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Research article

2-complex symmetric weighted composition operators on Fock space

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Abstract: The aim of the present paper is to completely characterize 2-complex symmetric weighted composition operators $W_{e^{\bar{p}z},az+b}$ with the conjugations C and $C_{r,s,t}$ defined by $Cf(z) = \overline{f(\bar{z})}$ and $C_{r,s,t}f(z) = te^{sz}\overline{f(\overline{rz+s})}$ on Fock space by building the relations between the parameters a, b, p, r, s and t. Some examples of such operators are also given.

Keywords: Fock space; weighted composition operator; 2-complex symmetric operator; reproducing kernel function
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1. Introduction

Let H denote a separable complex Hilbert space and $\mathcal{B}(H)$ the set of all bounded linear operators on H. For an operator $T \in \mathcal{B}(H)$, let T^* denote the adjoint operator of T.

Many problems require research on non-Hermitian operators. Among them, the complex symmetric operators have become particularly important in both theoretic and application aspects [13]. We need the following definition in order to introduce the complex symmetric operators.

Definition 1.1. *An operator $C : H \to H$ is said to be a conjugation if it satisfies the following conditions:*

(a) anti-linear or conjugate-linear: $C(\alpha x + \beta y) = \bar{\alpha}C(x) + \bar{\beta}C(y)$, for $\alpha, \beta \in \mathbb{C}$ and $x, y \in H$;

(b) isometric: $\|C(x)\| = \|x\|$, for all $x \in H$;

(c) involutive: $C^2 = I_d$ where I_d is an identity operator.

For any conjugation C, there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for H such that $Ce_n = e_n$ for all $n \in \mathbb{N}$ [9]. One will see that there are many conjugations on some holomorphic function spaces. For example, the common conjugation of complex numbers $(Cf)(z) = \overline{f(\bar{z})}$ defines a conjugation on Fock space. This space will be defined later.

Definition 1.2. *Let C be a conjugation on H. An operator $T \in \mathcal{B}(H)$ is said to be complex symmetric if $CTC = T^*$.*

Interestingly, if an operator $T \in \mathcal{B}(H)$ is complex symmetric, then T can be written as a symmetric matrix relative to some orthonormal basis of H (see [9]). This shows that the complex symmetric operators can be regarded as a generalization of the symmetric matrices. The class of complex symmetric operators includes all normal operators, Hankel matrices, finite Toeplitz matrices, all truncated Toeplitz operators and some Volterra integration operators. The study of the complex symmetric operators was initiated by Garcia, Putinar, and Wogen in [9–12]. There has been many studies about complex symmetric operators on holomorphic function spaces [7, 14, 18, 20–22, 25].

In 1970, Helton in [16] initiated the study of operators $T \in \mathcal{B}(H)$ which satisfy an identity of the form

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}T^{m-j} = 0. \tag{1.1}$$

In light of complex symmetric operators, using the identity (1.1), Chō et al. in [3] introduced the definition of m-complex symmetric operators which were continuously studied in [4, 6].

Definition 1.3. *Let $m \in \mathbb{N}$ and C be a conjugation on H. An operator $T \in \mathcal{B}(H)$ is said to be m-complex symmetric if*

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^{m-j}C = 0.$$

It is clear that 1-complex symmetric operator is just the complex symmetric operator. Hence, m-complex symmetric operators can be regarded as a generalization of the complex symmetric operators. From the definition, we see that an operator $T \in \mathcal{B}(H)$ is 2-complex symmetric if and only if

$$CT^2 - 2T^*CT + T^{*2}C = 0.$$

Also from [3], it follows that all complex symmetric operators are 2-complex symmetric operators. This shows that the set of all 2-complex symmetric operators is larger than the set of all complex symmetric operators.

Recently, Hai et al. proved in [15] that the operator $Cf(z) = \overline{f(\bar{z})}$ is a conjugation on Fock space. They also proved in [15] that the operator $C_{r,s,t}f(z) = te^{sz}\overline{f(\overline{rz+s})}$ is a conjugation on Fock space if and only if

$$|r| = 1, \;\; \bar{r}s + \bar{s} = 0, \;\; |t|^2e^{|s|^2} = 1.$$

Since $C = C_{1,0,1}$, operator $C_{r,s,t}$ can be viewed as an extension of C.

Most recently, Hu et al. in [17] characterized 2-complex symmetric weighted composition operators on Hardy space. To the best of our knowledge, apart from the work of [17] there is almost no research about 2-complex symmetric weighted composition operators. Therefore, it is natural to study 2-complex symmetric weighted composition operators on some other holomorphic function spaces. Here, we consider such operators on Fock space. However, in the study we find that the proper description of the adjoint $W_{u,\varphi}^*$ of the operator $W_{u,\varphi}$ on Fock space is very difficult. This forces us to consider the weight function $u(z)$ as the kernel function $e^{\overline{p}z}$. At the same time, Le in [19] showed that the operator $W_{u,\varphi}$ is bounded on Fock space if and only if the weight function $u(z)$ belongs to Fock space, $\varphi(z) = az + b$ with $|a| \leq 1$ and

$$M(u, \varphi) = \sup\left\{|u(z)|^2 e^{(|\varphi(z)|^2 - |z|^2)} : z \in \mathbb{C}\right\} < +\infty. \tag{1.2}$$

Later, Zhao et al. in [29] continuously studied the condition (1.2) and proved that if $u(z) = e^{\overline{p}z}$ and $\varphi(z) = az + b$, then the operator $W_{u,\varphi}$ is bounded on Fock space if and only if one of the conditions holds: (i) $|a| < 1$; (ii) $|a| = 1$ and $p + \overline{a}b = 0$. From this result, it is easy to see that the operator $C_\varphi := W_{1,\varphi}$ is bounded on Fock space if and only if $\varphi(z) = az$ with $|a| \leq 1$.

Motivated by above-mentioned facts, in this paper we consider 2-complex symmetric weighted composition operator $W_{u,\varphi}$ on Fock space with the symbol $\varphi(z) = az + b$ and the weight function $u(z) = e^{\overline{p}z}$. We sometimes write the operator $W_{u,\varphi}$ as $W_{e^{\overline{p}z}, az+b}$ for such special symbol and the weight function. Our this work can be regarded as a continuous study of the weighted composition operators on Fock space.

2. Preliminaries

Here, denote by $\mathbb{N}$ the set of all nonzero integers, by $\mathbb{C}$ the complex plane, and by $H(\mathbb{C})$ the set of all holomorphic functions on $\mathbb{C}$. The Fock space $\mathcal{F}^2(\mathbb{C})$ is the Hilbert space of all holomorphic functions $f \in H(\mathbb{C})$ with the inner product

$$\langle f, g\rangle = \frac{1}{\pi}\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-|z|^2}d\nu(z),$$

where $\nu(z)$ denotes the Lebesgue measure on $\mathbb{C}$. To simplify notation we will often use $\mathcal{F}^2$ instead of $\mathcal{F}^2(\mathbb{C})$ and we will denote by $\|f\|$ the corresponding norm of f. The reproducing kernel functions of Fock space are given by

$$K_w(z) = e^{\overline{w}z}, \quad z \in \mathbb{C}.$$

It is easy to see that $\|K_w\| = e^{\frac{|w|^2}{2}}$. Let k_w be the normalization of K_w. Then,

$$k_w(z) = e^{\overline{w}z - \frac{|w|^2}{2}}, \quad z \in \mathbb{C}.$$

For more information of Fock space, see [30].

For a given holomorphic mapping $\varphi : \mathbb{C} \to \mathbb{C}$ and $u \in H(\mathbb{C})$, the weighted composition operator, usually denoted by $W_{u,\varphi}$, on or between some subspaces of $H(\mathbb{C})$ is defined by

$$W_{u,\varphi}f(z) = u(z)f(\varphi(z)).$$

When $u = 1$, it is the composition operator usually denoted by C_φ. While $\varphi(z) = z$, it is the multiplication operator usually denoted by M_u. It is a well-known fact that Forelli in [8] proved that the isometries on the Hardy space H^p defined on the open unit disk $\mathbb{D}$ (for $p \neq 2$) are certain weighted composition operators which can be regarded as the earliest presence of the weighted composition operators. Weighted composition operators have also been used in descriptions of adjoints of composition operators [5].

It is important to provide function-theoretic characterizations of when the symbols u and φ induce a bounded or compact weighted composition operator on various holomorphic function spaces. Recently, several authors have worked on the composition operators and weighted composition operators on Fock space. For the one-variable case, Ueki in [24] characterized the boundedness and compactness of weighted composition operators on Fock space. In addition to studying the boundedness and compactness, Bhuia in [1] studied C-normality of the weighted composition operators on Fock space. Zhao in [26–28] studied the unitary, invertible and normal weighted composition operators $W_{u,\varphi}$ with the symbol $\varphi(z) = az + b$ and the weight function $u(z) = e^{\overline{p}z}$ on Fock space.

3. 2-complex symmetric weighted composition operators

Since the linear span of the reproducing kernel functions $\{K_w : w \in \mathbb{C}\}$ is dense in $\mathcal{F}^2$, we have the following direct result.

Lemma 3.1. *Let $T \in \mathcal{B}(\mathcal{F}^2)$ and C be a conjugation on $\mathcal{F}^2$. Then, the operator T is 2-complex symmetric on $\mathcal{F}^2$ with the conjugation C if and only if*

$$(CT^2 - 2T^*CT + T^{*2}C)K_w = 0.$$

In order to study 2-complex symmetry of weighted composition operators on Fock space, we need the formula for the adjoint of the weighted composition operator $W_{e^{\overline{p}z},az+b}$.

Lemma 3.2. *Let $\varphi(z) = az + b$ and $u(z) = e^{\overline{p}z}$ such that the operator $W_{u,\varphi}$ is bounded on $\mathcal{F}^2$. Then, it follows that*

$$W^*_{u,\varphi} = W_{e^{\bar{b}z},\bar{a}z+p}.$$

Proof. For each $f \in \mathcal{F}^2$, we have

$$\begin{aligned}
W^*_{u,\varphi}f(z) &= \langle W^*_{u,\varphi}f, K_z\rangle = \langle f, W_{u,\varphi}K_z\rangle \\
&= \frac{1}{\pi}\int_{\mathbb{C}} f(w)\overline{W_{u,\varphi}K_z(w)}e^{-|w|^2}d\nu(w) \\
&= \frac{1}{\pi}\int_{\mathbb{C}} f(w)\overline{u(w)K_z(\varphi(w))}e^{-|w|^2}d\nu(w) \\
&= \frac{1}{\pi}\int_{\mathbb{C}} f(w)\overline{e^{\overline{p}w}K_z(aw+b)}e^{-|w|^2}d\nu(w) \\
&= \frac{1}{\pi}\int_{\mathbb{C}} f(w)e^{p\overline{w}}e^{z(\overline{aw}+\overline{b})}e^{-|w|^2}d\nu(w)
\end{aligned}$$

$$= e^{\bar{b}z}\frac{1}{\pi}\int_{\mathbb{C}} f(w)e^{(\overline{a}z+p)\overline{w}}e^{-|w|^2}dv(w)$$
$$= e^{\bar{b}z}\langle f, K_{\bar{a}z+p}\rangle = e^{\bar{b}z}f(\bar{a}z+p) = W_{e^{\bar{b}z},\bar{a}z+p}f(z),$$

from which the desired result follows. The proof is complete. □

We first characterize 2-complex symmetric weighted composition operator $W_{e^{\overline{p}z},az+b}$ on Fock space when the conjugation is of the form $Cf(z) = \overline{f(\overline{z})}$. To this end, we need to assume that $p\bar{b} - p^2 \neq 2k\pi i$ for each $k \in \mathbb{N}$.

Theorem 3.1. *Let $\varphi(z) = az + b$ and $u(z) = e^{\overline{p}z}$ such that the operator $W_{u,\varphi}$ is bounded on $\mathcal{F}^2$. Then, the operator $W_{u,\varphi}$ is 2-complex symmetric on $\mathcal{F}^2$ with the conjugation $Cf(z) = \overline{f(\overline{z})}$ if and only if $p = \overline{b}$.*

Proof. For each $w, z \in \mathbb{C}$, we have the following calculations:

$$\begin{aligned} CW_{u,\varphi}^2K_w(z) &= CW_{u,\varphi}(e^{\overline{p}z}K_w(az+b)) \\ &= Ce^{\overline{p}z}e^{\overline{p}(az+b)}e^{\overline{w}(a(az+b)+b)} \\ &= Ce^{\overline{p}z}e^{\overline{p}(az+b)}e^{\overline{w}(a^2z+ab+b)} \\ &= e^{p\bar{b}}e^{(\bar{a}\bar{b}+\bar{b})w}e^{(p+p\bar{a}+w\bar{a}^2)z}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} W_{u,\varphi}^*CW_{u,\varphi}K_w(z) &= W_{u,\varphi}^*Cu(z)K_w(\varphi(z)) = W_{u,\varphi}^*Ce^{\overline{p}z}e^{\overline{w}(az+b)} \\ &= W_{u,\varphi}^*e^{pz}e^{w(\bar{a}z+\bar{b})} = W_{e^{\bar{b}z},\bar{a}z+p}(e^{pz}e^{w(\bar{a}z+\bar{b})}) \\ &= e^{\bar{b}z}e^{p(\bar{a}z+p)}e^{w(\bar{a}^2z+\bar{a}p+\bar{b})} \\ &= e^{p^2}e^{(\bar{a}p+\bar{b})w}e^{(\bar{b}+p\bar{a}+w\bar{a}^2)z} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} W_{u,\varphi}^{*2}CK_w(z) &= W_{u,\varphi}^{*2}e^{wz} = W_{u,\varphi}^*W_{e^{\bar{b}z},\bar{a}z+p}e^{wz} \\ &= W_{u,\varphi}^*e^{\bar{b}z}e^{w(\bar{a}z+p)} = W_{e^{\bar{b}z},\bar{a}z+p}e^{\bar{b}z}e^{w(\bar{a}z+p)} \\ &= e^{\bar{b}z}e^{\bar{b}(\bar{a}z+p)}e^{w(\bar{a}^2z+\bar{a}p+p)} \\ &= e^{\bar{b}p}e^{(\bar{a}p+p)w}e^{(\bar{b}+\bar{a}\bar{b}+w\bar{a}^2)z}. \end{aligned} \tag{3.3}$$

From (3.1)–(3.3) and Lemma 3.1, it follows that the operator $W_{u,\varphi}$ is 2-complex symmetric on $\mathcal{F}^2$ with the conjugation $Cf(z) = \overline{f(\overline{z})}$ if and only if

$$e^{p\bar{b}}\left[e^{(\bar{a}\bar{b}+\bar{b})w}e^{(p+p\bar{a}+w\bar{a}^2)z} + e^{(\bar{a}p+p)w}e^{(\bar{b}+\bar{a}\bar{b}+w\bar{a}^2)z}\right] = 2e^{p^2}e^{(\bar{a}p+\bar{b})w}e^{(\bar{b}+p\bar{a}+w\bar{a}^2)z}. \tag{3.4}$$

By letting $z = w = 0$ in (3.4), we obtain that $e^{p\bar{b}} = e^{p^2}$. Since $p\bar{b} - p^2 \neq 2k\pi i$ for each $k \in \mathbb{N}$, we have $p\bar{b} = p^2$. We will divide into two cases.

Case 1. Assume that $p \neq 0$. Then, $\bar{b} = p$. We will prove that if $p \neq 0$ then (3.4) holds if and only if $\bar{b} = p$. Indeed, we just need to prove that if $p \neq 0$ and $\bar{b} = p$ then (3.4) holds. Since $\bar{b} = p$, we have

$$e^{(\bar{a}\bar{b}+\bar{b})w} = e^{(\bar{a}p+p)w}$$

and

$$e^{(\bar{b}+\bar{a}\bar{b}+w\bar{a}^2)z} = e^{(p+\bar{a}p+w\bar{a}^2)z}$$

which shows that (3.4) holds.

Case 2. Assume that $p = 0$. Then, (3.4) is reduced to

$$e^{(\bar{a}\bar{b}+\bar{b})w}e^{w\bar{a}^2z} + e^{(\bar{b}+\bar{a}\bar{b}+w\bar{a}^2)z} = 2e^{\bar{b}w}e^{(\bar{b}+w\bar{a}^2)z}. \tag{3.5}$$

By letting $z = w$ in (3.5), we obtain

$$e^{(\bar{a}\bar{b}+\bar{b})z+\bar{a}^2z^2} = e^{2\bar{b}z+\bar{a}^2z^2}$$

which implies that $\bar{a}\bar{b} + \bar{b} = 2\bar{b}$, that is, $(a-1)b = 0$. We also will divide into two subcases.

Subcase 1. Assume that $a \neq 1$. Then, $b = 0$. At this time, (3.5) holds clearly if and only $b = 0$.

Subcase 2. Assume that $a = 1$. Then, from a direct calculation we see that (3.5) becomes $e^{\bar{b}z} = e^{\bar{b}w}$ which forces b to be zero. Clearly, if $p = 0$ and $a = 1$ then (3.5) holds if and only $b = 0$.

Combining Subcases 1 and 2, we see that if $p = 0$ then (3.4) holds if and only if $b = 0$.

According to Cases 1 and 2, we finish the proof. □

Considering Theorem 3.1 and the discussions in Introduction, we have the following result.

Corollary 3.1. *The operator C_φ is 2-complex symmetric on $\mathcal{F}^2$ with the conjugation $Cf(z) = \overline{f(\bar{z})}$ if and only if $\varphi(z) = az$ with $|a| \leq 1$.*

By Theorem 3.1, we can give some examples of 2-complex symmetric operators $W_{u,\varphi}$ on $\mathcal{F}^2$ with the conjugation $Cf(z) = \overline{f(\bar{z})}$.

Example 3.1. (a) Let $\varphi(z) = \frac{1}{2}z + i$, $u(z) = K_{-i}(z)$. Then, the operator $W_{u,\varphi}$ is 2-complex symmetric on $\mathcal{F}^2$ with the conjugation C.

(b) Let $\varphi(z) = iz - \frac{\sqrt{2}}{2}i$, $u(z) = K_{\frac{\sqrt{2}}{2}i}(z)$. Then, the operator $W_{u,\varphi}$ is 2-complex symmetric on $\mathcal{F}^2$ with the conjugation C.

Proof. (a) It is clear that $p = \bar{b}$. From Theorem 3.1, it follows that the operator $W_{u,\varphi}$ is 2-complex symmetric on $\mathcal{F}^2$ with the conjugation C.

(b) It can be similarly proved. So, here the details are omitted. □

Now, we begin to characterize 2-complex symmetric weighted composition operator $W_{e^{\bar{p}z},az+b}$ on Fock space with the conjugation $C_{r,s,t}f(z) = te^{sz}\overline{f(\overline{rz+s})}$. We need to assume that $(\bar{a}s+\bar{b})p-(s+rp)p \neq 2k\pi i$ for each $k \in \mathbb{N}$.

Theorem 3.2. *Let $u(z) = e^{\overline{p}z}$ and $\varphi(z) = az+b$ such that the operator $W_{u,\varphi}$ is bounded on $\mathcal{F}^2$. Then, the operator $W_{u,\varphi}$ is 2-complex symmetric on $\mathcal{F}^2$ with the conjugation $C_{r,s,t}$ if and only if $(\bar{a}-1)s+\bar{b} = rp$.*

Proof. For each $w, z \in \mathbb{C}$, we have

$$\begin{aligned} C_{r,s,t}W_{u,\varphi}^2K_w(z) &= C_{r,s,t}W_{u,\varphi}(e^{\overline{p}z}K_w(az+b)) \\ &= C_{r,s,t}W_{u,\varphi}(e^{\overline{p}z}e^{\overline{w}(az+b)}) \\ &= C_{r,s,t}e^{\overline{p}z}e^{\overline{p}(az+b)}e^{\overline{w}[a(az+b)+b]} \end{aligned}$$

$$
\begin{aligned}
&= C_{r,s,t} e^{\overline{p}z} e^{\overline{p}(az+b)} e^{\overline{w}(a^2 z+ab+b)} \\
&= t e^{sz} e^{p(rz+s)} e^{p[\bar{a}(rz+s)+\bar{b}]} e^{w[\bar{a}^2(rz+s)+\bar{a}\bar{b}+\bar{b}]} \\
&= t e^{(s+\bar{a}s+\bar{b})p} e^{(\bar{a}^2 s+\bar{a}\bar{b}+\bar{b})w} e^{(s+pr+p\bar{a}r+w\bar{a}^2 r)z} \qquad (3.6)
\end{aligned}
$$

and

$$
\begin{aligned}
W_{u,\varphi}^{*} C_{r,s,t} W_{u,\varphi} K_w(z) &= W_{u,\varphi}^{*} C_{r,s,t} u(z) K_w(\varphi(z)) \\
&= W_{u,\varphi}^{*} C_{r,s,t} e^{\overline{p}z} e^{\overline{w}(az+b)} \\
&= W_{u,\varphi}^{*} t e^{sz} e^{p(rz+s)} e^{w[\bar{a}(rz+s)+\bar{b}]} \\
&= t W_{e^{\bar{b}z},\bar{a}z+p} \left(e^{sz} e^{p(rz+s)} e^{w[\bar{a}(rz+s)+\bar{b}]} \right) \\
&= t e^{\bar{b}z} e^{s(\bar{a}z+p)} e^{p[r(\bar{a}z+p)+s]} e^{w\{\bar{a}[r(\bar{a}z+p)+s]+\bar{b}\}} \\
&= t e^{(2s+rp)p} e^{(\bar{a}rp+\bar{a}s+\bar{b})w} e^{(\bar{b}+s\bar{a}+pr\bar{a}+wr\bar{a}^2)z}. \qquad (3.7)
\end{aligned}
$$

For each $w, z \in \mathbb{C}$, we also have

$$
\begin{aligned}
W_{u,\varphi}^{*2} C_{r,s,t} K_w(z) &= W_{u,\varphi}^{*2} t e^{sz} e^{w(rz+s)} \\
&= t W_{u,\varphi}^{*} W_{e^{\bar{b}z},\bar{a}z+p} e^{sz} e^{w(rz+s)} \\
&= t W_{u,\varphi}^{*} e^{\bar{b}z} e^{s(\bar{a}z+p)} e^{w[r(\bar{a}z+p)+s]} \\
&= t W_{e^{\bar{b}z},\bar{a}z+p} e^{\bar{b}z} e^{s(\bar{a}z+p)} e^{w[r(\bar{a}z+p)+s]} \\
&= t e^{\bar{b}z} e^{\bar{b}(\bar{a}z+p)} e^{s[\bar{a}(\bar{a}z+p)+p]} e^{w\{r[\bar{a}(\bar{a}z+p)+p]+s\}} \\
&= t e^{(\bar{b}+s\bar{a}+s)p} e^{(r\bar{a}p+rp+s)w} e^{(\bar{b}+\bar{b}\bar{a}+s\bar{a}^2+wr\bar{a}^2)z}. \qquad (3.8)
\end{aligned}
$$

Therefore, from (3.6)–(3.8) it follows that the operator $W_{u,\varphi}$ is 2-complex symmetric on $\mathcal{F}^2$ with the conjugation $C_{r,s,t}$ if and only if

$$
\begin{aligned}
& t e^{(s+\bar{a}s+\bar{b})p} \Big[e^{(\bar{a}^2 s+\bar{a}\bar{b}+\bar{b})w} e^{(s+pr+p\bar{a}r+w\bar{a}^2 r)z} + e^{(r\bar{a}p+rp+s)w} e^{(\bar{b}+\bar{b}\bar{a}+s\bar{a}^2+wr\bar{a}^2)z} \Big] \\
&= 2t e^{(2s+rp)p} e^{(\bar{a}rp+\bar{a}s+\bar{b})w} e^{(\bar{b}+s\bar{a}+pr\bar{a}+wr\bar{a}^2)z}. \qquad (3.9)
\end{aligned}
$$

Since $t \neq 0$, (3.9) is equivalent to

$$
\begin{aligned}
& e^{(s+\bar{a}s+\bar{b})p} \Big[e^{(\bar{a}^2 s+\bar{a}\bar{b}+\bar{b})w} e^{(s+pr+p\bar{a}r+w\bar{a}^2 r)z} + e^{(r\bar{a}p+rp+s)w} e^{(\bar{b}+\bar{b}\bar{a}+s\bar{a}^2+wr\bar{a}^2)z} \Big] \\
&= 2 e^{(2s+rp)p} e^{(\bar{a}rp+\bar{a}s+\bar{b})w} e^{(\bar{b}+s\bar{a}+pr\bar{a}+wr\bar{a}^2)z}. \qquad (3.10)
\end{aligned}
$$

Then, by letting $w = z = 0$ in (3.10) we obtain

$$
e^{(s+\bar{a}s+\bar{b})p} = e^{(2s+rp)p}. \qquad (3.11)
$$

Since $(\bar{a}s + \bar{b})p - (s + rp)p \neq 2k\pi i$ for each $k \in \mathbb{N}$, by (3.11) we have

$$
(\bar{a}s + \bar{b})p = (s + rp)p.
$$

We will divide into two cases.

Case 1. Assume that $p \neq 0$. Then, $\bar{a}s + \bar{b} = s + rp$. We will prove that if $p \neq 0$ then (3.10) holds if and only if $\bar{a}s + \bar{b} = s + rp$. Indeed, we just need to prove that if $p \neq 0$ and $\bar{a}s + \bar{b} = s + rp$ then (3.10) holds. Since $\bar{a}s + \bar{b} = s + rp$, we have

$$\bar{a}^2 s + \bar{a}\bar{b} = \bar{a}s + \bar{a}rp$$

which shows

$$\bar{a}s + \bar{a}rp + \bar{b} = \bar{a}^2 s + \bar{a}\bar{b} + \bar{b} \tag{3.12}$$

and

$$\bar{a}s + \bar{a}rp + \bar{b} = rp\bar{a} + rp + s. \tag{3.13}$$

From (3.12) and (3.13), we therefore have

$$e^{(\bar{a}^2 s + \bar{a}\bar{b} + \bar{b})w} = e^{(\bar{a}rp + \bar{a}s + \bar{b})w} = e^{(r\bar{a}p + rp + s)w}.$$

On the other hand, by $\bar{a}s + \bar{b} = s + rp$ we have

$$s + pr + p\bar{a}r + w\bar{a}^2 r = \bar{b} + s\bar{a} + p\bar{a}r + w\bar{a}^2 r \tag{3.14}$$

and

$$\bar{b} + \bar{a}\bar{b} + \bar{a}^2 s + w\bar{a}^2 r = \bar{b} + \bar{a}(rp + s) + r\bar{a}^2 w = \bar{b} + s\bar{a} + \bar{a}rp + wr\bar{a}^2. \tag{3.15}$$

From (3.14) and (3.15), we obtain

$$e^{(s + pr + p\bar{a}r + w\bar{a}^2 r)z} = e^{(\bar{b} + \bar{a}\bar{b} + s\bar{a}^2 + wr\bar{a}^2)z} = e^{(\bar{b} + s\bar{a} + pr\bar{a} + wr\bar{a}^2)z}.$$

Therefore, we prove that if $p \neq 0$ and $\bar{a}s + \bar{b} = s + rp$ then (3.10) holds.

Case 2. Assume that $p = 0$. Then, (3.10) is reduced to

$$e^{(\bar{a}^2 s + \bar{a}\bar{b} + \bar{b})w} e^{(s + w\bar{a}^2 r)z} + e^{sw} e^{(\bar{b} + \bar{a}\bar{b} + s\bar{a}^2 + wr\bar{a}^2)z} = 2e^{(\bar{a}s + \bar{b})w} e^{(\bar{b} + s\bar{a} + wr\bar{a}^2)z}. \tag{3.16}$$

By letting $z = w$ in (3.16), we obtain

$$e^{(s + \bar{a}^2 s + \bar{a}\bar{b} + \bar{b})w + r\bar{a}^2 w^2} = e^{2(\bar{b} + s\bar{a})w + r\bar{a}^2 w^2}.$$

From this, it follows that

$$s + \bar{a}^2 s + \bar{a}\bar{b} + \bar{b} = 2(\bar{b} + s\bar{a}),$$

that is,

$$s(\bar{a} - 1)^2 + \bar{b}(\bar{a} - 1) = 0. \tag{3.17}$$

We will divide into two subcases.

Subcase 1. Assume that $a \neq 1$. Then, from (3.17) we get that

$$\bar{a}s + \bar{b} = s. \tag{3.18}$$

It follows from (3.18) that

$$\bar{a}^2 s + \bar{a}\bar{b} + \bar{b} = \bar{a}(\bar{a}s + \bar{b}) + \bar{b} = \bar{a}s + \bar{b} = s. \tag{3.19}$$

Combining (3.18) and (3.19), we prove that if $p = 0$ and $a \neq 1$ then (3.16) holds if and only if $\bar{a}s+\bar{b} = s$.

Subcase 2. Assume that $a = 1$. Then, (3.16) becomes

$$e^{(s+2\bar{b})w}e^{(s+rw)z} + e^{sw}e^{(2\bar{b}+s+rw)z} = 2e^{(s+\bar{b})w}e^{(\bar{b}+s+rw)z}. \tag{3.20}$$

By letting $w = 0$ in (3.20), we have

$$e^{sz} + e^{(2\bar{b}+s)z} = 2e^{(\bar{b}+s)z}$$

which is equivalent to $b = 0$. Thus, if $p = 0$ and $a = 1$ then (3.16) holds if and only if $b = 0$.

Considering Subcases 1 and 2, we prove that if $p = 0$ then (3.10) holds if and only if $(\bar{a}-1)s+\bar{b} = 0$.

From Cases 1 and 2, the desired result follows. □

By using Theorem 3.2 and the discussions in Introduction, we obtain the following result.

Corollary 3.2. *The operator C_φ is 2-complex symmetric on $\mathcal{F}^2$ with the conjugation $C_{r,s,t}$ if and only if $\varphi(z) = az + b$ with $|a| \leq 1$ and $(a-1)\bar{s} + b = 0$.*

By Theorem 3.2, we can give an example as follows.

Example 3.2. Let $u(z) = e^{\overline{p}z}$ and $\varphi(z) = az+b$ where $a = 1 - \frac{\sqrt{3}}{6} - \frac{i}{2}$, $b = -\sqrt{3} + \frac{i}{2}$, $p = -i$, $r = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, $s = -\frac{3}{4}(1 + \sqrt{3}i)$, $t = e^{-\frac{9}{8}}(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$. Then, the operator $W_{u,\varphi}$ is 2-complex symmetric on $\mathcal{F}^2$ with the conjugation $C_{r,s,t}$.

Proof. As what we have discussed, we need to show that the following four conditions simultaneously hold:

$$|r| = 1, \ \ \overline{r}s + \bar{s} = 0, \ \ |t|^2 e^{|s|^2} = 1,$$

$$(\bar{a}s + \bar{b})p - (s + rp)p \neq 2k\pi i,$$

$$|a| \leq 1$$

and

$$(\bar{a} - 1)s + \bar{b} = rp.$$

It is easy to see that

$$|a|^2 = \frac{4 - \sqrt{3}}{3} \leq 1$$

which shows that $|a| \leq 1$. By a direct calculation, we have

$$\bar{r}s + \bar{s} = \Big(\frac{1}{2} + \frac{\sqrt{3}}{2}i\Big)\Big[-\frac{3}{4}(1 + \sqrt{3}i)\Big] - \frac{3}{4}(1 - \sqrt{3}i) = 0.$$

We also have $|s|^2 = \frac{9}{4}$ and $|t|^2 = e^{-\frac{9}{4}}$ which shows $|t|^2 e^{|s|^2} = 1$. It is clear that $|r| = 1$. So, the first condition holds.

Next, we will prove that the second condition holds. A direct calculation shows

$$\begin{aligned}\bar{a}s + \bar{b} &= -\frac{3}{4}\Big(1 - \frac{\sqrt{3}}{6} + \frac{i}{2}\Big)(1 + \sqrt{3}i) - \sqrt{3} - \frac{i}{2} \\ &= -\frac{3 + 2\sqrt{3}}{4} - \frac{3\sqrt{3} + 2}{4}i\end{aligned}$$

and

$$\begin{aligned}s + rp &= -\frac{3}{4}\Big(1 + \sqrt{3}i\Big) - \Big(\frac{1}{2} - \frac{\sqrt{3}}{2}i\Big)i \\ &= -\frac{3 + 2\sqrt{3}}{4} - \frac{3\sqrt{3} + 2}{4}i,\end{aligned}$$

from which we obtain $\bar{a}s + \bar{b} = s + rp$. So, we have

$$[(\bar{a}s + \bar{b}) - (s + rp)]p = 0 \neq 2k\pi i.$$

This shows that the second condition holds. Since $\bar{a}s + \bar{b} = s + rp$, we obtain

$$(\bar{a} - 1)s + \bar{b} = rp.$$

Thus, the last condition holds. □

4. Conclusions

In the preparation of this study, we find that the proper description of the adjoint $W_{u,\varphi}^*$ of the weighted composition operator $W_{u,\varphi}$ on Fock space is very difficult. So, in this paper we just obtain a description for the operator $W_{u,\varphi}$ with the special symbols $u(z) = e^{\bar{p}z}$ and $\varphi(z) = az + b$ on Fock space. By using the description, we completely characterize 2-complex symmetric operator $W_{u,\varphi}$ with $u(z) = e^{\bar{p}z}$ and $\varphi(z) = az + b$ under the conjugations C and $C_{r,s,t}$ defined by $Cf(z) = \overline{f(\bar{z})}$ and $C_{r,s,t}f(z) = te^{sz}\overline{f(\overline{rz + s})}$ on Fock space in terms of the relations of the parameters a, b, p, r, s and t. We also give some examples of such operators on Fock space. However, we still do not obtain any result for the general weight functions although we consider this problem for a long time. We therefore hope that the study can attract people's more attention for such a topic.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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