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Research article

Existence criteria for fractional differential equations using the topological degree method

Kottakkaran Sooppy Nisar[1,*], **Suliman Alsaeed**[1,2], **Kalimuthu Kaliraj**[3], **Chokkalingam Ravichandran**[4], **Wedad Albalawi**[5] **and Abdel-Haleem Abdel-Aty**[6]

[1] Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia
[2] Applied Sciences College, Department of Mathematical Sciences, Umm Al-Qura University, P.O. Box 715, Makkah 21955, Saudi Arabia
[3] Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai 600005, India
[4] Department of Mathematics, Kongunadu Arts and Science College, Coimbatore-641029, India
[5] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
[6] Department of Physics, College of Sciences, University of Bisha, P.O. Box 344, Bisha 61922, Saudi Arabia

* **Correspondence:** Email: n.sooppy@psau.edu.sa.

Abstract: In this work, we analyze the fractional order by using the Caputo-Hadamard fractional derivative under the Robin boundary condition. The topological degree method combined with the fixed point methodology produces the desired results. Finally to show how the key findings may be utilized, applications are presented.

Keywords: fractional calculus; integro-differential equation; fixed point techniques; topological degree method
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1. Introduction

Fractional calculus is a field of mathematics that expands the notion of differentiation and integration beyond integer orders. These operations are applicable to all real numbers, including non-integer values. Fractional calculus has found practical applications in a variety of physical systems, as can be seen in [2, 16, 17, 20, 30, 40, 41] and some classic books [22, 27, 29, 39].

Based on the literature, fixed point theory has been applied for many years to establish that differential equations have a solution [26, 31, 35, 46, 48, 49]. Mahwin [32] in their paper made use of the topological degree theory (TDT) to solve integral equations for the first time. Isaia [24] theoretically applied TDT to analyze some integral equations. Use of TDT can also be observed in [14, 42, 43, 47].

To date, a lot of good work with integro-differential equations has been conducted, including the studies described in [7, 34]. Zuo et al. [33] derived the following fractional integro-differential equations with impulsive and antiperiodic boundary conditions:

$$\begin{cases} D^{\gamma}\zeta(q) + \lambda\zeta(q) = f(q, \zeta(q), P\zeta(q), S\zeta(q)), \quad q \in J' \\ \delta\zeta(q_i) = I_i(\zeta_i), \quad i = 1, 2, \ldots, m \\ \zeta(0) = -\zeta(1) \end{cases}$$

where $J' = J \setminus \{q_1, q_2, \ldots, q_m\}, 0 < \gamma \leq 1, \lambda > 0$ and D^{γ} is denoted as CFD, $1 < \gamma \leq 2$. Here $f \in C(J \times \mathscr{R} \times \mathscr{R} \times \mathscr{R}, \mathscr{R}), J = [0, 1]$ is the integro-differential function and P and S are linear operators:

$$(P\zeta)(q) = \int_0^q k(q, s)\zeta(s)ds$$
$$(S\zeta)(q) = \int_0^1 h(q, s)\zeta(s)ds$$

where $q \in J,\ k \in C(D, \mathscr{R}),\ D = \{(q, s) \in J \times J : q \geq s\}, h \in C(J \times J, \mathscr{R})$.

It is observed from the literature that for last many years, fixed point theory has been used to prove the existence of a solution to the differential equations. However, the use of fixed point theory requires strong conditions, which severely limits its applicability. Also the uniqueness is proved via the Banach contraction principle, which is applied to find a unique solution for the defined problem. It is noticed that most of the work on the topic of fractional differential equations (FDEs) involves either the RL or CFD. While these derivatives are common place in the study of FDEs, the Hadamard fractional derivative (HFD) is another kind of fractional derivatives. This kind of derivative was introduced by Hadamard [21]. This fractional derivative differs from the other ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains the logarithmic function of an arbitrary exponent. In [25], we see the modification of the HFD into a more suitable one called the Caputo-Hadamard fractional derivative (CHFD). Applications of where Hadamard derivative and the Hadamard derivative integral can be found in papers by Butzer et al. [11–13]. Other important results dealing with studies on fractional calculus using Hadamard derivatives can be seen in [4, 6, 8, 9, 18, 19, 23, 36–38, 45].

A paper by Jarad et al. [25], deals with the CHFD by modifying the to be of caputo type. This is familiar with different kinds of boundary conditions like the Neumann boundary condition and the Dirichlet boundary condition. A weighed combination of these boundary conditions is called the Robin boundary condition. It finds its applications in fields such as physics. The Caputo-Hadamard (CH) derivative type of FDEs with boundary value problems are described in [1, 3, 5, 10].

TDT-based existence results for FIDEs with CH-derivatives have the following form:

$$\begin{cases} {}^{C}_{H}D^{\nu}\zeta(q) = g(q, \zeta(q), P\zeta(q), S\zeta(q)), & q \in J := [0, L] \\ a\zeta(1) + b^{C}_{H}D^{\gamma}\zeta(1) = c_1^{H}\mathcal{I}^{\nu_1}\zeta(\zeta_1), & 1 < \zeta_1 < L, \nu_1 > 0 \\ c\zeta(L) + d^{C}_{H}D^{\gamma}\zeta(L) = c_2^{H}\mathcal{I}^{\nu_2}\zeta(\zeta_2), & 1 < \zeta_2 < L, \nu_2 > 0 \end{cases} \tag{1.1}$$

where ${}^{C}_{H}D^{\nu}$, ${}^{C}_{H}D^{\gamma}$ are CH-derivatives of order ν, γ, respectively with $1 < \nu \leq 2, 0 < \gamma \leq 1$, HD integral of order ${}^{H}\mathcal{I}^{\nu_i}$, ν_i, $i \in [1, 2]$ and $f \in C(J \times \mathscr{R} \times \mathscr{R} \times \mathscr{R}, \mathscr{R})$ is the continuous function; P and S are linear operators;

$$(P\zeta)(q) = \int_0^q k(q, s)\zeta(s)ds$$

$$(S\zeta)(q) = \int_0^1 h(q, s)\zeta(s)ds.$$

Let $a, b, c, d \in \mathscr{R}$ such that

$$\chi = \Big(a - \frac{c_1(\log\zeta_1)^{\nu_1}}{\Gamma(\nu_1 + 1)}\Big)\Big(c\log L + \frac{d(\log L)^{1-\gamma}}{\Gamma(2 - \gamma)} - \frac{c_2(\log\zeta_2)^{\nu_2+1}}{\Gamma(\nu_2 + 2)}\Big) + \frac{c_1(\log\zeta_1)^{\nu_1+1}}{\Gamma(\nu_1 + 2)}\Big(c\log L - \frac{c_2(\log\zeta_2)^{\nu_2}}{\Gamma(\nu_2 + 1)}\Big) \neq 0. \tag{1.2}$$

In this paper, we determine the existence results via TDT in Section 2. Additionally, we discuss the FIDEs existence results under boundary conditions. An appropriate illustration and conclusion are provided in Sections 4 and 5.

2. Facts

Here, we shall establish basic results and definitions for our analysis. We shall refer to the notations and results from [15]. Let the Banach space (BS) be X and $\mathscr{B} \subset \mathscr{P}(X)$ be bounded subsets.

Definition 2.1. *[14] Let $\epsilon : \mathscr{B} \to \mathscr{R}_+$,*

$$\zeta(B) := \inf\{d > 0 : B \text{ permits finite cover by sets of diameter} \leq d\}$$

where, Kuratowski- measure of non compactness is $B \in \mathscr{B}$.

Let B be a compact set and set B of a space X is compact if and only if it is complete and totally bounded. The value of ϵ is the measured value, and the value of $\epsilon(B)$ is 0. The set is compact, when the value is 0. The larger the value of ϵ, the less it is like a compact set.

Proposition 2.2. *[14] For bounded subsets $\mathscr{D}$, $\mathscr{D}_1$, $\mathscr{D}_2$ on a BS,*

(1) $\epsilon(\mathscr{D}) = 0 \Leftrightarrow \overline{\mathscr{D}}$ *is compact,*

(2) $\epsilon(\lambda\mathscr{D}) = |\lambda|\epsilon(\mathscr{D}), \quad \lambda \in \mathscr{R}$,

(3) $\zeta(\mathscr{D}_1 + \mathscr{D}_2) \leq \epsilon(\mathscr{D}_1) + \epsilon(\mathscr{D}_2), \quad \mathscr{D}_1, \mathscr{D}_2 \in \mathscr{B}$,

(4) $\mathscr{D}_1 \subset \mathscr{D}_2 \Rightarrow \epsilon(\mathscr{D}_1) \leq \epsilon(\mathscr{D}_2)$,

(5) $\epsilon(\mathscr{D}_1 \cup \mathscr{D}_2) = \max\{\epsilon(\mathscr{D}_1), \epsilon(\mathscr{D}_2)\}$,

(6) $\epsilon(conv\ \mathscr{D}) = \epsilon(\mathscr{D})$,

(7) $\epsilon(\overline{\mathscr{D}}) = \epsilon(\mathscr{D})$.

Let $\Psi := \{\zeta : [0, \mathscr{T}] \to \mathscr{R} : \zeta \in C(I')\}$ and $(\Psi, \|\cdot\|)$ be a BS under $\|\zeta\| := \sup\{|\zeta(q)| : q \in [0, \mathscr{T}]\}$.

Definition 2.3. *[14] Suppose a continuous bounded map is $\mathscr{F} : \varpi \to \mathscr{Y}$ and $\varpi \subset \mathscr{Y}$ and $\exists\ k \geq 0$ such that $\epsilon(\mathscr{F}(\mathscr{H})) \leq k\epsilon(\mathscr{H})\forall \mathscr{H} \subset \varpi$. If $\mathscr{F}$ is an ϵ-contraction that implies $k < 1$.*

Definition 2.4. *[14, 15, 24] Let a $C_\epsilon(\varpi)$ be a class of all ϵ-condensing maps $\mathscr{F} : \varpi \to X$. An ϵ-condensing map $\mathscr{F} : \mathscr{Y} \to \mathscr{F}$ if $\forall A \in \mathscr{B}$, $\epsilon(\mathscr{F}(A)) \leq \epsilon(A)$.*

Theorem 2.5. *[14] Let map $\mathscr{F} : \mathscr{Y} \to \mathscr{Y}$ be ϵ-condensing; then,*

$$\mathscr{H} = \{\zeta \in \mathscr{Y} : \exists\ 1 \leq \lambda \leq \mathscr{L} \ni \zeta = \lambda T\zeta\},$$

such that $\mathscr{H} \subset \mathscr{B}_r(0)$, also $\mathscr{H}$ is a bounded set in Ψ, so $\exists\ r > 0$

$$D(I - \lambda T, \mathscr{B}_r(0), 0) = 1, \forall \lambda \in [1, \mathscr{L}].$$

Therefore, T has a fixed point.

Definition 2.6. *[29] Let $\nu > 0$ be an Hadamard derivative integral of order $\zeta \in L^1(J)$ is;*

$${}^H\mathscr{I}^\nu\zeta(q) = \frac{1}{\Gamma(\nu)}\int_1^q \log(q/s)^{\nu-1}\zeta(s)\frac{ds}{s}$$

where

$$\Gamma(\nu) = \int_0^\infty e^{-q}q^{\nu-1}dt, \quad \nu > 0.$$

Let $\delta = q\dfrac{d}{dt}, \quad \nu > 0, n = [\nu] + 1.$

Definition 2.7. *[29] Let $\nu > 0$ be the Hadamard derivative and $\zeta \in \Psi$ is;*

$${}^HD^\nu\zeta(q) = \delta^n({}^H\mathscr{I}^{n-\nu}\zeta(q)).$$

Definition 2.8. *[25, 29] Let $\nu > 0$ be the CH-derivative and $\zeta \in \Psi$ is;*

$${}^H_CD^\nu\zeta(q) = {}^H\mathscr{I}^{n-\nu}\delta^n\zeta(q).$$

Lemma 2.9. *[25, 29] Let $\nu > 0, r > 0$ and $n = [\nu] + 1$.*

(1) ${}^H\mathscr{I}^\nu(\log\frac{q}{a})^{r-1} = \dfrac{\Gamma(r)}{\Gamma(\nu + r)}(\log\frac{q}{a})^{\nu+r+1}$.

Let $a = 1$ and $r = 1$; we get that ${}^H\mathscr{I}^\nu(1)(\nu) = \dfrac{1}{\Gamma(\nu + 1)}(\log(q))^{\nu+1}$.

(2) ${}^H_CD^\nu(\log\frac{q}{a})^{r-1} = \begin{cases} \dfrac{\Gamma(r)}{\Gamma(r-\nu)}(\log\frac{q}{a})^{r-\nu-1}, & r > n \\ 0, & r \in \{0, 1, \ldots, n-1\} \end{cases}$.

Lemma 2.10. *[25, 29] Let* $nu_1, \nu_2 > 0$ *and* $\zeta \in \Psi$*; then*

(1) ${}^{H}\mathscr{I}^{\nu_1}({}^{H}\mathscr{I}^{\nu_2}\zeta(q)) = ({}^{H}\mathscr{I}^{\nu_1+\nu_2}\zeta(q))$;

(2) ${}^{H}_{C}D^{\nu_1}({}^{H}\mathscr{I}^{\nu_2}\zeta(q)) = ({}^{H}\mathscr{I}^{\nu_2-\nu_1}\zeta(q))$;

(3) ${}^{H}_{C}D^{\nu_1}({}^{H}\mathscr{I}^{\nu_1}\zeta(q)) = \zeta(q)$.

Lemma 2.11. *[25, 29] Let* $\nu > 0$ *and* $n = [\nu] + 1$. *Let* $\zeta \in \Psi$ *be the CH derivative of the FDEs*

$$ {}^{H}_{C}D^{\nu}\zeta(q) = 0 $$

has a solution as

$$ \zeta(q) = \sum_{i=0}^{n} c_i(\log q)^i $$

$$ {}^{H}I^{\nu}({}^{H}_{C}D^{\nu}\zeta(q)) = \zeta(q) + \sum_{i=0}^{n-1} c_i(\log q)^i, \;\; c_i \in \mathscr{R}, i = 0, 1. \dots, n-1. $$

3. Main results

We shall define some hypotheses:

(A1) $\exists$ constants $A_1, A_2 > 0$ and $p \in [0, 1)$ such that

$$ |g(q, \zeta(q), P\zeta(q), S\zeta(q))| \leq (\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p), \;\; \forall\, \zeta \in \Psi. $$

(A2) $\exists$ constants α, β, χ such that

$$ \begin{aligned} |g(q, \zeta_1(q), P\zeta_1(q), S\zeta_1(q)) - g(q, \zeta_2(q), P\zeta_2(q), S\zeta_2(q))| \leq \alpha\|\zeta_1 - \zeta_2\| + \beta\|P\zeta_1 - P\zeta_2\| \\ + \chi\|S\zeta_1 - S\zeta_2\|. \end{aligned} $$

We define $k_{\max} = \sup\limits_{q\in J} \int_0^q |k(q, s)|ds$ and $h_{\max} = \sup\limits_{q\in J} \int_0^1 |h(q, s)|ds$.
Thus using (A2),

$$ \begin{aligned} &|g(q, \zeta_1(q), P\zeta_1(q), S\zeta_1(q)) - g(q, \zeta_2(q), P\zeta_2(q), S\zeta_2(q))| \\ &\leq \alpha\|\zeta_1 - \zeta_2\| + \beta\|P\zeta_1 - P\zeta_2\| + \chi\|S\zeta_1 - S\zeta_2\| \\ &\leq \alpha\|\zeta_1 - \zeta_2\| + \beta \int_0^q |k(q, s)||\zeta_1(q) - \zeta_2(q)|ds + \chi \int_0^1 |h(q, s)||\zeta_1(q) - \zeta_2(q)|ds \\ &\leq (\alpha + \beta k_{\max} + \chi h_{\max})\|\zeta_1 - \zeta_2\|. \end{aligned} $$

Now we prove the existence result:

Lemma 3.1. *[14] Let h be a continuous function on J; then, we have the following FIDE:*

$$ \begin{cases} {}^{C}_{H}D^{\nu}\zeta(q) = h(q), \quad q \in J := [0, L] \\ a\zeta(1) + b{}^{C}_{H}D^{\gamma}\zeta(1) = c^{H}_{1}\mathscr{I}^{\nu_1}\zeta(\zeta_1), \quad 1 < \zeta_1 < L, \nu_1 > 0 \\ c\zeta(L) + d{}^{C}_{H}D^{\gamma}\zeta(L) = c^{H}_{2}\mathscr{I}^{\nu_2}\zeta(\zeta_2), \quad 1 < \zeta_2 < L, \nu_2 > 0 \end{cases} \tag{3.1} $$

have a unique solution given by

$$\zeta(q) = {}^H\mathscr{I}^{\nu}h(q) + K_1(q){}^H\mathscr{I}^{\nu_1+\nu}h(\zeta_1) + K_2(q)(c_2{}^H\mathscr{I}^{\nu_2+\nu}h(\zeta_2) - (c{}^H\mathscr{I}^{\nu}h(L) + d{}^H\mathscr{I}^{\nu-\gamma}h(L))) \tag{3.2}$$

where,

- $K_1(q) = c_1(\chi_1 - \chi_2 q), \quad K_2(q) = c_1\chi_3 + \chi_4 q$
- $\chi_1 = \frac{1}{\chi}\Big(c\log L + \frac{d(\log L)^{1-\gamma}}{\Gamma(2-\gamma)} - \frac{c_2(\log\zeta_2)^{\nu_2+1}}{\Gamma(\nu_2+2)}\Big)$
- $\chi_2 = \frac{1}{\chi}\Big(c\log L - \frac{c_2(\log\zeta_2)^{\nu_2}}{\Gamma(\nu_2+1)}\Big)$
- $\chi_3 = \frac{1}{\chi}\Big(\frac{c_1(\log\zeta_1)^{\nu_1+1}}{\Gamma(\nu_1+2)}\Big)$
- $\chi_4 = \frac{1}{\chi}\Big(a - \frac{c_1(\log\zeta_1)^{\nu_1}}{\Gamma(\nu_1+1)}\Big)$

and χ is given by (1.2).

Proof. By Lemma 2.11, Eq (3.1) becomes,

$$\zeta(q) = {}^H\mathscr{I}^{\nu}h(q) + k_0 + k_1\log(q), \quad k_0, k_1 \in \mathscr{R}.$$

By using boundary conditions, we have

$$\begin{aligned} {}^H\mathscr{I}^{\nu_i}\zeta(\zeta_i) &= {}^H\mathscr{I}^{\nu_1+\nu}h(\zeta_i) + k_0\frac{(\log\zeta_i)^{\nu_i}}{\Gamma(\nu_i+1)} + k_1\frac{(\log\zeta_i)^{\nu_i+1}}{\Gamma(\nu_i+2)}, \quad i = 1,2 \\ {}^C_HD^{\gamma}\zeta(L) &= {}^H\mathscr{I}^{\nu-\gamma}h(q) + k_1\frac{\Gamma(2)(\log L)^{1-\gamma}}{\Gamma(2-\gamma)}. \end{aligned}$$

Solving for k_0, k_1 we get the following solutions:

$$k_0 = c_1\chi_1{}^H\mathscr{I}^{\nu+\nu_1}\zeta(\zeta_1) + \chi_3(c_2{}^H\mathscr{I}^{\nu+\nu_2}\zeta(\zeta_2) - (c{}^H\mathscr{I}^{\nu}h(L) + d{}^H\mathscr{I}^{\nu-\gamma}h(L)))$$

and,

$$k_1 = c_1\chi_4({}^H\mathscr{I}^{\nu+\nu_2}h(\zeta_2) - (c{}^H\mathscr{I}^{\nu}h(L) + d{}^H\mathscr{I}^{\nu-\gamma}h(L))) - c_1\chi_2{}^H\mathscr{I}^{\nu}h(\zeta_1).$$

Substituting for k_0 and k_1 we get (3.2). □

In view of the problem (1.1), by Lemma 3.1, we get,

$$\begin{aligned} \zeta(q) = {}^H\mathscr{I}^{\nu}g_{\zeta}(q) &+ K_1(q){}^H\mathscr{I}^{\nu_1+\nu}g_{\zeta}(\zeta_1) \\ &+ K_2(q)(c_2{}^H\mathscr{I}^{\nu_2+\nu}g_{\zeta}(\zeta_2) - (c{}^H\mathscr{I}^{\nu}g_{\zeta}(L) + d{}^H\mathscr{I}^{\nu-\gamma}g_{\zeta}(L))) \end{aligned} \tag{3.3}$$

we denote $g(q\zeta(q), P\zeta(q), S\zeta(q))$ by g_{ζ}; then, we have

$$\begin{aligned} &K_1(q) = c_1(\chi_1 - \chi_2 q), \quad K_2(q) = c_1\chi_3 + \chi_4 q \\ &\chi_1 = \frac{1}{\chi}\Big(c\log L + \frac{d(\log L)^{1-\gamma}}{\Gamma(2-\gamma)} - \frac{c_2(\log\zeta_2)^{\nu_2+1}}{\Gamma(\nu_2+2)}\Big), \;\; \chi_2 = \frac{1}{\chi}\Big(c\log L - \frac{c_2(\log\zeta_2)^{\nu_2}}{\Gamma(\nu_2+1)}\Big), \\ &\chi_3 = \frac{1}{\chi}\Big(\frac{c_1(\log\zeta_1)^{\nu_1+1}}{\Gamma(\nu_1+2)}\Big), \quad \chi_4 = \frac{1}{\chi}\Big(a - \frac{c_1(\log\zeta_1)^{\nu_1}}{\Gamma(\nu_1+1)}\Big) \end{aligned}$$

and χ is given by (1.2). The next steps are as follows:

(1) Define $T_1 : \Psi \to \Psi$ as $T_1\zeta(q) = {}^H \mathscr{I}^{\nu} g_{\zeta}(q)$.

(2) Define $T_2 : \Psi \to \Psi$ as $T_2\zeta(q) = K_1(q)^H \mathscr{I}^{\nu_1+\nu} g_{\zeta}(\zeta_1) + K_2(q) c_2^H \mathscr{I}^{\nu_2+\nu} g_{\zeta}(\zeta_2)$.

(3) Define $T_3 : \Psi \to \Psi$ as $T_3\zeta(q) = K_2(q)(c^H \mathscr{I}^{\nu} g_{\zeta}(L) + d^H \mathscr{I}^{\nu-\gamma} g_{\zeta}(L)))$.

Let $T : \Psi \to \Psi$ given that $T = T_1 + T_2 + T_3$. Thus the problem is reduced to finding the fixed points of the operator T.

Theorem 3.2. *$T_1 : \Psi \to \Psi$ is Lipschitz-continuous with the Lipschitz constant $\frac{(\log L)^{\nu}}{\Gamma(\nu+1)}(\alpha + \beta k_{\max} + \chi h_{\max})$. It also satisfies the following growth relation:*

$$\|T_1\zeta(q)\| \leq \frac{(\log L)^{\nu}}{\Gamma(\nu+1)}(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p).$$

Proof. Let $\zeta_1, \zeta_2 \in \Psi$; then

$$\begin{aligned}
|T_1\zeta_1(q) - T_1\zeta_2(q)| &\leq |^H \mathscr{I}^{\nu} g_{\zeta_1}(q) -^H \mathscr{I}^{\nu} g_{\zeta_2}(q)| \\
&\leq^H \mathscr{I}^{\nu} |g_{\zeta_1} - g_{\zeta_2}|(q) \\
&\leq^H \mathscr{I}^{\nu}(1)(T)(\alpha + \beta k_{\max} + \chi h_{\max})\|\zeta_1 - \zeta_2\| \\
&= \frac{(\log L)^{\nu}}{\Gamma(\nu+1)}(\alpha + \beta k_{\max} + \chi h_{\max})\|\zeta_1 - \zeta_2\|.
\end{aligned}$$

This is true for all $q \in J$. Thus when we take the supremum over $q \in J$,

$$\|T_1\zeta_1(q) - T_1\zeta_2(q)\| \leq \frac{(\log L)^{\nu}}{\Gamma(\nu+1)}(\alpha + \beta k_{\max} + \chi h_{\max})\|\zeta_1 - \zeta_2\|.$$

Hence, T_1 is a Lipschitz constant provided that

$$\frac{(\log L)^{\nu}}{\Gamma(\nu+1)}(\alpha + \beta k_{\max} + \chi h_{\max}).$$

For the growth relation, we have,

$$\begin{aligned}
|T_1\zeta(q)| &\leq |^H \mathscr{I}^{\nu} g_{\zeta}(q)| \\
&= \frac{(\log L)^{\nu}}{\Gamma(\nu+1)}(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p).
\end{aligned}$$

Since this is true $\forall \ \ q \in J$, taking the supremum over all q, we have

$$\|T_1\zeta(q)\| \leq \frac{(\log L)^{\nu}}{\Gamma(\nu+1)}(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p).$$

□

Theorem 3.3. *Assume that the operator T_2 is continuous and fulfills the following growth relation:*

$$\|T_2\zeta(q)\| \leq C_{T_2}(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p)$$

where,

$$C_{T_2} = |c_1||\chi_1| + |c_2||\chi_2|L\frac{(\log \zeta_1)^{\nu+\nu_1}}{\Gamma(\nu_1+\nu+1)} + (|c_1||c_2||\chi_3| + |\chi_4|L)\frac{(\log \zeta_2)^{\nu+\nu_2}}{\Gamma(\nu_2+\nu+1)}.$$

Proof. Let ζ_n be a sequence in Ψ that converges to $\zeta \in \Psi$. Let g_ζ by continuous; it follows that $g_{\zeta_n} \to g_\zeta$. So, T_2 is continuous according to the Lebesgue dominated convergence theorem (LDCT).

$$\begin{aligned}|T_2\zeta(q)| &= |K_1(q)^H\mathscr{I}^{\nu_1+\nu}g_\zeta(\zeta_1) + K_2(q)(c_2^H\mathscr{I}^{\nu_2+\nu}g_\zeta(\zeta_2))| \\ &\leq C_{T_2}(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p).\end{aligned}$$

Hence,

$$\|T_2\zeta(q)\| \leq C_{T_2}(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p).$$

□

Theorem 3.4. *Assume that the operator T_3 is continuous and the following growth relation is satisfied:*

$$\|T_2\zeta(q)\| \leq C_{T_3}(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p),$$

where

$$C_{T_3} = (|c_1||\chi_3| + |\chi_4|)\Big[\frac{|c|(\log L)^\nu}{\Gamma(\nu+1)} + \frac{|d|(\log L)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\Big].$$

Proof. Let ζ_n be a sequence in Ψ that converges to $\zeta \in \Psi$. Since g_ζ is continuous, it follows that $g_{\zeta_n} \to g_\zeta$. Thus by the LDCT, it follows that T_3 is continuous.

$$\begin{aligned}|T_2\zeta(q)| &= |K_2(q)(c^H\mathscr{I}^{\nu}g_\zeta(L) + d^H\mathscr{I}^{\nu-\gamma}g_\zeta(L))| \\ &\leq C_{T_1}(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p).\end{aligned}$$

Hence,

$$\|T_3\zeta(q)\| \leq C_{T_3}(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p).$$

□

Theorem 3.5. *Suppose that T_2 is a compact map implying that T_2 is Lipschitz constant zero.*

Proof. Let $\zeta \subset B(r)$ be a bounded set. In order to prove that T_2 is a compact map. By Theorem 3.3, for $\zeta \in \zeta$,

$$\|T_2\zeta(q)\| \leq C_{T_2}(\mathcal{W}_1 + \mathcal{W}_2 r^p + \mathcal{W}_3 r^p + \mathcal{W}_4 r^p).$$

Hence $T_2(\zeta)$ is uniformly bounded.

Now, for any $\zeta \in \Psi$,

$$\begin{aligned}|T_2\zeta'(q)| &= |K_1'(q)^H\mathscr{I}^{\nu_1+\nu}g_\zeta(\zeta_1) + K_2'(q)(c_2^H\mathscr{I}^{\nu_2+\nu}g_\zeta(\zeta_2))| \\ &\leq |K_1'(q)|^H\mathscr{I}^{\nu_1+\nu}|g_\zeta(\zeta_1)| + |K_2'(q)c_2|^H\mathscr{I}^{\nu_2+\nu}|g_\zeta(\zeta_2)| \\ &\Big(|c_1||\chi_2|\frac{(\log\zeta_1)^{\nu+\nu_1}}{\Gamma(\nu_1+\nu+1)} + |\chi_4||c_2|\frac{(\log\zeta_2)^{\nu+\nu_2}}{\Gamma(\nu_2+\nu+1)}\Big)(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p) \\ &\leq \overline{\varpi_1}(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p).\end{aligned}$$

Now, for $q_1, q_2 \in J$,

$$|T_2\zeta(q_2) - T_2\zeta(q_1)| \leq \int_{q_1}^{q_2}|T_2\zeta'(q)|dt \leq \overline{\varpi_1}(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p)(q_2 - q_1).$$

Thus, because $q_2 \to q_1$, $|T_2\zeta(q_2) - T_2\zeta(q_1)| \to 0$ which implies that T_2 is equicontinuous. T_2 is compact according to the Arzela-Ascoli theorem. Hence T_2 is LC zero. □

Theorem 3.6. *If T_3 is a compact then T_3 is Lipschitz constant zero.*

Proof. Assume that $\zeta \subset B(r)$ is a bounded set. In order to prove that T_3 is a compact map. From Theorem 3.4, for $\zeta \in \zeta$,

$$\|T_3\zeta(q)\| \leq C_{T_3}(\mathcal{W}_1 + \mathcal{W}_2 r^p + \mathcal{W}_3 r^p + \mathcal{W}_4 r^p).$$

Hence $T_3(\zeta)$ is uniformly bounded.
Now, for any $\zeta \in \Psi$,

$$\begin{aligned}|T_2\zeta'(q)| &\leq |K_2'(q)|\Big(|c|^H \mathscr{I}^\nu |g_\zeta(L)| + |d|^H \mathscr{I}^\nu |g_\zeta(L)|\Big)\\ &\leq \overline{\varpi_2}(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p).\end{aligned}$$

Now, for $q_1, q_2 \in J$, we have,

$$|T_3\zeta(q_2) - T_3\zeta(q_1)| \leq \int_{q_1}^{q_2} |T_2\zeta'(q)|dt \leq \overline{\varpi_1}(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p)(q_2 - q_1).$$

Thus, because $q_2 \to q_1$, $|T_3\zeta(q_2) - T_3\zeta(q_1)| \to 0 \Rightarrow T_3$ is equicontinuous. T_3 is compact according to the ArzelÃ -Ascoli theorem. Hence T_3 is LC zero. □

Since $T = T_1 + T_2 + T_3$ and T_1 is LC $\dfrac{(\log L)^\nu}{\Gamma(\nu+1)}(\alpha + \beta k_{\max} + \chi h_{\max})$ and T_2, T_3 are LC 0, it follows that T is LC $\dfrac{(\log L)^\nu}{\Gamma(\nu+1)}(\alpha + \beta k_{\max} + \chi h_{\max})$.

If we assume that $\dfrac{(\log L)^\nu}{\Gamma(\nu+1)}(\alpha + \beta k_{\max} + \chi h_{\max}) < 1$, then by Definition 2.4, T is ϵ-condensing.

Theorem 3.7 (Existence). *Let FODE (1.1) have at least one solution if*

$$\Big(\frac{(\log L)^\nu}{\Gamma(\nu+1)} + C_{T_2} + C_{T_3}\Big) < 1.$$

Proof. Consider the set

$$\mathscr{H} = \{\zeta \in \Psi : \exists\, \lambda \in [0,1] \ni \lambda T\zeta = \zeta\}.$$

Let $\zeta \in \mathscr{H}$ such that $\lambda T\zeta = \zeta$; then,

$$\zeta = \lambda(T_1(\zeta) + T_2(\zeta) + T_3(\zeta)).$$

Taking $\|\cdot\|$ on both sides,

$$\begin{aligned}\|\zeta\| &=\leq \lambda\|T_1(\zeta)\| + \|T_2(\zeta)\| + \|T_3(\zeta)\|\\ &\leq \Big(\frac{(\log L)^\nu}{\Gamma(\nu+1)} + C_{T_2} + C_{T_3}\Big)(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p)\\ 1 &\leq \Big(\frac{(\log L)^\nu}{\Gamma(\nu+1)} + C_{T_2} + C_{T_3}\Big)\frac{(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p)}{\|\zeta\|}.\end{aligned}$$

Letting $\|\zeta\| \to \infty$, and by using $p \in [0,1)$ by Theorem 2.5, (1.1) has a solution. □

Theorem 3.8 **(Uniqueness).** *The FODE has a unique solution if*

$$\Big(\frac{(\log L)^{\nu}}{\Gamma(\nu+1)}+C_{T_2}+C_{T_3}\Big)(\alpha+\beta k_{\max}+\chi h_{\max})<1.$$

Proof. Let $\zeta_1,\zeta_2\in\Psi$ be arbitrary and $q\in J$; then

$$\begin{aligned}|T\zeta_1(q)-T\zeta_2(q)| &= |(T_1\zeta_1(q)-T_1\zeta_2(q))+(T_2\zeta_1(q)-T_2\zeta_2(q))+(T_3\zeta_1(q)-T_3\zeta_2(q))|\\ &\leq |(T_1\zeta_1(q)-T_1\zeta_2(q))|+|(T_2\zeta_1(q)-T_2\zeta_2(q))|\\ &+|(T_3\zeta_1(q)-T_3\zeta_2(q))|\\ &\leq |(T_1\zeta_1-T_1\zeta_2)|+|(T_2\zeta_1-T_2\zeta_2)|+|(T_3\zeta_1-T_3\zeta_2)|.\end{aligned}$$

Taking the supremum over all $q\in J$, we have

$$\|T\zeta_1-T\zeta_2\|\leq|(T_1\zeta_1-T_1\zeta_2)|+|(T_2\zeta_1-T_2\zeta_2)|+|(T_3\zeta_1-T_3\zeta_2)|.$$

From Theorems 3.1–3.3,

$$\|T_1\zeta_1(q)-T_1\zeta_2(q)\|\leq\frac{(\log L)^{\nu}}{\Gamma(\nu+1)}(\alpha+\beta k_{\max}+\chi h_{\max})\|\zeta_1-\zeta_2\|.$$

We have the following from the definition of T_2:

$$\begin{aligned}|T_2\zeta_1(q)-T_2\zeta_2(q)| &\leq |K_1(q)|^{H}\mathscr{I}^{\nu_1+\nu}|g_{\zeta_1}(\zeta_1)-g_{\zeta_2}(\zeta_1)|\\ &+|K_2(q)||c_2|^{H}\mathscr{I}^{\nu_2+\nu}|g_{\zeta_1}(\zeta_2)-g_{\zeta_2}(\zeta_2)|\\ &\leq\Big(|K_1(q)|^{H}\mathscr{I}^{\nu_1+\nu}(1)(\nu_1+\nu)+|K_2(q)||c_2|^{H}\mathscr{I}^{\nu_2+\nu}(1)(\nu_2+\nu)\Big)\\ &\times(\alpha+\beta k_{\max}+\chi h_{\max})\|\zeta_1-\zeta_2\|\\ &=C_{T_2}(\alpha+\beta k_{\max}+\chi h_{\max})\|\zeta_1-\zeta_2\|.\end{aligned}$$

Thus,

$$\|T_2\zeta_1-T_2\zeta_2\|\leq C_{T_2}(\alpha+\beta k_{\max}+\chi h_{\max})\|\zeta_1-\zeta_2\|.$$

We have the following from the definition of T_3:

$$\begin{aligned}|T_3\zeta_1(q)-T_3\zeta_2(q)| &\leq |K_2(q)|\Big(|c|^{H}\mathscr{I}^{\nu}|g_{\zeta_1}(L)-g_{\zeta_2}(L)|+|d|^{H}\mathscr{I}^{\nu-\gamma}|g_{\zeta_1}(L)-g_{\zeta_2}(L)|\Big)\\ &\leq |K_2(q)|\Big(|c|^{H}\mathscr{I}^{\nu}(1)(\nu)+|d||c_2|^{H}\mathscr{I}^{\nu-\gamma}(1)(\nu-\gamma)\Big)\\ &\times(\alpha+\beta k_{\max}+\chi h_{\max})\|\zeta_1-\zeta_2\|\\ &=C_{T_3}(\alpha+\beta k_{\max}+\chi h_{\max})\|\zeta_1-\zeta_2\|.\end{aligned}$$

Thus,

$$\|T_3\zeta_1-T_3\zeta_2\|\leq C_{T_3}(\alpha+\beta k_{\max}+\chi h_{\max})\|\zeta_1-\zeta_2\|.$$

Hence, we have,

$$\begin{aligned}\|T\zeta_1-T\zeta_2\| &\leq |(T_1\zeta_1-T_1\zeta_2)|+|(T_2\zeta_1-T_2\zeta_2)|+|(T_3\zeta_1-T_3\zeta_2)|\\ &\leq\Big(\frac{(\log L)^{\nu}}{\Gamma(\nu+1)}+C_{T_2}+C_{T_3}\Big)(\alpha+\beta k_{\max}+\chi h_{\max})\|\zeta_1-\zeta_2\|.\end{aligned}$$

Since $\Big(\frac{(\log L)^{\nu}}{\Gamma(\nu+1)}+C_{T_2}+C_{T_3}\Big)(\alpha+\beta k_{\max}+\chi h_{\max})<1$, the FODE (1.1), has a unique solution. □

4. Example

The FIDEs with boundary value conditions are as follows:

$$\begin{cases} {}^{C}_{H}D^{\frac{3}{2}}\zeta(q) = \frac{1}{12\pi}\sin(2\pi\zeta) + \frac{1}{30}\zeta + q^2 + 2 + \frac{1}{8}\int_0^1 e^{2s-2}\zeta(s)ds, \quad q \in [0,1] \\ \zeta(0) + 3({}^{C}_{H}D^{\frac{1}{3}}\zeta(0)) = \int_0^{\frac{1}{4}} \frac{(\frac{1}{4}-s)^{\frac{-1}{2}}}{\Gamma(\frac{1}{2})}\zeta(s)ds, \\ \zeta(1) + 2({}^{C}_{H}D^{\frac{1}{3}}\zeta(1)) = 3\int_0^{\frac{3}{4}} \frac{(\frac{3}{4}-s)^{\frac{-1}{2}}}{\Gamma(\frac{1}{2})}\zeta(s)ds, \end{cases} \tag{4.1}$$

where $g(\zeta) = \frac{1}{8}\int_0^1 e^{2s-2}\zeta(s)ds$. Here $\nu = \frac{3}{2}, \gamma = 13, a, c = 1, b = 3, d = 2, c_1 = 3, \nu_1 = \frac{1}{4}$, $\nu_2 = \frac{3}{4}, \zeta_1 = \frac{1}{4}, \zeta_2 = \frac{3}{4}, L = e$ and

$$C_{T_2} = |c_1||\chi_1| + |c_2||\chi_2|L\frac{(\log\zeta_1)^{\nu+\nu_1}}{\Gamma(\nu_1+\nu+1)} + (|c_1||c_2||\chi_3| + |\chi_4|L)\frac{(\log\zeta_2)^{\nu+\nu_2}}{\Gamma(\nu_2+\nu+1)},$$
$$C_{T_3} = (|c_1||\chi_3| + |\chi_4|)\Big[\frac{|c|(\log L)^{\nu}}{\Gamma(\nu+1)} + \frac{|d|(\log L)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}\Big].$$

By the above parameters in C_{T_2} and C_{T_3}, we get

$$\Big(\frac{(\log L)^{\nu}}{\Gamma(\nu+1)} + C_{T_2} + C_{T_3}\Big) = 3.2498.$$

To prove Theorem 3.6, we take

$$f(q,\zeta,g(\zeta)) = \frac{1}{12\pi}\sin(2\pi\zeta) + \frac{1}{30}\zeta + q^2 + 2 - \frac{1}{8}\int_0^1 e^{2s-2}\zeta(s)ds$$

in (1.1) and then

$$\begin{aligned} |f(q,\zeta_1,g(\zeta_1)) - f(q,\zeta_2,g(\zeta_2))| &= \frac{1}{12\pi}|\sin(2\pi\zeta_1) - \sin(2\pi\zeta_2)| + \frac{1}{30}|\zeta_1 - \zeta_|| + 0.054|\zeta_1 - \zeta_2| \\ &= 0.087|\zeta_1 - \zeta_2|. \end{aligned}$$

Hence the condition (A1) holds with $P = 0.087$, where $P = \alpha + \beta k_{\max} + \chi h_{\max}$. We use the following equation to calculate q_f from the given data is;

$$q_f = \frac{P(\log L)^{\nu}}{\Gamma(\nu+1)} = 0.0655.$$

Let $q \in J, \zeta \in \mathscr{R}$ and

$$\begin{aligned} |f(q,\zeta,g(\zeta))| &= |\frac{1}{12\pi}\sin(2\pi\zeta) + \frac{1}{30}\zeta + q^2 + 2 - \frac{1}{8}\int_0^1 e^{2s-2}\zeta(s)ds| \\ &= \frac{1}{30}|\zeta| + 3 + 0.054|\zeta|. \end{aligned}$$

Hence the condition (A2) holds with $\mathcal{W}_1 = 3, \mathcal{W}_2 = \frac{1}{30}, \mathcal{W}_3 = 0.054$ and $\mathcal{W}_4 = 0$.
By Theorem 3.6,

$$\mathscr{H} = \{\zeta \in \Psi : \exists\, 0 \le \lambda \le 1 \ni \lambda T\zeta = \zeta\}$$

has the following solution set;

$$\begin{aligned}\|\zeta\| &= \|\lambda(T_1(\zeta) + T_2(\zeta) + T_3(\zeta))\| \\ &\leq \Big(\frac{(\log L)^\nu}{\Gamma(\nu+1)} + C_{T_2} + C_{T_3}\Big)(\mathcal{W}_1 + \mathcal{W}_2\|\zeta\|^p + \mathcal{W}_3\|\zeta\|^p + \mathcal{W}_4\|\zeta\|^p).\end{aligned}$$

Thus,

$$\|\zeta\| \leq \frac{\Big(\frac{(\log L)^\nu}{\Gamma(\nu+1)} + C_{T_2} + C_{T_3}\Big)A_1}{1 - \Big(\frac{(\log L)^\nu}{\Gamma(\nu+1)} + C_{T_2} + C_{T_3}\Big)(A_2 + A_3 + A_4)} = 13.6109.$$

Hence, $\Big[\frac{(\log L)^\nu}{\Gamma(\nu+1)} + C_{T_2} + C_{T_3}\Big]P = 0.2827 < 1.$

5. Conclusions

This work was performed to investigate the existence and uniqueness of FIDEs with CH derivatives of fractional order by using the Robin boundary condition, TDT and fixed point theorem have been used to accomplish the analysis. The fundamental idea is shown with an efficient example.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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