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Research article

Existence and uniqueness of radial solution for the elliptic equation system in an annulus

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Abstract: This article discusses the existence and uniqueness of radial solution for the elliptic equation system

$$\begin{cases} -\triangle u = f(|x|,\ u,\ v,\ |\nabla u|), \quad x \in \Omega, \\ -\triangle v = g(|x|,\ u,\ v,\ |\nabla v|), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0,\ v|_{\partial\Omega} = 0, \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^N :\ r_1 < |x| < r_2\},\ N \geq 3,\ f,\ g : [r_1,\ r_2] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ are continuous. Due to the appearance of the gradient term in the nonlinearity, the equation system has no variational structure and the variational method cannot be applied to it directly. We will give the correlation conditions of f and g, that is, f and g are superlinear or sublinear, and prove the existence and uniqueness of radial solutions by using Leray-Schauder fixed point theorem.

Keywords: elliptic equation system; gradient term; radial solution; annular domain; Leray-Schauder fixed point theorem
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1. Introduction

In this article we discuss the existence and uniqueness of radial solution for the elliptic equation system

$$\begin{cases} -\triangle u = f(|x|,\ u,\ v,\ |\nabla u|), \quad x \in \Omega, \\ -\triangle v = g(|x|,\ u,\ v,\ |\nabla v|), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0,\ v|_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

in an annular domain $\Omega = \{x \in \mathbb{R}^N : r_1 < |x| < r_2\}$, where $N \geq 3$, $0 < r_1 < r_2 < \infty$, f, g : $[r_1, r_2] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ are continuous.

This problem arises in many different areas of applied mathematics and physics, for instance, incineration theory of gases, solid state physics, variational methods and optimal control. Therefore, there have been many research results, see [1–25] and references therein.

The authors of [1] considered the Dirichlet elliptic system

$$\begin{cases} \triangle u + \lambda k_1(|x|)f(u, v) = 0, \\ \triangle v + \lambda k_2(|x|)g(u, v) = 0, \quad in\ \Omega, \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$, $R_1, R_2 > 0$, $f, g : [0, \infty) \times [0, \infty) \to (0, \infty)$, λ is a positive real parameter. By establishing the strong maximum principle, applying upper and lower solutions method and fixed point index results proved the existence of positive radial solutions in the condition (A).

$(A) \quad f_\infty \equiv \lim\limits_{(u,v)\to\infty} \dfrac{f(u,v)}{u+v} = \infty, \quad g_\infty \equiv \lim\limits_{(u,v)\to\infty} \dfrac{g(u,v)}{u+v} = \infty.$

In [2], Lee replaced the annular domain with an exterior domain.

In [4], the authors used topological methods to prove the existence of positive solutions for semilinear elliptic systems of the form

$$\begin{cases} -\triangle u = g(x, u, v), \quad x \in \Omega, \\ -\triangle v = f(x, u, v), \quad x \in \Omega, \\ u > 0, \ v > 0, \quad in\ \Omega, \\ u|_{\partial\Omega} = 0, \ v|_{\partial\Omega} = 0, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$, $f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are continuous. Similarly, in [8], the authors also obtain a priori estimates, and then use Leray-Schauder topological degree theory to establish the existence of positive radial solutions vanishing at infinity.

In addition to the above domain, there are ball domain, see [3, 12, 13, 17, 18, 20, 21]. In [3], Hai considered the boundary value problem

$$\begin{cases} \triangle u = -\lambda f(v), \\ \triangle v = -\mu g(u), \quad in\ B, \\ u = v = 0, \quad on\ \partial B, \end{cases}$$

where B is the open unit ball in $\mathbb{R}^N$, $f, g : \mathbb{R}^+ \to \mathbb{R}^+$. They establish upper and lower estimates, and the existence and uniqueness of positive solutions are obtained in the case of f and g superlinear. In [17], the above authors proved the existence and multiplicity of positive radial solutions for the infinite semipositone/positone superlinear systems.

Recently, in [23], the authors used the fixed point index theory to study the existence of positive radial solutions for a system of boundary value problems with semipositone second order elliptic equations

$$\begin{cases} \triangle\varphi + k(|z|)f(\varphi,\ \phi) = 0, & z \in \Omega, \\ \triangle\phi + k(|z|)g(\varphi,\ \phi) = 0, & z \in \Omega, \\ \alpha\varphi + \beta\frac{\partial\varphi}{\partial n} = 0,\ \alpha\phi + \beta\frac{\partial\phi}{\partial n} = 0, & |z| = R_1 \\ \gamma\varphi + \delta\frac{\partial\varphi}{\partial n} = 0,\ \gamma\phi + \delta\frac{\partial\phi}{\partial n} = 0, & |z| = R_2, \end{cases}$$

where $\alpha,\ \beta,\ \gamma,\ \delta \geq 0,\ f,\ g : C(\mathbb{R}^+ \times \mathbb{R}^+,\ \mathbb{R})$ and satisfy

$$f(u, v),\ g(u, v) \geq -M,\ \forall\ u,\ v \in \mathbb{R}^+.$$

In [24], Li discussed the existence of positive radial solutions of single elliptic equation. Inspired by the aforementioned article, we extend the results of [24] to the equation system.

The purpose of this article is to obtain existence and uniqueness results of radial solution for the elliptic equation system. However, we note that in most of the article on nonlinear differential equations the nonlinear terms are usually assumed to be nonnegative, see [1–3, 17, 18, 21–23]. However, in this article, we do not assume that the nonlinear terms are nonnegative, $f,\ g \in C([r_1,\ r_2] \times \mathbb{R}^2 \times \mathbb{R}^+, \mathbb{R})$. Using Leray-Schauder fixed point theorem, we prove the main results in the case of f and g superlinear or sublinear.

As usual, writing $r = |x|$, BVP (1.1) becomes the ordinary differential equation system boundary value problem

$$\begin{cases} -u''(r) - \frac{N-1}{r}u'(r) = f(r,\ u(r),\ v(r),\ |u'(r)|), & r \in [r_1,\ r_2], \\ -v''(r) - \frac{N-1}{r}v'(r) = g(r,\ u(r),\ v(r),\ |v'(r)|), & r \in [r_1,\ r_2], \\ u(r_1) = u(r_2) = 0,\ v(r_1) = v(r_2) = 0. \end{cases} \tag{1.2}$$

By discussing BVP (1.2) we will obtain radial solution of BVP (1.1).

Our main results are as follows:

Theorem 1.1. *Let $f,\ g : [r_1,\ r_2] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ be continuous. If f and g satisfy the following conditions:*

(F0) *for any $M > 0$, there exists a positive monotone nondecreasing continuous function $G_M : [0, +\infty] \to (0, +\infty)$ satisfying*

$$\int_0^{+\infty} \frac{\rho\ d\rho}{G_M(\rho)} = +\infty, \tag{1.3}$$

such that

$$|f(r,\ u,\ v,\ \xi)| \leq G_M(|\xi|)\ ,\ |g(r,\ u,\ v,\ \eta)| \leq G_M(|\eta|), \tag{1.4}$$

where $r \in [r_1, r_2],\ |u| \leq M,\ |v| \leq M,\ \xi,\ \eta \in \mathbb{R}^+$;

(F1) *there exist positive constants $a,\ b,\ c,\ d \geq 0$, satisfying $\frac{r_2^{N-1}}{r_1^{N-1}}\left(\frac{(r_1-r_2)^2}{2}(a+b) + c + d\right) < 1$ and $e > 0$, such that*

$$f(r,\ u,\ v,\ \xi)u + g(r,\ u,\ v,\ \eta)v \leq au^2 + bv^2 + c\xi^2 + d\eta^2 + e, \tag{1.5}$$

where $(r, u, v) \in [r_1, r_2] \times \mathbb{R} \times \mathbb{R},\ \xi,\ \eta \in \mathbb{R}^+$. Then BVP (1.1) has at least one radial solution.

Remark 1.1. Condition (F1) allows $f(r,\ u,\ v,\ \xi)$ and $g(r,\ u,\ v,\ \eta)$ to grow superlinearly with respect to $u,\ v,\ \xi,\ \eta$, while the Nagumo-type condition (F0) restricts $f(r,\ u,\ v,\ \xi)$ and $g(r,\ u,\ v,\ \eta)$ to grow at most quadratically with respect to ξ and η, respectively.

Next we give the uniqueness condition.

Theorem 1.2. *Let* $f,\ g : [r_1,\ r_2] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ *be continuous. If* f *and* g *satisfy (F0) and the following condition:*

(F2) *there exist positive constants* $a,\ b,\ c,\ d \geq 0$*, satisfying* $\frac{r_2^{N-1}}{r_1^{N-1}}\Big(\frac{(r_1-r_2)^2}{2}(a+b)+c+d\Big) < 1$*, such that*

$$(f(r,u_2,v_2,\xi_2) - f(r,u_1,v_1,\xi_1))(u_2-u_1) + (g(r,u_2,v_2,\eta_2) - g(r,u_1,v_1,\eta_1))(v_2-v_1)$$

$$\leq a(u_2-u_1)^2 + b(v_2-v_1)^2 + c(\xi_2-\xi_1)^2 + d(\eta_2-\eta_1)^2, \tag{1.6}$$

where $(r,u_i,v_i) \in [r_1,r_2] \times \mathbb{R} \times \mathbb{R},\ \xi_i,\eta_i \in \mathbb{R}^+,\ i = 1,\ 2$*. Then BVP (1.1) has a unique radial solution.*

The main innovations of this article are as follows: First, the nonlinearities are sign-changing. Second, we replace the previous independent conditions with the correlation conditions of f and g, which can better reflect the characteristics of the equations. Finally, as far as we know, there are few articles discussing the elliptic equation system of the nonlinear terms with gradient term, and this article is one of them.

In Section 2, we will present some preliminaries. The proofs of Theorems 1.1 and 1.2 are based on the Leray-Schauder fixed point theorem, which will be given in Section 3.

2. Preliminaries

Let $I = [r_1,\ r_2]$. $C(I)$ denote the Banach space of all continuous function on I with norm $\|u\|_C = \max_{t\in I}|u(t)|$. $C^1(I)$ denote the Banach space of all 1-order continuous differentiable function on I with norm $\|u\|_{C^1} = \max_{t\in I}\{\|u\|_C,\ \|u'\|_C\}$. $L^2(I)$ denote the Hilbert space composed of all Lebesgue square integrable functions on I with inner product $(u,\ v) = \int_0^1 u(t)v(t)dt$, and its inner product norm is $\|u\|_2 = (\int_0^1 |u(t)|^2dt)^{\frac{1}{2}}$. Let $H^1(I) = \{u \in C(I) : u$ be absolutely continuous on I, and $u' \in L^2(I)\}$.

Let X and Y be Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively. $X \times Y$ denotes the product space of X and Y, forming the Banach space with norm $\|(x,\ y)\| = \max\{\|x\|_X,\ \|y\|_Y\}$.

For the case of a single equation, given $h \in C(I)$, we consider the linear boundary value problem (LBVP)

$$\begin{cases} -u''(r) - \frac{N-1}{r}u'(r) = h(r), & r \in I, \\ u(r_1) = u(r_2) = 0. \end{cases} \tag{2.1}$$

Lemma 2.1. *If* $h \in C(I)$*, then the solution of LBVP (2.1) satisfies*

$$\|u\|_2^2 \leq \frac{(r_1-r_2)^2}{2}\|u'\|_2^2.$$

Proof. Set $u \in C^2(I)$ is the solution of LBVP (2.1), then from the Hölder inequality, we have

$$\|u\|_2^2 = \int_{r_1}^{r_2}\left|\int_{r_1}^{r} u'(s)ds\right|^2 dr \leq \int_{r_1}^{r_2}(r-r_1)dr\|u'\|_2^2 \leq \frac{(r_1-r_2)^2}{2}\|u'\|_2^2.$$

The proof of Lemma 2.1 is completed. □

Given $(h_1,\ h_2) \in C(I) \times C(I)$, we consider the linear boundary value problem corresponding to BVP (1.2)

$$\begin{cases} -u''(r) - \frac{N-1}{r} u'(r) = h_1(r), & r \in [r_1,\ r_2], \\ -v''(r) - \frac{N-1}{r} v'(r) = h_2(r), & r \in [r_1,\ r_2], \\ u(r_1) = u(r_2) = 0,\ v(r_1) = v(r_2) = 0. \end{cases} \tag{2.2}$$

Lemma 2.2. *For every $(h_1,\ h_2) \in C(I) \times C(I)$, LBVP (2.2) has a unique solution $(u,\ v) := S(h_1,\ h_2) \in C^2(I) \times C^2(I)$. Moreover, the solution operator $S : C(I) \times C(I) \to C^1(I) \times C^1(I)$ is a completely continuous linear operator.*

Proof. The case of a single space is known, see [24] Lemma 2.1. We give the proof of the solution operator is completely continuous in product space.

Set

$$\phi(r) = \frac{1}{N-2}\Big[\frac{1}{{r_1}^{N-2}} - \frac{1}{r^{N-2}}\Big],\ \psi(r) = \frac{1}{N-2}\Big[\frac{1}{r^{N-2}} - \frac{1}{{r_2}^{N-2}}\Big],\ \ r \in I.$$

By direct computing we have

$$(r^{N-1}\phi'(r))' = 0,\ (r^{N-1}\psi'(r))' = 0,\ \ r \in I.$$

$$r^{N-1}\Big(\phi'(r)\psi(r) - \phi(r)\psi'(r)\Big) = \frac{1}{N-2}\Big(\frac{1}{{r_1}^{N-2}} - \frac{1}{{r_2}^{N-2}}\Big) \triangleq \rho > 0,\ \ r \in I.$$

We define a function $G : I \times I \to \mathbb{R}^+$ by

$$G(r,\ s) = \begin{cases} \frac{1}{\rho}\, \phi(r)\, \psi(s), & r_1 \le r \le s \le r_2, \\ \frac{1}{\rho}\, \phi(s)\, \psi(r), & r_1 \le s \le r \le r_2. \end{cases} \tag{2.3}$$

Then $G \in C(I \times I)$. We verify that $G(r,\ s)$ is the Green function of the LBVP (2.2), namely

$$(u(r), v(r)) = \Big(\int_{r_1}^{r_2} G(r,s)\, h_1(s)ds, \int_{r_1}^{r_2} G(r,s)\, h_2(s)ds\Big) \triangleq S(h_1,\ h_2)(r),\ \ r \in I \tag{2.4}$$

is the unique solution of LBVP (2.2). By the above and the definition of G, we have

$$u(r) = \frac{1}{\rho}\int_{r_1}^{r} \phi(s)\, \psi(r)\, h_1(s)ds + \frac{1}{\rho}\int_{r}^{r_2} \phi(r)\, \psi(s)\, h_1(s)ds,$$

$$v(r) = \frac{1}{\rho}\int_{r_1}^{r} \phi(s)\, \psi(r)\, h_2(s)ds + \frac{1}{\rho}\int_{r}^{r_2} \phi(r)\, \psi(s)\, h_2(s)ds.$$

By differentiating, we get that

$$u'(r) = \frac{1}{\rho}\int_{r_1}^{r} \phi(s)\, \psi'(r)\, h_1(s)ds + \frac{1}{\rho}\int_{r}^{r_2} \phi'(r)\, \psi(s)\, h_1(s)ds, \tag{2.5}$$

$$v'(r) = \frac{1}{\rho}\int_{r_1}^{r} \phi(s)\, \psi'(r)\, h_2(s)ds + \frac{1}{\rho}\int_{r}^{r_2} \phi'(r)\, \psi(s)\, h_2(s)ds. \tag{2.6}$$

Hence, we see that $(u(r),\ v(r))$ is a solution of LBVP (2.2) by direct calculation. By the maximum principle, LBVP (2.2) has only one solution. From (2.4)–(2.6), we see that the solution operator $S : C(I)\times C(I) \to C^1(I)\times C^1(I)$ is a completely continuous linear operator.

The proof of Lemma 2.2 is completed. $\square$

Lemma 2.3. *Let $f,\ g : [r_1,\ r_2]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ be continuous and satisty (F0). For all $M > 0$, there exist constants $M_1 = M_1(M) > 0,\ M_2 = M_2(M) > 0$, such that if the solution $(u,\ v)$ of BVP (1.2) satisfis $\|(u,\ v)\|_C \le M$, then we have*

$$\|(u',\ v')\|_C \le \max\{M_1,\ M_2\}.$$

Proof. Set $M > 0$. By (1.3), there exist constants $M_1,\ M_2 > 0$, such that

$$\int_0^{M_1} \frac{\rho\, d\rho}{G_M(\rho)} > 2M;\ \int_0^{M_2} \frac{\sigma\, d\sigma}{G_M(\sigma)} > 2M. \tag{2.7}$$

Let $(u,\ v) \in C^2(I)\times C^2(I)$ is a solution of BVP (1.2) which satisfies $\|(u,\ v)\|_C \le M$, the following proof that $\|(u',\ v')\|_C \le \max\{M_1,\ M_2\}$. Suppose $(u'(r),\ v'(r))$ is not equal to 0, then there exists $t_0 \in (r_1,\ r_2)$ and $t_1 \in I,\ t_0 \ne t_1$, such that $(u'(t_0),\ v'(t_0)) = (0,\ 0),\ \|(u',\ v')\|_C = \max\{|u'(t_1)|,\ |v'(t_1)|\} > 0$. There are eight cases as follows:
1) $u'(t_1) > 0,\ v'(t_1) > 0,\ t_0 < t_1$;
2) $u'(t_1) > 0,\ v'(t_1) < 0,\ t_0 < t_1$;
3) $u'(t_1) < 0,\ v'(t_1) > 0,\ t_0 < t_1$;
4) $u'(t_1) < 0,\ v'(t_1) < 0,\ t_0 < t_1$;
5) $u'(t_1) > 0,\ v'(t_1) > 0,\ t_1 < t_0$;
6) $u'(t_1) > 0,\ v'(t_1) < 0,\ t_1 < t_0$;
7) $u'(t_1) < 0,\ v'(t_1) > 0,\ t_1 < t_0$;
8) $u'(t_1) < 0,\ v'(t_1) < 0,\ t_1 < t_0$.
We only prove case 1), other cases are similar. Set

$$s_1 = \sup\{r' \in [t_0,\ t_1)|\ u'(r') = 0,\ v'(r') = 0\},$$

then $s_1 < t_1$, and $(u'(s_1),\ v'(s_1)) = (0,\ 0)$. When $r \in (s_1,\ t_1]$, we have $u'(r) > 0,\ v'(r) > 0$. Hence,

$$\begin{cases} u''(r) + \frac{N-1}{r}u'(r) = -f(r,\ u(r),\ v(r),\ |u'(r)|) \le G_M(|u'(r)|), & r \in [s_1,\ t_1], \\ v''(r) + \frac{N-1}{r}v'(r) = -g(r,\ u(r),\ v(r),\ |v'(r)|) \le G_M(|v'(r)|), & r \in [s_1,\ t_1]. \end{cases}$$

Hence, for all $r \in [s_1,\ t_1]$, we have

$$\frac{u''(r)|u'(r)| + \frac{N-1}{r}u'^2(r)}{G_M(|u'(r)|)} \le |u'(r)|\,, \quad \frac{v''(r)|v'(r)| + \frac{N-1}{r}v'^2(r)}{G_M(|v'(r)|)} \le |v'(r)|.$$

Integrating both sides of this inequality on $[s_1,\ t_1]$, and variable substitution $\rho = |u'(r)|,\ \sigma = |v'(r)|$, we obtain that

$$\int_0^{|u'(t_1)|} \frac{\rho\, d\rho}{G_M(\rho)} = \int_{s_1}^{t_1} \frac{u''(r)|u'(r)|}{G_M(|u'(r)|)}\, dr$$

$$
\begin{aligned}
&\leq \int_{s_1}^{t_1} \frac{u''(r)|u'(r)|}{G_M(|u'(r)|)}\,dr + \int_{s_1}^{t_1} \frac{\frac{N-1}{r}u'^2(r)}{G_M(|u'(r)|)}\,dr \\
&\leq \int_{s_1}^{t_1} |u'(r)|\,dr \\
&= |u(t_1)| - |u(s_1)| \leq 2M.
\end{aligned}
$$

By (2.7) it follows that

$$|u'(t_1)| < M_1.$$

Similarly, it can be obtained

$$|v'(t_1)| < M_2.$$

Therefore,

$$\|(u',\ v')\|_C = \max\{\|u'\|_C,\ \|v'\|_C\} = \max\{|u'(t_1)|,\ |v'(t_1)|\} \leq \max\{M_1,\ M_2\}.$$

The proof of Lemma 2.3 is completed. □

Theorem 2.1. (Leray-Schauder fixed point theorem) [26,27] *Let E be a Banach space, $A : E\times E \to E \times E$ be a completely continuous mapping. If the solution set of the equation*

$$(u,\ v) = \lambda A(u,\ v),\ \ 0 < \lambda < 1$$

is bounded in $E \times E$, then A has a fixed point.

3. Proofs of the main results

Proof of Theorem 1.1. We known LBVP (2.2) has a unique solution $(u,\ v) \in C^2(I) \times C^2(I)$ by Lemma 2.2

$$(u(r),\ v(r)) = \Big(\int_{r_1}^{r_2} G(r, s)\, h_1(s)ds, \int_{r_1}^{r_2} G(r, s)\, h_2(s)ds \Big),\ \ r,\ s \in I,$$

where $G(r,\ s)$ defined by (2.3). We make integral operator $A : C^1(I)\times C^1(I) \to C^1(I)\times C^1(I)$ as follows:

$$
\begin{aligned}
A(u,\ v) = \Big(&\int_{r_1}^{r_2} G(r, s) f(r,\ u(r),\ v(r),\ |u'(r)|)ds, \\
&\int_{r_1}^{r_2} G(r, s) g(r,\ u(r),\ v(r),\ |v'(r)|)ds \Big),\ \ r \in I,
\end{aligned}
$$

then, A is a completely continuous linear operator. The solution of BVP (1.2) is equivalent to the fixed point of A. Next we prove that A has fixed point. We consider the equation

$$(u,\ v) = \lambda A(u,\ v),\ \ \lambda \in (0,\ 1). \tag{3.1}$$

Let $(u,\ v) \in C^1(I) \times C^1(I)$ be the solution of (3.1), then, $(u,\ v) \in C^2(I) \times C^2(I)$ satisfies the equations

$$
\begin{cases}
-u''(r) - \frac{N-1}{r}u'(r) = \lambda f(r,\ u(r),\ v(r),\ |u'(r)|),\ \ r \in I, \\
-v''(r) - \frac{N-1}{r}v'(r) = \lambda g(r,\ u(r),\ v(r),\ |v'(r)|),\ \ r \in I, \\
u(r_1) = u(r_2) = 0,\ v(r_1) = v(r_2) = 0.
\end{cases} \tag{3.2}
$$

Multiply both sides of the first formula of Eq (3.2) by $u(r)$, and multiply both sides of the second formula by $v(r)$. Then, add the two formulas together, by condition (F1) we have

$$\begin{aligned}
& -u''(r)u(r) - \frac{N-1}{r}u'(r)u(r) - v''(r)v(r) - \frac{N-1}{r}v'(r)v(r) \\
= \ & \lambda(f(r,\ u(r),\ v(r),\ |u'(r)|)u(r) + g(r,\ u(r),\ v(r),\ |v'(r)|)v(r)) \\
\leq \ & au^2(r) + bv^2(r) + cu'^2(r) + dv'^2(r) + e, \quad r \in I.
\end{aligned}$$

Multiply both sides of the above formula by r^{N-1}, we have

$$\begin{aligned}
& -(r^{N-1}u'(r))'u(r) - (r^{N-1}v'(r))'v(r) \\
\leq \ & r^{N-1}(au^2(r) + bv^2(r) + cu'^2(r) + dv'^2(r) + e) \\
\leq \ & {r_2}^{N-1}(au^2(r) + bv^2(r) + cu'^2(r) + dv'^2(r) + e), \quad r \in I.
\end{aligned}$$

By integrating on I, by Lemma 2.1 we have

$$\begin{aligned}
{r_1}^{N-1}(\|u'\|_2^2 + \|v'\|_2^2) &= {r_1}^{N-1}\Big(\int_{r_1}^{r_2} u'^2(r)dr + \int_{r_1}^{r_2} v'^2(r)dr\Big) \\
&\leq \int_{r_1}^{r_2} r^{N-1}u'^2(r)dr + \int_{r_1}^{r_2} r^{N-1}v'^2(r)dr \\
&\leq {r_2}^{N-1}(a\|u\|_2^2 + b\|v\|_2^2 + c\|u'\|_2^2 + d\|v'\|_2^2 + e(r_2 - r_1)) \\
&\leq {r_2}^{N-1}\Big(\frac{(r_1 - r_2)^2}{2}(a + b) + c + d\Big)(\|u'\|_2^2 + \|v'\|_2^2) + e{r_2}^{N-1}(r_2 - r_1),
\end{aligned}$$

namely,

$$\Big(1 - \frac{{r_2}^{N-1}}{{r_1}^{N-1}}\Big(\frac{(r_1 - r_2)^2}{2}(a + b) + c + d\Big)\Big)(\|u'\|_2^2 + \|v'\|_2^2) \leq \frac{{r_2}^{N-1}}{{r_1}^{N-1}}e(r_2 - r_1).$$

Hence,

$$\|u'\|_2^2 + \|v'\|_2^2 \leq \frac{\frac{{r_2}^{N-1}}{{r_1}^{N-1}}e(r_2 - r_1)}{1 - \frac{{r_2}^{N-1}}{{r_1}^{N-1}}\Big(\frac{(r_1-r_2)^2}{2}(a + b) + c + d\Big)} \triangleq C.$$

Then,

$$\|u'\|_2 \leq \sqrt{C},\ \|v'\|_2 \leq \sqrt{C}.$$

For all $r \in I$, we have

$$|u(r)| = \Big|\int_{r_1}^{r} u'(s)ds\Big| \leq \int_{r_1}^{r_2} |u'(s)|ds \leq \sqrt{r_2 - r_1}\|u'\|_2 \leq \sqrt{C(r_2 - r_1)},$$

namely,

$$\|u\|_C \leq \sqrt{C(r_2 - r_1)}.$$

Similarly, it can be obtained

$$\|v\|_C \leq \sqrt{C(r_2 - r_1)}.$$

Therefore,

$$\|(u,\ v)\|_C = \max\{\|u\|_C,\ \|v\|_C\} \leq \sqrt{C(r_2 - r_1)}.$$

By condition (F0), we have

$$|\lambda f(r,\ u,\ v,\ \xi)| \le |f(r,\ u,\ v,\ \xi)| \le G_M(|\xi|),\quad r \in I,$$

$$|\lambda g(r,\ u,\ v,\ \eta)| \le |g(r,\ u,\ v,\ \eta)| \le G_M(|\eta|),\quad r \in I.$$

Hence, λf and λg satisfy condition (F0). By Lemma 2.3, there exist constants $M_1 = M_1(M) > 0$ and $M_2 = M_2(M) > 0$, such that

$$\|(u',\ v')\|_C \le \max\{M_1,\ M_2\} := M_0.$$

Therefore,

$$\|(u,\ v)\|_{C^1} = \max\{\|(u,\ v)\|_C,\ \|(u',\ v')\|_C\} \le \max\{\sqrt{C(r_2 - r_1)},\ M_0\}.$$

Hence, the solution set of the Eq (3.1) is bounded in $C^1(I) \times C^1(I)$. By the Leray-Schauder fixed point, we know that A has fixed point $(u,\ v) \in C^1(I) \times C^1(I)$. By the definition of A, $(u,\ v)$ is a solution of BVP (1.2), namely, $(u(|x|),\ v(|x|))$ is a radial solution of BVP (1.1).

The proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2. First, we prove that $(F2) \Rightarrow (F1)$. For all $(r,\ u,\ v) \in I \times \mathbb{R} \times \mathbb{R},\ \xi,\ \eta \in \mathbb{R}^+$, we take $u_2 = u,\ v_2 = v,\ \xi_2 = \xi,\ \eta_2 = \eta,\ u_1 = v_1 = \xi_1 = \eta_1 = 0$ in (F2). Set

$$C_0 = \max_{r \in I}\{|f(r,\ 0,\ 0,\ 0)|,\ |g(r,\ 0,\ 0,\ 0)|\} + 1.$$

By condition (F2), we have

$$\begin{aligned}
& f(r,\ u,\ v,\ \xi)u + g(r,\ u,\ v,\ \eta)v \\
&= (f(r,\ u,\ v,\ \xi) - f(r,\ 0,\ 0,\ 0))u + (g(r,\ u,\ v,\ \eta) - g(r,\ 0,\ 0,\ 0))v \\
&\quad + f(r,\ 0,\ 0,\ 0)u + g(r,\ 0,\ 0,\ 0)v \\
&\le au^2 + bv^2 + c\xi^2 + d\eta^2 + |f(r,\ 0,\ 0,\ 0)u| + |g(r,\ 0,\ 0,\ 0)v| \\
&\le au^2 + bv^2 + c\xi^2 + d\eta^2 + C_0|u| + C_0|v| \\
&= au^2 + bv^2 + c\xi^2 + d\eta^2 + 2 \cdot \frac{\sqrt{\frac{2}{(r_1-r_2)^2} - (a+b) - \frac{2}{(r_1-r_2)^2}(c+d)}}{2}|u| \\
&\quad \cdot \frac{C_0}{\sqrt{\frac{2}{(r_1-r_2)^2} - (a+b) - \frac{2}{(r_1-r_2)^2}(c+d)}} + 2 \cdot \frac{\sqrt{\frac{2}{(r_1-r_2)^2} - (a+b) - \frac{2}{(r_1-r_2)^2}(c+d)}}{2}|v| \\
&\quad \cdot \frac{C_0}{\sqrt{\frac{2}{(r_1-r_2)^2} - (a+b) - \frac{2}{(r_1-r_2)^2}(c+d)}} \\
&\le au^2 + bv^2 + c\xi^2 + d\eta^2 + \frac{\frac{2}{(r_1-r_2)^2} - (a+b) - \frac{2}{(r_1-r_2)^2}(c+d)}{4}u^2 \\
&\quad + \frac{\frac{2}{(r_1-r_2)^2} - (a+b) - \frac{2}{(r_1-r_2)^2}(c+d)}{4}v^2 + \frac{2C_0{}^2}{\frac{2}{(r_1-r_2)^2} - (a+b) - \frac{2}{(r_1-r_2)^2}(c+d)}
\end{aligned}$$

$$= \left(a + \frac{\frac{2}{(r_1 - r_2)^2} - (a+b) - \frac{2}{(r_1 - r_2)^2}(c+d)}{4}\right)u^2 + \left(b + \frac{\frac{2}{(r_1 - r_2)^2} - (a+b) - \frac{2}{(r_1 - r_2)^2}(c+d)}{4}\right)v^2$$

$$+ c\xi^2 + d\eta^2 + \frac{2{C_0}^2}{\frac{2}{(r_1 - r_2)^2} - (a+b) - \frac{2}{(r_1 - r_2)^2}(c+d)}.$$

Let

$$a_1 = a + \frac{\frac{2}{(r_1 - r_2)^2} - (a+b) - \frac{2}{(r_1 - r_2)^2}(c+d)}{4} \geq 0,$$

$$b_1 = b + \frac{\frac{2}{(r_1 - r_2)^2} - (a+b) - \frac{2}{(r_1 - r_2)^2}(c+d)}{4} \geq 0,$$

$$c_1 = c \geq 0\,, \quad d_1 = d \geq 0,$$

$$e_1 = \frac{2{C_0}^2}{\frac{2}{(r_1 - r_2)^2} - (a+b) - \frac{2}{(r_1 - r_2)^2}(c+d)} \geq 0,$$

we have

$$f(r,\ u,\ v,\ \xi)u + g(r,\ u,\ v,\ \eta)v \leq a_1u^2 + b_1v^2 + c_1\xi^2 + d_1\eta^2 + e_1,$$

where $(r,\ u,\ v) \in I \times \mathbb{R} \times \mathbb{R},\ \xi,\ \eta \in \mathbb{R}^+$ and $\frac{(r_1 - r_2)^2}{2}(a_1 + b_1) + c_1 + d_1 = \frac{\frac{(r_1 - r_2)^2}{2}(a+b) + c + d + 1}{2} < 1$.

Hence, f and g satisfy condition (F1), by Theorem 1.1, BVP (1.1) has at least one radial solution.

Next, we prove the uniqueness. Set $(u_1,\ v_1),\ (u_2,\ v_2) \in C^2(I) \times C^2(I)$ are the solution of BVP (1.1), then

$$\begin{cases} -u_1''(r) - \frac{N-1}{r}u_1'(r) = f(r,\ u_1(r),\ v_1(r),\ |u_1'(r)|), & r \in I, \\ -v_1''(r) - \frac{N-1}{r}v_1'(r) = g(r,\ u_1(r),\ v_1(r),\ |v_1'(r)|), & r \in I, \\ u_1(r_1) = u_1(r_2) = 0,\ v_1(r_1) = v_1(r_2) = 0. \end{cases} \tag{3.3}$$

$$\begin{cases} -u_2''(r) - \frac{N-1}{r}u_2'(r) = f(r,\ u_2(r),\ v_2(r),\ |u_2'(r)|), & r \in I, \\ -v_2''(r) - \frac{N-1}{r}v_2'(r) = g(r,\ u_2(r),\ v_2(r),\ |v_2'(r)|), & r \in I, \\ u_2(r_1) = u_2(r_2) = 0,\ v_2(r_1) = v_2(r_2) = 0. \end{cases} \tag{3.4}$$

Subtract the first formula of Eq (3.4) and the first formula of Eq (3.3), we get

$$-(u_2''(r) - u_1''(r)) - \frac{N-1}{r}(u_2'(r) - u_1'(r))$$

$$= f(r,\ u_2(r),\ v_2(r),\ |u_2'(r)|) - f(r,\ u_1(r),\ v_1(r),\ |u_1'(r)|), \quad r \in I. \tag{3.5}$$

Similarly, it can be obtained

$$-(v_2''(r) - v_1''(r)) - \frac{N-1}{r}(v_2'(r) - v_1'(r))$$

$$= g(r,\ u_2(r),\ v_2(r),\ |v_2'(r)|) - g(r,\ u_1(r),\ v_1(r),\ |v_1'(r)|), \quad r \in I. \tag{3.6}$$

Multiply both sides of Eq (3.5) by $u_2(r) - u_1(r)$, and multiply both sides of Eq (3.6) by $v_2(r) - v_1(r)$. Then, add the two formulas together, by condition (F2), for all $r \in I$, we have

$$\begin{aligned}
&-(u_2''(r) - u_1''(r))(u_2(r) - u_1(r)) - \frac{N-1}{r}(u_2'(r) - u_1'(r))(u_2(r) - u_1(r)) \\
&-(v_2''(r) - v_1''(r))(v_2(r) - v_1(r)) - \frac{N-1}{r}(v_2'(r) - v_1'(r))(v_2(r) - v_1(r)) \\
=& (f(r,\ u_2(r),\ v_2(r),\ |u_2'(r)|) - f(r,\ u_1(r),\ v_1(r),\ |u_1'(r)|))(u_2(r) - u_1(r)) \\
&+ (g(r,\ u_2(r),\ v_2(r),\ |v_2'(r)|) - g(r,\ u_1(r),\ v_1(r),\ |v_1'(r)|))(v_2(r) - v_1(r)) \\
\le& a(u_2(r)-u_1(r))^2 + b(v_2(r)-v_1(r))^2 + c(|u_2'(r)|-|u_1'(r)|)^2 + d(|v_2'(r)|-|v_1'(r)|)^2.
\end{aligned}$$

Multiply both sides of the above formula by r^{N-1}, we have

$$\begin{aligned}
&-\Big(r^{N-1}(u_2'(r) - u_1'(r))\Big)'(u_2(r) - u_1(r)) - \Big(r^{N-1}(v_2'(r) - v_1'(r))\Big)'(v_2(r) - v_1(r)) \\
\le\ & r^{N-1}\Big(a(u_2(r)-u_1(r))^2 + b(v_2(r)-v_1(r))^2 + c(|u_2'(r)|-|u_1'(r)|)^2 + d(|v_2'(r)|-|v_1'(r)|)^2\Big) \\
\le\ & {r_2}^{N-1}\Big(a(u_2(r)-u_1(r))^2 + b(v_2(r)-v_1(r))^2 + c(|u_2'(r)|-|u_1'(r)|)^2 + d(|v_2'(r)|-|v_1'(r)|)^2\Big).
\end{aligned}$$

By integrating on I, by Lemma 2.1 we have

$$\begin{aligned}
& {r_1}^{N-1}(\|u_2' - u_1'\|_2^2 + \|v_2' - v_1'\|_2^2) \\
=\ & {r_1}^{N-1}\Big(\int_{r_1}^{r_2}(u_2'(r) - u_1'(r))^2 dr + \int_{r_1}^{r_2}(v_2'(r) - v_1'(r))^2 dr\Big) \\
\le\ & \int_{r_1}^{r_2} r^{N-1}(u_2'(r) - u_1'(r))^2 dr + \int_{r_1}^{r_2} r^{N-1}(v_2'(r) - v_1'(r))^2 dr \\
\le\ & {r_2}^{N-1}\Big(a\|u_2 - u_1\|_2^2 + b\|v_2 - v_1\|_2^2 + c\|u_2' - u_1'\|_2^2 + d\|v_2' - v_1'\|_2^2\Big) \\
\le\ & {r_2}^{N-1}\Big(\frac{(r_1 - r_2)^2}{2}(a + b) + c + d\Big)(\|u_2' - u_1'\|_2^2 + \|v_2' - v_1'\|_2^2),
\end{aligned}$$

namely,

$$0 \le \Big(1 - \frac{{r_2}^{N-1}}{{r_1}^{N-1}}\Big(\frac{(r_1 - r_2)^2}{2}(a + b) + c + d\Big)\Big)(\|u_2' - u_1'\|_2^2 + \|v_2' - v_1'\|_2^2) \le 0.$$

Hence,

$$\|u_2' - u_1'\|_2^2 + \|v_2' - v_1'\|_2^2 = 0,$$

namely $u_2' - u_1' = 0,\ v_2' - v_1' = 0$, then, $u_2 - u_1 = C_1,\ v_2 - v_1 = C_2$, where $C_1,\ C_2$ are constants. From the boundary conditions, $C_1 = C_2 = 0$, namely, $u_2 = u_1,\ v_2 = v_1$. Thus, BVP (1.1) has a unique radial solution.

The proof of Theorem 1.2 is completed. □

Example 3.1. Consider the elliptic boundary value problem

$$\begin{cases}
-\triangle u = -u^3 v^2 + u - u|\nabla u|^2 + \sin|x|, \quad x \in \Omega, \\
-\triangle v = -v^3 - u^2 v + 3v - 2v|\nabla v|^2 + 1, \\
u|_{\partial\Omega} = 0,\ v|_{\partial\Omega} = 0.
\end{cases} \tag{3.7}$$

The corresponding nonlinear term of Eq (3.7) are

$$f(r,\ u,\ v,\ \xi) = -u^3v^2 + u - u\xi^2 + \sin r, \quad g(r,\ u,\ v,\ \eta) = -v^3 - u^2v + 3v - 2v\eta^2 + 1.$$

It is easy to see that f and g are quadratic growth with respect to ξ and η respectively, satisfying condition (F0). We next verify that f and g satisfy condition (F1), take $r_1 = \frac{1}{2},\ r_2 = 1,\ a = 1 + \varepsilon,\ b = 3 + \varepsilon,\ c = d = 0,\ e = \frac{1}{2\varepsilon}$. When $\varepsilon < 2$, we have $\frac{(r_1-r_2)^2}{2}(a + b) < 1$, f and g satisfy

$$\begin{aligned}
&f(r,\ u,\ v,\ \xi)u + g(r,\ u,\ v,\ \eta)v \\
=&\ -u^4v^2 + u^2 - u^2\xi^2 + u\sin r - v^4 - u^2v^2 + 3v^2 - 2v^2\eta^2 + v \\
\leq&\ u^2 + 3v^2 + |u|\sin r + |v| \\
=&\ u^2 + 3v^2 + 2\cdot\sqrt{\varepsilon}|u|\cdot\frac{\sin r}{2\sqrt{\varepsilon}} + 2\cdot\sqrt{\varepsilon}|v|\cdot\frac{1}{2\sqrt{\varepsilon}} \\
\leq&\ u^2 + 3v^2 + \varepsilon u^2 + \frac{\sin^2 r}{4\varepsilon} + \varepsilon v^2 + \frac{1}{4\varepsilon} \\
\leq&\ (1+\varepsilon)u^2 + (3+\varepsilon)v^2 + \frac{1}{2\varepsilon} \\
=&\ au^2 + bv^2 + e.
\end{aligned}$$

Thus, $f(r,\ u,\ v,\ \xi)$ and $g(r,\ u,\ v,\ \eta)$ satisfy condition (F1). By Theorem 1.1, BVP (3.7) has at least one radial solution.

4. Conclusions

It is well known that elliptic equations arises in many different areas of applied mathematics and physics, for instance, incineration theory of gases, solid state physics, variational methods and optimal control. Due to the appearance of the gradient term in the nonlinearity, the equation system has no variational structure and the variational method cannot be applied to it directly. Therefore, we given existence and uniqueness results of radial solution in the case of f and g superlinear or sublinear, we replace the previous independent conditions with the correlation conditions of f and g. In this paper, we just consider the existence of solutions. However, the properties of the solution have not been fully discussed.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare that they have no competing interests.

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