\Upsilon_1(\varrho_k,k_1(\varrho_k),k_2(\varrho_k))-\lambda_1\sum_{k=1}^{r}\aleph_k\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(\varrho_k)\\
&+\sum_{l=1}^{q}\Theta_l\, {}^{p}I_{\mathtt{w}_1}^{\alpha_1+\upsilon_l,\rho,\psi}\Upsilon_1(\vartheta_l,k_1(\vartheta_l),k_2(\vartheta_l))-\lambda_1\sum_{l=1}^{q}\Theta_l\, {}^{p}I_{\mathtt{w}_1}^{1+\upsilon_l,\rho,\psi}k_1(\vartheta_l)\\
&-{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}\Upsilon_1(\mathtt{w}_2,k_1(\mathtt{w}_2),k_2(\mathtt{w}_2))+\lambda_2\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(\mathtt{w}_2)\Big)\Bigg]+{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}\Upsilon_2(t,k_1(t),k_2(t))\\
&-\lambda_2\, {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(t).
\end{aligned}
$$

Then, we introduce the following notation for computational convenience.

Notation 3.1. *Let $\mathcal{A}_i,\mathcal{B}_i,C_i$ for $i=1,2$ be the constants:*

$$
\begin{aligned}
\mathcal{A}_1 &= \frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\gamma_1-1}}{|\ddot{\Lambda}|\rho^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg[Y_1\Big(\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}\Big)+X_2\Big(\sum_{k=1}^{r}|\aleph_k|\frac{(\psi(\varrho_k)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}\\
&+\sum_{l=1}^{q}|\Theta_l|\,\frac{(\psi(\vartheta_l)-\psi(\mathtt{w}_1))^{\alpha_1+\upsilon_l}}{\rho^{\alpha_1+\upsilon_l}\Gamma(\alpha_1+\upsilon_l+1)}\Big)\Bigg]+\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)},\\
\mathcal{B}_1 &= \frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\gamma_1-1}}{|\ddot{\Lambda}|\rho^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg[Y_1\Big(\sum_{i=1}^{n}|\eta_i|\,\frac{(\psi(\xi_i)-\psi(\mathtt{w}_1))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\\
&+\sum_{j=1}^{m}|\zeta_j|\,\frac{(\psi(\theta_j)-\psi(\mathtt{w}_1))^{\alpha_2+\Phi_j}}{\rho^{\alpha_2+\Phi_j}\Gamma(\alpha_2+\Phi_j+1)}\Big)+X_2\Big(\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\Big)\Bigg],\\
C_1 &= \frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\gamma_1-1}}{|\ddot{\Lambda}|\rho^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg[Y_1\Big(|\lambda_2|\sum_{i=1}^{n}|\eta_i|\,\frac{(\psi(\xi_i)-\psi(\mathtt{w}_1))}{\rho}+|\lambda_1|\,\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}\\
&+|\lambda_2|\sum_{j=1}^{m}|\zeta_j|\frac{(\psi(\theta_j)-\psi(\mathtt{w}_1))^{1+\Phi_j}}{\rho^{1+\Phi_j}\Gamma(2+\Phi_j)}\Big)+X_2\Big(|\lambda_1|\,\sum_{k=1}^{r}|\aleph_k|\frac{(\psi(\varrho_k)-\psi(\mathtt{w}_1))}{\rho}\\
&+|\lambda_2|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}+|\lambda_1|\,\sum_{l=1}^{q}|\Theta_l|\,\frac{(\psi(\vartheta_l)-\psi(\mathtt{w}_1))^{1+\upsilon_l}}{\rho^{1+\upsilon_l}\Gamma(1+\upsilon_l+1)}\Big)\Bigg]+|\lambda_1|\,\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho},\\
\mathcal{A}_2 &= \frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\gamma_2-1}}{|\ddot{\Lambda}|\rho^{\gamma_2-1}\Gamma(\gamma_2)}\Bigg[Y_2\Big(\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{+\alpha_1}\Gamma(\alpha_1+1)}\Big)+X_1\Big(\sum_{k=1}^{r}\frac{|\aleph_k|(\psi(\varrho_k)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}\\
&+\sum_{l=1}^{q}|\Theta_l|\frac{(\psi(\vartheta_l)-\psi(\mathtt{w}_1))^{\alpha_1+\upsilon_l}}{\rho^{\alpha_1+\upsilon_l}\Gamma(\alpha_1+\upsilon_l+1)}+\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\Big)\Bigg],
\end{aligned}
$$

$$\mathcal{B}_2 = \frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\gamma_2-1}}{|\ddot{\Lambda}|\rho^{\gamma_2-1}\Gamma(\gamma_2)}\Bigg[Y_2\Big(\sum_{i=1}^{n}|\eta_i|\,\frac{(\psi(\xi_i)-\psi(\mathtt{w}_1))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}$$
$$+\sum_{j=1}^{m}|\zeta_j|\,\frac{(\psi(\theta_j)-\psi(\mathtt{w}_1))^{\alpha_2+\Phi_j}}{\rho^{\alpha_2+\Phi_j}\Gamma(\alpha_2+\Phi_j+1)}\Big)\Bigg]+\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)},$$

$$C_2 = \frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\gamma_2-1}}{|\ddot{\Lambda}|\rho^{\gamma_2-1}\Gamma(\gamma_2)}\Bigg[Y_2\Big(|\lambda_2|\sum_{i=1}^{n}|\eta_i|\,\frac{(\psi(\xi_i)-\psi(\mathtt{w}_1))}{\rho}$$
$$+|\lambda_2|\sum_{j=1}^{m}|\zeta_j|\frac{(\psi(\theta_j)-\psi(\mathtt{w}_1))^{1+\Phi_j}}{\rho^{1+\Phi_j}\Gamma(2+\Phi_j)}+|\lambda_1|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}\Big)$$
$$+X_1(|\lambda_1|\sum_{k=1}^{r}|\aleph_k|\frac{(\psi(\varrho_k)-\psi(\mathtt{w}_1))}{\rho}+|\lambda_1|\sum_{l=1}^{q}|\Theta_l|\,\frac{(\psi(\vartheta_l)-\psi(\mathtt{w}_1))^{1+\upsilon_l}}{\rho^{1+\upsilon_l}\Gamma(2+\upsilon_l)}$$
$$+|\lambda_2|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}\Big)\Bigg]+|\lambda_2|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}.$$

Now we prove our first existence result via Leray-Schauder alternative [28].

Theorem 3.1. *Let* Υ_1, Υ_2*:* $[\mathtt{w}_1, \mathtt{w}_2]\times\mathbb{R}^2 \to \mathbb{R}$ *be continuous functions. Suppose that:*

(H_1) *There exist* $u_i, v_i \geq 0$ *for* $i = 1, 2$ *and* $u_0, v_0 > 0$ *such that for each* $k_1, k_2 \in \mathbb{R}$, $t \in [\mathtt{w}_1, \mathtt{w}_2]$,

$$|\Upsilon_1(t,k_1,k_2)| \leq u_0 + u_1|k_1| + u_2|k_2|,$$
$$|\Upsilon_2(t,k_1,k_2)| \leq v_0 + v_1|k_1| + v_2|k_2|.$$

If

$$(\mathcal{A}_1+\mathcal{A}_2)u_1 + (\mathcal{B}_1+\mathcal{B}_2)v_1 + (C_1+C_2) < 1$$

and

$$(\mathcal{A}_1+\mathcal{A}_2)u_2 + (\mathcal{B}_1+\mathcal{B}_2)v_2 + (C_1+C_2) < 1,$$

where the constants $\mathcal{A}_i, \mathcal{B}_i, C_i$ *for* $i = 1, 2$ *are defined in the Notation 3.1, then the problem* (1.4) *has at lest one solution on* $[\mathtt{w}_1, \mathtt{w}_2]$.

Proof. Since f and g are continuous functions, then T is a continuous operator. We prove that TB_r is uniformly bounded, where B_r is the closed ball

$$B_r = \{(k_1,k_2) \in X \times Y : \|(k_1,k_2)\| \leq r\}.$$

For all $(k_1, k_2) \in B_r$, by (H_1) we have

$$\begin{aligned}|\Upsilon_1(t,k_1,k_2)| &\leq u_0 + u_1|k_1| + u_2|k_2|\\ &\leq u_0 + u_1\|k_1\| + u_2\|k_2\|\\ &\leq u_0 + (u_1+u_2)r\\ &:= P_1\end{aligned}$$

and similarly

$$|\Upsilon_2(t,k_1,k_2)| \leq v_0 + (v_1+v_2)r := P_2.$$

So, for any $k_1, k_2 \in B_r$, we have

$$
\begin{aligned}
|T_1(k_1,k_2)(t)| \leq\ & \frac{1}{|\ddot{\Lambda}|\rho^{\gamma_1-1}\Gamma(\gamma_1)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_1-1}\Big[Y_1\Big(\sum_{i=1}^{n}|\eta_i|\ {}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}|\Upsilon_2(\xi_i,k_2(\xi_i),k_1(\xi_i))| \\
& +|\lambda_2|\sum_{i=1}^{n}|\eta_i|\ {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}|k_2(\xi_i)| + \sum_{j=1}^{m}|\zeta_j|\ {}^{p}I_{\mathtt{w}_1}^{\alpha_2+\Phi_j,\rho,\psi}|\Upsilon_2(\theta_j,k_2(\theta_j),k_1(\theta_j))| \\
& +|\lambda_2|\sum_{j=1}^{m}|\zeta_j|{}^{p}I_{\mathtt{w}_1}^{1+\Phi_j,\rho,\psi}|k_2(\theta_j)| + {}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}|\Upsilon_1(\mathtt{w}_2,k_1(\mathtt{w}_2),k_2(\mathtt{w}_2))| + |\lambda_1|\ {}^{p}I_{\mathtt{w}_1}^{1}|k_1(\mathtt{w}_2)|\Big) \\
& +X_2\Big(\sum_{k=1}^{r}|\aleph_k|\ {}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}|\Upsilon_1(\varrho_k,k_1(\varrho_k),k_2(\varrho_k))| + |\lambda_1|\ \sum_{k=1}^{r}|\aleph_k|\ {}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}|k_1(\varrho_k)| \\
& +\sum_{l=1}^{q}|\Theta_l|\ {}^{p}I_{\mathtt{w}_1}^{\alpha_1+\upsilon_l,\rho,\psi}|\Upsilon_1(\vartheta_l,k_1(\vartheta_l),k_2(\vartheta_l))| + |\lambda_1|\ \sum_{l=1}^{q}|\Theta_l|\ {}^{p}I_{\mathtt{w}_1}^{1+\upsilon_l,\rho,\psi}|k_1(\vartheta_l)| \\
& +{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}|\Upsilon_2(\mathtt{w}_2,k_2(\mathtt{w}_2),k_1(\mathtt{w}_2))| + |\lambda_2|{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}|k_2(\mathtt{w}_2)|\Big)\Big] + {}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}|\Upsilon_1(t,k_1(t),k_2(t))| \\
& +|\lambda_1|{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}|k_1(t)| \\
\leq\ & \frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\gamma_1-1}}{|\ddot{\Lambda}|\rho^{\gamma_1-1}\Gamma(\gamma_1)}\Big[Y_1\Big(P_2\sum_{i=1}^{n}|\eta_i|\ \frac{(\psi(\xi_i)-\psi(\mathtt{w}_1))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} \\
& +P_2\sum_{j=1}^{m}|\zeta_j|\ \frac{(\psi(\theta_j)-\psi(\mathtt{w}_1))^{\alpha_2+\Phi_j}}{\rho^{\alpha_2+\Phi_j}\Gamma(\alpha_2+\Phi_j+1)} + |\lambda_2|\|k_2\|\sum_{i=1}^{n}|\eta_i|\ \frac{(\psi(\xi_i)-\psi(\mathtt{w}_1))}{\rho} \\
& +|\lambda_2|\|k_2\|\sum_{j=1}^{m}|\zeta_j|\frac{(\psi(\theta_j)-\psi(\mathtt{w}_1))^{1+\Phi_j}}{\rho^{1+\Phi_j}\Gamma(2+\Phi_j)} + P_1\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)} + |\lambda_1|\ \|k_1\|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}\Big) \\
& +X_2\Big(P_1\sum_{k=1}^{r}|\aleph_k|\frac{(\psi(\varrho_k)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)} + |\lambda_1|\ \|k_1\|\sum_{k=1}^{r}|\aleph_k|\frac{(\psi(\varrho_k)-\psi(\mathtt{w}_1))}{\rho} \\
& +|\lambda_1|\ \|k_1\|\sum_{l=1}^{q}|\Theta_l|\ \frac{(\psi(\vartheta_l)-\psi(\mathtt{w}_1))^{1+\upsilon_l}}{\rho^{1+\upsilon_l}\Gamma(1+\upsilon_l+1)} + P_1\sum_{l=1}^{q}|\Theta_l|\ \frac{(\psi(\vartheta_l)-\psi(\mathtt{w}_1))^{\alpha_1+\upsilon_l}}{\rho^{\alpha_1+\upsilon_l}\Gamma(\alpha_1+\upsilon_l+1)} \\
& +P_2\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} + |\lambda_2|\|k_2\|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}\Big)\Big] \\
& +P_1\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)} + |\lambda_1|\ \|k_1\|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho} \\
& \leq P_1\mathcal{A}_1 + P_2\mathcal{B}_1 + rC_1,
\end{aligned}
$$

and hence

$$\|T_1(k_1,k_2)\| \leq P_1\mathcal{A}_1 + P_2\mathcal{B}_1 + rC_1.$$

In the same way, we can obtain that

$$\|T_2(k_1,k_2)\| \leq P_1\mathcal{A}_2 + P_2\mathcal{B}_2 + rC_2.$$

Consequently,

$$\|T(k_1,k_2)\| \leq (\mathcal{A}_1+\mathcal{A}_2)P_1 + (\mathcal{B}_1+\mathcal{B}_2)P_2 + (C_1+C_2)r.$$

Therefore, the set TB_r is uniformly bounded.

Next, it is proven that TB_r is equicontinuous. Let $(k_1,k_2)\in B_r$ and $t_1,t_2\in[\mathtt{w}_1,\mathtt{w}_2]$ with $t_1<t_2$. Then, we have

$$\begin{aligned}
|T_1(k_1,k_2)(t_2)-T_1(k_1,k_2)(t_1)| \le{}& \frac{(\psi(t_2)-\psi(\mathtt{w}_1))^{\gamma_1-1}-(\psi(t_1)-\psi(\mathtt{w}_1))^{\gamma_1-1}}{|\ddot{\Lambda}|\rho^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg[Y_1\Big(\sum_{i=1}^{n}|\eta_i|\,{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}|\Upsilon_2(\xi_i,k_2(\xi_i),k_1(\xi_i))| \\
&+|\lambda_2|\sum_{i=1}^{n}|\eta_i|\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}|k_2(\xi_i)|+\sum_{j=1}^{m}|\zeta_j|\,{}^{p}I_{\mathtt{w}_1}^{\alpha_2+\Phi_j,\rho,\psi}|\Upsilon_2(\theta_j,k_2(\theta_j),k_1(\theta_j))| \\
&+|\lambda_2|\sum_{j=1}^{m}|\zeta_j|\,{}^{p}I_{\mathtt{w}_1}^{1+\Phi_j,\rho,\psi}|k_2(\theta_j)|+{}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}|\Upsilon_1(\mathtt{w}_2,k_1(\mathtt{w}_2),k_2(\mathtt{w}_2))|+|\lambda_1|\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}|k_1(\mathtt{w}_2)|\Big) \\
&+X_2\Big(\sum_{k=1}^{r}|\aleph_k|\,{}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}|\Upsilon_1(\varrho_k,k_1(\varrho_k),k_2(\varrho_k))|+|\lambda_1|\sum_{k=1}^{r}|\aleph_k|\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}|k_1(\varrho_k)| \\
&+\sum_{l=1}^{q}|\Theta_l|\,{}^{p}I_{\mathtt{w}_1}^{\alpha_1+\upsilon_l,\rho,\psi}|\Upsilon_1(\vartheta_l,k_1(\vartheta_l),k_2(\vartheta_l))|+|\lambda_1|\sum_{l=1}^{q}|\Theta_l|\,{}^{p}I_{\mathtt{w}_1}^{1+\upsilon_l,\rho,\psi}|k_1(\vartheta_l)| \\
&+{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}|\Upsilon_2(\mathtt{w}_2,k_2(\mathtt{w}_2),k_1(\mathtt{w}_2))|+|\lambda_2|{}^{p}I_{\mathtt{w}_1}^{1}|k_2(\mathtt{w}_2)|\Big)\Bigg] \\
&+\frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)}\Big|\int_{\mathtt{w}_1}^{t_1}\Big[(\psi(t_2)-\psi(s))^{\alpha-1}-(\psi(t_1)-\psi(s))^{\alpha-1}\Big]\psi'(s)\Upsilon_1(x,k_1(s),k_2(s)ds\Big| \\
&+\frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)}\Big|\int_{t_1}^{t_2}(\psi(t_2)-\psi(s))^{\alpha-1}\psi'(s)\Upsilon_1(x,k_1(s),k_2(s)ds\Big| \\
&+|\lambda_1|\,|{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(t_2)-{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(t_1)| \\
\le{}& \frac{(\psi(t_2)-\psi(\mathtt{w}_1))^{\gamma_1-1}-(\psi(t_1)-\psi(\mathtt{w}_1))^{\gamma_1-1}}{|\ddot{\Lambda}|\rho^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg[Y_1\Big(P_2\sum_{i=1}^{n}|\eta_i|\,\frac{(\psi(\xi_i)-\psi(\mathtt{w}_1))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} \\
&+|\lambda_2|r\sum_{i=1}^{n}|\eta_i|\,\frac{(\psi(\xi_i)-\psi(\mathtt{w}_1))}{\rho}+P_2\sum_{j=1}^{m}|\zeta_j|\,\frac{(\psi(\theta_j)-\psi(\mathtt{w}_1))^{\alpha_2+\Phi_j}}{\rho^{\alpha_2+\Phi_j}\Gamma(\alpha_2+\Phi_j+1)} \\
&+|\lambda_2|r\sum_{j=1}^{m}|\zeta_j|\,\frac{(\psi(\theta_j)-\psi(\mathtt{w}_1))^{1+\Phi_j}}{\rho^{1+\Phi_j}\Gamma(2+\Phi_j)}+P_1\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}+|\lambda_1|\,r\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho\Gamma(2)}\Big) \\
&+X_2\Big(P_1\sum_{k=1}^{r}|\aleph_k|\,\frac{(\psi(\varrho_k)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}+|\lambda_1|\,r\sum_{k=1}^{r}|\aleph_k|\frac{(\psi(\varrho_k)-\psi(\mathtt{w}_1))}{\rho} \\
&+P_1\sum_{l=1}^{q}|\Theta_l|\,\frac{(\psi(\vartheta_l)-\psi(\mathtt{w}_1))^{\alpha_1+\vartheta_l}}{\rho^{\alpha_1+\vartheta_l}\Gamma(\alpha_1+\vartheta_l+1)}+|\lambda_1|\,r\sum_{l=1}^{q}|\Theta_l|\,\frac{(\psi(\vartheta_l)-\psi(\mathtt{w}_1))^{1+\vartheta_l}}{\rho^{1+\vartheta_l}\Gamma(2+\vartheta_l)} \\
&+P_2\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}+|\lambda_2|r\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}\Big)\Bigg] \\
&+\frac{P_1}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}\Big[(\psi(t_2)-\psi(\mathtt{w}_1))^{\alpha_1}-(\psi(t_1)-\psi(\mathtt{w}_1))^{\alpha_1}+2(\psi(t_2)-\psi(t_1))^{\alpha_1}\Big] \\
&+\frac{|\lambda_1|}{\rho}r(\psi(t_2)-\psi(t_1)).
\end{aligned}$$

Then, we obtain that

$$|T_1(k_1,k_2)(t_2)-T_1(k_1,k_2)(t_1)|\to 0,$$

when $t_2\to t_1$, independently of k_1 and k_2. Similarly,

$$|T_2(k_1,k_2)(t_2)-T_2(k_1,k_2)(t_1)|\to 0,$$

as $t_2 \to t_1$. Therefore TB_r is equicontinuous on $[\mathtt{w}_1, \mathtt{w}_2]$. From the above three steps and Arzelá-Ascoli theorem, we conclude that T is completely continuous.

Let

$$U = \{(k_1, k_2) \in X \times Y : (k_1, k_2) = \mu T(k_1, k_2),\ 0 \leq \mu \leq 1\}.$$

We prove that U is bounded. Let

$$(k_1, k_2) \in C([\mathtt{w}_1, \mathtt{w}_2], \mathbb{R})$$

be any solution of $(k_1, k_2) = \mu T(k_1, k_2)$. For each $t \in [\mathtt{w}_1, \mathtt{w}_2]$, we have

$$k_1(t) = \mu T_1(k_1, k_2),\ k_2(t) = \mu T_2(k_1, k_2).$$

Then

$$\begin{aligned}
|k_1(t)| &= \mu |T_1(k_1,k_2)(t)| \\
&\leq |T_1(k_1,k_2)(t)| \\
&\leq \frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\gamma_1-1}}{|\ddot{\Lambda}|\rho^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg[Y_1\Big((v_0+v_1|k_1|+v_2|k_2|)\sum_{i=1}^{n}|\eta_i|\,\frac{(\psi(\xi_i)-\psi(\mathtt{w}_1))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} \\
&\quad +(v_0+v_1|k_1|+v_2|k_2|)\sum_{j=1}^{m}|\zeta_j|\,\frac{(\psi(\theta_j)-\psi(\mathtt{w}_1))^{\alpha_2+\Phi_j}}{\rho^{\alpha_2+\Phi_j}\Gamma(\alpha_2+\Phi_j+1)} \\
&\quad +|\lambda_2|\|k_2\|\sum_{i=1}^{n}|\eta_i|\,\frac{(\psi(\xi_i)-\psi(\mathtt{w}_1))}{\rho}+|\lambda_2|\|k_2\|\sum_{j=1}^{m}|\zeta_j|\frac{(\psi(\theta_j)-\psi(\mathtt{w}_1))^{1+\Phi_j}}{\rho^{1+\Phi_j}\Gamma(2+\Phi_j)} \\
&\quad +(u_0+u_1|k_1|+u_2|k_2|)\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}+|\lambda_1|\,\|k_1\|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}\Big) \\
&\quad +X_2\Big((u_0+u_1|k_1|+u_2|k_2|)\sum_{k=1}^{r}|\aleph_k|\frac{(\psi(\varrho_k)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)} \\
&\quad +|\lambda_1|\,\|k_1\|\sum_{k=1}^{r}|\aleph_k|\frac{(\psi(\varrho_k)-\psi(\mathtt{w}_1))}{\rho}+|\lambda_1|\,\|k_1\|\sum_{l=1}^{q}|\Theta_l|\,\frac{(\psi(\vartheta_l)-\psi(\mathtt{w}_1))^{1+\upsilon_l}}{\rho^{1+\upsilon_l}\Gamma(2+\upsilon_l)} \\
&\quad +(u_0+u_1|k_1|+u_2|k_2|)\sum_{l=1}^{q}|\Theta_l|\,\frac{(\psi(\vartheta_l)-\psi(\mathtt{w}_1))^{\alpha_1+\upsilon_l}}{\rho^{\alpha_1+\upsilon_l}\Gamma(\alpha_1+\upsilon_l+1)} \\
&\quad +(v_0+v_1|k_1|+v_2|k_2|)\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}+|\lambda_2|\|k_2\|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}\Big)\Bigg] \\
&\quad +(u_0+u_1|k_1|+u_2|k_2|)\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}+|\lambda_1|\,\|k_1\|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho} \\
&\leq (u_0+u_1\|k_1\|+u_2\|k_2\|)\mathcal{A}_1+(v_0+v_1\|k_1\|+v_2\|k_2\|)\mathcal{B}_1+(\|k_1\|+\|k_2\|)C_1,
\end{aligned}$$

and thus

$$\|k_1\| \leq (u_0+u_1\|k_1\|+u_2\|k_2\|)\mathcal{A}_1+(v_0+v_1\|k_1\|+v_2\|k_2\|)\mathcal{B}_1+(\|k_1\|+\|k_2\|)C_1. \tag{3.1}$$

Similarly, we have

$$\|k_2\| \leq (u_0+u_1\|k_1\|+u_2\|k_2\|)\mathcal{A}_2+(v_0+v_1\|k_1\|+v_2\|k_2\|)\mathcal{B}_2+(\|k_1\|+\|k_2\|)C_2. \tag{3.2}$$

Thus we obtain

$$\begin{aligned}\|k_1\|+\|k_2\| &\le (\mathcal{A}_1+\mathcal{A}_2)u_0+(\mathcal{B}_1+\mathcal{B}_2)v_0+[(\mathcal{A}_1+\mathcal{A}_2)u_1+(\mathcal{B}_1+\mathcal{B}_2)v_1+(C_1+C_2)]\|k_1\|\\ &\quad +[(\mathcal{A}_1+\mathcal{A}_2)u_2+(\mathcal{B}_1+\mathcal{B}_2)v_2+(C_1+C_2)]\|k_2\|.\end{aligned}$$

This imply that,

$$\|(k_1,k_2)\|\le \frac{(\mathcal{A}_1+\mathcal{A}_2)u_0+(\mathcal{B}_1+\mathcal{B}_2)v_0}{1-P^*},$$

where

$$P^*=\max\{(\mathcal{A}_1+\mathcal{A}_2)u_1+(\mathcal{B}_1+\mathcal{B}_2)v_1+(C_1+C_2),(\mathcal{A}_1+\mathcal{A}_2)u_2+(\mathcal{B}_1+\mathcal{B}_2)v_2+(C_1+C_2)\}.$$

Then, the set U is bounded. Therefore, by Leray-Schauder alternative the problem (1.4) has at least one solution on $[\mathtt{w}_1,\mathtt{w}_2]$. □

Now, we prove the second existence of results by applying Krasnosel'skii point theorem [29].

Theorem 3.2. *Suppose* Υ_1,Υ_2*:* $[\mathtt{w}_1,\mathtt{w}_2]\times\mathbb{R}^2\to\mathbb{R}$ *are continuous functions. In addition we assume that:*

(H_2) *There exist positive functions* $\varphi_1,\varphi_2\in C([\mathtt{w}_1,\mathtt{w}_2],\mathbb{R}^+)$, *such that*

$$|\Upsilon_1(t,k_1,k_2)|\le\varphi_1(t),\ \ |\Upsilon_2(t,k_1,k_2)|\le\varphi_2(t),\ \text{for all } t\in[\mathtt{w}_1,\mathtt{w}_2].$$

If

$$C_1+C_2<1, \tag{3.3}$$

then, the problem (1.4) *has at least one solution on* $[\mathtt{w}_1,\mathtt{w}_2]$.

Proof. First, we separate the operator T as

$$\begin{aligned}T_1(k_1,k_2)(t)&=T_{11}(k_1,k_2)(t)+T_{12}(k_1,k_2)(t),\\ T_2(k_1,k_2)(t)&=T_{21}(k_1,k_2)(t)+T_{22}(k_1,k_2)(t)\end{aligned}$$

with

$$\begin{aligned}T_{11}(k_1,k_2)(t) &= \frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\ddot{\Lambda}\rho^{\gamma_1-1}\Gamma(\gamma_1)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_1-1}\Bigg[Y_1\Big(\sum_{i=1}^{n}\eta_i\,{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}\Upsilon_2(\xi_i,k_1(\xi_i),k_2(\xi_i))\\ &\quad+\sum_{j=1}^{m}\zeta_j\,{}^{p}I_{\mathtt{w}_1}^{\alpha_2+\Phi_j,\rho,\psi}\Upsilon_2(\theta_j,k_1(\theta_j),k_2(\theta_j))-{}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}\Upsilon_1(\mathtt{w}_2,k_1(\mathtt{w}_2),k_2(\mathtt{w}_2))\Big)\\ &\quad+X_2\Big(\sum_{k=1}^{r}\aleph_k\,{}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}\Upsilon_1(\varrho_k,k_1(\varrho_k),k_2(\varrho_k))\\ &\quad+\sum_{l=1}^{q}\Theta_l\,{}^{p}I_{\mathtt{w}_1}^{\alpha_1+\upsilon_l,\rho,\psi}\Upsilon_1(\vartheta_l,k_1(\vartheta_l),k_2(\vartheta_l))-{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}\Upsilon_2(\mathtt{w}_2,k_1(\mathtt{w}_2),k_2(\mathtt{w}_2))\Big)\Bigg]\\ &\quad+{}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}\Upsilon_1(t,k_1(t),k_2(t)),\quad t\in[\mathtt{w}_1,\mathtt{w}_2],\end{aligned}$$

$$
\begin{aligned}
T_{12}(k_1,k_2)(t) &= \frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\ddot{\Lambda}\rho^{\gamma_1-1}\Gamma(\gamma_1)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_1-1}\Bigg[Y_1\Big(-\lambda_2\sum_{i=1}^{n}\eta_i\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(\xi_i)\\
&\quad -\lambda_2\sum_{j=1}^{m}\zeta_j\,{}^{p}I_{\mathtt{w}_1}^{1+\Phi_j,\rho,\psi}k_2(\theta_j)+\lambda_1\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(\mathtt{w}_2)\Big)\\
&\quad +X_2\Big(-\lambda_1\sum_{k=1}^{r}\aleph_k\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(\varrho_k)-\lambda_1\sum_{l=1}^{q}\Theta_l\,{}^{p}I_{\mathtt{w}_1}^{1+\upsilon_l,\rho,\psi}k_1(\vartheta_l)\\
&\quad +\lambda_2\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(\mathtt{w}_2)\Big)\Bigg]-\lambda_1\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(t),\quad t\in[\mathtt{w}_1,\mathtt{w}_2],\\
T_{21}(k_1,k_2)(t) &= \frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\ddot{\Lambda}\rho^{\gamma_2-1}\Gamma(\gamma_2)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_2-1}\Bigg[Y_2\Big(\sum_{i=1}^{n}\eta_i\,{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}\Upsilon_2(\xi_i,k_1(\xi_i),k_2(\xi_i))\\
&\quad +\sum_{j=1}^{m}\zeta_j\,{}^{p}I_{\mathtt{w}_1}^{\alpha_2+\Phi_j,\rho,\psi}\Upsilon_2(\theta_j,k_1(\theta_j),k_2(\theta_j))-{}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}\Upsilon_1(\mathtt{w}_2,k_1(\mathtt{w}_2),k_2(\mathtt{w}_2))\Big)\\
&\quad +X_1\Big(\sum_{k=1}^{r}\aleph_k\,{}^{p}I_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}\Upsilon_1(\varrho_k,k_1(\varrho_k),k_2(\varrho_k))+\sum_{l=1}^{q}\Theta_l\,{}^{p}I_{\mathtt{w}_1}^{\alpha_1+\upsilon_l,\rho,\psi}\Upsilon_1(\vartheta_l,k_1(\vartheta_l),k_2(\vartheta_l))\\
&\quad -{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}\Upsilon_1(\mathtt{w}_2,k_1(\mathtt{w}_2),k_2(\mathtt{w}_2))\Big)\Bigg]+{}^{p}I_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}\Upsilon_2(t,k_1(t),k_2(t)),\quad t\in[\mathtt{w}_1,\mathtt{w}_2],\\
T_{22}(k_1,k_2)(t) &= \frac{e^{\frac{\rho-1}{\rho}(\psi(t)-\psi(\mathtt{w}_1))}}{\ddot{\Lambda}\rho^{\gamma_2-1}\Gamma(\gamma_2)}(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_2-1}\Bigg[Y_2\Big(-\lambda_2\sum_{i=1}^{n}\eta_i\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(\xi_i)\\
&\quad -\lambda_2\sum_{j=1}^{m}\zeta_j\,{}^{p}I_{\mathtt{w}_1}^{1+\Phi_j,\rho,\psi}k_2(\theta_j)+\lambda_1\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(\mathtt{w}_2)\Big)\\
&\quad +X_1\Big(-\lambda_1\sum_{k=1}^{r}\aleph_k\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_1(\varrho_k)-\lambda_1\sum_{l=1}^{q}\Theta_l\,{}^{p}I_{\mathtt{w}_1}^{1+\upsilon_l,\rho,\psi}k_1(\vartheta_l)\\
&\quad +\lambda_2\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(\mathtt{w}_2)\Big)\Bigg]-\lambda_2\,{}^{p}I_{\mathtt{w}_1}^{1,\rho,\psi}k_2(t),\quad t\in[\mathtt{w}_1,\mathtt{w}_2].
\end{aligned}
$$

We claim that $TB_r\subset B_r$ where

$$B_r=\{(k_1,k_2)\in X\times Y:\|(k_1,k_2)\|\le r\}.$$

We set

$$\sup_{t\in[\mathtt{w}_1,\mathtt{w}_2]}\varphi_i(t)=\|\varphi_i\|$$

for $i=1,2$ and choose

$$r\ge\frac{\|\varphi_1\|(\mathcal{A}_1+\mathcal{A}_2)+\|\varphi_2\|(\mathcal{B}_1+\mathcal{B}_2)}{1-(C_1+C_2)}.$$

Let $(k_1,k_2),(\bar{k}_1,\bar{k}_2)\in B_r$. Acconding the proof of Theorem 3.1, the following inequalities are obtained

$$
\begin{aligned}
|T_{11}(k_1,k_2)(t)+T_{12}(\bar{k}_1,\bar{k}_2)(t)| &\le \|\varphi_1\|\mathcal{A}_1+\|\varphi_2\|\mathcal{B}_1+rC_1,\\
|T_{21}(k_1,k_2)(t)+T_{22}(\bar{k}_1,\bar{k}_2)(t)| &\le \|\varphi_1\|\mathcal{A}_2+\|\varphi_2\|\mathcal{B}_2+rC_2,
\end{aligned}
$$

and therefore

$$\|T_1(k_1,k_2)+T_2(k_1,k_2)\| \leq \|\varphi_1\|(\mathcal{A}_1+\mathcal{A}_2)+\|\varphi_2\|(\mathcal{B}_1+\mathcal{B}_2)+r(C_1+C_2)\leq r.$$

Hence,

$$T_{11}(k_1,k_2)(t)+T_{12}(\bar{k}_1,\bar{k}_2)(t)\subset B_r$$

and

$$T_{21}(k_1,k_2)(t)+T_{22}(\bar{k}_1,\bar{k}_2)(t)\subset B_r.$$

Consider the operators T_{11} and T_{21}. By continuity of f and g, T_{11} and T_{21} are continuous operators. For any $(k_1,k_2)\in B_r$, we have

$$\begin{aligned}
|T_{11}(k_1,k_2)(t)| &\leq \frac{(\psi(t)-\psi(\mathtt{w}_1))^{\gamma_1-1}}{|\ddot{\Lambda}|\rho^{\gamma_1-1}\Gamma(\gamma_1)}\Big[Y_1\Big(\sum_{i=1}^{n}|\eta_i|{}^pI_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}|\Upsilon_2(\xi_i,k_1(\xi_i),k_2(\xi_i))|\\
&\quad+\sum_{j=1}^{m}|\zeta_j|{}^pI_{\mathtt{w}_1}^{\alpha_2+\Phi_j,\rho,\psi}|\Upsilon_2(\theta_j,k_1(\theta_j),k_2(\theta_j))|+{}^pI_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}|\Upsilon_1(\mathtt{w}_2,k_1(\mathtt{w}_2),k_2(\mathtt{w}_2))|\Big)\\
&\quad+X_2\Big(\sum_{k=1}^{r}|\aleph_k|{}^pI_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}|\Upsilon_1(\varrho_k,k_1(\varrho_k),k_2(\varrho_k))|+\sum_{l=1}^{q}|\Theta_l|{}^pI_{\mathtt{w}_1}^{\alpha_1+\upsilon_l,\rho,\psi}|\Upsilon_1(\vartheta_l,k_1(\vartheta_l),k_2(\vartheta_l))|\\
&\quad+{}^pI_{\mathtt{w}_1}^{\alpha_2,\rho,\psi}|\Upsilon_2(\mathtt{w}_2,k_1(\mathtt{w}_2),k_2(\mathtt{w}_2))|\Big)\Big]+{}^pI_{\mathtt{w}_1}^{\alpha_1,\rho,\psi}|\Upsilon_1(t,k_1(t),k_2(t))|\\
&\leq \|\varphi_1\|\mathcal{A}_1+\|\varphi_2\|\mathcal{B}_1.
\end{aligned}$$

In a similar way, we can get

$$|T_{21}(k_1,k_2)(t)|\leq\|\varphi_1\|\mathcal{A}_2+\|\varphi_2\|\mathcal{B}_2.$$

Hence, we obtain that

$$\|(T_{11},T_{21})(k_1,k_2)\|\leq\|\varphi_1\|(\mathcal{A}_1+\mathcal{A}_2)+\|\varphi_2\|(\mathcal{B}_1+\mathcal{B}_2),$$

which yields that $(T_{11},T_{12})B_r$ is uniformly bounded. For any t_1, $t_2\in[\mathtt{w}_1,\mathtt{w}_2]$, $t_2>t_1$ and for all $(k_1,k_2)\in B_r$ the operators $(T_{11},T_{12})B_r$ are equicontinuous by the proof of Theorem 3.1.

Lastly, it is proven that the operators T_{12} and T_{22} are contraction mappings. For all $(k_1,k_2),(\bar{k}_1,\bar{k}_2)\in B_r$, we have:

$$\begin{aligned}
|T_{12}(\bar{k}_1,\bar{k}_2)(t)-T_{12}(k_1,k_2)(t)| &\leq \frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\gamma_1-1}}{|\ddot{\Lambda}|\rho^{\gamma_1-1}\Gamma(\gamma_1)}\Big[Y_1\Big(|\lambda_2|\sum_{i=1}^{n}|\eta_i|{}^pI_{\mathtt{w}_1}^{1,\rho,\psi}|\bar{k}_2(\xi_i)-k_2(\xi_i)|\\
&+|\lambda_2|\sum_{j=1}^{m}|\zeta_j|{}^pI_{\mathtt{w}_1}^{1+\Phi_j,\rho,\psi}|\bar{k}_2(\theta_j)-k_2(\theta_j)|+|\lambda_1|{}^pI_{\mathtt{w}_1}^{1,\rho,\psi}|\bar{k}_1(\mathtt{w}_2)-k_1(\mathtt{w}_2)|\Big)\\
&+X_2\Big(|\lambda_1|\sum_{k=1}^{r}|\aleph_k|{}^pI_{\mathtt{w}_1}^{1,\rho,\psi}|\bar{k}_1(\varrho_k)-k_1(\varrho_k)|+|\lambda_1|\sum_{l=1}^{q}|\Theta_l|{}^pI_{\mathtt{w}_1}^{1+\upsilon_l,\rho,\psi}|\bar{k}_1(\vartheta_l)-k_1(\vartheta_l)|\\
&+|\lambda_2|{}^pI_{\mathtt{w}_1}^{1,\rho,\psi}|\bar{k}_2(\mathtt{w}_2)-k_2(\mathtt{w}_2)|\Big)\Big]+|\lambda_1|{}^pI_{\mathtt{w}_1}^{1,\rho,\psi}|\bar{k}_1(t)-k_1(t)|
\end{aligned}$$

$$\leq \frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\gamma_1-1}}{|\ddot{\Lambda}|\rho^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg[Y_1\Big(|\lambda_2|\|\bar{k}_2-k_2\|\sum_{i=1}^{n}|\eta_i|\frac{(\psi(\xi_i)-\psi(\mathtt{w}_1))}{\rho}$$
$$+|\lambda_2|\|\bar{k}_2-k_2\|\sum_{j=1}^{m}|\zeta_j|\frac{(\psi(\theta_j)-\psi(\mathtt{w}_1))^{1+\Phi_j}}{\rho^{1+\Phi_j}\Gamma(2+\Phi_j)}+|\lambda_1|\|\bar{k}_1-k_1\|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}\Big)$$
$$+X_2\Big(|\lambda_1|\|\bar{k}_1-k_1\|\sum_{k=1}^{r}|\aleph_k|\frac{(\psi(\varrho_k)-\psi(\mathtt{w}_1))}{\rho}+|\lambda_1|\|\bar{k}_1-k_1\|\sum_{l=1}^{q}|\Theta_l|\frac{(\psi(\vartheta_l)-\psi(\mathtt{w}_1))^{1+\upsilon_l}}{\rho^{1+\upsilon_l}\Gamma(1+\upsilon_l+1)}$$
$$+|\lambda_2|\|\bar{k}_2-k_2\|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}\Big)\Bigg]+|\lambda_1|\|\bar{k}_1-k_1\|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}$$
$$\leq C_1(\|\bar{k}_1-k_1\|+\|\bar{k}_2-k_2\|).$$

Additionally, we also obtain that

$$|T_{21}(\bar{k}_1,\bar{k}_2)(t)-T_{21}(k_1,k_2)(t)|\leq C_2(\|\bar{k}_1-k_1\|+\|\bar{k}_2-k_2\|).$$

Combining the above inequalities, we have

$$\|(T_{11},T_{21})(k_1,k_2)\|\leq (C_1+C_2)\Big(\|\bar{k}_1-k_1\|+\|\bar{k}_2-k_2\|\Big).$$

We have (T_{11},T_{21}) is a contraction. Consequently, by Krasnosel'skii's fixed point theorem, the problem (1.4) has at least one solution on $[\mathtt{w}_1,\mathtt{w}_2]$. □

Banach's fixed point theorem [30] is applied to obtain our uniqueness and existence result.

Theorem 3.3. *Let* Υ_1,Υ_2*:* $[\mathtt{w}_1,\mathtt{w}_2]\times\mathbb{R}^2\to\mathbb{R}$ *such that:*
(H_3) *There exist positive constants* L_1,L_2*, such that, for all* $t\in[\mathtt{w}_1,\mathtt{w}_2]$ *and* $o_i,\bar{o}_i\in\mathbb{R}$, $i=1,2$, *we have*

$$\begin{aligned}|\Upsilon_1(t,o_2,\bar{o}_2)-\Upsilon_1(t,o_1,\bar{o}_1)| &\leq L_1(|o_2-o_1|+|\bar{o}_2-\bar{o}_1|),\\ |\Upsilon_2(t,o_2,\bar{o}_2)-\Upsilon_2(t,o_1,\bar{o}_1)| &\leq L_2(|o_2-o_1|+|\bar{o}_2-\bar{o}_1|).\end{aligned}$$

Then, the problem (1.4) *has a unique solution, provided that*

$$L_1(\mathcal{A}_1+\mathcal{A}_2)+L_2(\mathcal{B}_1+\mathcal{B}_2)+C_1+C_2<1. \tag{3.4}$$

Proof. Let

$$\sup_{t\in[\mathtt{w}_1,\mathtt{w}_2]}|\Upsilon_1(t,0,0)|=M_1<\infty,$$
$$\sup_{t\in[\mathtt{w}_1,\mathtt{w}_2]}|\Upsilon_2(t,0,0)|=M_2<\infty$$

and

$$B_r=\{(k_1,k_2)\in X\times Y:\|(k_1,k_2)\|\leq r\}$$

with

$$r\geq\frac{M_1(\mathcal{A}_1+\mathcal{A}_2)+M_2(\mathcal{B}_1+\mathcal{B}_2)}{1-[L_1(\mathcal{A}_1+\mathcal{A}_2)+L_2(\mathcal{B}_1+\mathcal{B}_2)+C_1+C_2]}.$$

For all $(k_1, k_2) \in B_r$ and $t \in [\mathtt{w}_1, \mathtt{w}_2]$. By applying (H_3), we obtain the following inequalities

$$\begin{aligned}
|\Upsilon_1(t, k_1(t), k_2(t))| &\le |\Upsilon_1(t, k_1(t), k_2(t)) - \Upsilon_1(t, 0, 0)| + |\Upsilon_1(t, 0, 0)| \\
&\le L_1(|k_1(t)| + |k_2(t)|) + M_1 \le L_1(\|k_1\| + \|k_2\|) \le L_1 r + M_1, \\
|\Upsilon_2(t, k_1(t), k_2(t))| &\le |\Upsilon_2(t, k_1(t), k_2(t)) - \Upsilon_2(t, 0, 0)| + |\Upsilon_2(t, 0, 0)| \\
&\le L_2(|k_1(t)| + |k_2(t)|) + M_2 \le L_2 r + M_2.
\end{aligned}$$

We have

$$\begin{aligned}
|T_1(k_1, k_2)(t)| \le\ & \frac{(\psi(\mathtt{w}_2) - \psi(\mathtt{w}_1))^{\gamma_1 - 1}}{|\ddot{\Lambda}|\rho^{\gamma_1 - 1}\Gamma(\gamma_1)} \Bigg[Y_1 \Big((L_2 r + M_2) \sum_{i=1}^{n} |\eta_i| \frac{(\psi(\xi_i) - \psi(\mathtt{w}_1))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2 + 1)} \\
& + (L_2 r + M_2) \sum_{j=1}^{m} |\zeta_j| \frac{(\psi(\theta_j) - \psi(\mathtt{w}_1))^{\alpha_2 + \Phi_j}}{\rho^{\alpha_2 + \Phi_j}\Gamma(\alpha_2 + \Phi_j + 1)} \\
& + |\lambda_2| \|k_2\| \sum_{i=1}^{n} |\eta_i| \frac{(\psi(\xi_i) - \psi(\mathtt{w}_1))}{\rho} + |\lambda_2| \|k_2\| \sum_{j=1}^{m} |\zeta_j| \frac{(\psi(\theta_j) - \psi(\mathtt{w}_1))^{1 + \Phi_j}}{\rho^{1 + \Phi_j}\Gamma(2 + \Phi_j)} \\
& + (L_1 r + M_1) \frac{(\psi(\mathtt{w}_2) - \psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1 + 1)} + |\lambda_1| \, \|k_1\| \frac{(\psi(\mathtt{w}_2) - \psi(\mathtt{w}_1))}{\rho} \Big) \\
& + X_2 \big((L_1 r + M_1) \sum_{k=1}^{r} |\aleph_k| \frac{(\psi(\varrho_k) - \psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1 + 1)} \\
& + |\lambda_1| \, \|k_1\| \sum_{k=1}^{r} |\aleph_k| \frac{(\psi(\varrho_k) - \psi(\mathtt{w}_1))}{\rho} + |\lambda_1| \, \|k_1\| \sum_{l=1}^{q} |\Theta_l| \frac{(\psi(\vartheta_l) - \psi(\mathtt{w}_1))^{1 + \upsilon_l}}{\rho^{1 + \upsilon_l}\Gamma(1 + \upsilon_l + 1)} \\
& + (L_1 r + M_1) \sum_{l=1}^{q} |\Theta_l| \frac{(\psi(\vartheta_l) - \psi(\mathtt{w}_1))^{\alpha_1 + \upsilon_l}}{\rho^{\alpha_1 + \upsilon_l}\Gamma(\alpha_1 + \upsilon_l + 1)} \\
& + (L_2 r + M_2) \frac{(\psi(\mathtt{w}_2) - \psi(\mathtt{w}_1))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2 + 1)} + |\lambda_2| \|k_2\| \frac{(\psi(\mathtt{w}_2) - \psi(\mathtt{w}_1))}{\rho} \Big) \Bigg] \\
& + (L_1 r + M_1) \frac{(\psi(\mathtt{w}_2) - \psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1 + 1)} + |\lambda_1| \, \|k_1\| \frac{(\psi(\mathtt{w}_2) - \psi(\mathtt{w}_1))}{\rho} \\
\le\ & (L_1 r + M_1)\mathcal{A}_1 + (L_2 r + M_2)\mathcal{B}_1 + r C_1.
\end{aligned}$$

Thus,

$$\|T_1(k_1, k_2)\| \le (L_1 r + M_1)\mathcal{A}_1 + (L_2 r + M_2)\mathcal{B}_1 + r C_1.$$

We also have

$$\|T_2(k_1, k_2)\| \le (L_1 r + M_1)\mathcal{A}_2 + (L_2 r + M_2)\mathcal{B}_2 + r C_2.$$

Therefore

$$\|T(k_1, k_2)\| \le M_1(\mathcal{A}_1 + \mathcal{A}_2) + M_2(\mathcal{B}_1 + \mathcal{B}_2) + [L_1(\mathcal{A}_1 + \mathcal{A}_2) + L_2(\mathcal{B}_1 + \mathcal{B}_2) + C_1 + C_2] r.$$

Hence, $T(B_r) \subset B_r$. Now, it is shown that (T_1, T_2) is a contraction mapping. For all $(k_1, k_2), (\bar{k}_1, \bar{k}_2) \in B_r$, we have

$$
\begin{aligned}
|T_1(\bar{k}_1,\bar{k}_2)(t)-T_1(k_1,k_2)(t)| \le{}& \frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\gamma_1-1}}{|\ddot{\Lambda}|\rho^{\gamma_1-1}\Gamma(\gamma_1)}\Bigg[Y_1\Big(L_2(\|\bar{k}_1-k_1\|+\|\bar{k}_2-k_2\|)\sum_{i=1}^{n}|\eta_i|\frac{(\psi(\xi_i)-\psi(\mathtt{w}_1))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\\
&+L_2(\|\bar{k}_1-k_1\|+\|\bar{k}_2-k_2\|)\sum_{j=1}^{m}|\zeta_j|\frac{(\psi(\theta_j)-\psi(\mathtt{w}_1))^{\alpha_2+\Phi_j}}{\rho^{\alpha_2+\Phi_j}\Gamma(\alpha_2+\Phi_j+1)}\\
&+|\lambda_2|\|\bar{k}_2-k_2\|\sum_{i=1}^{n}|\eta_i|\frac{(\psi(\xi_i)-\psi(\mathtt{w}_1))}{\rho}+|\lambda_2|\|\bar{k}_2-k_2\|\sum_{j=1}^{m}|\zeta_j|\frac{(\psi(\theta_j)-\psi(\mathtt{w}_1))^{1+\Phi_j}}{\rho^{1+\Phi_j}\Gamma(2+\Phi_j)}\\
&+L_1(\|\bar{k}_1-k_1\|+\|\bar{k}_2-k_2\|)\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}+|\lambda_1|\|\bar{k}_1-k_1\|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}\Big)\\
&+X_2(L_1(\|\bar{k}_1-k_1\|+\|\bar{k}_2-k_2\|)\sum_{k=1}^{r}|\aleph_k|\frac{(\psi(\varrho_k)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}\\
&+|\lambda_1|\|\bar{k}_1-k_1\|\sum_{k=1}^{r}|\aleph_k|\frac{(\psi(\varrho_k)-\psi(\mathtt{w}_1))}{\rho}+|\lambda_1|\|\bar{k}_1-k_1\|\sum_{l=1}^{q}|\Theta_l|\,\frac{(\psi(\vartheta_l)-\psi(\mathtt{w}_1))^{1+\upsilon_l}}{\rho^{1+\upsilon_l}\Gamma(2+\upsilon_l)}\\
&+L_1(\|\bar{k}_1-k_1\|+\|\bar{k}_2-k_2\|)\sum_{l=1}^{q}|\Theta_l|\frac{(\psi(\vartheta_l)-\psi(\mathtt{w}_1))^{\alpha_1+\upsilon_l}}{\rho^{\alpha_1+\upsilon_l}\Gamma(\alpha_1+\upsilon_l+1)}\\
&+L_2(\|\bar{k}_1-k_1\|+\|\bar{k}_2-k_2\|)\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}+|\lambda_2|\|\bar{k}_2-k_2\|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}\Big)\Bigg]\\
&+L_1(\|\bar{k}_1-k_1\|+\|\bar{k}_2-k_2\|)\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}+|\lambda_1|\|\bar{k}_1-k_1\|\frac{(\psi(\mathtt{w}_2)-\psi(\mathtt{w}_1))}{\rho}\\
\le{}& [L_1\mathcal{A}_1+L_2\mathcal{B}_1+C_1](\|\bar{k}_1-k_1\|+\|\bar{k}_2-k_2\|).
\end{aligned}
$$

By a similar argument, we have

$$
\begin{aligned}
|T_2(\bar{k}_1,\bar{k}_2)(t)-T_2(k_1,k_2)(t)| \quad\le\quad & L_1(\|\bar{k}_1-k_1\|+\|\bar{k}_2-k_2\|)\mathcal{A}_2+L_2(\|\bar{k}_1-k_1\|+\|\bar{k}_2-k_2\|)\mathcal{B}_2\\
&+(\|\bar{k}_1-k_1\|+\|\bar{k}_2-k_2\|)C_2.
\end{aligned}
$$

Hence, we obtain that

$$
\|T(\bar{k}_1,\bar{k}_2)-T(k_1,k_2)\|\le[L_1(\mathcal{A}_1+\mathcal{A}_2)+L_2(\mathcal{B}_1+\mathcal{B}_2)+C_1+C_2](\|\bar{k}_1-k_1\|+\|\bar{k}_2-k_2\|).
$$

By assumption (3.4), the operator T is a contraction mapping. By Banach's fixed point theorem the problem (1.4) has a unique solution. □

4. Examples

Example 4.1. *Consider the following coupled system of ψ-Hilfer genneralized proportional fractional differential equation,*

$$
\begin{cases}
({}^{H}D_{0.1}^{\frac{3}{2},\frac{1}{2},\frac{3}{4},\frac{\log(t)}{9}}k_1)(t)+\dfrac{1}{12}({}^{H}D_{0.1}^{\frac{3}{2}-1,\frac{1}{2},\frac{3}{4},\frac{\log(t)}{9}}k_1)(t)=\Upsilon_1(t,k_1(t),k_2(t)),\ t\in\left[\dfrac{1}{10},3\right],\\
({}^{H}D_{0.1}^{\frac{5}{4},\frac{1}{4},\frac{3}{4},\frac{\log(t)}{9}}k_2)(t)+\frac{1}{15}({}^{H}D_{0.1}^{\frac{5}{4}-1,\frac{1}{4},\frac{3}{4},\frac{\log(t)}{9}}k_2)(t)=\Upsilon_2(t,k_2(t),k_1(t)),\ \ t\in\left[\dfrac{1}{10},3\right],\\
k_1\left(\dfrac{1}{10}\right)=0,\ k_1(3)=\dfrac{2}{21}\,k_2\left(\dfrac{3}{11}\right)+\dfrac{4}{23}\,k_2\left(\dfrac{7}{11}\right)+\dfrac{4}{23}\,{}^{p}I_{0.1}^{\frac{1}{2},\frac{3}{4},\frac{\log(t)}{9}}k_2\left(\dfrac{10}{11}\right)+\dfrac{4}{25}\,{}^{p}I_{0.1}^{\frac{3}{2},\frac{3}{4},\frac{\log(t)}{9}}k_2\left(\dfrac{8}{12}\right),\\
k_2\left(\dfrac{1}{10}\right)=0,\ k_2(3)=\dfrac{1}{31}\,k_1\left(\dfrac{1}{6}\right)+\dfrac{3}{41}\,k_1\left(\dfrac{5}{11}\right)+\dfrac{4}{23}\,{}^{p}I_{0.1}^{\frac{4}{3},\frac{3}{4},\frac{\log(t)}{9}}k_1\left(\dfrac{2}{3}\right).
\end{cases}
\tag{4.1}
$$

Here, we take

$$\psi(t)=\frac{\log(t)}{9},\ \mathtt{w}_1=1/10,\ \mathtt{w}_2=3,\ \alpha_1=3/2,\ \alpha_2=5/4,\ \beta_1=1/2,\ \beta_2=1/4,\ \rho=3/4,$$
$$\lambda_1=1/12,\ \lambda_2=1/15,\ \eta_1=2/21,\ \eta_2=4/23,\ \xi_1=3/11,\ \xi_2=7/11,$$
$$\zeta_1=4/23,\ \zeta_2=6/25,\ \theta_1=10/11,\ \theta_2=8/12,\ \Phi_1=1/2,\ \Phi_2=3/2,$$
$$\aleph_1=1/31,\ \aleph_2=3/41,\ \varrho_1=1/6,\ \varrho_2=5/11,\ \upsilon_1=4/3,\ \vartheta_1=2/3.$$

From the above data, we obtain

$$k_1\approx 1.1096,\ \bar{k}_1\approx 0.9391,\ k_2\approx 0.2570,\ \bar{k}_2\approx 0.7302,\ \ddot{\Lambda}\approx -0.4005,$$

$$\mathcal{A}_1\approx 0.3385,\ \mathcal{A}_2\approx 3.6140,\ \mathcal{B}_1\approx 0.4068,\ \mathcal{B}_2\approx 0.6287,\ C_1\approx 0.0891,\ C_2\approx 0.3720.$$

(i) In order to illustrate Theorem 3.1, consider the functions f and g, defined by

$$\begin{aligned}\Upsilon_1(t,k_1,k_2) &= \frac{\cos^2 k_1}{4}+\frac{|k_1|^3e^{-k_2^2}}{32\sqrt[5]{t}(1+2|k_1(t)|+|k_1(t)|^2)}+\frac{e^{-10t}k_2^7\sin^2 k_1}{25(1+k_2^6)},\\ \Upsilon_2(t,k_1,k_2) &= \frac{1}{12}e^{-|k_2|}+\frac{3k_1\sin^2|k_1k_2|}{10+2t^2}+\frac{3k_2\cos^4 k_1}{\sqrt{(t+4)^3}}.\end{aligned} \tag{4.2}$$

Then, we have

$$\begin{aligned}|\Upsilon_1(t,k_1,k_2)| &\le \frac{1}{4}+\frac{1}{32\sqrt[5]{t}}|k_1|+\frac{1}{25}|k_2|,\\ |\Upsilon_2(t,k_1,k_2)| &\le \frac{1}{12}+\frac{3}{10+2t^2}|k_1|+\frac{3}{\sqrt{(t+4)^3}}|k_2|.\end{aligned}$$

Thus, (H_1) is satisfied with

$$u_0=1/4,\ u_1=0.05,\ u_2=1/25,\ v_0=1/12,\ v_1=0.2994,\ v_2=0.3613.$$

Then

$$(\mathcal{A}_1+\mathcal{A}_2)u_1+(\mathcal{B}_1+\mathcal{B}_2)v_1+(C_1+C_2)\approx 0.9687<1$$

and

$$(\mathcal{A}_1+\mathcal{A}_2)u_2+(\mathcal{B}_1+\mathcal{B}_2)v_2+(C_1+C_2)\approx 0.9933<1.$$

Thus, all assumptions of Theorem 3.1 are satisfied. Hence, the proplem (4.1), with f and g, given by (4.2), has at least one solution on $[1/10,3]$.

(ii) Consider now the following functions f and g,

$$\begin{aligned}\Upsilon_1(t,k_1,k_2) &= \tan^{-1}\left(\sqrt{1+k_1^2}\right)+\frac{t^3+4t}{2}+\frac{\cos^2(k_2)}{5},\\ \Upsilon_2(t,k_1,k_2) &= \frac{(k_1k_2)^4}{1+(k_1k_2)^4}+\sin t+\frac{e^{-|k_1|^3}}{3}.\end{aligned} \tag{4.3}$$

It is obvious to check that the above functions satisfy

$$|\Upsilon_1(t,k_1,k_2)| \le \frac{5\pi+2}{10}+\frac{1}{2}t^3+2t:=\varphi_1(t),$$

$$|\Upsilon_2(t,k_1,k_2)| \leq \sin t + \frac{4}{3} := \varphi_2(t).$$

Then we find that $C_1 + C_2 \approx 0.4610 < 1$. Hence, by Theorem 3.2, the coupled system (4.1), with f and g, given by (4.3), has at lest one solution on an interval $[1/10, 3]$.
(iii) To illustrate Theorem 3.3, we consider the functions f and g as

$$\begin{aligned}\Upsilon_1(t,k_1,k_2) &= \frac{1}{2} + (\log t^2) + \frac{2(k_1)^2 + |k_1|}{5(1+4|k_1|)} + \frac{\sin|k_2|}{10t+11},\\ \Upsilon_2(t,k_1,k_2) &= \pi + \frac{t^2+2t}{3} + \frac{\tan^{-1}|k_1|}{100e^{2\log 3t}+1} + \frac{k_2^2+|k_2|}{(7+2^{10t})(1+|k_2|)}.\end{aligned} \tag{4.4}$$

We have

$$\begin{aligned}|\Upsilon_1(t,\bar{k}_1,\bar{k}_2) - \Upsilon_1(t,k_1,k_2)| &\leq \frac{1}{10}|\bar{k}_1 - k_1| + \frac{1}{12}|\bar{k}_2 - k_2|,\\ |\Upsilon_2(t,\bar{k}_1,\bar{k}_2) - \Upsilon_2(t,k_1,k_2)| &\leq \frac{1}{10}|\bar{k}_1 - k_1| + \frac{1}{9}|\bar{k}_2 - k_2|,\end{aligned}$$

and therefore the Lipschitz condition for f and g is satisfied with $L_1 = 1/10$ and $L_2 = 1/9$. In addition, we find that

$$L_1(\mathcal{A}_1 + \mathcal{A}_2) + L_2(\mathcal{B}_1 + \mathcal{B}_2) + C_1 + C_2 \approx 0.9714 < 1.$$

Thus, by Theorem 3.3, problem (4.1), with f and g, given by (4.4), has a unique solution on $[1/10, 3]$.

Example 4.2. *We investigate the behavior of solutions by replacing the values of proportional constant ρ by $0.1, 0.2, \cdots, 0.9$, in the following coupled linear system of ψ-Hilfer generalized proportional fractional differential equations of the form:*

$$\begin{cases} ({}^H D_0^{\frac{3}{2},\frac{1}{2},\rho,t^2} k_1)(t) + \frac{1}{10}({}^H D_0^{\frac{1}{2},\frac{1}{2},\rho,t^2} k_1)(t) = e^{\frac{\rho-1}{\rho}t^2} \cdot (t^2)^{-\frac{1}{2}}, \ t \in (0,1],\\ ({}^H D_0^{\frac{11}{10},\frac{1}{2},\rho,t^2} k_2)(t) + \frac{1}{5}({}^H D_0^{\frac{1}{10},\frac{1}{2},\rho,t^2} k_2)(t) = e^{\frac{2\rho-2}{\rho}t^2} \cdot (t^2)^{-\frac{1}{2}}, \ t \in (0,1],\\ k_1(0) = 0, \qquad k_1(1) = \frac{1}{50} k_2\left(\frac{1}{5}\right) + \frac{1}{5}\, {}^\rho I_0^{\frac{3}{2},\rho,t^2} k_2\left(\frac{1}{20}\right),\\ k_2(0) = 0, \qquad k_2(1) = k_1\left(\frac{7}{10}\right) + \frac{1}{2}\, {}^\rho I_0^{\frac{11}{5},\rho,t^2} k_1\left(\frac{3}{25}\right). \end{cases} \tag{4.5}$$

Here, we set

$$\psi(t) = t^2,\ \mathtt{w}_1 = 0,\ \mathtt{w}_2 = 1,\ \alpha_1 = 3/2,\ \alpha_2 = 11/10,\ \beta_1 = \beta_2 := \beta = 1/2,$$

$$\lambda_1 = 1/10,\ \lambda_2 = 1/5,\ \eta_1 = 1/50,\ \xi_1 = 1/5,\ \zeta_1 = 1/5,\ \theta_1 = 1/20,$$

$$\Phi_1 = 3/2,\ \aleph_1 = 1,\ \varrho_1 = 7/10,\ \Theta_1 = 1/2,\ \upsilon_1 = 11/5,\ \vartheta_1 = 3/25.$$

By using integrating factor technique, we can obtain

$$k_1(t) = \frac{2\Gamma(\beta)}{\rho^{\alpha_1}\Gamma(\alpha_1+\beta-1)} \int_0^t e^{[\frac{1-\rho+\lambda_1}{\rho}](-t^2+s^2)+[\frac{\rho-1}{\rho}](s^2)} \cdot s^{2\alpha_1-2} ds$$

$$+\frac{2c_1}{\rho^{\gamma_1}\Gamma(\gamma_1)}\int_0^t e^{[1-\frac{1}{\rho}](t^2)+\frac{\lambda_1}{\rho}(s^2-t^2)}\cdot s^{2\gamma_1-1}ds, \tag{4.6}$$

$$k_2(t) = \frac{2\Gamma(\beta)}{\rho^{\alpha_2}\Gamma(\alpha_2+\beta-1)}\int_0^t e^{[\frac{1-\rho+\lambda_2}{\rho}](-t^2+s^2)+[\frac{\rho-1}{\rho}](s^2)}\cdot s^{2\alpha_2-2}ds$$
$$+\frac{2c_3}{\rho^{\gamma_2}\Gamma(\gamma_2)}\int_0^t e^{[1-\frac{1}{\rho}](t^2)+\frac{\lambda_2}{\rho}(s^2-t^2)}\cdot s^{2\gamma_2-1}ds, \tag{4.7}$$

where $\gamma_1 = 7/4$, $\gamma_2 = 31/20$ *and*

$$c_1 = \frac{R(M-O)+P(Q-S)}{PT-RN}, \qquad c_3 = \frac{T(M-O)+N(Q-S)}{PT-RN}, \qquad PT-RN \neq 0, \tag{4.8}$$

with the canstants M, N, O, P, Q, R, S *and* T *are defined by*

$$M = \frac{2\Gamma(\beta)}{\rho^{\alpha_1}\Gamma(\alpha_1+\beta-1)}\int_0^1 e^{[\frac{1-\rho-\lambda_1}{\rho}](-s^2)+[\frac{\rho-1}{\rho}](s^2)}\cdot s^{2\alpha_1+2\beta-3}ds,$$

$$N = \frac{2}{\rho^{\gamma_1}\Gamma(\gamma_1)}\int_0^1 e^{[1-\frac{1}{\rho}]+\frac{\lambda_1}{\rho}(s^2)}\cdot s^{2\gamma_1-1}ds,$$

$$O = \frac{\Gamma(\beta)}{\Gamma(\alpha_2+\beta-1)}\Bigg[\frac{2\eta_1}{\rho^{\alpha_2}}\int_0^{\xi_1} e^{[\frac{1-\rho+\lambda_2}{\rho}](\xi_1^2+s^2)+2[\frac{\rho-1}{\rho}](s^2)}\cdot s^{2\alpha_2+2\beta-3}ds$$
$$+\frac{4\xi_1}{\rho^{\Phi_1+\alpha_2}\Gamma(\Phi_1)}\int_0^{\theta_1}\int_0^r e^{[\frac{\rho-1}{\rho}](\theta_1^2+r^2)+[\frac{1-\rho+\lambda_2}{\rho}](-r^2+s^2)+2[\frac{\rho-1}{\rho}]s^2}\cdot r(\theta_1^2-r^2)^{\Phi_1-1}\cdot s^{2\alpha_2+2\beta-3}dsdr\Bigg],$$

$$P = \frac{2}{\rho^{\gamma_2}\Gamma(\gamma_2)}\Bigg[\eta_1\int_0^{\xi_1} e^{[1-\frac{1}{\rho}]\xi_1^2+\frac{\lambda 1}{\rho}[s^2-\xi_1^2]}\cdot s^{2\gamma_2-1}ds$$
$$+\frac{2\zeta_1}{\rho^{\Phi_1}\Gamma(\Phi_1)}\int_0^{\theta_1}\int_0^r e^{[\frac{\rho-1}{\rho}](\theta_1^2+r^2)+(1-\frac{1}{\rho})r^2+\frac{\lambda_1}{\rho}(s^2-r^2)}\cdot r(\theta_1^2-r^2)^{\Phi_1-1}\cdot s^{2\gamma_2-1}\Bigg]dsdr,$$

$$Q = \frac{2\Gamma(\beta)}{\rho^{\alpha_2}\Gamma(\alpha_2+\beta-1)}\int_0^1 e^{[\frac{1-\rho+\lambda_2}{\rho}](-s^2)+2[\frac{\rho-1}{\rho}]s^2}\cdot s^{2\alpha_2+2\beta-3}ds,$$

$$R = \frac{2}{\rho^{\gamma_2}\Gamma(\gamma_2)}\Bigg[\int_0^1 e^{[1-\frac{1}{\rho}]+\frac{\lambda_1}{\rho}(s^2)}\cdot s^{2\gamma_2-1}ds$$
$$+\frac{2\Theta_1}{\rho^{\upsilon_1}\Gamma(\upsilon_1)}\int_0^{\vartheta_1}\int_0^r e^{[\frac{\rho-1}{\rho}](\vartheta_1^2-r^2)+[1-\frac{1}{\rho}]r^2+\frac{\lambda_1}{\rho}(s^2-r^2)}\cdot r(\vartheta_1^2-r^2)^{\upsilon_1-1}\cdot s^{2\gamma_2-1}dsdr\Bigg],$$

$$S = \frac{\Gamma(\beta)}{\Gamma(\alpha_1+\beta-1)}\Bigg[\int_0^{\varrho_1} e^{[\frac{1-\rho+\lambda_1}{\rho}](-\varrho_1^2+s^2)+[\frac{\rho-1}{\rho}]s^2}\cdot s^{2\alpha_1+2\beta-3}ds$$
$$+\frac{2\Theta_1}{\rho^{\upsilon_1+\alpha_2-1}\Gamma(\upsilon_1)}\int_0^{\vartheta_1}\int_0^r e^{[\frac{\rho-1}{\rho}](\vartheta_1^2-r^2)+[\frac{1-\rho+\lambda_2}{\rho}](-r^2-s^2)+2[\frac{\rho-1}{\rho}]s^2}\cdot r(\vartheta_1^2-r^2)^{\upsilon_1-1}\cdot s^{2\alpha_2+2\beta-3}dsdr\Bigg],$$

$$T = \frac{2}{\rho^{\gamma_1}\Gamma(\gamma_1)}\int_0^{\varrho_1} e^{[1-\frac{1}{\rho}]\varrho_1^2+\frac{\lambda_1}{\rho}s^2}\cdot s^{2\gamma_1-1}ds.$$

For finding the analytic solutions, we use two constants c_1 *and* c_3 *from Table 1 and substitute them in Eqs* (4.6) *and* (4.7)*, respectively.*

Table 1. Table of two constants c_1 and c_3 with varying the values of ρ.

No.	ρ	c_1	c_3
1	0.1	−0.2753	−0.9029
2	0.2	−0.3394	−1.0376
3	0.3	−0.7134	−1.1597
4	0.4	−1.0612	−1.3129
5	0.5	−1.4429	−1.4882
6	0.6	−1.8446	− 1.6497
7	0.7	−2.2518	−1.7895
8	0.8	−2.6547	−1.9073
9	0.9	−3.0472	−2.0051

Next, using the Matlab program, we can find the approximate analytical solutions of $k_1(t)$ *and* $k_2(t)$ *with different values of* ρ *as* $0.1, 0.2, 0.3, \cdots, 0.9$. *Two graphs of* $k_1(t)$ *and* $k_2(t)$ *can be drawn.*

From Figure 1, if the value ρ *is increasing then the value of* $k_1(t)$ *is decreasing for each point* $t \in [0, 1]$. *From the Figure 2, we see that if the value of* ρ *increases, then the value of* $k_2(t)$ *decreases for each* $t \in [0, 1]$. *The lower and upper bounds for the above two curves correspond to* $\rho = 0.1$ *and* $\rho = 0.9$*, respectively, when the value of t increases.*

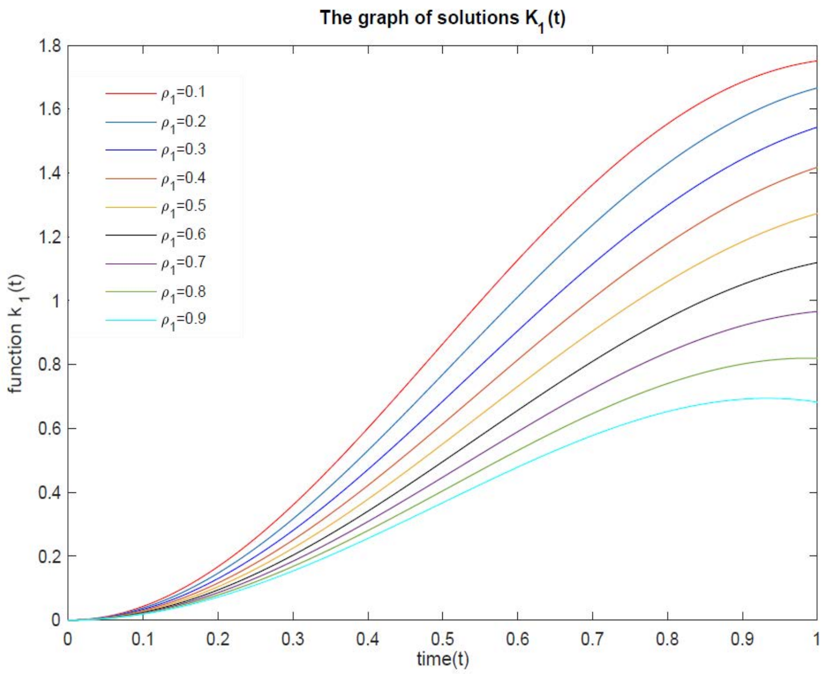

Figure 1. The graph of solutions $k_1(t)$ with different values of ρ.

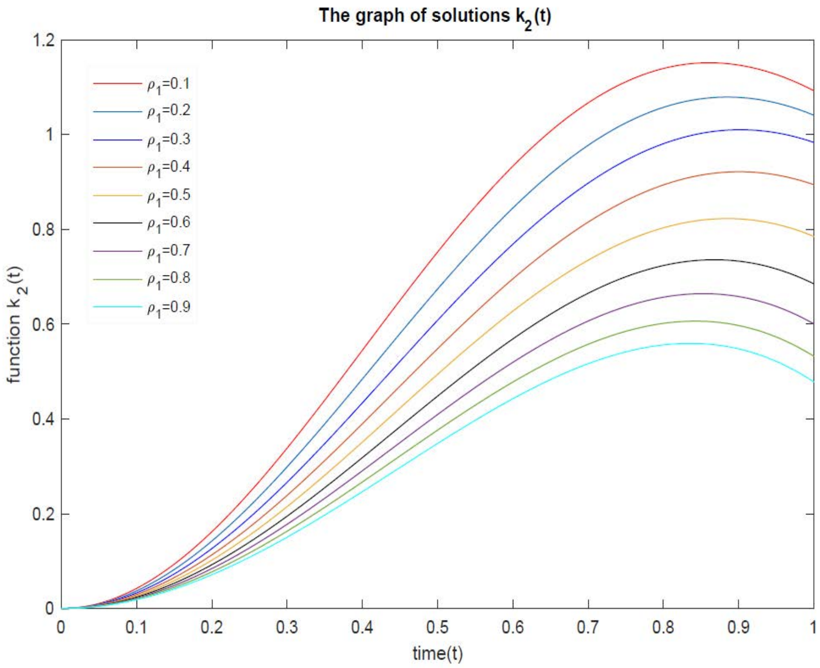

Figure 2. The figraph of solutions $k_2(t)$ with varying values of ρ.

5. Conclusions

In this paper, we investigated a coupled system of ψ-Hilfer fractional proportional differential equations supplemented with nonlocal integro-multipoint boundary conditions. We rely on standard fixed point theorems, Banach, Krasnosel'skii and Leray-Schauder alternative to establish the desired existence and uniqueness results. The obtained theoretical results are well illustrated by numerical examples. Our results are new and contribute significialy to the existing results in the literature concerning ψ-Hilfer fractional proportional nonlocal integro-multi-point coupled systems.

Our results are novel and contribute to the existing literature on nonlocal systems of nonlinear ψ-Hilfer generalized fractional proportional differential equations. Note that the results presented in this paper are wider in scope and produced a variety of new results as special cases. For instance, fixing the parameters in the nonlocal integro-multi-point ψ Hilfer generalized proportional fractional system in (1.4), we obtained some new results as special cases associated with the following:

- Nonlocal ψ-Hilfer generalized proportional fractional systems of order in $(1, 2]$ if $\zeta_j = 0,\ \Theta_l = 0,$ $j = 1, 2, \cdots, m,\ l = 1, 2, \cdots, q.$
- Integro-multi-point nonlocal ψ-Hilfer generalized proportional fractional systems of order in $(1, 2]$ if $\eta_i = 0,\ \aleph_k = 0,\ i = 1, 2, \cdots, n,\ k = 1, 2, \cdots, r.$
- Nonlocal Integro-multi-point Hilfer generalized proportional fractional systems of order in $(1, 2]$ if $\psi(t) = t$.
- Integro-multi-point nonlocal Hilfer generalized fractional systems of order in $(1, 2]$ if $\rho = 1$.

Furthermore, additional new results can be recorded as special cases for different combinations of the parameters $\zeta_j, \Theta_l,\ j = 1, 2, \cdots, m,\ \ l = 1, 2, \cdots, q,\ \ \eta_i = 0, \aleph_k = 0,\ \ i = 1, 2, \cdots, n,\ \ k = 1, 2, \cdots, r$

involved in the system (1.4). For example, by taking all values where $\eta_j = 0,\ j = 1, 2, \cdots, m$, we obtain the results for a coupled system of nonlinear ψ-Hilfer generalized proportional fractional differential equations supplemented by the following nonlocal boundary conditions:

$$\begin{aligned} k_1(\mathtt{w}_1) &= 0, \qquad k_1(\mathtt{w}_2) = \sum_{j=1}^{m} \zeta_j \, {}^{p}I^{\Phi_j,\rho,\psi} k_2(\theta_j), \\ k_2(\mathtt{w}_1) &= 0, \qquad k_2(\mathtt{w}_2) = \sum_{k=1}^{r} \aleph_k \, k_1(\varrho_k) + \sum_{l=1}^{q} \Theta_l \, {}^{p}I^{\upsilon_l,\rho,\psi} k_1(\vartheta_l). \end{aligned}$$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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