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Research article

Hankel inequalities for bounded turning functions in the domain of cosine Hyperbolic function

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Abstract: In the present article, we define and investigate a new subfamily of holomorphic functions connected with the cosine hyperbolic function with bounded turning. Further some interesting results like sharp coefficients bounds, sharp Fekete-Szegö estimate, sharp 2^{nd} Hankel determinant and non-sharp 3^{rd} order Hankel determinant. Moreover, the same estimates have been investigated for 2-fold, 3-fold symmetric functions, the first four initial sharp bounds of logarithmic coefficient and sharp second Hankel determinant of logarithmic coefficients fort his defined function family.

Keywords: holomorphic functions; Hankel determinant; Subordination; Hyperbolic function
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1. Introduction

The class of all analytic functions $u(\varepsilon)$ defined in the open unit disk

$$\mathbf{U} = \{\varepsilon : \varepsilon \in \mathbb{C} \quad \text{and} \quad |\varepsilon| < 1\},$$

is denoted by $\mathcal{A}$ and normalized also by the conditions

$$u(0) = 0 \quad \text{and} \quad u'(0) = 1.$$

Thus, the Taylor series expansion of each $u(\varepsilon) \in \mathcal{A}$ is as follows:

$$u(\varepsilon) = \varepsilon + \sum_{n=2}^{\infty} d_n \varepsilon^n \quad \varepsilon \in \mathbf{U}. \tag{1.1}$$

Furthermore, let $\mathcal{S}$ denotes a subfamily of $\mathcal{A}$, which are univalent in $\mathbf{U}$. For two functions $h_1, h_2 \in \mathcal{A}$, we say that the function h_1 is subordinate to the function h_2 (written as $h_1 \prec h_2$) if there exists an holomorphic function w with the property $|w(\varepsilon)| \leq |\varepsilon|$ and $w(0) = 0$ such that $h_1(\varepsilon) = h_2(w(\varepsilon))$ for $\varepsilon \in \mathbf{U}$. Moreover, if $h_2 \in \mathcal{S}$, then the above conditions can be written as:

$$h_1 \prec h_2 \Leftrightarrow h_1(0) = h_2(0) \text{ and } h_1(\mathbf{U}) \subset h_2(\mathbf{U}).$$

The family $\mathcal{S}^*(\Psi)$, given by

$$\mathcal{S}^*(\Psi) = \left\{ u \in \mathcal{A} : \frac{\varepsilon u'(\varepsilon)}{u(\varepsilon)} \prec \Psi(\varepsilon) \right\}, \tag{1.2}$$

is introduced by Ma and Minda [1] in 1992, where Ψ is an univalent function in $\mathbf{U}$ having the properties

$$\Psi(0) = 1 \quad \text{and} \quad \Re(\Psi) > 0.$$

Many useful and intrusting properties of these classes have been obtained by them. If specifically, we take $\Psi(\varepsilon) = (1+\varepsilon)/(1-\varepsilon)$, then we have the family $\mathcal{S}^*(\Psi)$ of starlike functions. For different choice of Ψ involved in the right hand side of (1.2), one can get a number of known subclasses of starlike functions. Some of them are listed as follows:

(1) If we choose

$$\Psi(\varepsilon) = 1 + \sinh^{-1}(\varepsilon),$$

then we get the family given by

$$\mathcal{S}^*_{pet} = \mathcal{S}^*\left(1 + \sinh^{-1}(\varepsilon)\right).$$

The function $\Psi(\varepsilon)$ maps open unit disc onto the image domain which is bounded by petal shape and was established by Kumar et al. [2].

(2) If we take

$$\Psi(\varepsilon) = \frac{2}{1 + e^{-\varepsilon}},$$

then we obtain the follow family

$$\mathcal{S}^*_{sig} = \mathcal{S}^*\left(\frac{2}{1 + e^{-\varepsilon}}\right),$$

this family starlike functions based on modified sigmoid functions was established and investigated by Geol et al. [3].

(3) If we take

$$\Psi(\varepsilon) = \cos\varepsilon$$

then we obtain the follow family

$$\mathcal{S}^*_{\cos} = \mathcal{S}^*(\cos\varepsilon),$$

this family was established and investigated by Tang et al. [4].

(4) If we pick

$$\Psi(\varepsilon) = 1 + \sin \varepsilon$$

then we obtain the follow family

$$\mathcal{S}^*_{\sin} = \mathcal{S}^* (1 + \sin \varepsilon),$$

the function $\Psi(\varepsilon)$ has image under $\mathbf{U}$ is eight shaped and was established and studied by Cho et al. [5] .

(5) If we take

$$\Psi(\varepsilon) = 1 + \varepsilon - \frac{1}{3}\varepsilon^3,$$

then we obtain the follow family (see [6])

$$\mathcal{S}^*_{nep} = \mathcal{S}^* \left(1 + \varepsilon - \frac{1}{3}\varepsilon^3\right),$$

(6) If we take $\mathcal{S}^*(\varphi)$ with

$$\Psi(\varepsilon) = \sqrt{1 + \varepsilon},$$

then the functions family lead to the family

$$\mathcal{S}^*_{\mathcal{L}} = \mathcal{S}^* \left(\sqrt{1 + \varepsilon}\right),$$

which is described as the functions of starlike functions, bounded by lemniscate of Bernoulli (see [7]).

(7) Moreover, if we take

$$\Psi(\varepsilon) = 1 + \frac{4}{3}\varepsilon + \frac{2}{3}\varepsilon^2,$$

then we obtain the follow family

$$\mathcal{S}^*_{car} = \mathcal{S}^* \left(1 + \frac{4}{3}\varepsilon + \frac{2}{3}\varepsilon^2\right),$$

which was studied by Sharma et al. [8].

(8) Furthermore if we pick $\Psi(\varepsilon) = e^{\varepsilon}$ we get the family $\mathcal{S}^*_{\exp} = \mathcal{S}^* (e^{\varepsilon})$, which was introduced and studied by Mendiratta et al. [9]. On the other side, if we take $\Psi(\varepsilon) = \varepsilon + \sqrt{1 + \varepsilon^2}$, we get the family $\mathcal{S}^*_{\mathbf{l}} = \mathcal{S}^* \left(\varepsilon + \sqrt{1 + \varepsilon^2}\right)$, which maps $\mathbf{U}$ to crescent shaped region and was introduced by Raina and Sokól [10].

Beside these, numerous subfamilies of the family of starlike functions were introduced in different domains (see [11–13]).

The Hankel determinant $H_{q,n}(u)$ for function $u \in \mathcal{S}$ of the form (1.1),was given firstly by Pommerenke [14, 15] as follows:

$$H_{q,n}(u) = \begin{vmatrix} d_n & d_{n+1} & \dots & d_{n+q-1} \\ d_{n+1} & d_{n+2} & \dots & d_{n+q} \\ \vdots & \vdots & \dots & \vdots \\ d_{n+q-1} & d_{n+q} & \dots & d_{n+2q-2} \end{vmatrix} \quad q, n \in \mathbb{N} = \{1, 2, 3, \cdots\}. \tag{1.3}$$

For particular values, for example $q = 2$ and $n = 1$, we get the first order Hankel determinant is

$$\begin{aligned}\left|H_{2,1}(u)\right| &= \begin{vmatrix} d_1 & d_2 \\ d_2 & d_3 \end{vmatrix} \\ &= \left|d_3 - d_2^2\right|, \text{ where } d_1 = 1.\end{aligned}$$

And for $q = 2$ and $n = 2$, in (1.3) we get the second order Hankel determinant

$$\begin{aligned}H_{2,2}(u) &= \begin{vmatrix} d_2 & d_3 \\ d_3 & d_4 \end{vmatrix} \\ &= d_2 d_4 - d_3^2.\end{aligned}$$

For the third order Hankel determinant we take $q = 3$ and $n = 1$, and get the following

$$\left|H_{3,1}(u)\right| = \begin{vmatrix} 1 & d_2 & d_3 \\ d_2 & d_3 & d_4 \\ d_3 & d_4 & d_5 \end{vmatrix}.$$

Note that $H_{2,1}(u) = d_3 - d_2^2$, is the particular case of Fekete-Szegö approximations. The sharp upper bounds for $\left|H_{2,1}(u)\right|$ for different subfamilies of holomorphic functions was investigated by different authors (see [16–18] for details). Moreover, the second Hankel determinant and the sharp upper bound of this has been studied and investigated by several authors from many different directions and perspectives. For few of them are, Hayman [19], the Ohran et al. [20], Noonan and Thomas [21] and Shi et al. [22]. Furthermore, the bounds for the third Hankel determinant for subfamilies of holomorphic functions was first investigated by Babalola [23]. Some recent and interested works on this topic may be found in [24–26] and the reference therein. Recently, Mundalia et al. [27] defined the family of holomorphic starlike functions based on the trigonometric cosine hyperbolic function as follows:

$$\mathcal{S}^*_{Cosh} = \left\{u \in \mathcal{A} : \frac{\varepsilon u'(\varepsilon)}{u(\varepsilon)} \prec \cosh\sqrt{\varepsilon}\right\} \quad (\varepsilon \in \mathbf{U}).$$

For more about this study, we may refer the readers to see [28–30].

By taking motivation from above cited work we introduce the following family of holomorphic function:

$$\mathcal{R}_{Cosh} = \left\{u \in \mathcal{A} : u'(\varepsilon) \prec \cosh\sqrt{\varepsilon}\right\} \quad (\varepsilon \in \mathbf{U}). \tag{1.4}$$

In this paper we evaluate first three initial sharp coefficient bounds, sharp Fekete-Szegö functional, sharp second Hankel determinant non-sharp third Hankel determinant, third Hankel for 2, 3-fold symmetric function and Krushkal inequality for functions belonging to this family. Further, sharp initial four logarithmic coefficients bounds and second Hankel determinant are investigated.

2. A set of Lemmas

We next denote by $\mathcal{P}$ the family of holomorphic functions p which are normalized by $p(0) = 1$, with $Re(p(\varepsilon)) > 0$, $\varepsilon \in \mathbf{U}$ and have the following form:

$$p(\varepsilon) = 1 + \sum_{n=1}^{\infty} c_n \varepsilon^n \quad \varepsilon \in \mathbf{U}. \tag{2.1}$$

Lemma 2.1. *If $p \in \mathcal{P}$ and has the form* (2.1)*. Then, for x and δ with $|x| \leq 1, |\delta| \leq 1$, such that*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{2.2}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\delta. \tag{2.3}$$

We note that (2.2) and (2.3) are taken from [31].

Lemma 2.2. *If $p \in \mathcal{P}$ and has the form (2.1), then we get following estimates*

$$\begin{aligned} |c_k| &\leq 2 \text{ for } k \geq 1, & (2.4)\\ |c_{n+k} - \mu c_n c_k| &< 2 \text{ for } 0 \leq \mu \leq 1, & (2.5)\\ \left| c_2 - \frac{c_1^2}{2} \right| &\leq 2 - \frac{\left|c_1^2\right|}{2}, & (2.6) \end{aligned}$$

and for complex number η, we have

$$\left| c_2 - \eta c_1^2 \right| < 2 \max\{1, |2\eta - 1|\}. \tag{2.7}$$

For the inequalities (2.4)–(2.6) see [16] and (2.7) is given in [32].

Lemma 2.3. *[33] If $p \in \mathcal{P}$ and has the form* (2.1)*, then*

$$|\Lambda_1 c_1^3 - \Lambda_2 c_1 c_2 + \Lambda_3 c_3| \leq 2|\Lambda_1| + 2|\Lambda_2 - 2\Lambda_1| + 2|\Lambda_1 - \Lambda_2 + \Lambda_3|, \tag{2.8}$$

where Λ_1, Λ_2 and Λ_3 are real numbers.

Lemma 2.4. *[34] Let α, β, t and s satisfy the conditions $0 < \alpha < 1,\ 0 < s < 1$ and*

$$8s(1-s)\left[(\alpha\beta - 2t)^2 + (\alpha(s+\alpha) - \beta)^2\right] + \alpha(1-\alpha)(\beta - 2s\alpha)^2 \leq 4\alpha^2(1-\alpha)^2 s(1-s).$$

If $h \in \mathcal{P}$ and of the form (2.1)*, then*

$$\left| tc_1^4 + sc_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4 \right| \leq 2.$$

3. Main results

Theorem 3.1. *If $u(\varepsilon) \in \mathcal{R}_{Cosh}$ and it has the form given in (1.1), then*

$$\begin{aligned} |d_2| &\leq \frac{1}{4}, & (3.1)\\ |d_3| &\leq \frac{1}{6}, & (3.2)\\ |d_4| &\leq \frac{1}{8}, & (3.3)\\ |d_5| &\leq \frac{1}{10}. & (3.4) \end{aligned}$$

Equalities in these inequalities are obtained for functions defined as follow:

$$u_1(\varepsilon) = \int_0^{\varepsilon} \cosh \sqrt{t} dt = \varepsilon + \frac{1}{4}\varepsilon^2 + \cdots, \quad (3.5)$$

$$u_2(\varepsilon) = \int_0^{\varepsilon} \cosh \left(t^2\right) dt = \varepsilon + \frac{1}{6}\varepsilon^3 + \cdots, \quad (3.6)$$

$$u_3(\varepsilon) = \int_0^{\varepsilon} \cosh \sqrt{t^3} dt = \varepsilon + \frac{1}{8}\varepsilon^4 + \cdots, \quad (3.7)$$

$$u_4(\varepsilon) = \int_0^{\varepsilon} \cosh \sqrt{t^4} dt = \varepsilon + \frac{1}{10}\varepsilon^4 + \cdots, \quad (3.8)$$

respectively.

Proof. Let $u(\varepsilon) \in \mathcal{R}_{Cosh}$ then the function $w(\varepsilon)$ with conditions that $w(0) = 0$ and $|w(\varepsilon)| < 1$, such that consider

$$u'(\varepsilon) = \cosh \sqrt{w(\varepsilon)}. \quad (3.9)$$

Let $p \in \mathcal{P}$, then above (3.9) , can be written in the form of Schwarz function as:

$$p(\varepsilon) = \frac{1 + w(\varepsilon)}{1 - w(\varepsilon)} = 1 + c_1\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3 + \cdots. \quad (3.10)$$

Or

$$\begin{aligned} w(\varepsilon) &= \frac{p(\varepsilon) - 1}{p(\varepsilon) + 1} = \frac{c_1\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3 + \cdots}{2 + c_1\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3 + \cdots} \\ &= \frac{1}{2}c_1\varepsilon + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)\varepsilon^2 + \left(\frac{1}{8}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3\right)\varepsilon^3 + \cdots. \end{aligned}$$

Now from (3.9) , we have

$$u'(\varepsilon) = 1 + 2d_2\varepsilon + 3d_3\varepsilon^2 + 4d_4\varepsilon^3 + 5d_5\varepsilon^4 + \cdots. \quad (3.11)$$

And

$$\begin{aligned} \cosh \sqrt{w(\varepsilon)} &= 1 + \frac{1}{4}\varepsilon c_1 + \left(\frac{1}{4}c_2 - \frac{11}{96}c_1^2\right)\varepsilon^2 + \left(\frac{301}{5760}c_1^3 - \frac{11}{48}c_2c_1 + \frac{1}{4}c_3\right)\varepsilon^3 \\ &\quad + \left(-\frac{91}{3840}c_1^4 + \frac{301}{1920}c_1^2c_2 - \frac{11}{48}c_3c_1 - \frac{11}{96}c_2^2 + \frac{1}{4}c_4\right)\varepsilon^4 + \cdots. \end{aligned} \quad (3.12)$$

Comparing (3.11) and (3.12) , we get

$$d_2 = \frac{1}{8}c_1, \quad (3.13)$$

$$d_3 = \frac{1}{12}\left(c_2 - \frac{11}{24}c_1^2\right), \quad (3.14)$$

$$d_4 = \frac{301}{23\,040}c_1^3 - \frac{11}{192}c_2c_1 + \frac{1}{16}c_3, \quad (3.15)$$

$$d_5 = \frac{-1}{20}\left(\frac{91}{960}c_1^4 - \frac{301}{480}c_1^2c_2 + \frac{11}{12}c_3c_1 + \frac{11}{24}c_2^2 - c_4\right). \tag{3.16}$$

Applying (2.4) , to (3.13) , we get

$$|d_2| \leq \frac{1}{4}.$$

From (3.14) , using (2.5) with $n = k = 1$, we have

$$|d_3| \leq \frac{1}{6}.$$

Applying Lemma 2.3 , to Eq (3.15) , we get

$$|d_4| \leq \frac{1}{8}.$$

From Lemma 2.4, the Eq (3.16) , where $t = \frac{91}{960}$, $s = \frac{11}{24}, \beta = \frac{301}{720}$, $\alpha = \frac{11}{24}$, then

$$\begin{aligned} & 8s(1-s)\left[(\alpha\beta - 2t)^2 + (\alpha(s+\alpha) - \beta)^2\right] + \alpha(1-\alpha)(\beta - 2s\alpha)^2 \\ = & \frac{383\,669}{21\,499\,084\,800} \simeq 0.0000178, \end{aligned}$$

and

$$4\alpha^2(1-\alpha)^2 s(1-s) = \frac{2924\,207}{47\,775\,744} \simeq 0.0612069,$$

satisfies the condition Lemma 2.4, so we get

$$|d_5| \leq \frac{1}{10}.$$

Thus we obtain the desired result. □

Theorem 3.2. *If $u(\varepsilon) \in \mathcal{R}_{Cosh}$ and it has the form given in (1.1), then*

$$\left|d_3 - \lambda d_2^2\right| \leq \frac{1}{6}\max\left\{1, \frac{|9\lambda + 2|}{24}\right\}. \tag{3.17}$$

Equalities of this inequalities is obtained for functions u_2 defined in (3.6).

Proof. From (3.13) and (3.14) , we get

$$\left|d_3 - \lambda d_2^2\right| = \frac{1}{12}\left|c_2 - \frac{22 - 9\lambda}{48}c_1^2\right|.$$

Applying (2.7) , to above we get the required results. □

Corollary 3.3. *If $u(\varepsilon) \in \mathcal{R}_{Cosh}$ and it has the form given in (1.1), then*

$$\left|d_3 - d_2^2\right| \leq \frac{1}{6}. \tag{3.18}$$

Equalities of this inequalities is obtained for function u_2 defined in (3.6) .

Theorem 3.4. *If $u(\varepsilon) \in \mathcal{R}_{Cosh}$ and it has the form given in (1.1), then*

$$|d_2 d_3 - d_4| \le \frac{1}{8}. \tag{3.19}$$

Equalities of this inequalities is obtained for function u_3 defined in (3.7) .

Proof. From (3.13)–(3.15) , we get

$$|d_2 d_3 - d_4| = \left| \frac{137}{7680} c_1^3 - \frac{13}{192} c_2 c_1 + \frac{1}{16} c_3 \right|.$$

Applications of Lemma 2.3, lead us to required results. □

Theorem 3.5. *If $u(\varepsilon) \in \mathcal{R}_{Cosh}$ and it has the form given in (1.1), then*

$$\left| d_2 d_4 - d_3^2 \right| \le \frac{1}{9}. \tag{3.20}$$

Equalities of this inequalities is obtained for function u_2 defined in (3.6) .

Proof. From (3.13)–(3.15) , we get

$$d_2 d_4 - d_3^2 = \frac{289}{1658\,880} c_1^4 - \frac{11}{13\,824} c_1^2 c_2 + \frac{1}{128} c_3 c_1 - \frac{1}{144} c_2^2.$$

Applying (2.2) and (2.3) to write c_2 and c_3 in term of $c_1 = c \in [0, 2]$, we get

$$\begin{aligned} d_2 d_4 - d_3^2 &= -\frac{11}{1658\,880} c^4 - \frac{1}{576}(4 - c^2)^2 x^2 - \frac{1}{512} c^2 (4 - c^2) x^2 \\ &\quad + \frac{1}{27\,648} c^2 (4 - c^2) x + \frac{1}{256} c (4 - c^2)(1 - |x|^2)\delta. \end{aligned}$$

By implementing triangle inequality along with $|\delta| \le 1$ and $|x| = k \le 1$, we get

$$\begin{aligned} \left| d_2 d_4 - d_3^2 \right| &\le \frac{11}{1658\,880} c^4 + \frac{1}{576}(4 - c^2)^2 k^2 + \frac{1}{512} c^2 (4 - c^2) k^2 \\ &\quad + \frac{1}{27\,648} c^2 (4 - c^2) k + \frac{1}{256} c (4 - c^2)(1 - k^2) = \Upsilon(c, k) \text{ say.} \end{aligned}$$

Now differentiating partially with respect to k, we get

$$\frac{\partial \Upsilon(c, y)}{\partial y} = \frac{1}{288} \left(4 - c^2\right)^2 k + \frac{1}{258} c^2 \left(4 - c^2\right) k - \frac{1}{128} c \left(4 - c^2\right) k.$$

Clearly, $\frac{\partial \Upsilon(c,k)}{\partial y} > 0$ increasing function so maximum at $k = 1$, so that

$$\begin{aligned} \Upsilon(c, k) &\le \Upsilon(c, 1) = \frac{11}{1658\,880} c^4 + \frac{1}{576}(4 - c^2)^2 \\ &\quad + \frac{1}{512} c^2 (4 - c^2) + \frac{1}{27\,648} c^2 (4 - c^2) \\ &= -\frac{409}{1658\,880} c^4 - \frac{41}{6912} c^2 + \frac{1}{36}. \end{aligned}$$

Now taking derivative with reference to c, we get

$$\Upsilon'(c,1) = -\frac{409}{414720}c^3 - \frac{41}{2304}c.$$

Obviously $\Upsilon'(c,1) \leq 0$, is decreasing function, so maximum value attained at $c = 2$, that is

$$\left|d_2d_4 - d_3^2\right| \leq \frac{1}{36}.$$

□

Theorem 3.6. *If $u(\varepsilon) \in \mathcal{R}_{Cosh}$ and it has the form given in (1.1), then*

$$\left|H_{3,1}(u)\right| \leq \frac{319}{8640} \simeq 0.0369.$$

Proof. Since

$$\left|H_{3,1}(u)\right| \leq |d_3|\left|d_2d_4 - d_3^2\right| + |d_4|\,|d_2d_3 - d_4| + |d_5|\left|d_3 - d_2^2\right|.$$

Putting values of (3.2)–(3.4) and (3.18)–(3.20), we get the required result. □

4. Bounds of $\left|H_{3,1}(u)\right|$ for two and three fold symmetric functions

Let us consider that $m \in N = 1, 2, \ldots$. The rotation of domain Ω through origin can be get by an angle $\frac{2\pi}{m}$ and in this case a domain Ω is called m-fold symmetric. An holomorphic function λ is m-fold symmetric in $\mathbf{U}$, if

$$u\left(e^{\frac{2\pi}{m}}\varepsilon\right) = e^{\frac{2\pi}{m}}\ u(\omega),\ \varepsilon \in \mathbf{U}.$$

The family of all m-fold symmetric functions belong to well-known family $\mathcal{S}$, and denoted by $\mathcal{S}^m$ having the following Taylor series form:

$$u(\varepsilon) = \varepsilon + \sum_{n=1}^{\infty} d_{mn+1}\varepsilon^{mn+1} \quad \varepsilon \in \mathbf{U}. \tag{4.1}$$

The holomorphic functions of the form (4.1) is in the family $\mathcal{R}_{Cosh}^m$, if and only if

$$u'(\varepsilon) = \cosh\sqrt{\frac{p(\varepsilon)-1}{p(\varepsilon)+1}},\ \ \varepsilon \in \mathbf{U}. \tag{4.2}$$

Where $p(\varepsilon)$ belong to the family $\mathcal{P}^{(m)}$ is defined by:

$$\mathcal{P}^{(m)} = \left\{p \in \mathcal{P}:\ p(\varepsilon) = 1 + \sum_{n=1}^{\infty} c_{mn}\varepsilon^{mn} \quad \varepsilon \in \mathbf{U}.\ \right\}. \tag{4.3}$$

Theorem 4.1. *If $u(\varepsilon) \in \mathcal{R}_{Cosh}^2$ and it has the form given in (4.1), then*

$$\left|H_{3,1}(u)\right| \leq \frac{1}{60}. \tag{4.4}$$

Proof. Let $u(\varepsilon) \in \mathcal{R}^2_{Cosh}$. Then, there exists a function $p \in \mathcal{P}^{(2)}$, using the series form (4.1) and (4.3), when $m = 2$ in the above relation (4.2), we obtain

$$u'(\varepsilon) = 1 + 3d_3\varepsilon^2 + 5d_5\varepsilon^4 + \cdots. \tag{4.5}$$

Consider

$$\cosh\sqrt{\frac{p(\varepsilon)-1}{p(\varepsilon)+1}} = 1 + \frac{1}{4}c_2\varepsilon^2 + \left(\frac{1}{4}c_4 - \frac{11}{96}c_2^2\right)\varepsilon^4 + \cdots. \tag{4.6}$$

Comparing (4.5) and (4.6) we obtained

$$\begin{aligned} d_3 &= \frac{c_2}{12}, \\ d_5 &= \frac{1}{20}c_4 - \frac{11}{480}c_2^2. \end{aligned}$$

Now

$$\begin{aligned} H_{3,1}(u) &= d_3d_5 - d_3^3 \\ &= \frac{1}{240}c_2c_4 - \frac{43}{17\,280}c_2^3 \\ &= \frac{1}{240}c_2\left(c_4 - \frac{43}{72}c_2^2\right). \end{aligned}$$

Applying the trigonometric inequality to (2.4) and (2.5), we get

$$\left|H_{3,1}(u)\right| \leq \frac{1}{60}.$$

Hence, the proof is complete. □

Theorem 4.2. *If $u(\varepsilon) \in \mathcal{R}_{Cosh}$ and it has the form given in (4.1), then*

$$\left|H_{3,1}(u)\right| \leq \frac{1}{64}. \tag{4.7}$$

Equalities of this inequalities is obtained for function defined as:

$$u(\varepsilon) = \int_0^{\varepsilon} \cosh\sqrt{t^3}dt = \varepsilon + \frac{1}{6}\varepsilon^4 + \cdots.$$

Proof. As $u \in \mathcal{R}^3_{Cosh}$, therefore there exists a function $p \in \mathcal{P}^{(3)}$, such that

$$u'(\varepsilon) = \cosh\sqrt{\frac{p(\varepsilon)-1}{p(\varepsilon)+1}}.$$

For $m = 3$ and form (4.1) and (4.3), the above condition become as:

$$1 + 4d_4\varepsilon^3 + \cdots = 1 + \frac{c_3}{4}\varepsilon^3 + \cdots. \tag{4.8}$$

Comparing the coefficients of (4.8), we obtained

$$d_4 = \frac{c_3}{16},$$

then

$$H_{3,1}(u) = -d_4^2 = -\frac{c_3^2}{256}.$$

Utilizing (2.4) and triangle inequality, we have

$$\left|H_{3,1}(u)\right| \leq \frac{1}{64}.$$

Thus the complete the proof. □

5. Logarithmic coefficients for the family $\mathcal{R}_{Cosh}$

The logarithmic coefficients of $u \in \mathcal{S}$ denoted by $\gamma_n = \gamma_n(u)$, are defined by with the following series expansion:

$$\log \frac{u(\varepsilon)}{\varepsilon} = 2\sum_{n=1}^{\infty} \gamma_n \varepsilon^n.$$

For function u given by (1.1), the logarithmic coefficients are as follow:

$$\gamma_1 = \frac{1}{2}d_2, \tag{5.1}$$

$$\gamma_2 = \frac{1}{2}\left(d_3 - \frac{1}{2}d_2^2\right), \tag{5.2}$$

$$\gamma_3 = \frac{1}{2}\left(d_4 - d_2 d_3 + \frac{1}{3}d_2^3\right), \tag{5.3}$$

$$\gamma_4 = \frac{1}{2}\left(d_5 - d_2 d_4 - d_2^2 d_3 - \frac{1}{2}d_3^2 - \frac{1}{4}d_2^4\right). \tag{5.4}$$

Theorem 5.1. *If $u(\varepsilon) \in \mathcal{R}_{Cosh}$ and it has the form given in (1.1), then*

$$|\gamma_1| \leq \frac{1}{8},$$
$$|\gamma_2| \leq \frac{1}{12},$$
$$|\gamma_3| \leq \frac{1}{16},$$
$$|\gamma_4| \leq \frac{1}{20}.$$

Equalities in these inequalities are obtained for function

$$u_n(\varepsilon) = \int_0^{\varepsilon} \cosh \sqrt{t^n} dt = \varepsilon + \frac{1}{2n+2}\varepsilon^{n+1} + \cdots, \textit{ for } n = 1, 2, 3, 4. \tag{5.5}$$

Proof. Now from (5.1) to (5.4) and (3.13) to (3.16) , we get

$$\gamma_1 = \frac{1}{16}c_1, \tag{5.6}$$

$$\gamma_2 = \frac{1}{24}c_2 - \frac{53}{2304}c_1^2, \tag{5.7}$$

$$\gamma_3 = \frac{71}{7680}c_1^3 - \frac{13}{384}c_2c_1 + \frac{1}{32}c_3, \tag{5.8}$$

$$\gamma_4 = -\frac{1802\,099}{464\,486\,400}c_1^4 + \frac{14\,861}{691\,200}c_1^2c_2 - \frac{103}{3840}c_3c_1 - \frac{19}{1440}c_2^2 + \frac{1}{40}c_4. \tag{5.9}$$

Applying (2.4) , to (5.6) , we get

$$|\gamma_1| \leq \frac{1}{8}.$$

From (5.7) , using (2.5) , we get

$$|\gamma_2| \leq \frac{1}{12}.$$

Applying Lemma 2.3, to Eq (5.8) , we get

$$|\gamma_3| \leq \frac{1}{16}.$$

Also, using Lemma 2.4 to (5.9) , we get

$$|\gamma_4| \leq \frac{1}{20}.$$

Proof for sharpness: Since

$$\log\frac{u_1(\varepsilon)}{\varepsilon} = 2\sum_{n=2}^{\infty}\gamma(u_1)\varepsilon^n = \frac{1}{4}\varepsilon + \cdots,$$

$$\log\frac{u_2(\varepsilon)}{\varepsilon} = 2\sum_{n=2}^{\infty}\gamma(u_2)\varepsilon^n = \frac{1}{6}\varepsilon^2 + \cdots,$$

$$\log\frac{u_3(\varepsilon)}{\varepsilon} = 2\sum_{n=2}^{\infty}\gamma(u_2)\varepsilon^n = \frac{1}{8}\varepsilon^3 + \cdots,$$

$$\log\frac{u_4(\varepsilon)}{\varepsilon} = 2\sum_{n=2}^{\infty}\gamma(u_2)\varepsilon^n = \frac{1}{10}\varepsilon^4 + \cdots,$$

it follows that these inequalities are obtained for the functions $u_n(\varepsilon)$ for $n = 1, 2, 3, 4$ defined in (5.5). □

Theorem 5.2. *If* $u(\varepsilon) \in \mathcal{R}_{Cosh}$ *and it has the form given in (1.1), then*

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq \frac{1}{36}.$$

Equalities in this inequalities are obtained for function u_2 *in* (5.5) .

Proof. From (5.6)–(5.8) we have

$$\gamma_1\gamma_3-\gamma_2^2=\frac{1291}{26\,542\,080}c_1^4-\frac{11}{55\,296}c_1^2c_2+\frac{1}{512}c_3c_1-\frac{1}{576}c_2^2.$$

Applying (2.2) and (2.3) to write c_2 and c_3 in term of $c_1=c\in[0,2]$, we get

$$\begin{aligned}\gamma_1\gamma_3-\gamma_2^2 &= \frac{91}{26\,542\,080}c_1^4-\frac{1}{2304}\left(4-c_1^2\right)^2x^2-\frac{1}{2048}c_1^2\left(4-c_1^2\right)x^2\\ &\quad+\frac{1}{110\,592}c_1^2\left(4-c_1^2\right)x+\frac{1}{1024}c_1\left(4-c_1^2\right)\left(1-|x|^2\right)\delta.\end{aligned}$$

By applying triangle inequality along with $|\delta|\leq 1$ and $|x|=k\leq 1$, we get

$$\begin{aligned}\left|\gamma_1\gamma_3-\gamma_2^2\right| &\leq \frac{91}{26\,542\,080}c_1^4+\frac{1}{2304}\left(4-c_1^2\right)^2k^2+\frac{1}{2048}c_1^2\left(4-c_1^2\right)k^2\\ &\quad+\frac{1}{110\,592}c_1^2\left(4-c_1^2\right)k+\frac{1}{1024}c_1\left(4-c_1^2\right)\left(1-k^2\right)=\Upsilon(c,k).\end{aligned}$$

If we differentiate the above inequlity partially with respect to k, we have

$$\frac{\partial\Upsilon(c,k)}{\partial k}=\frac{1}{2304}k\,(c-2)^2\left(-c^2+14c+32\right).$$

It is easy to observe that

$$\frac{\partial\Upsilon(c,k)}{\partial k}\geq 0$$

in interval $[0,1]$, so maximum attained at $k=1$, thus

$$\begin{aligned}\Upsilon(c,k) &\leq \Upsilon(c,1)=\frac{7}{12\,288}c^4+\frac{1}{576}\left(4-c^2\right)^2+\frac{1}{512}c^2\left(4-c^2\right)\\ &= \frac{13}{36\,864}c^4-\frac{7}{1152}c^2+\frac{1}{36}.\end{aligned}$$

Taking derivative with reference to c, we get

$$\Upsilon'(c,1)=c\left(\frac{1}{9216}c^2-\frac{1}{576}\right).$$

Obviously $\Upsilon'(c,1)=0$, has three roots namely 0, ± 4 the only root lies in interval $[0,2]$ is 0, so

$$\Upsilon''(c,1)=\frac{1}{3072}c^2-\frac{1}{576}.$$

Thus $\Upsilon''(0,1)\leq 0$, so the function has maximum at $c=0$, that is

$$\left|\gamma_1\gamma_3-\gamma_2^2\right|\leq\frac{1}{36}.$$

□

6. Conclusions

Recently the investigations of the Hankel determinant got attractions of many researchers, due to its applications in many diverse areas of mathematics and other sciences. Here in this paper, we have defined a new subfamily of holomorphic functions connected with the Tan hyperbolic function with bounded boundary rotation. We have then investigated the upper bound of the third Hankel determinant for this newly defined functions family. On the other hand, we have obtained the same bounds for 2-fold, 3-fold symmetric functions,The first four initial sharp bounds of logarithmic coefficient and sharp second Hankel determinant of logarithmic coefficients for this defined function family.

Here, we passing to remark the fact that one can extend the suggested results investigated in this article, for some other subfamilies of holomorphic functions and also the interested can use the D_q derivative operator (see for example [35–38]) and can generalize the work presented here.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of Interest

The authors declare that they have no competing interests

Author's Contributions

All authors jointly worked on the results, and they read and approved the final manuscript

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