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Research article

Some results on the space of bounded second κ-variation functions

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Abstract: In this paper, we prove that if a globally Lipschitz non-autonomous superposition operator maps the space of functions of bounded second κ-variation into itself then its generator function satisfies a Matkowski condition. We also provide conditions for the existence and uniqueness of solutions of the Hammerstein and Volterra equations in this space.

Keywords: superposition operator; integral equation; bounded second κ-variation; Hammerstein; Hammerstein-Volterra; Volterra existence and uniqueness
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1. Introduction

A function of bounded variation is a real-valued function in which total variation is finite. This class of functions was first introduced by Camile Jordan in [8] when he was dealing with the convergence of the Fourier series through a critical revision of Dirichlet's famous flawed proof. An interesting feature of bounded variation functions is that they form an algebra of functions whose first derivative exists almost everywhere. So, they have approximation properties and provide a natural framework for studying the theory of integration for a wider class of functions. The concept of bounded variation has become diversified and generalized in many ways in order to study more complex structures and properties (see e.g., [6, 9, 13]).

Ereú, López and Merentes [6] introduced the class of all functions of bounded second κ-variation on a closed interval $[a, b]$ which is denoted by $\kappa BV^2([a, b])$ and is a generalization of the notion of bounded second variation in the sense of De la Vallée Poussin. It was obtained by distorting the classical notion through a function κ that measures lengths in the domain of functions and not in the range. Among other things, they equipped the set of functions of bounded second κ-variation with a norm and proved that $\kappa BV^2([a,b])$ is a Banach space under that norm.

The functions with certain notions of bounded variation are commonly employed to define generalized solutions for nonlinear problems involving functionals, integral equations and variational problems. Specifically, the fact that solutions of integral equations belong to a space of functions with some notion of bounded variation provides valuable insights into the properties of regularity, stability, good behavior and approximation associated with these solutions. These properties assume a vital role in theoretical analysis and numerical methods.

The equations of Hammerstein and Volterra type

$$x(t) = g(t) + \lambda \int_I K(t,s) f(x(s))ds, \;\; \lambda \in I, t \in I = [0,b], \tag{1.1}$$

$$x(t) = g(t) + \int_a^t K(t,s) f(x(s))ds, \;\; t \in I = [a,b], \tag{1.2}$$

where the integration is taken in the Lebesgue sense serve as models for a wide range of phenomena in diverse disciplines such as physics, engineering, economics and biology (see [5]). Consequently, there has been an escalating interest in exploring solutions for these integral equations in spaces of functions that incorporate specific notions of bounded variation or in spaces of absolutely continuous functions which is important since there the fundamental theorem of calculus is satisfied (see [2, 3, 11, 14], among others).

In [14] conditions are studied in such a manner that Eqs (1.1) and (1.2) have solutions in the space of functions of bounded κ-variation. In this paper, we find conditions on f, g and K so that Eqs (1.1) and (1.2) have solutions in the space of functions bounded second κ-variation.

On the other hand, in problems related to existence of solutions of differential, integral or functional equations in spaces of functions it is required to consider the so-called Nemytskii operator or superposition operator in order to make use of the basic principles of nonlinear analysis. The natural problem related to the superposition operator H generated by a function h, consists in finding both necessary and sufficient conditions for the function h under which the superposition operator H maps a class of functions $f\colon [a,b] \to \mathbb{R}$ into itself.

In 1982, J. Matkowski [12] showed that a composition operator maps the function space $\mathrm{Lip}([0,1],\mathbb{R})$ into itself and is globally Lipschitz if and only if its generator h has the form

$$h(x,y) = h_1(x)y + h_0(x), \quad x \in [0,1], \; y \in \mathbb{R}, \tag{1.3}$$

for some $h_0, h_1 \in \mathrm{Lip}([0,1],\mathbb{R})$. This result was extended to a lot of spaces such as spaces of functions with some notion of bounded variation (see e.g., [1, 4, 13]). In this paper, we establish that if the (non-autonomous) superposition operator maps the function space $\kappa BV^2([a,b])$ into itself and is globally Lipschitz then the generating function h satisfies Eq (1.3), with both h_0 and h_1 belonging to $\kappa BV^2([a,b])$.

This document is structured as follows. Section 2 on preliminaries provides the necessary results for the proof of the main theorems. Here it is shown that $\kappa BV^2([a,b])$ is contained in the space of absolutely continuous functions $AC([a,b])$. Section 3 studies the action of the superposition operator on $\kappa BV^2([a,b])$. Section 4 presents the existence and uniqueness results for the Hammerstein and Volterra integral equations on $\kappa BV^2([a,b])$, the proofs of which are based on the Banach Contraction Principle. Some applications are presented in Section 5. Finally, the conclusions in the last section.

2. Preliminaries

In 1975, Korenblum [9] introduced the notion of bounded κ-variation of a function by studying the problem of representation for harmonic functions defined on the unit disk of the complex plane. This notion differs from other known variations in the fact that it maximizes ratios between Jordan's sums and the so-called κ-entropies generated by a distortion function κ.

Definition 2.1 ([9]). *A function $\kappa\colon [0,1] \to [0,1]$ is said to be a distortion function if it satisfies the following properties:*

i) *κ is continuous with $\kappa(0) = 0$ and $\kappa(1) = 1$.*
ii) *κ is nondecreasing and concave.*
iii) $\lim_{x\to 0^+} \dfrac{\kappa(x)}{x} = +\infty$.

All distortion function is subadditive; that is, if $x, y \in [0,1]$ are such that $x + y \in [0,1]$ then

$$\kappa(x+y) \le \kappa(x) + \kappa(y).$$

Some examples of distortion functions are the following:

(1) $\kappa(s) = s^{\alpha}$, $0 < \alpha < 1$ is the simplest example.
(2) $\kappa(s) = s(1 - \log s)$, $0 < t \le 1$ is another typical example.
(3) $\kappa(s) = (1 - \frac{1}{2}\log s)^{-1}$ is another one that arises in entropy theory.

Let $[a,b] \subseteq \mathbb{R}$ be an interval. From now on, $\Pi([a,b])$ denote the set of all partitions $\xi = \{t_i\}_{i=0}^n$ of the interval $[a,b]$ such that $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$. For $m > 2$, $\Pi_m([a,b])$ denote the set of all partitions $\xi = \{t_i\}_{i=0}^n$ of the interval $[a,b]$ such that $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ and $n \ge m-1$.

Definition 2.2 ([9]). *Let κ be a distortion function. The quantity*

$$\kappa(\xi;[a,b]) := \sum_{i=1}^{n} \kappa\left(\frac{t_i - t_{i-1}}{b-a}\right)$$

is called the κ-entropy of ξ relative to $[a,b]$, where $\xi = \{t_i\}_{i=0}^n \in \Pi([a,b])$.

Notice that, by the subadditivity of κ, for any $\xi = \{t_i\}_{i=0}^n \in \Pi([a,b])$, we have

$$1 = \kappa(1) \le \sum_{i=1}^{n} \kappa\left(\frac{t_i - t_{i-1}}{b-a}\right) = \kappa(\xi;[a,b]). \tag{2.1}$$

Additionally, it can be easily derived that

$$\kappa(\xi;[a,b]) \le n\,\kappa\left(\frac{1}{n}\right), \tag{2.2}$$

where the estimate is sharp and attained for equidistant partitions ξ (see [9]).

Definition 2.3 ([9]). *Let κ be a distortion function. A function $f\colon [a,b] \to \mathbb{R}$ is said to be of bounded κ-variation if*

$$\kappa V(f;[a,b]) := \sup_{\xi\in\Pi([a,b])} \frac{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|}{\kappa(\xi;[a,b])} < \infty,$$

where the supremum is taken over the set of all partitions $\xi = \{t_i\}_{i=0}^n \in \Pi([a,b])$.

In [6], we are provided with the concept of second κ-variation of a function which generalizes the notion of function of bounded second variation given by De la Vallée Poussin in 1908 while considering a distortion function in the same way as Korenblum did.

Definition 2.4 ([6]). *A function $f\colon [a,b] \to \mathbb{R}$ is said to be of bounded second κ-variation on $[a,b]$ if*

$$\kappa V^2(f;[a,b]) := \sup_{\xi\in\Pi_3([a,b])} \frac{\sum_{i=0}^{n-2} |f[t_{i+1},t_{i+2}] - f[t_i,t_{i+1}]|}{\kappa(\xi;[a,b])} < \infty,$$

where the supremum is taken over the set of all partitions $\xi = \{t_i\}_{i=0}^n \in \Pi_3([a,b])$ and

$$f[t_{i+1},t_{i+2}] = \frac{f(t_{i+2}) - f(t_{i+1})}{t_{i+2} - t_{i+1}}, \quad i = 0,\ldots,n-2.$$

The class of all functions defined on $[a,b]$ of bounded second κ-variation is denoted by $\kappa BV^2([a,b])$. The following theorem characterizes those functions which second κ-variation is null.

Theorem 2.5 ([6]). *Let $f\colon [a,b] \to \mathbb{R}$ be a function. Then, $\kappa V^2(f;[a,b]) = 0$ if and only if there exist constants α, $\beta \in \mathbb{R}$ such that $f(t) = \alpha t + \beta$.*

The following remark guarantees that $\kappa BV^2([a,b])$ is known to be a vector space.

Remark 2.6 ([6]). *Let f and g in $\kappa BV^2([a,b])$ and $\lambda \in \mathbb{R}$. Then*

$$\kappa V^2(f + \lambda g;[a,b]) \le \kappa V^2(f;[a,b]) + |\lambda| \kappa V^2(g;[a,b]).$$

Furthermore, $\kappa BV^2([a,b])$ is a Banach space.

Theorem 2.7 ([6]). *$\kappa BV^2([a,b])$ equipped with the norm*

$$\|f\|_{\kappa BV^2} = |f(a)| + |f(b)| + \kappa V^2(f;[a,b]),$$

is a Banach space.

In [6], it is also shown that the functions in the unit ball of the space $(\kappa BV^2([a,b]), \|\cdot\|)$ are uniformly majorized by a certain fixed continuous function. That is,

Theorem 2.8 ([6]). *The continuous function $p_\kappa\colon [a,b] \to \mathbb{R}$ given by*

$$p_\kappa(t) = 1 + \frac{1}{b-a}(b-t)(t-a)\left(\kappa\left(\frac{b-t}{b-a}\right) + \kappa\left(\frac{t-a}{b-a}\right)\right), \tag{2.3}$$

is such that for all $f \in \kappa BV^2([a,b])$,

$$|f(t)| \leq p_k(t)\,\|f\|_{\kappa BV^2},\ \textit{for all } t \in [a,b].$$

In particular, $\kappa BV^2([a,b])$ *is a subspace of* $\mathcal{B}([a,b])$ *the Banach space of all bounded functions on* $[a,b]$ *with the sup norm.*

Below, we show that $\kappa BV^2([a,b]) \subseteq AC([a,b])$. Also we show that $\kappa BV^2([a,b])$ is closed under the pointwise product operation and in consequence the set of polynomial functions is contained in $\kappa BV^2([a,b])$.

A similar technique to the one used to prove lemma 4 in [15] and property (2.2) enable us to show the following property of functions of bounded second κ-variation.

Lemma 2.1. *If* $f \in \kappa BV^2([a,b])$*, then* $f[y_0,y_1]$ *is bounded for all* y_0*,* y_1 *in the interval* $[a,b]$.

Proof. First, consider a partition $\xi = \{t_i\}_{i=0}^n \in \Pi_4([a,b])$. Then,

$$\begin{aligned}
&\left|f[t_{i+2},t_{i+3}] - f[t_i,t_{i+1}]\right| \\
&= \left|f[t_{i+2},t_{i+3}] - f[t_{i+1},t_{i+2}] + f[t_{i+1},t_{i+2}] - f[t_i,t_{i+1}]\right| \\
&\leq |f[t_{i+2},t_{i+3}] - f[t_{i+1},t_{i+2}]| + |f[t_{i+1},t_{i+2}] - f[t_i,t_{i+1}]| \\
&\leq \kappa(\xi,[a,b])\,\kappa V^2(f;[a,b]) + \kappa(\xi,[a,b])\,\kappa V^2(f;[a,b]) \\
&= 2\kappa(\xi,[a,b])\,\kappa V^2(f;[a,b]).
\end{aligned} \tag{2.4}$$

Let c be chosen arbitrarily in (a,b) and let $A = \left|f[a,c]\right|$. The proof of this lemma depends on the position of y_0 and y_1 with respect to the points a, b, and c. So we consider the following six cases:

Case 1. $a < y_0 < c < y_1 < b$, **Case 4.** $a < c \leq y_0 < y_1 < b$,
Case 2. $a < y_0 < c < y_1 = b$, **Case 5.** $a = y_0 < c < y_1 < b$,
Case 3. $a < y_0 < y_1 \leq c < b$, **Case 6.** $a = y_0 < c < y_1 = b$.

Case 1. Suppose $a < y_0 < c < y_1 < b$. If y_2 is any point such that $y_1 < y_2 < b$ then $a < y_0 < c < y_1 < y_2 < b$. By (2.4) and (2.2) we have

$$\begin{aligned}
&\left|f[y_0,y_1]\right| \\
&= \left|f[y_0,y_1] - f[y_1,y_2] + f[y_1,y_2] - f[a,c] + f[a,c]\right| \\
&\leq \left|f[y_0,y_1] - f[y_1,y_2]\right| + \left|f[y_1,y_2] - f[a,c]\right| + \left|f[a,c]\right| \\
&\leq \kappa(\xi_1,[a,b])\,\kappa V^2(f;[a,b]) + 2\kappa(\xi_2,[a,b])\,\kappa V^2(f;[a,b]) + A \\
&\leq 4\kappa\left(\frac{1}{4}\right)\kappa V^2(f;[a,b]) + 8\kappa\left(\frac{1}{4}\right)\kappa V^2(f;[a,b]) + A \\
&= 12\kappa\left(\frac{1}{4}\right)\kappa V^2(f;[a,b]) + A := K_1
\end{aligned}$$

where ξ_1 y ξ_2 are the partitions $a < y_0 < y_1 < y_2 < b$ and $a < c < y_1 < y_2 < b$, respectively.

Case 2. Suppose $a < y_0 < c < y_1 = b$. If y_2 is any point such that $c < y_2 < y_1$ then $a < y_0 < c < y_2 < y_1 = b$. It follows from (2.4) and (2.2) that

$$
\begin{aligned}
\left|f[y_0,y_1]\right| &= \left|f[y_0,y_1]-f[a,y_0]+f[a,y_0]-f[y_2,y_1]+f[y_2,y_1]-f[a,c]+f[a,c]\right|\\
&\le \left|f[y_0,y_1]-f[a,y_0]\right|+\left|f[a,y_0]-f[y_2,y_1]\right|+\left|f[y_2,y_1]-f[a,c]\right|+\left|f[a,c]\right|\\
&\le \kappa(\xi_3,[a,b])\,\kappa V^2(f;[a,b])+2\kappa(\xi_4,[a,b])\,\kappa V^2(f;[a,b])\\
&\quad +2\kappa(\xi_5,[a,b])\,\kappa V^2(f;[a,b])+A\\
&\le \left[2\kappa\left(\frac{1}{2}\right)+12\kappa\left(\frac{1}{3}\right)\right]\kappa V^2(f;[a,b])+A := K_2
\end{aligned}
$$

where the partitions ξ_3,ξ_4, and ξ_5 are $a<y_0<y_1=b$, $a<y_0<y_2<y_1=b$, and $a<c<y_2<y_1=b$, respectively. Consequently, $\left|f[y_0,y_1]\right|\le K_2$.

Case 3. Suppose $a<y_0<y_1\le c<b$. If y_2 is any point such that $c<y_2<b$ then $a<y_0<y_1\le c<y_2<b$. By (2.4) and (2.2) we have

$$
\begin{aligned}
\left|f[y_0,y_1]\right| &= \left|f[y_0,y_1]-f[y_1,y_2]+f[y_1,y_2]-f[y_2,b]+f[y_2,b]-f[a,c]+f[a,c]\right|\\
&\le \left|f[y_0,y_1]-f[y_1,y_2]\right|+\left|f[y_1,y_2]-f[y_2,b]\right|+\left|f[y_2,b]-f[a,c]\right|+\left|f[a,c]\right|\\
&\le \kappa(\xi_6,[a,b])\,\kappa V^2(f;[a,b])+\kappa(\xi_6,[a,b])\,\kappa V^2(f;[a,b])\\
&\quad +2\kappa(\xi_7,[a,b])\,\kappa V^2(f;[a,b])+A\\
&\le= \left[8\kappa\left(\frac{1}{4}\right)+6\kappa\left(\frac{1}{3}\right)\right]\kappa V^2(f;[a,b])+A := K_3,
\end{aligned}
$$

where ξ_6 and ξ_7 are the respective partitions $a<y_0<y_1<y_2<b$ and $a<c<y_2<b$. So, $\left|f[y_0,y_1]\right|\le K_3$.

The proof of the other cases is handled in a similar fashion and we omit the details. Since there is a finite number of cases we can ensure that $f[y_0,y_1]$ is bounded. □

Theorem 2.9. *If $f\in\kappa BV^2([a,b])$ then f is Lipschitz on $[a,b]$ and in consequence absolutely continuous on $[a,b]$.*

Proof. By lemma 2.1, there exists $K>0$ such that

$$
\left|\frac{f(y_1)-f(y_0)}{y_1-y_0}\right| = \left|f[y_0,y_1]\right|\le K
$$

for all $y_0,y_1\in[a,b]$ with $y_0\neq y_1$ which implies that f is Lipschitz continuous on $[a,b]$ and thus f is absolutely continuous on the interval $[a,b]$. □

From now on, denote by $L_a^b(f)$ the Lipschitz constant of a function $f\colon[a,b]\to\mathbb{R}$, i.e.,

$$
L_a^b(f)=\sup\left\{\left|\frac{f(x_2)-f(x_1)}{x_2-x_1}\right| : x_1,x_2\in[a,b], x_1\neq x_2\right\}.
$$

Proposition 2.1. *If $f,\ g\in\kappa BV^2([a,b])$ then $fg\in\kappa BV^2([a,b])$.*

Proof. Let $\xi = \{t_i\}_{i=0}^n \in \Pi_3([a,b])$. We begin by considering the expression

$$(f\,g)\,[t_{i+1},t_{i+2}] - (f\,g)\,[t_i,t_{i+1}] = \frac{f(t_{i+2})\,g(t_{i+2}) - f(t_{i+1})\,g(t_{i+1})}{t_{i+2}-t_{i+1}} - \frac{f(t_{i+1})\,g(t_{i+1}) - f(t_i)\,g(t_i)}{t_{i+1}-t_i}.$$

By adding the terms $\pm f(t_{i+1})\,g(t_{i+2}),\ \pm f(t_i)\,g(t_{i+1})$ we get

$$\begin{aligned}
&(f\,g)\,[t_{i+1},t_{i+2}] - (f\,g)\,[t_i,t_{i+1}]\\
&= \frac{[f(t_{i+2})-f(t_{i+1})]\ g(t_{i+2}) + f(t_{i+1})\ [g(t_{i+2})-g(t_{i+1})]}{t_{i+2}-t_{i+1}}\\
&\quad - \frac{[f(t_{i+1})-f(t_i)]\ g(t_{i+1}) + f(t_i)\ [g(t_{i+1})-g(t_i)]}{t_{i+1}-t_i}\\
&= \frac{f(t_{i+2})-f(t_{i+1})}{t_{i+2}-t_{i+1}}\,g(t_{i+2}) - \frac{f(t_{i+1})-f(t_i)}{t_{i+1}-t_i}\,g(t_{i+1})\\
&\quad + f(t_{i+1})\,\frac{g(t_{i+2})-g(t_{i+1})}{t_{i+2}-t_{i+1}} - f(t_i)\,\frac{g(t_{i+1})-g(t_i)}{t_{i+1}-t_i}\\
&= f\,[t_{i+1},t_{i+2}]\ g(t_{i+2}) - f\,[t_i,t_{i+1}]\ g(t_{i+1}) + f(t_{i+1})\,g\,[t_{i+1},t_{i+2}]\\
&\quad - f(t_i)\,g\,[t_i,t_{i+1}]\,.
\end{aligned}$$

Adding the following terms $\pm\, g(t_{i+2})\,f[t_i,t_{i+1}],\ \pm\, f(t_{i+1})\,g[t_i,t_{i+1}]$ and combining them yields

$$\begin{aligned}
&(f\,g)\,[t_{i+1},t_{i+2}] - (f\,g)\,[t_i,t_{i+1}]\\
&= g(t_{i+2})\,\{f[t_{i+1},t_{i+2}] - f[t_i,t_{i+1}]\} + f(t_{i+1})\,\{g\,[t_{i+1},t_{i+2}] - g\,[t_i,t_{i+1}]\}\\
&\quad + f[t_i,t_{i+1}]\,\{g(t_{i+2}) - g(t_{i+1})\} + g\,[t_i,t_{i+1}]\ \{f(t_{i+1}) - f(t_i)\}\\
&= g(t_{i+2})\,\{f[t_{i+1},t_{i+2}] - f[t_i,t_{i+1}]\} + f(t_{i+1})\,\{g\,[t_{i+1},t_{i+2}] - g\,[t_i,t_{i+1}]\}\\
&\quad + f[t_i,t_{i+1}]\ \{g(t_{i+2}) - g(t_i)\}\,.
\end{aligned}$$

By theorem 2.9, $f[y_0,y_1] \le L_a^b(f)$ and $g[y_0,y_1] \le L_a^b(g)$ for all $y_0,y_1 \in [a,b]$ (lemma 2.1). Then,

$$\begin{aligned}
&|(f\,g)\,[t_{i+1},t_{i+2}] - (f\,g)\,[t_i,t_{i+1}]|\\
&\le \|g\|_\infty \left|f[t_{i+1},t_{i+2}] - f[t_i,t_{i+1}]\right| + \|f\|_\infty \left|g\,[t_{i+1},t_{i+2}] - g\,[t_i,t_{i+1}]\right|\\
&\quad + \left|f[t_i,t_{i+1}]\right| \left|g\,[t_i,t_{i+2}]\right| |t_{i+2}-t_i|\\
&\le \|g\|_\infty \left|f[t_{i+1},t_{i+2}] - f[t_i,t_{i+1}]\right| + \|f\|_\infty \left|g\,[t_{i+1},t_{i+2}] - g\,[t_i,t_{i+1}]\right|\\
&+ L_a^b(f)L_a^b(g)\,|t_{i+2}-t_i|.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{i=0}^{n-2} \frac{|(f\,g)\,[t_{i+1},t_{i+2}] - (f\,g)\,[t_i,t_{i+1}]|}{\kappa(\xi;[a,b])}\\
&< \|g\|_\infty \sum_{i=0}^{n-2} \frac{|f[t_{i+1},t_{i+2}] - f[t_i,t_{i+1}]|}{\kappa(\xi;[a,b])}\\
&+ \|f\|_\infty \sum_{i=0}^{n-2} \frac{|g[t_{i+1},t_{i+2}] - g[t_i,t_{i+1}]|}{\kappa(\xi;[a,b])} + 2L_a^b(f)L_a^b(g)(b-a)
\end{aligned}$$

$$\leq \|g\|_\infty \kappa V^2(f;[a,b]) + \|f\|_\infty \kappa V^2(g;[a,b]) + 2L_a^b(f)L_a^b(g)(b-a).$$

By theorem 2.8, $\kappa V^2(f\,g;[a,b]) < \infty$ and so $fg \in \kappa BV^2([a,b])$. □

Corollary 2.1. *If f is a polynomial function then $f \in \kappa BV^2([a,b])$.*

Proof. Immediate consequence of theorem 2.5, proposition 2.1 and remark 2.6. □

3. Superposition operators on $\kappa BV^2([a,b])$

Let $I, J \subseteq \mathbb{R}$ be intervals and let J^I denote the set of all functions $\varphi\colon I \to J$. For a given function $h\colon I \times J \to \mathbb{R}$ the mapping $H\colon J^I \to \mathbb{R}^I$ defined by

$$H(\varphi)(x) = h(x, \varphi(x)),\ \varphi \in J^I$$

is called a non-autonomous superposition operator (or Nemytskii operator) of a generator h.

In this section, we present a characterization of the action of a non-autonomous superposition operator on the space $\kappa BV^2([a,b])$ showing that if a superposition operator applies such space to itself and is globally Lipschitz then its generating function satisfies a so-called Matkowski condition.

Lemma 3.1. *Let $\alpha, \beta \in [a,b]$ be such that $\alpha < \beta$. Consider the function*

$$\eta_{\alpha,\beta}(t) := \begin{cases} 0, & \text{if } a \leq t \leq \alpha \\ \dfrac{t-\alpha}{\beta-\alpha}, & \text{if } \alpha \leq t \leq \beta \\ 1, & \text{if } \beta \leq t \leq b \end{cases}.$$

For $y_1, y_2 \in \mathbb{R}$, with $y_1 \neq y_2$, define the functions $\varphi_j\colon [a,b] \to \mathbb{R}$, $j = 1, 2$, by

$$\varphi_j(t) = \frac{1}{2}\left[\eta_{\alpha,\beta}(t)(y_1 - y_2) + y_j + y_2\right], \quad j = 1, 2.$$

Then the functions φ_1, φ_2 are both of bounded second κ-variation.

Proof. Let $\xi = \{t_i\}_{i=0}^n \in \Pi_3([a,b])$ such that $a = t_0 < \cdots < t_r \leq \alpha < t_{r+1} < \cdots < t_{r+s} \leq \beta < t_{r+s+1} < \cdots < t_{n-1} < t_n = b$.

Notice that by (2.1)

$$\frac{\sum_{i=0}^{n-2}\left|\varphi_1[t_{i+1}, t_{i+2}] - \varphi_1[t_i, t_{i+1}]\right|}{\kappa(\xi;[a,b])} \leq \sum_{i=0}^{n-2}\left|\varphi_1[t_{i+1}, t_{i+2}] - \varphi_1[t_i, t_{i+1}]\right|.$$

We proceed to bound the right side of this inequality, by theorem 2.5, in the following way:

$$\begin{aligned}&\sum_{i=0}^{n-2}\left|\varphi_1[t_{i+1}, t_{i+2}] - \varphi_1[t_i, t_{i+1}]\right| \\ &\leq \left|\frac{\varphi_1(t_{r+1}) - \varphi_1(t_r)}{t_{r+1} - t_r}\right| + \left|\frac{\varphi_1(t_{r+2}) - \varphi_1(t_{r+1})}{t_{r+2} - t_{r+1}} - \frac{\varphi_1(t_{r+1}) - \varphi_1(t_r)}{t_{r+1} - t_r}\right|\end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{\varphi_1(t_{r+s+1})-\varphi_1(t_{r+s})}{t_{r+s+1}-t_{r+s}}-\frac{\varphi_1(t_{r+s})-\varphi_1(t_{r+s-1})}{t_{r+s}-t_{r+s-1}}\right| \\
&+\left|\frac{\varphi_1(t_{r+s})-\varphi_1(t_{r+s-1})}{t_{r+s1}-t_{r+s-1}}\right| \\
\leq\ & 2\left|\frac{\varphi_1(t_{r+1})-\varphi_1(t_r)}{t_{r+1}-t_r}\right|+\left|\frac{\varphi_1(t_{r+2})-\varphi_1(t_{r+1})}{t_{r+2}-t_{r+1}}\right|+2\left|\frac{\varphi_1(t_{r+s+1})-\varphi_1(t_{r+s})}{t_{r+s+1}-t_{r+s}}\right| \\
&+\left|\frac{\varphi_1(t_{r+s})-\varphi_1(t_{r+s-1})}{t_{r+s}-t_{r+s-1}}\right| \\
\leq\ & \frac{\left|\frac{y_1-y_2}{\beta-\alpha}\right|(t_{r+1}-\alpha)}{t_{r+1}-t_r}+\frac{1}{2}\left|\frac{y_1-y_2}{\beta-\alpha}\right|+\frac{\left|\frac{y_1-y_2}{2}\right|\left|1-\frac{t_{r+s}-\alpha}{\beta-\alpha}\right|}{t_{r+s+1}-t_{r+s}}+\frac{1}{2}\left|\frac{y_1-y_2}{\beta-\alpha}\right| \\
\leq\ & 2\left|\frac{y_1-y_2}{\beta-\alpha}\right|+\frac{\left|\frac{y_1-y_2}{2}\right|\left|1-\frac{t_{r+s}-\alpha}{\beta-\alpha}\right|}{t_{r+s+1}-t_{r+s}} \\
\leq\ & 2\left|\frac{y_1-y_2}{\beta-\alpha}\right|+\frac{\frac{1}{2}\left|\frac{y_1-y_2}{\beta-\alpha}\right||\beta-t_{r+s}|}{t_{r+s+1}-t_{r+s}}\leq\frac{5}{2}\left|\frac{y_1-y_2}{\beta-\alpha}\right|<\infty.
\end{aligned}
$$

Thus, $\varphi_1\in\kappa BV^2([a,b])$. It can be proved analogously that $\varphi_2\in\kappa BV^2([a,b])$. □

Theorem 3.1. *Let $h\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ be a function such that for each $x\in[a,b]$, $h(x,\cdot)$ is a continuous function. Suposse that the superposition operator H generated by h is a globally Lipschitzian map i.e., there is $L\geq 0$ such that*

$$
\|H(\varphi_1)-H(\varphi_2)\|_{\kappa BV^2}\leq L\|\varphi_1-\varphi_2\|_{\kappa BV^2},\qquad \varphi_1,\varphi_2\in\kappa BV^2([a,b]). \tag{3.1}
$$

Then, H transforms the space $\kappa BV^2([a,b])$ into itself if and only if h satisfies the Matkowski's condition

$$
h(x,y)=h_1(x)y+h_0(x),\quad x\in[a,b],\ y\in\mathbb{R},
$$

with $h_0,h_1\in\kappa BV^2([a,b])$.

Proof. Let us suppose that H transforms the space $\kappa BV^2([a,b])$ into itself. Let $\alpha,\beta\in[a,b]$ be such that $a<\alpha<\beta$. For $y_1,y_2\in\mathbb{R}$, with $y_1\neq y_2$, consider the functions in lemma 3.1 given by

$$
\varphi_j(t)=\frac{1}{2}\left[\eta_{\alpha,\beta}(t)(y_1-y_2)+y_j+y_2\right],\quad j=1,2.
$$

Notice that

$$
\varphi_1(\alpha)=\varphi_2(\beta)=\frac{1}{2}(y_1+y_2),\quad \varphi_2(\alpha)=y_2,\quad\text{and}\quad \varphi_1(\beta)=y_1.
$$

Also,

$$
(\varphi_1-\varphi_2)(t)=\frac{y_1-y_2}{2},\qquad t\in[a,b],
$$

is a constant function. So, by theorem 2.5

$$
\|\varphi_1-\varphi_2\|_{\kappa BV^2}=|(\varphi_1-\varphi_2)(a)|+|(\varphi_1-\varphi_2)(b)|+\kappa V^2(\varphi_1-\varphi_2;[a,b])=|y_1-y_2|.
$$

By lemma 3.1 $\varphi_1,\varphi_2\in\kappa BV^2([a,b])$. Then, by (3.1) we have

$$
\begin{aligned}
&\|H(\varphi_1) - H(\varphi_2)\|_{\kappa VB^2} \\
&= |h(a,\varphi_1(a)) - h(a,\varphi_2(a))| + |h(b,\varphi_1(b)) - h(b,\varphi_2(b))| \\
&\quad + \kappa V^2\left(h(\cdot,\varphi_1(\cdot)) - h(\cdot,\varphi_2(\cdot)); [a,b]\right) \\
&\leq L\left(\|\varphi_1 - \varphi_2\|_{\kappa BV^2}\right) \\
&= L|y_1 - y_2|. && (3.2)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\kappa V^2\left(h(\cdot,\varphi_1(\cdot)) - h(\cdot,\varphi_2(\cdot)); [a,b]\right) \\
&\geq \left|\frac{(h(\beta,\varphi_1(\beta)) - h(\beta,\varphi_2(\beta))) - (h(\alpha,\varphi_1(\alpha)) - h(\alpha,\varphi_2(\alpha)))}{\beta-\alpha}\right. \\
&\quad \left. - \frac{(h(\alpha,\varphi_1(\alpha)) - h(\alpha,\varphi_2(\alpha))) - (h(a,\varphi_1(a)) - h(a,\varphi_2(a)))}{\alpha - a}\right| \\
&\geq \left|\frac{h(\beta,\varphi_1(\beta)) - h(\beta,\varphi_2(\beta)) - h(\alpha,\varphi_1(\alpha) + h(\alpha,\varphi_2(\alpha))}{\beta-\alpha}\right| \\
&\quad - \left|\frac{h(\alpha,\varphi_1(\alpha)) - h(\alpha,\varphi_2(\alpha)) - h(a,\varphi_1(a)) + h(a,\varphi_2(a))}{\alpha - a}\right|.
\end{aligned}
$$

By (3.2) and the inequality above

$$
\begin{aligned}
&\left|\frac{h(\beta,\varphi_1(\beta)) - h(\beta,\varphi_2(\beta)) - h(\alpha,\varphi_1(\alpha) + h(\alpha,\varphi_2(\alpha))}{\beta-\alpha}\right| \\
&- \left|\frac{h(\alpha,\varphi_1(\alpha)) - h(\alpha,\varphi_2(\alpha)) - h(a,\varphi_1(a)) + h(a,\varphi_2(a))}{\alpha - a}\right| \\
&+ |h(a,\varphi_1(a)) - h(a,\varphi_2(a))| + |h(b,\varphi_1(b)) - h(b,\varphi_2(b))| \\
\leq\ & L|y_1 - y_2|
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&\left|h(\beta,y_1) - h\left(\beta,\frac{y_1+y_2}{2}\right) - h\left(\alpha,\frac{y_1+y_2}{2}\right) + h(\alpha,y_2)\right| \\
&- \left|\frac{h\left(\alpha,\frac{y_1+y_2}{2}\right) - h(\alpha,y_2) - h\left(a,\frac{y_1+y_2}{2}\right) + h(a,y_2)}{\alpha - a}\right| |\beta-\alpha| \\
&+ \left|h\left(a,\frac{y_1+y_2}{2}\right) - h(a,y_2)\right| |\beta-\alpha| \\
&+ \left|h(b,y_1) - h\left(b,\frac{y_1+y_2}{2}\right)\right| |\beta-\alpha| \\
\leq\ & L|y_1-y_2||\beta-\alpha|. && (3.3)
\end{aligned}
$$

Since $H(\varphi) \in \kappa BV^2([a,b])$ for every function $\varphi \in \kappa BV^2([a,b])$, putting $\varphi(t) = y$ for $t \in [a,b]$ we see that for every $y \in \mathbb{R}$ the function $h(\cdot,y) \in \kappa BV^2([a,b])$. Then, $h(\cdot,y)$ is a continuous function. Taking limit as $\beta \to \alpha$ in (3.3) we obtain

$$
\left|h(\alpha,y_1) - h\left(\alpha,\frac{y_1+y_2}{2}\right) - h\left(\alpha,\frac{y_1+y_2}{2}\right) + h(\alpha,y_2)\right| \leq 0.
$$

Since α is arbitrary (and $h(\cdot, y)$ is continuos),

$$2h\Big(x, \frac{y_1 + y_2}{2}\Big) = h(x, y_1) + h(x, y_2), \quad \forall x \in [a, b].$$

This shows that $h(x, \cdot)$ satisfies the Jensen's equation (see [10], p. 351). But $h(x, \cdot)\colon \mathbb{R} \to \mathbb{R}$ is continuous, the theorem 13.2.2 of [10] (p. 354) asserts that there exist h_0, $h_1\colon [a, b] \to \mathbb{R}$ such that

$$h(x, y) = h_1(x)y + h_0(x), \quad x \in [a, b], \quad y \in \mathbb{R}. \tag{3.4}$$

Note that for each $y \in \mathbb{R}$, we obtain that $h(\cdot, y) \in \kappa BV^2([a, b])$. So, by equation (3.4) we have for $y = 0$ and $y = 1$

$$h_0(x) = h(x, 0) \quad \text{and} \quad h_1(x) = h(x, 1) - h_0(x), \quad \forall x \in [a, b].$$

Therefore, h_0, $h_1 \in \kappa BV^2([a, b])$.

Now, suppose that $h_0, h_1 \in \kappa BV^2([a, b])$ and that the operator H is generated by the function h given by $h(x, y) = h_1(x)y + h_0(x)$, $x \in [a, b]$, $y \in \mathbb{R}$. It follows from proposition 2.1 and remark 2.6 that H transforms the space $\kappa BV^2([a, b])$ into itself which completes the proof. □

4. Existence and uniqueness of solutions of the Hammerstein and Volterra equations in $\kappa BV^2([a, b])$

In this section, we study the existence and uniqueness of solutions in the class of functions of bounded second κ-variation of the nonlinear integral equations (1.1) and (1.2).

Note that the case $a = 0$ in the Eq (1.2) is the Volterra-Hammerstein equation with $\lambda = 1$.

The Banach Contraction Principle is used in the proofs of the main results to guarantee the existence and uniqueness of solutions of the integral equations.

Theorem 4.1 (Banach Contraction Principle)**.** *Let X be a complete metric space and let $f\colon X \to X$ be a contraction mapping. Then, f has a unique fixed point in X. Moreover, if $B \subseteq X$ is a closed subset such that $f(B) \subseteq B$ then $f\colon B \to B$ has a unique fixed point in B.*

As conditions of existence and uniqueness for the solutions of the Eqs (1.1) and (1.2) on $\kappa BV^2([a, b])$ we set the following hypotheses: Let $I = [a, b]$.

(H_1) $f\colon \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function.
(H_2) $g\colon I \to \mathbb{R}$ is a function of bounded second κ-variation on I, that is, $g \in \kappa BV^2(I)$.
(H_3) $K\colon I \times I \to \mathbb{R}$ is a function such that $\kappa V^2(K(\cdot, s); I) \leq M(s)$, for each $s \in I$, where $M\colon I \to \mathbb{R}$ is a Lebesgue integrable function. In addition, $K(t, \cdot)$ is Lebesgue integrable for each $t \in I$, and both $K(a, \cdot)$ and $K(b, \cdot)$ are L^1 integrable functions.

Next, we estimate the second κ-variation of certain auxiliary functions.

Lemma 4.1. *Let f and K satisfy hypotheses (H_1) and (H_3), respectively (with $a = 0$). For every $x \in \kappa BV^2([0, b])$ let $F(x)$ be the integral function given by $F(x)(t) = \int_0^b K(t, s)f(x(s))ds$, for all $t \in [0, b]$. Then,*

$$\kappa V^2(F(x); [0, b]) \leq \|f\|_\infty \int_0^b M(s)ds < +\infty.$$

Proof. The function $f \circ x$ is continuous on $[0, b]$ by (H_1) and theorem 2.9, so that it is Lebesgue integrable. Since $K(t, \cdot)$ is Lebesgue integrable for all $t \in [0, b]$ we have that $K(t, \cdot) f(x(\cdot))$ is Lebesgue integrable for all $t \in [0, b]$. Thus, the function $F(x)$ is well defined.

Let $\xi = \{t_i\}_{i=0}^n \in \pi_3([0, b])$. Then,

$$\begin{aligned}
&|F(x)[t_{i+1}, t_{i+2}] - F(x)[t_i, t_{i+1}]| \\
&= \left| \frac{F(x)(t_{i+2}) - F(x)(t_{i+1})}{t_{i+2} - t_{i+1}} - \frac{F(x)(t_{i+1}) - F(x)(t_i)}{t_{i+1} - t_i} \right| \\
&= \left| \frac{\int_0^b [K(t_{i+2}, s) - K(t_{i+1}, s)] f(x(s)) ds}{t_{i+2} - t_{i+1}} - \frac{\int_0^b [K(t_{i+1}, s) - K(t_i, s)] f(x(s)) ds}{t_{i+1} - t_i} \right| \\
&\leq \sup_{s \in [0,b]} |f(x(s))| \left| \int_0^b [K(\cdot, s)[t_{i+2}, t_{i+1}] - K(\cdot, s)[t_i, t_{i+1}]] \, ds \right|.
\end{aligned}$$

Consequently, by (H_3) we have

$$\begin{aligned}
&\frac{\sum_{i=0}^{n-2} |F(x)[t_{i+1}, t_{i+2}] - F(x)[t_i, t_{i+1}]|}{\kappa(\xi, [0, b])} \\
&\leq \sup_{s \in [0,b]} |f(x(s))| \int_0^b \frac{\sum_{i=0}^{n-2} |(K(\cdot, s)[t_{i+1}, t_{i+2}] - K(\cdot, s)[t_i, t_{i+1}]|}{\kappa(\xi, [0, b])} ds \\
&\leq \sup_{s \in [0,b]} |f(x(s))| \int_0^b M(s) ds.
\end{aligned}$$

Taking supremum over the set of all partitions $\xi \in \pi_3([0, b])$ yields

$$\kappa V^2(F(x); [0, b]) \leq \|f\|_\infty \int_0^b M(s) ds < +\infty. \qquad \square$$

The following result can be obtained by a similar technique to that used in the proofs of lemma above and theorem 2.9 to estimate $\kappa V^2((F(x) - F(y)) ; [0, b])$ and by remark 2.6 and theorem 2.8.

Lemma 4.2. *Let f and K satisfy hypotheses (H_1) and (H_3), respectively (with $a = 0$). For every $x \in \kappa BV^2([0, b])$ let $F(x)$ be defined as in the preceding lemma. Then, for all $x, y \in \kappa BV^2([0, b])$,*

$$\kappa V^2(\lambda (F(x) - F(y)) ; [0, b]) \leq C_{p_\kappa} L_0^b(f) |\lambda| \, \|x - y\|_{\kappa BV^2} \int_0^b M(s) ds, \; \lambda \in [0, b]$$

where $C_{p_\kappa} = \sup_{t \in [0,b]} |p_\kappa(t)|$ (see (2.3)).

The following remark is an immediate consequence of the lemma 2.1.

Remark 4.2. *Let $T = \{(t, s) : a \leq t \leq b, a \leq s \leq t\}$, and let $K \colon T \to \mathbb{R}$ be a function. If $K(\cdot, s) \in \kappa BV^2([s, b])$ for each $s \in [a, b]$ then there exists a function C which depends on s such that*

$$K(\cdot, s)[y_0, y_1] \leq C(s), \text{ for all } y_0, y_1 \in [s, b].$$

Lemma 4.3. *Let $T = \{(t,s)\colon a \le t \le b, a \le s \le t\}$ and let $K\colon T \to \mathbb{R}$ be a function such that $K(\cdot,s) \in \kappa BV^2([s,b])$. If $K(s,s) = 0$ for all $s \in [a,b]$ or if there exists a function $L\colon [a,b] \to [0,+\infty)$ such that $\left|\frac{K(t,s)}{t-s}\right| \le L(s)$ for all $t \in [a,b]$ with $t \ne s$. Then for*

$$\widehat{K}(t,s) = \begin{cases} K(t,s), & a \le s \le t \\ 0 & t < s \le b \end{cases}$$

we have

$$\kappa V^2\left(\widehat{K}(\cdot,s);[a,b]\right) \le 2L(s) + 3C(s) + \kappa V^2\left(K(\cdot,s),[s,b]\right) \tag{4.1}$$

where $C(s)$ is guaranteed by remark 4.2. If $K(s,s) = 0$ we put $L(s) = 0$ in (4.1).

Proof. Let $s \in [a,b]$, let $\xi = \{t_i\}_{i=0}^n \in \Pi_3([a,b])$ and let $0 \le r \le n-1$ such that $s \in [t_r, t_{r+1}]$ and $s \notin [t_{r-1}, t_r]$. Then, by applying remark 4.2 we have

$$\begin{aligned}
&\sum_{i=0}^{n-2}\left|\widehat{K}(\cdot,s)[t_{i+1},t_{i+2}] - \widehat{K}(\cdot,s)[t_i,t_{i+1}]\right| \\
&= \left|\frac{K(t_{r+1},s)}{t_{r+1}-t_r}\right| + \left|\frac{K(t_{r+2},s)-K(t_{r+1},s)}{t_{r+2}-t_{r+1}} - \frac{K(t_{r+1},s)}{t_{r+1}-t_r}\right| \\
&\quad + \sum_{i=r+1}^{n-2}\left|K(\cdot,s)[t_{i+1},t_{i+2}] - K(\cdot,s)[t_i,t_{i+1}]\right| \\
&\le 2\left|\frac{K(t_{r+1},s)}{t_{r+1}-s}\right| + C(s) + \sum_{i=r+1}^{n-2}\left|K(\cdot,s)[t_{i+1},t_{i+2}] - K(\cdot,s)[t_i,t_{i+1}]\right|.
\end{aligned}$$

If $K(s,s) = 0$ for all $s \in [a,b]$ then

$$\begin{aligned}
&\sum_{i=0}^{n-2}\left|\widehat{K}(\cdot,s)[t_i,t_{i+1}] - \widehat{K}(\cdot,s)[t_i,t_{i+1}]\right| \\
&\le 2\left|\frac{K(t_{r+1},s)-K(s,s)}{t_{r+1}-s}\right| + \sum_{i=r+2}^{n-1}\left|K(\cdot,s)[t_i,t_{i+1}] - K(\cdot,s)[t_i,t_{i+1}]\right| + C(s) \\
&\le 3C(s) + \sum_{i=r+1}^{n-2}\left|K(\cdot,s)[t_{i+1},t_{i+2}] - K(\cdot,s)[t_i,t_{i+1}]\right|.
\end{aligned}$$

So,

$$\frac{\sum\limits_{i=0}^{n-2}\left|\widehat{K}(\cdot,s)[t_i,t_{i+1}] - \widehat{K}(\cdot,s)[t_i,t_{i+1}]\right|}{\kappa(\xi,[a,b])} \le 3C(s) + \kappa V^2(K(\cdot,s),[s,b]).$$

Taking supremum over the set of all partitions $\xi \in \pi_3([a,b])$ we have

$$\kappa V^2\left(\widehat{K}(\cdot,s),[a,b]\right) \le 3C(s) + \kappa V^2\left(K(\cdot,s),[s,b]\right).$$

Suppose there exists a function L such that $\left|\frac{K(t,s)}{t-s}\right| \le L(s)$ for each $t \in [a,b]$ with $t \ne s$. Then,

$$\sum_{i=0}^{n-2} \left| \widehat{K}(\cdot, s)[t_i, t_{i+1}] - \widehat{K}(\cdot, s)[t_i, t_{i+1}] \right|$$
$$\leq C(s) + 2L(s) + \sum_{i=r+2}^{n-2} |K(\cdot, s)[t_i, t_{i+1}] - K(\cdot, s)[t_i, t_{i+1}]| ,$$

which implies

$$\frac{\sum_{i=0}^{n-2} \left| \widehat{K}(\cdot, s)[t_i, t_{i+1}] - \widehat{K}(\cdot, s)[t_i, t_{i+1}] \right|}{\kappa(\xi, [a, b])} \leq C(s) + 2L(s) + \kappa V^2(K(\cdot, s), [s, b]).$$

Taking supremum over the set of all partitions $\xi \in \pi_3([a, b])$ yields

$$\kappa V^2 \left(\widehat{K}(\cdot, s); [a, b] \right) \leq C(s) + 2L(s) + \kappa V^2 \left(K(\cdot, s), [s, b] \right). \qquad \square$$

Now we prove the main theorems of this section.

Theorem 4.3. *Suppose that f, g and K satisfy hypotheses (H_1), (H_2), and (H_3), respectively (with $a = 0$). Then, there is a number $\tau > 0$ such that, for every λ with $|\lambda| < \tau$, the Eq (1.1) has a unique solution in $\kappa BV^2([0, b])$.*

Proof. Let $\widehat{C} = \int_0^b [|K(0, s)| + |K(b, s)|] \, ds < \infty$ (see (H_3)) and let C_{p_κ} be as in lemma 4.2. Let $r > 0$ be such that $\|g\|_{\kappa BV^2} < r$ and choose a number $\tau > 0$ such that

$$\|g\|_{\kappa V^2} + \tau \|f(x)\|_\infty \left[\widehat{C} + \int_0^b M(s) ds \right] < r \quad \text{and} \tag{4.2}$$

$$\tau C_{p_\kappa} L_0^b(f) \left[\widehat{C} + \int_0^b M(s) ds \right] < 1. \tag{4.3}$$

For $\lambda \in \mathbb{R}$ with $|\lambda| < \tau$ define the function $G \colon \kappa BV^2([0, b]) \to \kappa BV^2([0, b])$ by

$$G(x) = g(x) + \lambda F(x),$$

where $F(x)(t) = \int_0^b K(t, s) f(x(s)) ds$. By lemma 4.1 and remark 2.6 G is well defined.

Let $\overline{B}_r$ be the closed ball with center 0 and radius r in the space $\kappa BV^2([0, b])$. We begin by showing that $G(\overline{B}_r) \subset \overline{B}_r$. Let $x \in \overline{B}_r$. By remark 2.6 and the triangular inequality we have

$$\|G(x)\|_{\kappa BV^2} \leq \|g\|_{\kappa BV^2} + |\lambda| \, \|F(x)\|_{\kappa BV^2} . \tag{4.4}$$

Now,

$$\begin{aligned} \|F(x)\|_{\kappa BV^2} &= |F(x)(0)| + |F(x)(b)| + \kappa V^2 \left(F(x) \right) \\ &= \left| \int_0^b K(0, s) f(x(s)) ds \right| + \left| \int_0^b K(b, s) f(x(s)) ds \right| \end{aligned}$$

$$
\begin{aligned}
&+\ \kappa V^2\left(F(x);[0,b]\right)\\
&\leq \|f(x)\|_\infty \widehat{C} + \kappa V^2\left(F(x);[0,b]\right). \qquad (4.5)
\end{aligned}
$$

By lemma 4.1 the inequality (4.5) turns into the following

$$
\|F(x)\|_{\kappa BV^2} \leq \|f(x)\|_\infty \left[\widehat{C} + \int_0^b M(s)ds\right].
$$

It follows from (4.4) and (4.2) that

$$
\begin{aligned}
\|G(x)\|_{\kappa BV^2} &\leq \|g\|_{\kappa BV^2} + |\lambda|\,\|f(x)\|_\infty \left[\widehat{C} + \int_0^b M(s)ds\right]\\
&\leq \|g\|_{\kappa BV^2} + \tau\,\|f(x)\|_\infty \left[\widehat{C} + \int_0^b M(s)ds\right]\\
&< r.
\end{aligned}
$$

Thus, $G(\overline{B}_r) \subset \overline{B}_r$. Next we show that G is a contraction mapping. Let $x, y \in \overline{B}_r$. Then,

$$
\begin{aligned}
\|G(x)-G(y)\|_{\kappa BV^2} &= |G(x(0)) - G(y(0))| + |G(x(b)) - G(y(b))|\\
&\quad +\ \kappa V^2\left(G(x)-G(y);[0,b]\right)\\
&\leq |\lambda| \int_0^b \left(|K(0,s)| + |K(b,s)|\right)|f(x(s)) - f(y(s))|\,ds\\
&\quad + \kappa V^2\left(\lambda\left(F(x)-F(y)\right),[0,b]\right).
\end{aligned}
$$

Because $f \in \kappa BV^2([a,b])$ is Lipschitz (H_1) we infer from theorem 2.8 and lemma 4.2 that

$$
\begin{aligned}
\|G(x)-G(y)\|_{\kappa BV^2} &\leq L_0^b(f)|\lambda|\,\|x-y\|_\infty \widehat{C} + C_{p_\kappa} L_0^b(f)|\lambda|\,\|x-y\|_{\kappa BV^2} \int_0^b M(s)ds\\
&\leq C_{p_\kappa} L_0^b(f)|\lambda| \left[\widehat{C} + \int_0^b M(s)ds\right] \|x-y\|_{\kappa BV^2}\\
&\leq \tau C_{p_\kappa} L_0^b(f) \left[\widehat{C} + \int_0^b M(s)ds\right] \|x-y\|_{\kappa BV^2}.
\end{aligned}
$$

By inequality (4.3) G is a contraction mapping and by theorem 4.1 G has a unique fixed point in $\overline{B}_r$ which is to say that there exists a unique $x \in \overline{B}_r$ such that

$$
g(t) + \lambda \int_0^b K(t,s) f(x(s))ds = x(t).
$$

Since r can be chosen as large as necessary x is the only solution of Eq (1.1) in $\kappa BV^2([0,b])$. □

In addition to hypotheses H_1 and H_2, we consider

(H_4) Let $T = \{(t,s) : a \leq t \leq b, a \leq s \leq t\}$ and let $K\colon T \to \mathbb{R}$ be a function such that $K(b,\cdot)$ is L^1 integrable, $K(t,\cdot)$ is Lebesgue integrable for each $t \in [a,b]$, $K(\cdot,s)$ satisfies the hypotheses of lemma 4.3 and

$$
2L(s) + 3C(s) + \kappa V^2\left(K(\cdot,s);[s,b]\right) \leq m(s),
$$

where $m\colon [a,b] \to \mathbb{R}$ is Lebesgue integrable.

Theorem 4.4. *Let f, g and K be functions satisfying hypotheses (H_1), (H_2) and (H_4), respectively. If*

$$C_{p_\kappa} L_a^b(f)\left[\int_a^b |K(b,s)|\,ds + \int_a^b m(s)ds\right] < 1 \tag{4.6}$$

with $C_{p_\kappa} = \sup_{t\in[a,b]} |p_\kappa(t)|$ (see (2.3)) then Eq (1.2) has only one solution in $\kappa BV^2([a,b])$.

Proof. Let $r > 0$ such that

$$\|g\|_{\kappa BV^2} + \|f(x)\|_\infty \left[\int_a^b |K(b,s)|\,ds + \int_a^b m(s)ds\right] < r. \tag{4.7}$$

Define the function $\widetilde{G}\colon \kappa BV^2([a,b]) \to \kappa BV^2([a,b])$ by $\widetilde{G}(x) = g(x) + \widetilde{F}(x)$ where $\widetilde{F}(x)(t) = \int_a^t K(t,s)f(x(s))ds$ with $t \in [a,b]$, and consider the function

$$\widehat{K}(t,s) = \begin{cases} K(t,s), & a \le s \le t \\ 0 & t < s \le b. \end{cases}$$

By an argument similar to that used for $F(x)$ in lemma 4.1 it follows that $\widetilde{F}(x)$ is well defined. To see that $\widetilde{G}$ is well defined we choose $x \in \kappa BV^2([a,b])$ and we estimate $\kappa V^2(\widetilde{F}(x);[a,b])$. We will proceed in a similar way to that in the proof of lemma 4.1. Let $\xi = \{t_i\}_{i=0}^n \in \pi_3([a,b])$. Then,

$$\begin{aligned}
\left|\widetilde{F}(x)[t_{i+1},t_{i+2}] - \widetilde{F}(x)[t_i,t_{i+1}]\right| &= \left|\frac{\widetilde{F}(x)(t_{i+2}) - \widetilde{F}(x)(t_{i+1})}{t_{i+2}-t_{i+1}} - \frac{\widetilde{F}(x)(t_{i+1}) - \widetilde{F}(x)(t_i)}{t_{i+1}-t_i}\right| \\
&= \left|\frac{\int_a^{t_{i+2}} K(t_{i+2},s)f(x(s))ds - \int_a^{t_{i+1}} K(t_{i+1},s)f(x(s))ds}{t_{i+2}-t_{i+1}}\right. \\
&\qquad \left. - \frac{\int_a^{t_{i+1}} K(t_{i+1},s) - \int_a^{t_i} K(t_i,s)f(x(s))ds}{t_{i+1}-t_i}\right| \\
&= \left|\frac{\int_a^b \widehat{K}(t_{i+2},s)f(x(s))ds \ - \int_a^b \widehat{K}(t_{i+1},s)f(x(s))ds}{t_{i+2}-t_{i+1}}\right. \\
&\qquad \left. - \frac{\int_a^b \widehat{K}(t_{i+1},s)f(x(s))ds - \int_a^b \widehat{K}(t_i,s)ds}{t_{i+1}-t_i}\right| \\
&\le \sup_{s\in[a,b]} |f(x(s))| \left|\int_a^b \left[\widehat{K}(\cdot,s)[t_{i+2},t_{i+1}] - \widehat{K}(\cdot,s)[t_i,t_{i+1}]\right] ds\right|.
\end{aligned}$$

Consequently, by lemma 4.3 and hypothesis (H_4) we have

$$\begin{aligned}
&\frac{\sum_{i=0}^{n-2} |\widetilde{F}(x)[t_{i+1},t_{i+2}] - \widetilde{F}(x)[t_i,t_{i+1}]|}{\kappa(\xi,[a,b])} \\
&\le \sup_{s\in[a,b]} |f(x(s))| \int_a^b \frac{\sum_{i=0}^{n-2} \left|(\widehat{K}(\cdot,s)[t_{i+1},t_{i+2}] - \widehat{K}(\cdot,s)[t_i,t_{i+1}]\right|}{\kappa(\xi,[a,b])} ds
\end{aligned}$$

$$\leq \sup_{s\in[a,b]} |f(x(s))| \int_a^b m(s)ds.$$

Taking supremum over the set of all partitions $\xi \in \pi_3([a,b])$ yields

$$\kappa V^2(\widetilde{F}(x)) \leq \|f\|_\infty \left(\int_a^b m(s)ds \right) < +\infty. \tag{4.8}$$

Therefore, $\widetilde{G}$ is well defined.

Note that if $x, y \in \kappa BV^2([a,b])$ then we can use the ideas above, (H_1) and theorem 2.8 to deduce that

$$\kappa V^2(\widetilde{F}(x) - \widetilde{F}(y); [a,b]) \leq C_{p_\kappa} L_a^b(f) \|x-y\|_{\kappa BV^2} \int_a^b m(s)ds. \tag{4.9}$$

We first show that $\widetilde{G}(\overline{B}_r) \subset \overline{B}_r$. By remark 2.6 and the triangular inequality we have

$$\left\|\widetilde{G}(x)\right\|_{\kappa BV^2} \leq \|g\|_{\kappa BV^2} + \left\|\widetilde{F}(x)\right\|_{\kappa BV^2} \tag{4.10}$$

but

$$\begin{aligned} \left\|\widetilde{F}(x)\right\|_{\kappa BV^2} &= |\widetilde{F}(x(a))| + |\widetilde{F}(x(b))| + \kappa V^2(\widetilde{F}(x); [a,b]) \\ &\leq \|f\|_\infty \int_a^b |K(b,s)| ds + \kappa V^2(\widetilde{F}(x); [a,b]). \end{aligned} \tag{4.11}$$

Substituting (4.11) and (4.8) into the inequality (4.10) and by (4.7) we obtain

$$\begin{aligned} \left\|\widetilde{G}(x)\right\|_{\kappa BV^2} &\leq \|g\|_{\kappa BV^2} + \|f\|_\infty \left[\int_a^b |K(b,s)|\, ds + \int_a^b m(s)ds \right] \\ &< r. \end{aligned}$$

Now, let $x, y \in \overline{B}_r$. By inequality (4.9), theorem 2.8, hypothesis (H_1) and (4.6) we have

$$\begin{aligned} &\left\|\widetilde{G}(x) - \widetilde{G}(y)\right\|_{\kappa BV^2} \\ &= \left|\widetilde{F}(x)(b) - \widetilde{F}(y)(b)\right| + \kappa V^2\left(\widetilde{F}(x) - \widetilde{F}(y); [a,b]\right) \\ &\leq C_{p_\kappa} L_a^b(f) \left[\int_a^b |K(b,s)|\, ds + \int_a^b m(s)ds \right] \|x-y\|_{\kappa BV^2} \\ &< \|x-y\|_{\kappa BV^2}. \end{aligned}$$

So, $\widetilde{G}$ is a contraction and by theorem 4.1 $\widetilde{G}$ has a single fixed point at $\overline{B}_r$ that is there is a unique $\widetilde{x} \in \overline{B}_r$ such that

$$g(t) + \int_a^t K(t,s) f(\widetilde{x}(s))ds = \widetilde{x}(t).$$

Since r can be chosen arbitrarily large $\widetilde{x}$ is the only solution of Eq (1.2) in $\kappa BV^2([a,b])$.

□

Note the fact that the solutions of the Eqs (1.1) and (1.2) are in $\kappa BV^2(I)$, implying that (1.1) and (1.2) are continuous on I by virtue of theorem 2.9.

5. Applications

In this section we give examples of application of theorem 4.3 and theorem 4.4.

Example 5.1. *Several methods have been used to solve the Hammerstein integral equation. In [7], the nonlinear Hammerstein integral equation*

$$x(t) = -\frac{5}{12}t + 1 + \int_0^1 stx^2(s)ds, \quad \textit{with } t \in [0,1], \tag{5.1}$$

is considered with an exact solution $x(t) = 1 + \frac{1}{3}t$. *In that paper it can be observed that the Sinc-collocation method yields an indistinguishable approximation from the exact solution so the method is fairly accurate. In order to complement this approximate method we show that such solution is unique in the space of functions of bounded second* κ*-variation. Indeed, let* $f\colon \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = x^2$ *and* $g\colon [0,1] \to \mathbb{R}$ *be defined by* $g(t) = -\frac{5}{12}t + 1$.
(1) It is clear that f *is locally Lipschitz.*
(2) By theorem 2.5 we have that $g \in \kappa BV^2([0,1])$.
(3) Let $K(t,s) = ts$ *for* $t,s \in [0,1]$. *Clearly,* $K(t,\cdot)$ *is Lebesgue integrable for all* $t \in [0,1]$. *By theorem 2.5*

$$\kappa V^2\left(K(\cdot,s),[0,1]\right) = 0 \le s = M(s),$$

with M *Lebesgue integrable. Also,*

$$\int_0^1 |K(t,s)|\,ds = \int_0^1 |ts|\,ds \le \int_0^1 |s|\,ds = \frac{1}{2}$$

then $K(0,\cdot)$ *and* $K(1,\cdot)$ *are* L^1 *integrable.*

Therefore, all conditions of theorem 4.3 hold and so Eq (5.1) *has a unique solution in* $\kappa BV^2([0,1])$.□

Example 5.2. *Consider the Volterra-Hammerstein integral equation*

$$x(t) = -\frac{1}{4}t^5 - \frac{2}{3}t^4 - \frac{5}{6}t^3 - t^2 + 1 + \int_0^t (ts - s^2)x^2(s)ds, \quad \textit{with } 0 \le t \le \frac{1}{2}. \tag{5.2}$$

We will show that there is a unique solution in $\kappa BV^2\left(\left[0,\frac{1}{2}\right]\right)$ *(which will be continuous). Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = x^2$ *and let* $g\colon \left[0,\frac{1}{2}\right] \to \mathbb{R}$ *be defined by* $g(t) = -\frac{1}{4}t^5 - \frac{2}{3}t^4 - \frac{5}{6}t^3 - t^2 + 1$.
(1) It is clear that f *is locally Lipschitz with* $L_0^{\frac{1}{2}}(f) = 1$.
(2) Corollary 2.1 guarantees that $g \in \kappa BV^2([0,\frac{1}{2}])$.
(3) Let $T = \{(t,s) : 0 \le t \le \frac{1}{2}, 0 \le s \le t\}$, *let* $K\colon T \to \mathbb{R}$ *be given by* $K(t,s) = ts - s^2$ *and let*

$$\widehat{K}(t,s) = \begin{cases} K(t,s), & 0 \le s \le t \\ 0, & t < s \le \frac{1}{2}. \end{cases}$$

Clearly, $K(t,\cdot)$ *is Lebesgue integrable for all* $t \in [0,\frac{1}{2}]$ *and* $K\left(\frac{1}{2},\cdot\right)$ *is* L^1*-integrable since*

$$\int_0^{\frac{1}{2}} \left|K\left(\frac{1}{2},s\right)\right| ds = \int_0^{\frac{1}{2}} \left|\frac{1}{2}s - s^2\right| ds = \frac{1}{48} < +\infty.$$

Note that by theorem 2.5,

$$K(\cdot,s)\in \kappa V^2\left(\left[0,\frac{1}{2}\right]\right)$$

and even more so $\kappa V^2\left(K(\cdot,s),\left[s,\frac{1}{2}\right]\right)=0$. *Furthermore,*

$$|K(\cdot,s)[x,y]|=\left|\frac{K(x,s)-K(y,s)}{x-y}\right|=|s|=s \quad \textit{and} \quad \left|\frac{K(t,s)}{t-s}\right|=|s|=s.$$

Then, for $C(s)=s$ *and* $L(s)=s$ *the hypothesis of lemma 4.3 are satisfied. In this way we have*

$$\kappa V^2\left(\widehat{K}(\cdot,s);\left[0,\frac{1}{2}\right]\right)\leq 2L(s)+3C(s)+\kappa V^2\left(K(\cdot,s),\left[s,\frac{1}{2}\right]\right)\leq 5s=m(s).$$

It is evident that m is Lebesgue integrable.
(4) Finally, consider the distortion function $\kappa_\alpha\colon [0,1]\to[0,1]$ *given by* $\kappa_\alpha(t)=t^\alpha$ *(the entropy function corresponding to this distortion function is called Lipschitz entropy function). In the particular case* $\alpha=\frac{1}{3}$,

$$p_\kappa(t)=1+2t\left(\frac{1}{2}-t\right)(\kappa(1-2t)+\kappa(2t))=1+(t-2t^2)\left[\sqrt[3]{1-2t}+\sqrt[3]{2t}\right].$$

So, p_κ *attains its maximum value on* $t=\frac{1}{4}$, *i.e.*

$$p_\kappa\left(\frac{1}{4}\right)=1+\frac{\sqrt[3]{4}}{8}=1.19842<1.2=C_{p_\kappa}.$$

Since

$$\int_0^{\frac{1}{2}}\left|K\left(\frac{1}{2},s\right)\right|ds+\int_0^{\frac{1}{2}}m(s)ds=\frac{31}{48}=0.64583$$

we have

$$\begin{aligned} & C_{p_\kappa}L_0^{\frac{1}{2}}(f)\left[\int_0^{\frac{1}{2}}\left|K\left(\frac{1}{2},s\right)\right|ds+\int_0^{\frac{1}{2}}m(s)ds\right] \\ < \ & (1.2).(1)\,(0.64583)=0.77499<1. \end{aligned}$$

Therefore, all conditions of theorem 4.4 are satisfied and the equation (5.2) *has a unique solution in* $\kappa BV^2([0,\frac{1}{2}])$. □

6. Conclusions

In this paper we proved that the space of polynomial functions is contained in the space of functions of bounded second κ-variation $\kappa BV^2([a,b])$ and that the latter is contained in the space of absolutely continuous functions which was a very useful result for the development of this work. In addition, we proved that the pointwise product is a closed operations in $\kappa BV^2([a,b])$. By using the technique of Kuczma for the Jensen's equation we also provided necessary conditions for the superposition operator to act on $\kappa BV^2([a,b])$. We then stated the theorem for the existence and uniqueness of the solutions of the Hammerstein and Volterra integral equations by means of Banach Contraction Principle. Finally, we presented some applications. We hope the ideas and techniques used in this paper may be an inspiration to readers that are interested in studying these nonlinear integral equations in some new spaces of generalized bounded variation and that these results may be also a contribution to different areas which applications are modeled by this type of integral equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of the paper.

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