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Research article

Stability and bifurcation of a delayed diffusive predator-prey model affected by toxins

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Abstract: In this work, a diffusive predator-prey model with the effects of toxins and delay is considered. Initially, we investigated the presence of solutions and the stability of the system. Then, we examined the local stability of the equilibria and Hopf bifurcation generated by delay, as well as the global stability of the equilibria using a Lyapunov function. In addition, we extract additional results regarding the presence and nonexistence of non-constant steady states in this model by taking into account the influence of diffusion. We show several numerical simulations to validate our theoretical findings.

Keywords: toxins; stability; diffusion; delay; bifurcation
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1. Introduction

From an ecological and financial perspective, toxicants have emerged as a major threat to terrestrial and aquatic environments. With increasing demand, industries are cranking out a flood of toxic chemicals. Toxic chemicals and substances, such as cadmium, arsenic, copper, lead, etc., are often dumped into lakes, rivers and oceans, where they can have a devastating effect on aquatic life [1]. Toxic oil, metals and synthetic organic chemicals are common contaminants of river, lake, and sea water [2]. A significant loss of biodiversity occurs in ecosystems where toxic contaminants are present [3]. Fish, birds and mammals that eat contaminated marine life can be exposed to the toxins themselves. Therefore, many species have become extinct, and many more are on the verge of extinction, due to the unchecked release of toxic substances into the environment. There are numerous species in the ocean that produce toxins, and these toxins, if released into the environment, can have serious consequences for the development of other organisms. For example, the grazing pressure of zooplankton can be greatly reduced by phytoplankton that are naturally toxic. Therefore, research into the effects of toxic substances on ecological communities is becoming increasingly significant from both an environmental and a conservationist point of view.

The mathematical modeling of the impact of toxicants on a population was a newly established area in the early 1980s [4–6]. In order to effectively estimate the qualitative impact of toxic substances on species, mathematical models are a great tool to use. Das et al. [7] investigated a predator-prey fishery model with harvesting and the effects of toxicants which are released by some other species. Chakraborty and Das [8] studied a two-zooplankton one-phytoplankton system in the presence of toxicity. Ang and Safuan [9] discussed an intraguild fishery model in which predators are thought to become infected through their prey, while fish are thought to be infected directly by an anthropogenic toxicant in the environment. Juneja and Agnihotri [10] addressed the issue of two competing fish species, each of which releases a harmful chemical for the other.

It is noted that in biological populations, delay plays an important role. In the last few years, theoretical and mathematical ecologists have paid a lot of attention to research on differential equations with time delays [11–15]. Even so, studying the effects of time delay on the dynamics of a system can be very complicated. For example, it can cause the system to lose its stability and lead to periodic solutions and chaotic behavior. Pal and Mahapatra [16] studied the combined effect of a toxicant and delay on the dynamical behaviors of a delayed two-species competitive system with imprecise biological parameters. Pal et al. [17] took into account two fish species that are in competition with one another, each of which releases a toxin that is poisonous to the other and each of which obeys the law of logistic growth. Meanwhile, in reality, species are spatially heterogeneous, so individuals seek out low population densities where they have a better chance of survival. As a result, the reaction-diffusion predator-prey model with toxic effects has been considered by some researchers. Zhang and Zhao [18] investigated a diffusive predator-prey model with toxins, and their research results show that toxic substances have a great impact on the dynamics of the model. Zhu et al. [19] investigated a delayed diffusive predator-prey model affected by a toxic substance. However, we find that the research results about the delayed diffusive predator-prey model with toxic substances are rare.

Motivated by these pioneer works, we hypothesize that prey produce toxins for predators, and that this process is not instantaneous but rather follows a discrete time lag that can be thought of as the species' maturation period. Toxic substances released by prey into the environment have a half-life of τ, which we introduce here. We consider a diffusive predator-prey model as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + ru(1-\dfrac{u}{K}) - \frac{\alpha uv}{a+u^2}, & (x,t)\in\Omega\times(0,+\infty), \\ \dfrac{\partial v}{\partial t} = d_2\Delta v + \dfrac{\beta uv}{a+u^2} - dv - \gamma u(t-\tau)v^2, & (x,t)\in\Omega\times(0,+\infty), \\ u(x,t) = u_0(x,t), v(x,t) = v_0(x,t), & x\in\overline{\Omega}, t\in[-\tau,0], \\ \dfrac{\partial u(t,x)}{\partial n} = \dfrac{\partial v(t,x)}{\partial n} = 0, t>0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $u(x,t)$, $v(x,t)$ denote the density of the prey and the predator, respectively. r is the birth rate of prey. $\frac{u}{a+u^2}$ is the Holling type-IV function. α is the maximum predator per capita consumption rate of u due to v. β is the conversion of the biomass constant. The parameter d is the death rate of predator. γv^2 is the functional response of the u population to the density of the v population.

To explore the dynamics of system (1.1), we first do the non-dimensionalization described below

$$\bar{t} = rt, \bar{u} = \frac{u}{K}, \bar{v} = \frac{\alpha v}{K^2 r}, \bar{a} = \frac{a}{K^2}, \bar{\beta} = \frac{\beta}{Kr}, \bar{d} = \frac{d}{r}, \bar{\gamma} = \frac{\gamma K^3}{\alpha},$$
$$\bar{d}_1 = \frac{d_1}{r}, \bar{d}_2 = \frac{d_2}{r}, \bar{\tau} = r\tau.$$

Thus, system (1.1) is simplified (by removing the bars) to be

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + u(1-u) - \dfrac{uv}{a+u^2}, & (x,t) \in \Omega \times (0, +\infty), \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + \dfrac{\beta uv}{a+u^2} - dv - \gamma u(t-\tau) v^2, & (x,t) \in \Omega \times (0, +\infty), \\ u(x,t) = u_0(x,t), v(x,t) = v_0(x,t), & x \in \overline{\Omega}, t \in [-\tau, 0], \\ \dfrac{\partial u(t,x)}{\partial n} = \dfrac{\partial v(t,x)}{\partial n} = 0, t > 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

In this study, we will study the dynamics of system (1.2), such as the existence of the solutions, local/global stability of the equilibria, and Hopf bifurcation induced by delay. In addition, we will also discuss the existence and non-existence of non-constant positive solutions of the following elliptic system

$$\begin{cases} -d_1 \Delta u = u(1-u) - \dfrac{uv}{a+u^2}, & x \in \Omega, \\ -d_2 \Delta v = \dfrac{\beta uv}{a+u^2} - dv - \gamma uv^2, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & \mathrm{x} \in \partial\Omega. \end{cases} \tag{1.3}$$

The structure of this paper is as follows. In Section 2, the existence of solutions and the persistence of system (1.2) are studied. In Section 3, the stability of the equilibria, Turing bifurcation, and Hopf bifurcation induced by delay are discussed. In Section 4, the global stability of the equilibria are investigated by using the Lyapunov functional method. In Section 5, the non-existence and existence of the non-constant steady state are studied. In Section 6, theoretical results are verified through numerical simulations. Finally, a brief conclusion is given in Section 7.

2. Basic dynamics

Theorem 2.1. *Assume that* $u_0(x,t) \geq 0$*,* $v_0(x,t) \geq 0$*, and* $u_0(x,t) \not\equiv 0$*,* $v_0(x,t) \not\equiv 0$*. There is a unique solution* $(u(x,t), v(x,t)) > (0,0)$*,* $(t > 0, x \in \bar{\Omega})$ *of system* (1.2)*, and*

$$\limsup_{t \to +\infty} \max_{x \in \bar{\Omega}} u(x,t) \leq 1, \quad \int_\Omega v(x,t) dx \leq \frac{eK(m+r)}{m} |\Omega|. \tag{2.1}$$

Additionally,

$$\|u(\cdot,t)\|_{C(\bar{\Omega})} \leq K_1, \|v(\cdot,t)\|_{C(\bar{\Omega})} \leq K_2, \tag{2.2}$$

where $\|u(x,t)\|_{C(\overline{\Omega})} = \max\limits_{x\in\overline{\Omega}, t\in[-\tau,0]} u(x,t)$, $\|v(x,t)\|_{C(\overline{\Omega})} = \max\limits_{x\in\overline{\Omega}} v(x,t)$, $K_1 = \max\{1, \max\limits_{\bar{\Omega}, t\in[-\tau,0]} u_0(x,t)\}$ *and* K_2 *depends upon* β, a, d, γ, $u_0(x,t)$, $v_0(x,t)$ *and* Ω.

Proof. We consider the following auxiliary system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + u(1-u) - \dfrac{uv}{a+u^2}, \\ \dfrac{\partial v}{\partial t} = d_2\Delta v + v(\dfrac{\beta u}{a} - d), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x). \end{cases} \tag{2.3}$$

Obviously, $(\underline{u}(x,t), \underline{v}(x,t)) = (0,0)$ and $(\bar{u}(x,t), \bar{v}(x,t)) = (\tilde{u}(t), \tilde{v}(t))$ are the lower and upper solutions of system (2.3), respectively, where $(\tilde{u}(t), \tilde{v}(t))$ is the unique solution of

$$\begin{cases} \dfrac{du}{dt} = u(1-u) - \dfrac{uv}{a+u^2}, \\ \dfrac{dv}{dt} = v(\dfrac{\beta u}{a} - d), \\ u(0) = \bar{u}_0, v(0) = \bar{v}_0, \end{cases} \tag{2.4}$$

where $\bar{u}_0 = \max\limits_{\overline{\Omega}} u_0(x)$, $\bar{v}_0 = \max\limits_{\overline{\Omega}} v_0(x)$. As a result, according to Theorem 8.3.3 in [20], we obtain that system (1.2) has a unique globally defined solution which satisfies $0 \le u(x,t) \le \tilde{u}(t), 0 \le v(x,t) \le \tilde{v}(t)$. By the strong maximum principle, we have that $u(x,t), v(x,t) > 0$ $(t > 0, x \in \bar{\Omega})$.

Evidently, from the first equation of system (2.4) we have that $\lim\limits_{t\to+\infty} u(t) \le 1$, which implies $\limsup\limits_{t\to+\infty} \max\limits_{x\in\bar{\Omega}} u(x,t) \le 1$. Therefore, $\|u(\cdot,t)\|_{C(\bar{\Omega})} \le K_1$ for all $t \ge 0$.

For $v(x,t)$, we let $U(t) = \int_\Omega u(x,t)dx$ and $V(t) = \int_\Omega v(x,t)dx$; then,

$$\frac{dU}{dt} = \int_\Omega u_t dx = d_1\int_\Omega \Delta u dx + \int_\Omega [u(1-u) - \frac{uv}{a+u^2}]dx = \int_\Omega [u(1-u) - \frac{uv}{a+u^2}]dx, \tag{2.5}$$

$$\frac{dV}{dt} = \int_\Omega v_t dx = d_2\int_\Omega \Delta v dx + \int_\Omega (v(\frac{\beta u}{a+u^2} - d - \gamma uv))dx = \int_\Omega v(\frac{\beta u}{a+u^2} - d - \gamma u(t-\tau)v)dx. \tag{2.6}$$

Multiplying Eq (2.5) by β, and then addin it to Eq (2.6), we have

$$(\beta U + V)_t = -dV + \beta\int_\Omega (u(1-u)dx - \gamma\int_\Omega u(t-\tau)v^2 dx \le -d(\beta U + V) + (1 + d\beta)U.$$

$\|u(\cdot,t)\|_{C(\bar{\Omega})} \le K_1$, so we have that $U(t) \le K_1|\Omega|$. Thus,

$$(\beta U + V)_t \le -d(\beta U + V) + M_2, t > 0, \tag{2.7}$$

where $M_2 = (1 + d\beta)|\Omega|$. Integrating the inequality (2.7), we obtain

$$\int_\Omega v(x,t)dx = V(t) < \beta U(t) + V(t) \leq (\beta U(0) + V(0))e^{-dt} + \frac{M_2}{d}(1 - e^{-dt}). \tag{2.8}$$

This means that $\|v(\cdot,t)\|_{L^1(\Omega)} \leq \beta\|u_0(\cdot)\|_{L^1(\Omega)} + \|v_0(\cdot)\|_{L^1(\Omega)} + \frac{M_2}{d}$. According to Theorem 3.1 [21], there is an M_3 such that $\|v(\cdot,t)\|_{L^\infty(\Omega)} \leq M_3$. Therefore, there exists a K_2 such that $\|v(\cdot,t)\|_{C(\overline{\Omega})} \leq K_2$. □

Theorem 2.2. *If* $1 - \frac{K_2}{a} > 0$ *and* $\frac{\beta(1-\frac{K_2}{a})}{\gamma K_2(a+K_1)} - \frac{d}{\gamma K_2} > 0$*, then system* (1.2) *has the persistence property.*

Proof. From system (1.2), we obtain

$$\frac{\partial u}{\partial t} = d_1\Delta u + u(1 - u - \frac{v}{a+u^2}) \geq d_1\Delta u + u(1 - u - \frac{K_2}{a}). \tag{2.9}$$

For small enough $\varepsilon > 0$, it holds that $1 - \frac{K_2}{a} - \varepsilon > 0$. Therefore, there is a t_1 such that

$$u(x,t) \geq 1 - \frac{K_2}{a} - \varepsilon := \underline{c}_1, t > t_1. \tag{2.10}$$

The second equation of system (1.2) is then solved using the upper and lower bounds of u, yielding

$$\frac{\partial v}{\partial t} = d_2\Delta v + \frac{\beta uv}{a+u^2} - dv - \gamma u(t-\tau)v^2, \geq d_2\Delta v + v(\frac{\beta \underline{c}_1}{a+K_1^2} - d - \gamma K_2 v), \tag{2.11}$$

for $t > t_1 + \tau$. Then there exists $t_2 > t_1 + \tau$ such that for any $t > t_2$,

$$v(x,t) \geq \frac{\beta \underline{c}_1}{\gamma K_2(a+K_1)} - \frac{d}{\gamma K_2} := \underline{c}_2. \tag{2.12}$$

From (2.10) and (2.12), we can easily obtain that

$$\liminf_{t\to+\infty} \max_{x\in\bar{\Omega}} u(.,t) \geq \underline{c}_1, \qquad \liminf_{t\to+\infty} \max_{x\in\bar{\Omega}} v(.,t) \geq \underline{c}_2. \tag{2.13}$$

They are evidence that the system (1.2) is persistent. Regardless of the diffusion coefficients, this indicates that, from a biological point of view, a predator and its prey will always coexist within the habitable domain at any given time and in any given location. □

3. Local stability and bifurcation analysis

Obviously, model (1.2) has a trivial equilibrium $E_0 = (0,0)$ and a predator-free equilibrium $E_1 = (1,0)$, and the interior equilibrium must simultaneously meet the two non-trivial prey and predator nullcline conditions below:

$$\Phi(u,v) = 1 - u - \frac{v}{a+u^2} = 0, \tag{3.1}$$

$$\Psi(u,v) = \frac{\beta u}{a+u^2} - d - \gamma uv = 0. \tag{3.2}$$

From (3.2), we obtain that $v = -\left(u^2 + a\right)(u - 1)$, and substituting this into (3.1), we have

$$\gamma u^6 + (-\gamma) u^5 + (2a\gamma) u^4 + (-2a\gamma) u^3 + \left(a^2\gamma - d\right) u^2 + \left(\beta - a^2\gamma\right) u - ad = 0. \tag{3.3}$$

Obviously, Eq (3.3) has at least a positive root. Therefore, system (1.2) has at least an interior equilibrium $E^* = (u^*, v^*)$. To illustrate this, isoclines (3.1) and (3.2) are shown in Figure 1. Figure 1(a) shows that isoclines (3.1) and (3.2) intersect uniquely in the interior of the positive quadrant, i.e., system (1.2) has a unique interior equilibrium, and Figure 1(b) shows that isoclines (3.1) and (3.2) intersect two times in the interior of the positive quadrant, i.e., the system has two equilibria.

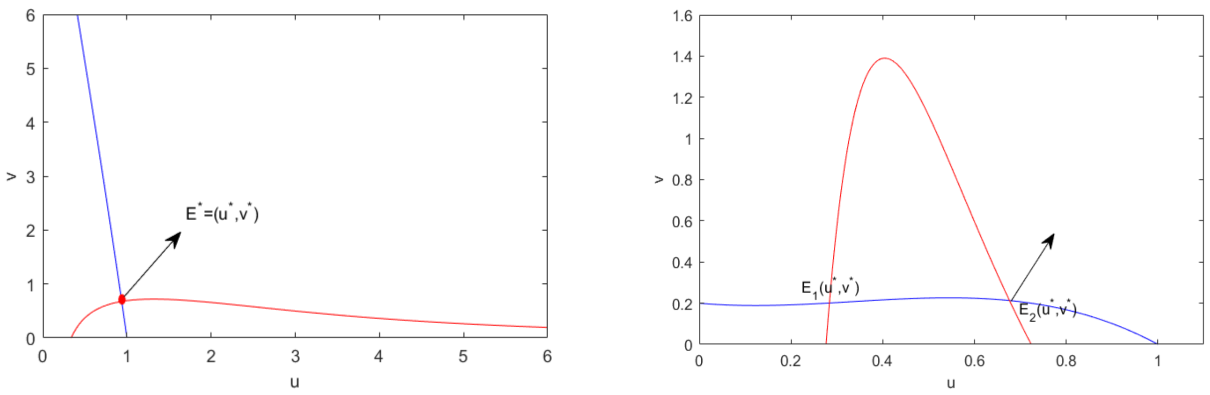


Figure 1. Intersection of isoclines (3.1) and (3.2). The parameters are set as follows: (a) $a = 10, \beta = 0.6, d = 0.02, \gamma = 0.05$; (b) $a = 0.2, \beta = 0.1, d = 0.1, \gamma = 0.02$.

In order to study the stability of the equilibria. First, we define the real-valued Sobolev space

$$X = \left\{(u, v)^T : u, v \in L^2(\Omega)\right\}, < u, v >=< u_1, v_1 >_{L^2} + < u_2, v_2 >_{L^2} .$$

Then $(X, < \cdot, \cdot >)$ is a Hilbert space. In $C([-\tau, 0], X)$, system (1.2) can be thought of as a functional differential equation in abstract form.

Let $U(t) = (u(\cdot, t), v(\cdot, t))^T$. Thus, we linearize system (1.2) around a constant solution $E = (u, v)$; we get

$$\dot{U} = D\Delta U(t) + L(U_t), \tag{3.4}$$

where $D = \mathrm{diag}(d_1, d_2)$,

$$dom(d\Delta) = \left\{(u, v)^T : u, v \in W^{2,2}(\Omega), \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0\right\},$$

and $U_t = \mathbf{col}(u(x, t), v(x, t)) \in C_\tau$ and $L : C([-\tau, 0] \to X$ is given by

$$L(\varphi) = \begin{pmatrix} a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{pmatrix} \varphi(0) + \begin{pmatrix} 0 & 0 \\ -b & 0 \end{pmatrix} \varphi(-1) \tag{3.5}$$

with $\varphi = (\varphi_1, \varphi_2)^T \in C([-\tau, 0]$ and

$$a_{11} = \frac{2u^2v}{(a+u^2)^2} - u, a_{12} = \frac{u}{u^2+a},$$
$$a_{21} = \frac{\beta v(a-u^2)}{(a+u^2)^2}, a_{22} = \gamma uv, b = \gamma v^2.$$

Therefore, the characteristic equation of system (3.4) is

$$\lambda y - D\Delta y - L(e^{\lambda \cdot} y) = 0, y \in dom(\Delta), y \neq 0. \tag{3.6}$$

We know that with the corresponding eigenfunctions $\psi_n(x)$ the problem

$$\begin{cases} -\Delta\psi = \lambda\psi, x \in \Omega, \\ \frac{\partial\psi}{\partial n} = 0, x \in \partial\Omega, \end{cases}$$

has eigenvalues $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \mu_{n+1} \leq \cdots$. Substituting

$$y = \sum_{n=0}^{\infty} \psi_n(x) \begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix}$$

into Eq (3.4), we have

$$\begin{pmatrix} a_{11} - d_1\mu_n & -a_{12} \\ a_{21} - be^{-\lambda\tau} & -a_{22} - d_2\mu_n \end{pmatrix} \begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix} = \lambda \begin{pmatrix} y_{1n} \\ y_{2n} \end{pmatrix}, \quad n \in \{0, 1, 2, \cdots\} := N_0.$$

Hence, Eq (3.6) equals

$$W_n(\lambda, \beta, \tau) = \lambda^2 + A_n\lambda + B_n + Ce^{-\lambda\tau} = 0, \tag{3.7}$$

where

$$\begin{aligned} A_n &= (d_1 + d_2)\mu_n + a_{22} - a_{11}, \\ B_n &= d_1d_2\mu_n^2 + (d_1a_{22} - d_2a_{11})\mu_n + (a_{12}a_{21} - a_{11}a_{22}), \\ C &= -a_{12}b. \end{aligned} \tag{3.8}$$

For the extinct equilibrium $E_0 = (0, 0)$, $J_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -d \end{pmatrix}$. When $n = 0$, the eigenvalue $\lambda = 1 > 0$, therefore, E_0 is a saddle point, which is always unstable.

For the predator free equilibrium $E_1 = (1, 0)$, $J_{(1,0)} = \begin{pmatrix} -1 & -\frac{1}{a+1} \\ 0 & \frac{\beta}{a+1} - d \end{pmatrix}$. Therefore, if $\frac{\beta}{a+1} - d < 0$, then E_1 is stable.

In what follows, we first discuss the stability of the interior equilibrium of system (1.2) with $\tau = 0$. When $\tau = 0$, the characteristic equation (3.7) becomes

$$\lambda^2 + A_n\lambda + B_n + C = 0. \tag{3.9}$$

Theorem 3.1. *Assume that the conditions*

$$a_{22} - a_{11} > 0, \tag{3.10}$$
$$a_{12}a_{21} - a_{11}a_{22} - a_{12}b > 0 \tag{3.11}$$

hold; then, the following results are true:

(i) If $d_1a_{22} - d_2a_{11} > 0$, then the interior equilibrium is locally asymptotically stable.

(ii) If $d_1a_{22} - d_2a_{11} < 0$ and $\Delta_1 < 0$, then the interior equilibrium is locally asymptotically stable, where

$$\Delta_1 = (d_1a_{22} + d_2a_{11})^2 - 4d_1d_2(a_{12}a_{21} - a_{12}b).$$

(iii) If $d_1a_{22} - d_2a_{11} < 0$ and $\Delta_1 > 0$, but there is no $n \in N_0$ such that $\mu_n \in (\mu^-, \mu^+)$, then the interior equilibrium is asymptotically stable, where

$$\mu^{\pm} = \frac{d_2a_{11} - d_1a_{22} \pm \sqrt{\Delta_1}}{2d_1d_2}.$$

Proof. Obviously, if $a_{22} - a_{11} > 0$, then $A_n \le A_0 > 0$ for $n \in N_0$. $a_{12}a_{21} - a_{11}a_{22} - a_{12}b > 0$ holds, so $B_0 + C = hb_3(b_2b_3 - b_1) > 0$.

(i) If $d_1a_{22} - d_2a_{11} > 0$, then $B_n + C \ge B_0 + C$, which implies that all roots of the characteristic equation (3.9) have negative real parts. Therefore, the interior equilibrium is locally asymptotically stable.

(ii) $d_1a_{22} - d_2a_{11} < 0$ and $\Delta_1 < 0$ hold, which implies that the equation

$$h(y) = d_1d_2y^2 + (d_1a_{22} - d_2a_{11})y + (a_{12}a_{21} - a_{11}a_{22}) - a_{12}b > 0$$

for any $y \ge 0$. That is $B_n + C > 0$ for any μ_n. Similar to the discussion in (i), we have that the interior equilibrium is locally asymptotically stable.

(iii) There is no $n \in N_0$ such that $\mu_n \in (\mu^-, \mu^+)$. So, $B_n + C > 0$ for any μ_n. Consequently, we have the results.

□

Theorem 3.2. *Suppose that the conditions* (3.10) *and* (3.11) *hold, and assume further that*

$$\frac{d_2}{d_1} > \frac{2a_2b_1 - a_1b_2 + \sqrt{(2a_2b_1 - a_1b_2)^2 - a_1^2b_2^2}}{a_1^2} \tag{3.12}$$

hold, then the Turing bifurcation occurs.

Proof. The condition (3.10) is satisfied, so E^* of the ODE model corresponding to model (1.2) is locally asymptotically stable.

The condition (3.10) holds, so $A_n < 0$. Therefore, when E^* of model (1.2) is unstable i.e., Eq (3.9) has at least one positive real root i.e., $B_n + C = 0$ has two true roots, one positive and one negative. Note that

$$B_n + C = d_1d_2\mu_n^2 + (d_1a_{22} - d_2a_{11})\mu_n + (a_{12}a_{21} - a_{11}a_{22}) - a_{12}b. \tag{3.13}$$

It is easy to see that $B_n + C$ reaches its minimal value at $\mu_n = \mu_{\min} = \frac{d_2a_{11}-d_1a_{22}}{2d_1d_2}$ with $d_1a_{22} - d_2a_{11} < 0$.
It implies that

$$a_{11}^2\frac{d_2^2}{d_1^2} - 2(2a_{12}b - 2a_{12}a_{21} - a_{11}a_{22})\frac{d_2}{d_1} + a_{22}^2 > 0. \tag{3.14}$$

Hence, D_n is negative when Eq (3.14) is met, and it applies for μ close to $\mu_{\min}$. By Eq (3.14), we obtain

$$\frac{d_2}{d_1} > \frac{2a_{12}b - 2a_{12}a_{21} - a_{11}a_{22} + \sqrt{(a_{12}b - a_{12}a_{21})^2 - a_{11}a_{22}(a_{12}b - a_{12}a_{21})}}{a_{11}^2}, \tag{3.15}$$

which completes the proof. □

Let $\lambda = \pm i\omega$ ($\omega > 0$) be a pure imaginary pair of eigenvalues of Eq (3.7). Thus, ω satisfies

$$-\omega^2 - iA_n\omega + B_n + C(cos(\omega\tau) - i\sin(\omega\tau) = 0. \tag{3.16}$$

Therefore, we have

$$\begin{cases} -\omega^2 + B_n = -C\cos(\omega\tau), \\ A_n\omega = C\sin(\omega\tau). \end{cases} \tag{3.17}$$

From the above equation, we obtain

$$\omega^4 + (A_n^2 - 2B_n)\omega^2 + B_n^2 - C^2 = 0. \tag{3.18}$$

From Eq (3.8), we know that if the condition (3.11) holds, then $B_n - C > B_n + C > 0$. Therefore, $B_n^2 - C^2 > 0$. We discuss the existence of roots of Eq (3.18) in two cases.

Case I. Suppose that

$$A_n^2 - 2B_n < 0, \Delta_n = (A_n^2 - 2B_n)^2 - 4(B_n^2 - C^2) > 0. \tag{3.19}$$

Thus, Eq (3.18) has two positive real roots $\omega^{\pm}(n) = \sqrt{\frac{-(A_n^2-2B_n)\pm\sqrt{\Delta_n}}{2}}$. Substituting $\omega^{\pm}(n)$ into Eq (3.17), we have

$$\tau_j^{\pm}(n) = \frac{1}{\omega^{\pm}(n)}\left[\arccos\left(\frac{\omega^{\pm^2}(n) - B_n}{C}\right) + 2j\pi\right], j \in \mathbb{N}. \tag{3.20}$$

Case II. Suppose that either of the following two conditions are met

$$(i) \quad \Delta_n < 0; (ii) \quad \Delta_n \geq 0, A_n^2 - 2B_n \geq 0. \tag{3.21}$$

So, Eq (3.18) does not have a positive real root.

We summarize the above discussions, and we have the following theorem.

Theorem 3.3. *Assume that the conditions* (3.10) *and* (3.11) *hold and the following is true:*
(i) When

$$A_n^2 - 2B_n < 0, \Delta_n = (A_n^2 - 2B_n)^2 - 4(B_n^2 - C^2) > 0,$$

Eq (3.18) *has two positive real roots $\omega^{\pm}(n)$; then, Eq* (3.7) *has a pair of pure imaginary roots $\pm i\omega^{\pm}(n)$, when $\tau = \tau_j^{\pm}(n)$.*
(ii) If either $\Delta_n < 0$ or $\Delta_n \geq 0, P_{1n} \geq 0$; then, Eq (3.7) *has no pure imaginary roots.*

Lemma 3.1. *Let $\lambda(\tau) = \xi(\tau) \pm i\eta(\tau)$ be the root of Eq (3.7) satisfying $\alpha(\tau_j^{\pm}(n)) = 0, \xi(\tau_j^{\pm}(n)) = \omega^{\pm}(n))$. Then the following transversality conditions are satisfied:*

$$\frac{d(\mathrm{Re}\lambda(\tau))}{d\tau}\Big|_{\tau=\tau_j^{\pm}(n)} = \pm\frac{\sqrt{\Delta_n}}{A_{3n}^2}. \tag{3.22}$$

Proof. When both sides of Eq (3.7) are differentiated with regard to λ, the result is

$$2\lambda + A_n - C(\tau + \lambda\frac{d\tau_1}{d\lambda})e^{-\lambda\tau} = 0.$$

Therefore,

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{(2\lambda + C)e^{\lambda\tau}}{\lambda C} - \frac{\tau}{\lambda}.$$

Combining with (3.17), we get

$$\mathrm{Re}\left(\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]^{-1}\right)_{\lambda=\pm\mathrm{i}\omega^{\pm}(\mathrm{n})} = \left[\frac{2\omega^2 + \mathrm{A_n^2} - 2\mathrm{B_n}}{\mathrm{C}^2}\right]_{\lambda=\pm\mathrm{i}\omega^{\pm}(\mathrm{n})} = \pm\frac{\sqrt{\Delta_n}}{\mathrm{C}^2}.$$

□

Denote

$$\Gamma = \left\{n \in \mathbb{N}_0 | A_n^2 - 2B_n < 0, \Delta_n = (A_n^2 - 2B_n)^2 - 4(B_n^2 - C^2) > 0\right\}.$$

For a given $n \in \Gamma$, it is seen that $\tau_j^+(n)$ grows with respect to j. As a result, we can deduce that $\tau_0^+(n) = \min_{j\in\mathbb{N}_0} \tau_j^+(n)$ for some fixed n. We define

$$\tau^* = \min_{n\in\Gamma}\{\tau_0^+(n)\}. \tag{3.23}$$

Theorem 3.4. *Assume that the conditions* (3.10) *and* (3.11) *hold; we have the following results*

(i) If either of the following two conditions are met

$$(a) \quad \Delta_n < 0; \;\; (b) \quad \Delta_n \geq 0, A_n^2 - 2B_n \geq 0, \tag{3.24}$$

then the interior equilibrium E^* is locally asymptotically stable for $\tau \geq 0$;

(ii) If $A_n^2 - 2B_n > 0$ and $\Delta_n > 0$, then the interior equilibrium E^* is locally asymptotically stable when $\tau \in [0, \tau^*)$ and system (1.2) *experiences a Hopf bifurcation at E^*. In addition, before it becomes unstable, the interior equilibrium $E*$ will go through k changes from stable to unstable and back to stable.*

4. Global stability

Theorem 4.1. *Suppose that $\frac{\beta K_1}{a} < d$ and $E_1 = (1, 0)$ is globally asymptotically stable.*

Proof. Define a Lyapunov functional as follows

$$V(t) = \beta \int_\Omega \int_1^u \frac{u-1}{u} d\xi dx + \int_\Omega v dx. \tag{4.1}$$

Differentiating $V(t)$ with respect to t, we have

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} =& \beta \int_\Omega \frac{u-1}{u}\left(d_1\Delta u + u(1-u) - \frac{uv}{a+u^2}\right) + \int_\Omega \left(d_2\Delta v + \frac{\beta uv}{a+u^2} - dv - \gamma u(t-\tau)v^2\right)dx \\
=& \beta d_1 \int_\Omega \frac{u-1}{u}\Delta u dx + \beta \int_\Omega (u-1)\left((1-u) - \frac{v}{a+u^2}\right) + d_2 \int_\Omega \Delta v dx \\
&+ \int_\Omega v\left(\frac{\beta u}{a+u^2} - d - \gamma u(t-\tau)v\right)dx \\
=& -\beta \int_\Omega (u-1)^2\, dx + \int_\Omega \left(\frac{\beta u}{a+u^2} - d - \gamma u(t-\tau)v\right)dx - \beta \int_\Omega \left(\frac{\nabla u}{u}\right)^2 dx \\
\leq& -\beta \int_\Omega (u-1)^2\, dx + \int_\Omega \left(\frac{\beta K_1}{a} - d\right) v dx - \beta \int_\Omega \left(\frac{\nabla u}{u}\right)^2 dx.
\end{aligned}$$

$\frac{\beta}{a} < d$, so $\frac{\mathrm{d}V}{\mathrm{d}t} \leq 0$, and $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$ if and only if $(u, v) = (1, 0)$. We conclude that $E_1 = (1, 0)$ is globally asymptotically stable. □

Theorem 4.2. *Assume that the conditions*

$$\begin{aligned}
&\frac{a\beta}{a+u^{*2}} - \frac{a\beta v^*(u^* + K_1)}{a(a+u^{*2})^2} - \frac{\beta u^* K_1}{2a(a+u^{*2})} > 0, \\
&\gamma \underline{c}_1 - \frac{\gamma v^*}{2} - \frac{\beta u^* k_1}{2a(a+u^{*2})} > 0, \frac{\gamma v^*}{2} - 1 > 0
\end{aligned} \tag{4.2}$$

hold. Then the unique interior equilibrium $E^* = (u^*, v^*)$ is globally asymptotically stable.

Proof. Define a Lyapunov function as follows

$$W(t) = \frac{a\beta}{a+u^{*2}} I_1(t) + I_2(t) + I_3(t),$$

where

$$I_1(t) = \int_\Omega \int_{u^*}^u \frac{\xi - u^*}{\xi} \mathrm{d}\xi \mathrm{d}x,\ I_2(t) = \int_\Omega \int_{v^*}^v \frac{\xi - v^*}{\xi} \mathrm{d}\xi \mathrm{d}x,\ I_3(t) = \int_\Omega \int_{t-\tau}^t (u(\xi) - u^*)^2 \mathrm{d}\xi \mathrm{d}x.$$

Thus, we have

$$\begin{aligned}
\frac{\mathrm{d}I_1(t)}{dt} &= d_1 \int_\Omega \frac{u-u^*}{u}\Delta u dx + \int_\Omega (u-u^*)\left(1 - u - \frac{v}{a+u^2} d\right) dx \\
&= -d_1 u^* \int_\Omega \left(\frac{\nabla u}{u}\right)^2 dx + \int_\Omega (u-u^*)\left(-(u-u^*) - \left(\frac{v}{a+u^2} - \frac{v^*}{a+u^{*2}}\right)\right) dx \\
&= -d_1 u^* \int_\Omega \left(\frac{\nabla u}{u}\right)^2 dx - \int_\Omega (u-u^*)^2 dx - \int_\Omega (u-u^*)\left(\frac{a(v-v^*) + u^{*2}v - u^2 v^*}{(a+u^2)(a+u^{*2})}\right) dx
\end{aligned}$$

$$= -d_1u^* \int_\Omega \left(\frac{\nabla u}{u}\right)^2 dx - \int_\Omega \left(1 + \frac{v^*(u^*+u)}{(a+u^2)(a+u^{*2})}\right)(u-u^*)^2 dx - \int_\Omega \frac{1}{a+u^2}(u-u^*)(v-v^*)\mathrm{d}x,$$

$$\begin{aligned}
\frac{\mathrm{d}I_2(t)}{dt} &= d_2 \int_\Omega \frac{v-v^*}{v}\Delta u dx + \int_\Omega (v-v^*)\Big(\frac{\beta u}{a+u^2} - d - \gamma u(t-\tau)v\Big) dx \\
&= -d_2v^* \int_\Omega \left(\frac{\nabla v}{v}\right)^2 dx + \int_\Omega (v-v^*)\frac{\beta u(a+u^{*2}) - \beta u^*(a+u^2)}{(a+u^2)(a+u^{*2})} dx - \gamma \int_\Omega (v-v^*)(u(t-\tau)v - u^*v^*)\mathrm{d}x \\
&= -d_2u^* \int_\Omega \left(\frac{\nabla v}{v}\right)^2 dx + \int_\Omega (a\beta - \beta u^* u)\frac{(u-u^*)(v-v^*)}{(a+u^2)(a+u^{*2})} dx - \gamma \int_\Omega u(t-\tau)(v-v^*)^2 \\
&\quad - \gamma v^* \int_\Omega (v-v^*)(u(t-\tau) - u^*)\mathrm{d}x \\
&\le -d_2u^* \int_\Omega \left(\frac{\nabla v}{v}\right)^2 dx + \int_\Omega (a\beta - \beta u^* u)\frac{(u-u^*)(v-v^*)}{(a+u^2)(a+u^{*2})} dx - \gamma \int_\Omega u(t-\tau)(v-v^*)^2 \\
&\quad + \frac{\gamma v^*}{2} \int_\Omega (v-v^*)^2 \mathrm{d}x + \int_\Omega (u(t-\tau) - u^*)^2 \mathrm{d}x,
\end{aligned}$$

$$\frac{\mathrm{d}I_3(t)}{dt} = \int_\Omega (u-u^*)^2 \mathrm{d}x - \int_\Omega (u(t-\tau) - u^*)^2 \mathrm{d}x.$$

Note that

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}I_1(t)}{dt} + \frac{\mathrm{d}I_2(t)}{dt} + \frac{\mathrm{d}I_3(t)}{dt}.$$

Therefore, we obtain

$$\begin{aligned}
\frac{\mathrm{d}W(t)}{\mathrm{d}t} &\le -\frac{a\beta u^* d_1}{a+u^{*2}} \int_\Omega \left(\frac{\nabla u}{u}\right)^2 dx - d_2v^* \int_\Omega \left(\frac{\nabla v}{v}\right)^2 dx \\
&\quad - \int_\Omega \left(\frac{a\beta}{a+u^{*2}} - \frac{a\beta v^*(u^*+u)}{(a+u^2)(a+u^{*2})^2} - \frac{\beta u^* u}{2(a+u^2)(a+u^{*2})}\right)(u-u^*)^2 dx \\
&\quad - \int_\Omega \left(\gamma u(t-\tau) - \frac{\gamma v^*}{2} - \frac{\beta u^* u}{2(a+u^2)(a+u^{*2})}\right)(v-v^*)^2 dx - (\frac{\gamma v^*}{2} - 1) \int_\Omega (u(t-\tau) - u^*)^2 dx.
\end{aligned}$$

From Eq (4.2), we know that $\frac{\mathrm{d}W}{\mathrm{d}t} \le 0$, and $\frac{\mathrm{d}W}{\mathrm{d}t} = 0$ if and only if $(u, v) = (u^*, v^*)$. Therefore, the interior equilibrium E^* is globally asymptotically stable. □

5. Non-constant positive steady states

In this section, we investigate the existence and non-existence of non-constant positive solutions of system (1.3). First, we will give a priori upper and lower bounds for the positive solutions of system (1.3).

5.1. A priori estimates

Lemma 5.1 (Harnack inequality)**.** *[22] Assume that $c(x) \in C(\bar{\Omega})$ and $\phi \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a positive solution to*

$$\Delta\phi + c(x)\phi, x \in \Omega, \frac{\partial\phi}{\partial n} = 0, x \in \partial\Omega.$$

Then there exists a positive constant $C_* = C_*(\|c\|_\infty)$ *such that*

$$\max_{\bar{\Omega}} \phi \le C_* \min_{\bar{\Omega}} \phi.$$

Theorem 5.1 (Upper bound). *Any positive solution* (u, v) *of system* (1.3) *satisfies*

$$0 < \max_{\bar{\Omega}} u(x) \le 1, 0 < \max_{\bar{\Omega}} v(x) \le \frac{\beta(d_2 + dd_1)}{dd_2}.$$

Proof. By the strong maximum principle, we know that if there is a $x_0 \in \bar{\Omega}$ such that $v(x_0) = 0$, then $v(x) \equiv 0$ and u satisfies

$$\begin{cases} -d_1 \Delta u = u(1-u), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{5.1}$$

From the well known result, $u \equiv 0$ or $u \equiv 1$. Hence, if (u, v) is not $(0, 0)$ or $(1, 0)$, then $u(x) > 0$ and $v(x) > 0$ for $x \in \bar{\Omega}$.

From the maximum principle, we easily obtained that $u(x) \le 1$ in Ω. Multiplying the first equation of Eq (1.3) by β and adding it to the second equation of Eq (1.3), we have

$$\begin{aligned} -(\beta d_1 \Delta u + d_2 \Delta v) &= \beta u(1-u) + \frac{d\beta d_1 u}{d_2} - \frac{d}{d_2}(\beta d_1 u + d_2 v) - \gamma u v^2 \\ &\le \beta + \frac{d\beta d_1}{d_2} - \frac{d}{d_2}(\beta d_1 u + d_2 v). \end{aligned} \tag{5.2}$$

Then from the maximum principle,

$$\beta d_1 u + d_2 v \le \frac{d_2\beta + d\beta d_1}{d},$$

which leads to

$$v \le \frac{\beta(d_2 + dd_1)}{dd_2}.$$

□

Theorem 5.2 (Lower bound). *There exists a positive constant* $\underline{C}$ *depending on* d_1, d_2, a, β, d *and* γ, *such that any positive solution* $(u(x), v(x))$ *of Eq* (1.3) *satisfies*

$$u(x), v(x) \ge \underline{C}, x \in \bar{\Omega}. \tag{5.3}$$

Proof.

$$u(x), v(x) \le \overline{C} := \max\left\{1, \frac{\beta(d_2 + dd_1)}{dd_2}\right\}. \tag{5.4}$$

Let

$$c_1(x) := 1 - u(x) - \frac{v(x)}{a + u^2(x)} \text{ and } c_2(x) := \frac{\beta u(x)}{a + u^2(x)} - d - \gamma u(x)v(x). \tag{5.5}$$

Then

$$|c_1(x)|2+\frac{\overline{C}}{a},|c_2(x)|d+\frac{\overline{C}}{a}+\gamma\overline{C}^2.$$

A positive constant C can be derived from Lemma 5.1 in such a way that

$$\sup_{\bar{\Omega}} u(x)\leq C\inf_{\bar{\Omega}} u(x), \sup_{\bar{\Omega}} v(x)\leq C\inf_{\bar{\Omega}} v(x).$$

Therefore, it must now show that there is some $c>0$ such that

$$\sup_{\bar{\Omega}} u(x)\geq c, \quad \sup_{\bar{\Omega}} v(x)\geq c. \tag{5.6}$$

On the other hand, suppose that the outcome is incorrect. Then there exists a series of affirmative solutions $(u_n(x), v_n(x))$ such that

$$\sup_{\bar{\Omega}} u_n\to 0 \text{ or } \sup_{\bar{\Omega}} v_n\to 0 \quad as\ n\to+\infty. \tag{5.7}$$

By standard elliptic regularity, we obtain that there exists a subsequence of $\{(u_n, v_n)\}$, which is again denoted by $\{(u_n, v_n)\}$, such that $\{(u_n, v_n)\}\to(u_\infty, v_\infty)$ in $C^2(\bar{\Omega})$ as $n\to+\infty$. Noting that $u_\infty\leq 1$, since Eq (5.7) holds, $u_\infty\equiv 0$ or $v_\infty\equiv 0$. So, we have the following:

(i) $u_\infty\equiv 0$, $v_\infty\not\equiv 0$; or $u_\infty\equiv 0$, $v_\infty\equiv 0$;

(ii) $u_\infty\not\equiv 0$, $v_\infty\equiv 0$.

Moreover, we get the following two integral equations:

$$\begin{cases} \int_\Omega u_n(1-u_n)-\dfrac{u_n v_n}{a+u_n^2})dx=0,\\ \int_\Omega v_n(-d+\dfrac{\beta u_n}{a+u_n^2}-\gamma uv^2)dx=0. \end{cases} \tag{5.8}$$

(i) Case: $u_\infty\equiv 0$, so we have

$$-d+\frac{\beta u_n}{a+u_n^2}-\gamma uv^2\to -d<0, n\to\infty,$$

and $v_n>0$. But, we integrate the equation of v_n; then, we have

$$\int_\Omega v_n(-\gamma+\frac{\beta u_n}{b+u_n})dx=0. \tag{5.9}$$

This is a contradiction.

(ii) $u_\infty\not\equiv 0$ and $v_\infty\equiv 0$, so u_∞ satisfies Eq (5.1). So, $u_\infty\equiv 1$, and for a large n, we have

$$-d+\frac{\beta u_n}{a+u_n^2}-\gamma uv^2\to -d+\frac{\beta}{a+1}>\varepsilon>0.$$

Thus, the second equation of Eq (5.9) does not hold, which is a contradiction. So, $\sup_{\bar{\Omega}} u(x)>0$, $\sup_{\bar{\Omega}} v(x)>0$, and consequently Eq (5.3) holds. □

Theorem 5.3. *There is a large constant d^* for which there is no non-constant positive solution to the problem* (1.3) *if $d_1, d_2 \geq d^*$.*

Proof. Suppose that $(u(x), v(x))$ is a non-constant positive solution of system (1.3). Denote

$$\overline{u} = |\Omega|^{-1} \int_\Omega u(x)dx \geq 0, \quad \overline{v} = |\Omega|^{-1} \int_\Omega v(x)dx \geq 0.$$

Then

$$\int_\Omega (u - \overline{u})dx = 0, \int_\Omega (v - \overline{v})dx = 0. \tag{5.10}$$

Multiplying the first equation in Eq (1.3) by $u - \bar{u}$ and applying system (1.3), we get

$$\begin{aligned}
\int_\Omega d_1 |\nabla (u - \bar{u})|^2 dx &= \int_\Omega (u - \overline{u}) u \left(1 - u - \frac{uv}{a + u^2}\right) dx \\
&= - \int_\Omega (u - \bar{u})^2 dx - \int_\Omega (u - \bar{u}) \frac{uv}{a + u^2} \mathrm{d}x \\
&= - \int_\Omega (u - \bar{u})^2 dx - \int_\Omega \frac{v(u - \bar{u})^2}{a + u^2} dx - \int_\Omega \frac{v\bar{u}(u - \overline{u})}{a + u^2} dx \\
&= - \int_\Omega (u - \bar{u})^2 dx - \int_\Omega \frac{v(u - \bar{u})^2}{a + u^2} dx - \int_\Omega (u - \overline{u}) \left(\frac{v\bar{u}}{a + u^2} - \frac{\bar{v}\bar{u}}{a + \bar{u}^2} \right) dx \\
&= - \int_\Omega (u - \bar{u})^2 dx - \int_\Omega \frac{v(u - \bar{u})^2}{a + u^2} dx - \int_\Omega \frac{\bar{u}}{a + u^2} (u - \bar{u})(v - \bar{v}) dx \\
&\quad + \int_\Omega \frac{\bar{u}\bar{v}(\bar{u} + u)}{(a + u^2)(a + \bar{u}^2)} (u - \bar{u})^2 dx \\
&\leq \left(\frac{\bar{C}}{2a} + \frac{2\bar{C}^3}{a^2} - \frac{\underline{C}}{a + \bar{C}^2} - 1 \right) \int_\Omega (u - \bar{u})^2 dx + \frac{\bar{C}}{2a} \int_\Omega (v - \bar{v})^2 dx.
\end{aligned} \tag{5.11}$$

Multiplying the second equation in system (1.3) by $v - \bar{v}$ and applying Theorem 5.1, we get

$$\begin{aligned}
d_2 \int_\Omega |\nabla (v - \bar{v})|^2 dx &= \delta \int_\Omega (v - \bar{v}) v \left(\frac{\beta u}{a + u^2} - d - \gamma uv \right) dx \\
&= -d \int_\Omega (v - \bar{v})^2 dx + \int_\Omega (v - \bar{v}) \frac{\beta uv}{a + u^2} dx - \int_\Omega \gamma uv^2 (v - \bar{v}) dx \\
&= -d \int_\Omega (v - \bar{v})^2 dx + \beta \int_\Omega \frac{a\bar{v}(u - \bar{u})(v - \bar{v})}{(a + u^2)(a + \bar{u}^2)} dx + \beta \int_\Omega \bar{v}(v - \bar{v}) \frac{u\bar{u}^2 - \bar{u}u^2}{(a + u^2)(a + \bar{u}^2)} dx \\
&\quad - \gamma \int_\Omega v(v - \bar{v})(u(v - \bar{v}) + u\bar{v}) dx \\
&\leq \frac{\beta \bar{C}(a + \bar{C}^2)}{2a^2} \int_\Omega (u - \bar{u})^2 dx + \left(\frac{\beta \bar{C}(a + \bar{C}^2)}{2a^2} - d \right) \int_\Omega (v - \bar{v})^2 dx - \gamma \int_\Omega uv(v - \bar{v})^2 dx \\
&\quad - \gamma \int \bar{v}v(u - \bar{u})(v - \bar{v}) dx - \gamma \int_\Omega \bar{v}\bar{u}(v - \bar{v})^2 dx \\
&\leq \left(\frac{\beta \bar{C}(a + \bar{C}^2)}{2a^2} + \frac{\gamma \bar{C}^2}{2} \right) \int_\Omega (u - \bar{u})^2 dx + \left(\frac{\beta \bar{C}(a + \bar{C}^2)}{2a^2} - d + \frac{\beta \bar{C}}{a} - \frac{3\gamma \underline{C}^2}{2} \right) \int_\Omega (v - \bar{v})^2 dx.
\end{aligned} \tag{5.12}$$

From Eqs (5.11) and (5.12), we get

$$\begin{aligned}&\int_\Omega d_1|\nabla(u-\bar{u})|^2dx+d_2\int_\Omega|\nabla(v-\bar{v})|^2dx\\ \leq&\left(\frac{\beta\bar{C}(a+\bar{C}^2)}{2a^2}+\frac{\gamma\bar{C}^2}{2}+\frac{\bar{C}}{2a}+\frac{2\bar{C}^3}{a^2}-\frac{\underline{C}}{a+\bar{C}^2}-1\right)\int_\Omega(u-\bar{u})^2dx\\ &+\left(\frac{\beta\bar{C}(a+\bar{C}^2)}{2a^2}-d+\frac{\beta\bar{C}}{a}-\frac{3\gamma\underline{C}^2}{2}+\frac{\bar{C}}{2a}\right)\int_\Omega(v-\bar{v})^2dx.\end{aligned} \tag{5.13}$$

By the *Poincaré* inequality, we have

$$\begin{aligned}&\int_\Omega d_1|\nabla(N-\bar{N})|^2dx+d_2\int_\Omega|\nabla(u-\bar{u})|^2dx\\ \leq&\frac{1}{\mu_1}\Big(A\int_\Omega(u-\bar{u})^2dx+B\int_\Omega(v-\bar{v})^2dx,\end{aligned} \tag{5.14}$$

where

$$\begin{aligned}A&=\frac{\beta\bar{C}(a+\bar{C}^2)}{2a^2}+\frac{\gamma\bar{C}^2}{2}+\frac{\bar{C}}{2a}+\frac{2\bar{C}^3}{a^2}-\frac{\underline{C}}{a+\bar{C}^2}-1,\\ B&=\frac{\beta\bar{C}(a+\bar{C}^2)}{2a^2}-d+\frac{\beta\bar{C}}{a}-\frac{3\gamma\underline{C}^2}{2}+\frac{\bar{C}}{2a}.\end{aligned}$$

These equations mean that if

$$\min\{d_1,d_2\}>\frac{1}{\mu_1}\max\{A,B\}$$

then

$$\nabla(u-\bar{u})=\nabla(v-\bar{v})=0,$$

and (u,v) must be a constant solution. □

5.2. *Existence of non-constant positive steady states*

For simplicity, denote

$$b_1=a_{21}-b,\mathbf{F_u}(\mathbf{u}^*)=\begin{pmatrix}a_{11}&-a_{12}\\ b_1&-a_{22}\end{pmatrix}.$$

By the maximum principle and the standard elliptic regularity, the embedding theorems, and the assumption that $\partial\Omega\in C^{2+\alpha}$, we obtain that $(u,v)\in C^2\times C^2$ for the elliptic system (1.3). Therefore, there is a positive constant M_1, such that $||\nabla u||_{C^1}\leq M_1$ and $||\nabla v||_{C^1}\leq M_1$. So, there exists a sufficiently large positive constant M such that $-d_1\Delta u-u(1-u-\frac{v}{a+u^2})+Mu$ and $-d_2\Delta v-\frac{\beta uv}{a+u^2}+dv+\gamma uv^2+Mv$ are monotonically increasing functions with respect to u and v.

Define $\mathcal{A}:[C^1(\overline{\Omega})]^2\to[C^1(\overline{\Omega})]^2$ by

$$\mathcal{A}(\mathbf{u})\triangleq\begin{pmatrix}(M-d_1\Delta)^{-1}[f_1(u,v)+Mu]\\ (M-d_2\Delta)^{-1}[f_2(u,v)+Mv]\end{pmatrix},$$

where $f_1(u,v) = u(1-u-\frac{v}{a+u^2})$ and $f_2(u,v) = \frac{\beta uv}{a+u^2} - dv - \gamma uv^2$.

It is worth noting that solving system (1.3) equates to finding positive solutions to the equation $(M\mathbf{I} - \mathcal{A})\mathbf{u} = 0$. We investigate the eigenvalue of the following problem using index theory.

$$-(M\mathbf{I} - \mathcal{A}_{\mathbf{u}}(\mathbf{u}^*))\Psi = \lambda\Psi, \Psi \neq 0, \tag{5.15}$$

where $\Psi = (\psi_1, \psi_2)^T$ and $\mathbf{u}^* = (u^*, v^*)$.

The following lemma is used to calculate the index of $(M\mathbf{I} - \mathcal{A}, \mathbf{u}^*)$.

Lemma 5.2. *[23] Assume that* $(M\mathbf{I} - \mathcal{A}_u(u^*)) \neq 0$. *Then* $\mathrm{index}(M\mathbf{I} - \mathcal{A}, \mathbf{u}^*) = (-1)^\sigma, \sigma = \sum_{\lambda>0} m_\lambda$, *where m_λ is the multiplicity of λ.*

By direct calculation, Eq (5.15) can be written as

$$\begin{cases} -(\lambda + M)d_1\Delta\psi_1 + (\lambda - a_{11})\psi_1 + a_{12}\psi_2 = 0, & x \in \Omega, \\ -(\lambda + M)d_2\Delta\psi_2 - b_1\psi_1 + (\lambda + a_{22})\psi_2 = 0, & x \in \Omega, \\ \frac{\partial\psi_1}{\partial n} = \frac{\partial\psi_2}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{5.16}$$

Notice that Eq (5.16) has a non-trivial solution if and only if $Q_n(\lambda; d_1, d_2) = 0$ for some $\lambda \geq 0$ and $n \geq 0$, where

$$Q_n(\lambda; d_1, d_2) \triangleq \det\begin{pmatrix} \lambda + \frac{Md_1\mu_n - a_{11}}{d_1\mu_n+1} & \frac{a_{12}}{d_1\mu_n+1} \\ -\frac{b_1}{d_2\mu_n+1} & \lambda + \frac{Md_2\mu_n + a_{22}}{d_2\mu_n+1} \end{pmatrix}.$$

Then, λ is an eigenvalue of Eqs (5.15) and (5.16) if and only if λ is a positive root of the characteristic equation $Q_n(\lambda; d_1, d_2) = 0$ for $n \geq 0$.

Lemma 5.3. *(i) When $n = 0$, $Q_0(\lambda, d_1, d_2) = 0$ may have no positive root, or exactly one positive root with a multiplicity of two, or two positive roots with a multiplicity of one.*

(ii) If $d_1 > \frac{a_{11}}{M\mu_1} := \hat{d}_1$, then $Q_n(\lambda, d_1, d_2) = 0$ has no positive root for $n \geq 1$.

Proof. (i) It is easily obtained that $Q_0(\lambda, d_1, d_2) = \lambda^2 - \mathrm{trace}(\mathbf{F}_{\mathbf{u}}(\mathbf{u}^*))\lambda + \det(\mathbf{F}_{\mathbf{u}}(\mathbf{u}^*))$. Obviously, the result holds.

(ii) It is clear that for $n \geq 1$,

$$\begin{aligned} Q_n(\lambda, d_1, d_2) =& \lambda^2 + \left(\frac{Md_2\mu_n + a_{12}}{d_2\mu_n + 1} + \frac{Md_1\mu_n - a_{11}}{d_1\mu_n + 1}\right)\lambda \\ &+ \frac{Md_2\mu_n + a_{12}}{d_2\mu_n + 1}\frac{Md_1\mu_n - a_{11}}{d_1\mu_n + 1} + \frac{b_1 a_{12}}{(d_2\mu_n + 1)(d_1\mu_n + 1)}. \end{aligned}$$

Since $a_{12} > 0$, $b_1 < 0$ and $a_{22} > 0$, then the polynomial $Q_n(\lambda, d_1, d_2) > 0$. So, if d_2 is big enough, the desired result is reached. □

Lemma 5.4. *Suppose that*

$$\frac{\beta v^*(a - u^{*2})}{(a + u^{*2})^2} - \gamma v^{*2} - \frac{2\gamma u^{*2}v^{*2}}{a + u^{*2}} < 0 \tag{5.17}$$

holds; then, the following is true:

(i) The quadratic equation $M^2 d_1 d_2 \mu_n^2 + (a_{22} M d_1 - a_{11} M d_2)\mu_n + b_1 a_{12} - a_{11} a_{22} = 0$ has two roots. One is positive, say μ_1^*, and the other is negative.

(ii) For some $n_1^* \geq 1$, suppose that $\mu_1^* \in (\mu_{n_1^*}, \mu_{n_1^*+1})$. Then, there is a $\hat{d}_1 := \hat{d}_1(\Gamma, d_1, d_2)$ such that the characteristic equation $Q_n(\lambda, d_1, d_2) = 0$ has a unique positive root for $1 \leq n \leq n_1^*$ and has no positive root for $n_1^* + 1 \leq n$ provided that $d_1 \geq \hat{d}_1$.

Proof. (i) The condition (5.17) holds; then, $b_1 a_{12} - a_{11} a_{22} < 0$, which implies that the result is true.

(ii) Obviously, according to the definitions of n_1^*, $Q_n(\lambda, d_1, d_2) = 0$ has a unique root with a multiplicity of one for $1 \leq b \leq n_1^*$, and it has no positive root for $n \geq n_1^* + 1$ if $d_1 \geq \hat{d}_1$. □

Theorem 5.4. *Assume that $\mu_1^* \in (\mu_{n_1^*}, \mu_{n_1^*+1})$ for some $n_1^* \geq 1$, and that $\sum_{k=1}^{n^*} n_k$ is even. Thus, there is a $\hat{d}_1$ such that for any $d_1 > \hat{d}_1$, system (1.3) has at least one non-constant positive solution.*

Proof. Assume, on the other hand, that the assertion is false for some $d_1 > \hat{d}_1$.

$$\mathcal{A}_t(\mathbf{u}) \triangleq \begin{pmatrix} (MI - (d_1^* + t(d_1 - d_1^*))\Delta)^{-1}[f_1(u, v) + Mu] \\ (MI - (d_2^* + t(d_2 - d_2^*))\Delta)^{-1}[f_2(u, v) + Mv] \end{pmatrix},$$

where d_1^* and d_2^* are constants that are positive and will be found out later.

Consider the problem

$$\mathcal{A}_t(\mathbf{u}) = \mathbf{u} \quad in \quad \Omega, \quad \frac{\partial \mathbf{u}}{\partial n} = 0 \quad on \quad \partial\Omega. \tag{5.18}$$

Its positive solution is contained in

$$\Lambda := \left\{ \mathbf{u} \in [C^1(\bar{\Omega})]^2 : \underline{C} \leq u, v < \overline{C} \right\}.$$

By the homotopy invariance of the Leray-Schauder degree,

$$deg(I - \mathcal{A}_0, \Lambda, 0) = deg(I - \mathcal{A}_1, \Lambda, 0). \tag{5.19}$$

Notice that $\mathbf{u}$ is a non-constant positive solution of Eq (1.3) if and only if it is such a solution of (5.18) for $t = 1$. And for any $t \in [0, 1]$, $\mathbf{u}^*$ is a constant solution of Eq (5.18).

Since we assumed that there are no non-constant positive solutions of system (1.3), $\mathcal{A}_t(\mathbf{u}) = \mathbf{u}$ has only the constant solution u^* in Λ. By Lemmas 5.3 and 5.4 , we can obtain

$$l_{\lambda_k} = \begin{cases} 0 \text{ or } 2, \text{if } n = 0, \\ 1, \qquad \text{if } 1 \leq n \leq n_1^*, \\ 0, \qquad \text{if } n \geq n_1^* + 1. \end{cases}$$

Thus, $\sigma = \sum_{k=1}^{k_1^*} n_k + 1$ (or 3) = an odd number. So that

$$deg(I - \mathcal{A}_1, \Lambda, 0) = \text{index}(\mathcal{A}_1, \mathbf{u}^*) = -1. \tag{5.20}$$

$$A = \frac{\beta\bar{C}(a + \bar{C}^2)}{2a^2} + \frac{\gamma\bar{C}^2}{2} + \frac{\bar{C}}{2a} + \frac{2\bar{C}^3}{a^2} - \frac{\underline{C}}{a + \bar{C}^2} - 1,$$

$$B = \frac{\beta\bar{C}(a+\bar{C}^2)}{2a^2} - d + \frac{\beta\bar{C}}{a} - \frac{3\gamma\underline{C}^2}{2} + \frac{\bar{C}}{2a}.$$

Let us take

$$d_1^* = \frac{1}{\mu_1}\left(\frac{\beta\bar{C}(a+\bar{C}^2)}{2a^2} + \frac{\gamma\bar{C}^2}{2} + \frac{\bar{C}}{2a} + \frac{2\bar{C}^3}{a^2} - \frac{\underline{C}}{a+\bar{C}^2} - 1\right),$$
$$d_2^* = \max\left\{\frac{1}{\mu_1}\left(\frac{\beta\bar{C}(a+\bar{C}^2)}{2a^2} - d + \frac{\beta\bar{C}}{a} - \frac{3\gamma\underline{C}^2}{2} + \frac{\bar{C}}{2a}\right), \hat{d}_2\right\} + 1,$$

where $\hat{d}_1$ and $\hat{d}_2$ are defined in Lemma 5.3. By Theorem 5.3, $\mathcal{A}_0(\mathbf{u}) = \mathbf{u}$ has only the positive constant solution u^*. In addition, by investigating the existence of positive roots λ_k of $Q_k(\lambda, D_0, D_I, D_P) = 0$, we get

$$\deg(I - \mathcal{A}_0, \Lambda, 0) = \mathrm{index}(\mathcal{A}_0, \mathbf{u}^*) = 1, \tag{5.21}$$

since Lemma 5.3 gives $\sigma = l_{\lambda_0} = 0$ or 2. Therefore, Eqs (5.19) and (5.20) contradict Eq (5.21). This completes the proof. □

6. Numerical simulations

In this section, we use numerical simulations of a few different scenarios to illustrate our theoretical findings.

6.1. The effect of delay τ

We choose the parameters $a = 1, \beta = 0.96$, $d = 0.01$, $\gamma = 0.3$, $d_1 = 0.2$, $d_2 = 2$ and $\Omega = (0, \pi)$. By a direct calculation, we get that system (1.2) has a unique equilibrium $E^* = (0.0151, 0.9852)$, and Eq (3.18) has two positive roots: $\omega_1 = 0.1336$ and $\omega_2 = 0.1018$. According to Eq (3.20), we obtain the critical values of τ

$$\tau_1^j = 28.1499, 75.1690, 122.1881, 169.2072, \cdots,$$

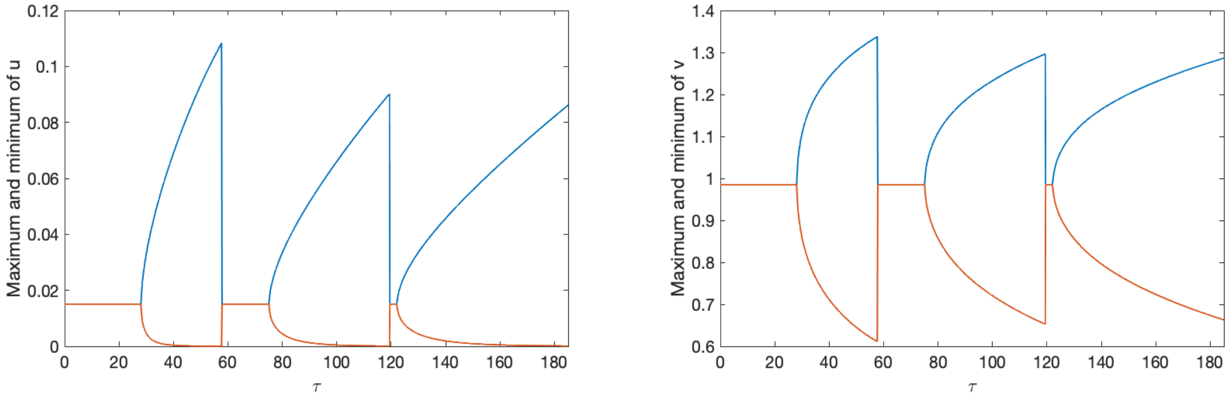
and

$$\tau_2^j = 57.2235, 118.9519, 180.6803, 242.4087, \cdots.$$

In addition, from Eq (3.1), we obtain

$$d\frac{\mathrm{Re}\lambda(\tau)}{d\tau}\bigg|_{\tau=\tau_1^{(j)}, \lambda=i\omega_1} > 0, \; d\frac{\mathrm{Re}\lambda(\tau)}{d\tau}\bigg|_{\tau=\tau_2^{(j)}, \lambda=i\omega_2} < 0.$$

When $\tau = \tau_1^j$, a pair of eigenvalues crosses the imaginary axis from left to right. Figure 2 shows the delay time histories for different locations. When $\tau \in [0, \tau_1^1) \bigcup (\tau_2^1, \tau_1^2) \bigcup (\tau_2^2, \tau_1^3)$, the equilibrium of system (1.2) is asymptotically stable, but it becomes unstable when $\tau \in (\tau_1^1, \tau_2^1) \bigcup (\tau_1^2, \tau_2^2) \bigcup (\tau_1^3, +\infty)$. In other words, the delay τ causes the system (1.2) to exhibit the phenomenon of multiple switching events, in which the state of system (1.2) alternates between stable and unstable and vice versa, and the equilibrium E^* is ultimately unstable.



(a) (b)

Figure 2. Bifurcation diagrams showing stability losses and gains: (a) Maximum and minimum of prey u; (b) maximum and minimum of prey v.

If we keep the other parameters unchanged, only changing the value of γ to 0.1, we find that the positive equilibrium E^* is locally stable for all $\tau \geq 0$ (see Figure 3).

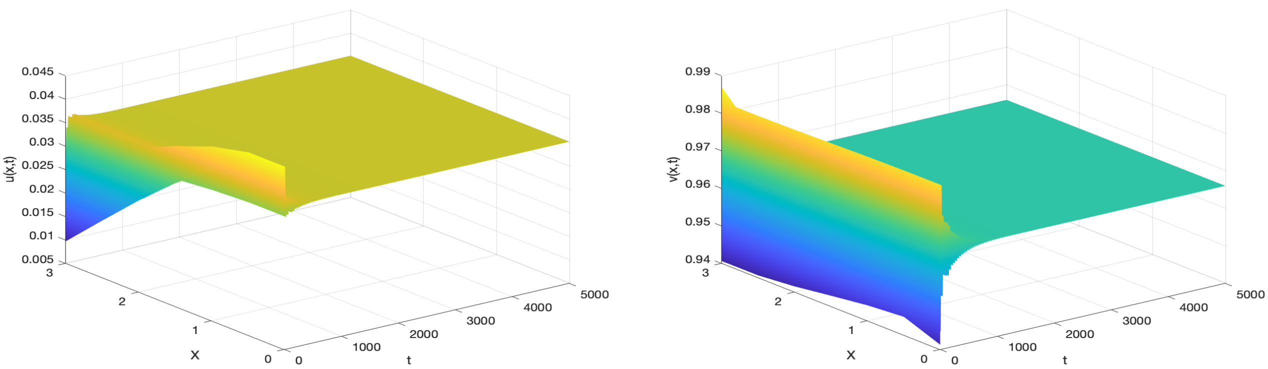


Figure 3. The positive equilibrium E^* is locally asymptotically stable with $\tau = 100$.

6.2. Turing instability

We apply the parameters $a = 0.3$, $\beta = 0.3$, $d = 0.2$, $\gamma = 0.3$, $d_1 = 0.002$, $d_2 = 4$ and $\Omega = (0, 60)$ By some calculations, we obtain that system (1.2) has a unique positive equilibrium $E^* = (0.2824, 0.2725)$. According to Theorem 3.1, by perturbing the initial value at the equilibrium E^*, we find that Turing bifurcation occurs (see Figure 4). Figure 4 shows that system (1.2) has a stable non-constant steady state. But, if we increase the value of τ to 12, we find that the stable non-constant steady state disappears, and that the system has a period solution (see Figure 5). However, if we further increase the value of τ to 81, we find that the system has chaotic behavior (see Figure 6); the bifurcation diagrams of system (1.2) are shown in Figure 7.

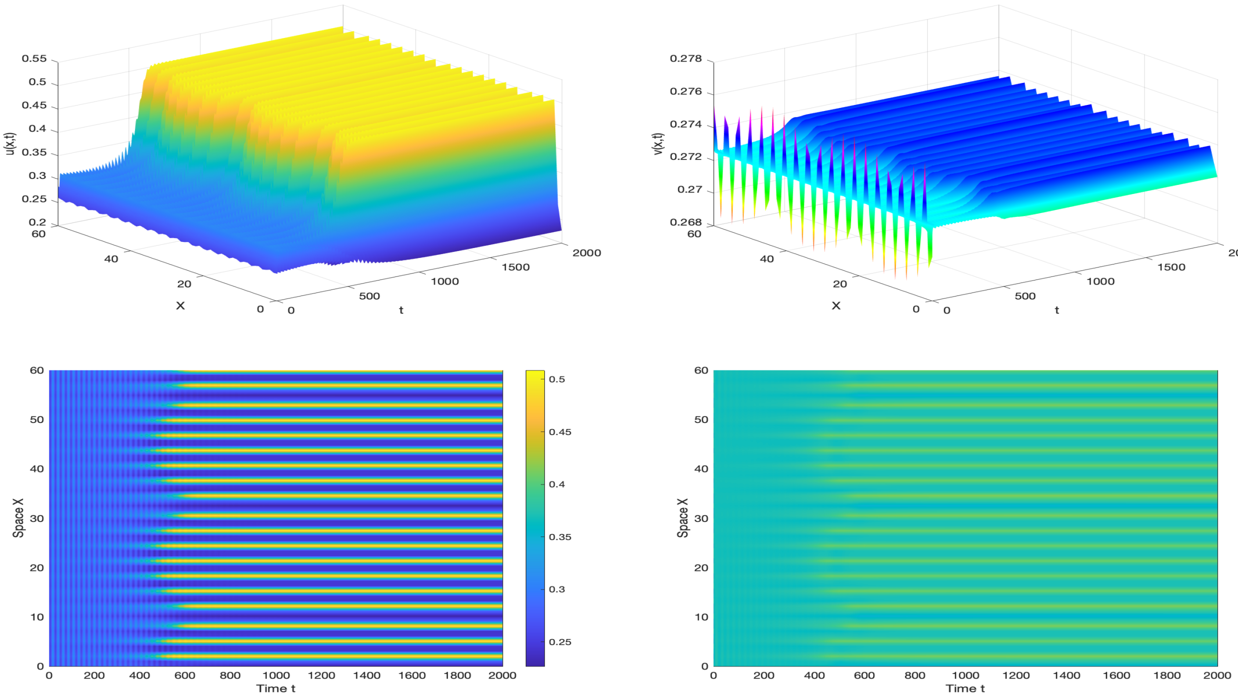


Figure 4. Turing bifurcation occurs with the initial conditions $u_0(x,t) = 0.2824 + 0.003\cos(2x)$, $v_0(x,t) = 0.2725 + 0.004\cos(2x)$.

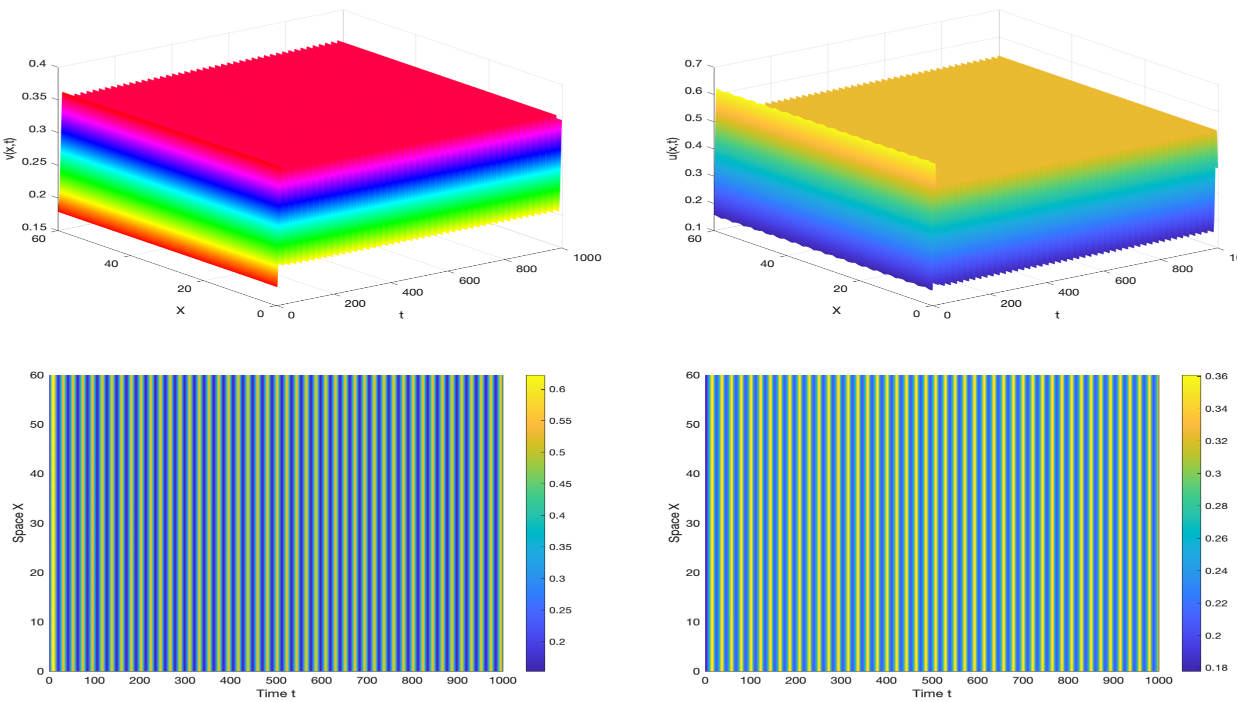


Figure 5. System (1.2) has a stable periodic solution with $\tau = 12$.

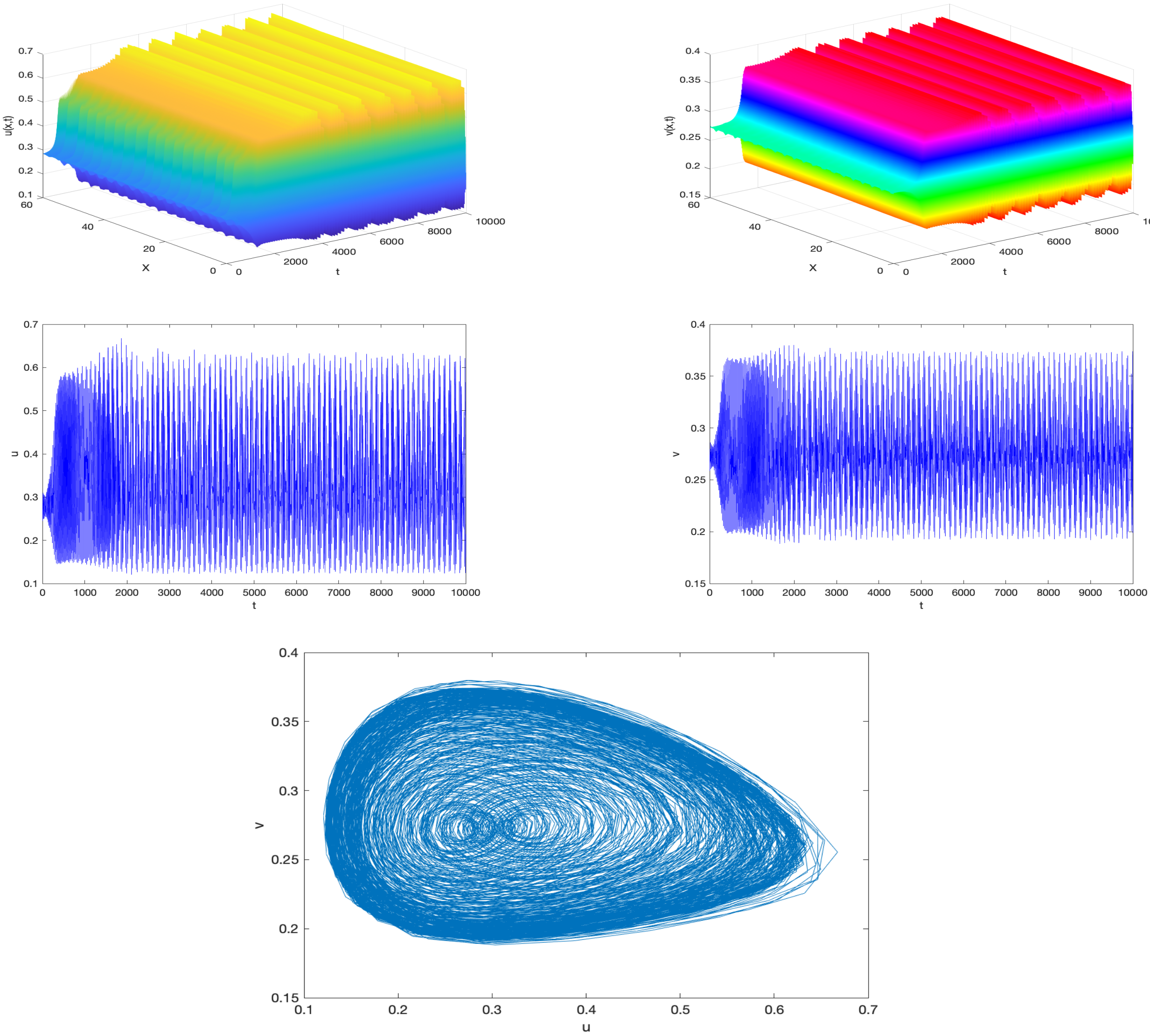


Figure 6. System (1.2) has chaotic behavior with $\tau = 81$.

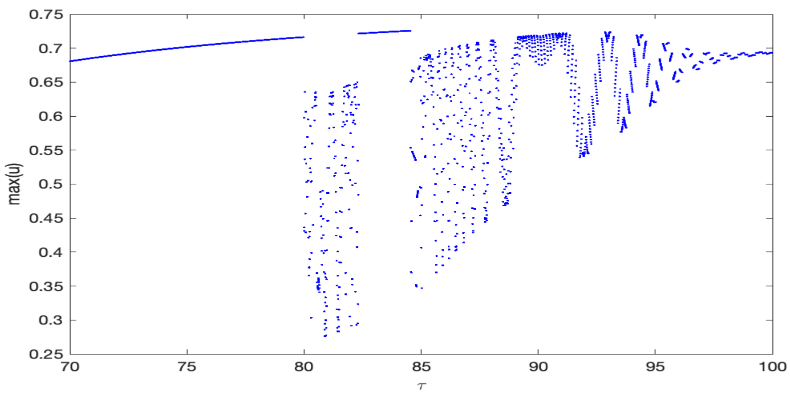


Figure 7. Bifurcation diagrams of system (1.2) for $\tau \in [70, 100]$.

7. Conclusions

The focus of this paper is on analyzing the effects of the toxins on a delayed diffusive predator-prey model. Overall, the paper provides a thorough analysis of the dynamic behavior of the system, considering various steady states and their stability. The incorporation of a delay in the model allows for an exploration of the effects of time lags in the predator-prey interaction, which adds realism to the study. The findings are interesting and reasonable.

Our system's dynamics were investigated in detail at and near all feasible steady states. We demonstrated that the system is persistent under specific conditions, where both the prey and the predator can survive. Conditions for the Turing bifurcation and global stability of the system at all equilibria are derived and presented. With respect to the delay τ, we found that the system displays a Hopf bifurcation near its interior equilibrium. Non-constant steady states were also discussed, along with the conditions under which they do and do not exist.

The theoretical findings are then illustrated by means of some numerical simulations. These results demonstrate that the system (1.2) displays spatial patterns and that a delay can lead to Hopf bifurcation and chaos. The findings might be useful for future qualitative research into a similar natural system.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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