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Research article

Approximation properties of Stancu type Szász-Mirakjan operators

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Abstract: In this paper, a class of Stancu type Szász-Mirakjan operators are introduced. The approximation properties of the operators are discussed using tools of modulus of continuity, modulus of smoothness and K-functional. The estimation of the Lipschitz function class by the operators is also studied. Later, the Voronvskaya type asymptotic expansion of the operators is established. Finally, we compare the convergence of these newly defined operators for certain functions with certain graphs.

Keywords: Szász-Mirakjan operators; modulus of continuity; K-functional; Voronovskaya-type asymptotic formula; rate of convergence
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1. Introduction

In recent years, approximation theory has attracted the attention of many mathematicians, especially in the field of mathematical analysis. In this context, many new positive and linear operators have been introduced and their approximate properties have been given. In this direction, Bernstein's main work on Bernstein polynomials has long maintained the primary position of approximation theory. More specifically, the Bernstein polynomial is defined as

$$B_n(\sigma, x) = \sum_{k=0}^{n} \sigma(\frac{k}{n}) C_n^k x^k (1-x)^{n-k}, x \in [0, 1]$$

for any $\sigma \in C[0, 1]$.

In 1950, Szász [1] introduced the generalization of Bernstein polynomials to infinite intervals called the classical Szász-Mirakyan operators which are defined by

$$S_n(\sigma, x) = \sum_{k=0}^{\infty} \sigma(\frac{k}{n}) s_{n,k}(x) \tag{1.1}$$

where $s_{n,k}(x) = e^{-nx} \cdot (nx)^k / k!$.

The operators S_n have always been a research hotspot as can be seen in [2–8].

In 1969, Stancu hoped to select nodes in a different way to achieve greater flexibility. In this way, he defined and studied the following linear and positive operators which are defined by [9]

$$B_n^{\gamma,\delta}(\sigma, x) = \sum_{k=0}^{n} \sigma(\frac{k+\gamma}{n+\delta}) C_n^k x^k (1-x)^{n-k}$$

where $x \in [0, 1], 0 \leq \gamma \leq \delta$. When $\gamma = \delta = 0$, $B_n^{\gamma,\delta}$ become to the classical Bernstein polynomials.

Many scholars have shown great interest in the study of Stancu type operators [10–12].

Bézier curve with shape parameters is one of the important research fields of computer graphics and computer aided geometric design (CAGD). Because of its simple and stable calculation, Bézier curve is widely used in fuselage design, numerical solution of partial differential equations, networks, animation, robotics and other fields. The selection of shape parameters is important so Bézier curves and surfaces can be represented by their control grids.

Recently, Ye [13] presented the Bézier basis with shape parameter $\lambda \in [-1, 1]$. In this article, the authors constructed a new class of basis functions for single shape parameter curves and used these basis functions to provide a practical curve modeling algorithm. Later, Cai [14] introduced λ-Bernstein operators and studied some approximation properties of the operators. Finally, Cai gave some graphs and numerical examples to show the convergence of λ-Bernstein operators. They have also shown that in some cases the errors were smaller than the classical Bernstein operators. Many scholars have done research on this type of operators [15–20].

Influenced by the construction of this type operators based on parameter λ, Qi [21] introduced the λ-Szász-Mirakjan operators as follows,

$$S_{n,\lambda}(\sigma, x) = \sum_{k=0}^{\infty} \sigma(\frac{k}{n}) s_{n,k}^{(\lambda)}(x) \tag{1.2}$$

where $\lambda \in [-1, 1]$ and

$$\begin{aligned} s_{n,0}^{(\lambda)}(x) &= s_{n,0}(x) - \frac{\lambda}{n+1} s_{n+1,1}(x), \\ s_{n,j}^{(\lambda)}(x) &= s_{n,j}(x) + \lambda \left(\frac{n-2j+1}{n^2-1} s_{n+1,j}(x) - \frac{n-2j-1}{n^2-1} s_{n+1,j+1}(x) \right), \end{aligned}$$

$j = 1, 2, \cdots, \infty, x \in [0, \infty)$.

In [21], the Korovkin type approximation theorem, the Voronovskaja-type asymptotic formula and the Grüss-Voronovskaja type theorem for the operators $S_{n,\lambda}$ were investigated. Then, Aslan [22,23] studied the Kantorovich type and Durrmeyer type of operators $S_{n,\lambda}$, respectively. In order to prove the accuracy and effectiveness of the discussed operators, Aslan provided a comparison of the convergence of the constructed operators for certain functions under certain parameters and provided some graphical explanations.

Based on the work of Qi and Aslan, we propose a new family of Stancu operators $S_{n,\lambda}^{\gamma,\delta}(\sigma, x)$ in the following way:

$$S_{n,\lambda}^{\gamma,\delta}(\sigma, x) = \sum_{k=0}^{\infty} \sigma(\frac{k+\gamma}{n+\delta}) s_{n,k}^{(\lambda)}(x). \tag{1.3}$$

Obviously, when $\gamma = \delta = 0$ the operators $S_{n,\lambda}^{\gamma,\delta}$ reduce to the operators $S_{n,\lambda}$ defined by (1.2). When $\alpha = \beta = \lambda = 0$, the operators $S_{n,\lambda}^{\gamma,\delta}$ reduce to the operators S_n defined by (1.1).

This article is organized in this way. First, the authors caculate the first to fourth moments of the operators. In the second section, by tools such as modulus of continuity and K-functional the approximation properties of the operators are discussed. The estimation of the Lipschitz function class by the operators is also studied. Later, the Voronvskaya type asymptotic expansion of the operators is established. Finally, we compare the convergence of these newly defined operators for certain functions with certain graphs.

2. Some lemmas

Our results are based on the following lemmas.

Lemma 2.1. ([21]) For $x \in [0, \infty)$, $e_i = t^i, i = 0, 1, 2$, we have the following equalities:

$$S_{n,\lambda}(e_0, x) = 1,$$

$$S_{n,\lambda}(e_1, x) = x + \lambda\left[\frac{1 - e^{-(n+1)x} - 2x}{n(n-1)}\right],$$

$$S_{n,\lambda}(e_2, x) = x^2 + \frac{x}{n} + \lambda\left[\frac{e^{-(n+1)x} - 1 + 2nx - 4(n+1)x^2}{n^2(n-1)}\right].$$

Lemma 2.2. Let the operators $S_{n,\lambda}^{\gamma,\delta}$ be defined by (1.3), we have

$$S_{n,\lambda}^{\gamma,\delta}(e_0, x) = 1,$$

$$S_{n,\lambda}^{\gamma,\delta}(e_1, x) = x + \frac{\gamma - \delta x}{n + \delta} + \lambda\left[\frac{1 - e^{-(n+1)x} - 2x}{(n+\delta)(n-1)}\right] = \varpi_{n,\gamma,\delta,\lambda}(x),$$

$$\begin{aligned} S_{n,\lambda}^{\gamma,\delta}(e_2, x) &= x^2 + \frac{\gamma^2 + n(2\gamma+1)x - (2n\delta + \delta^2)x^2}{(n+\delta)^2} \\ &+ \lambda\left[\frac{(2\gamma - 1)(1 - e^{-(n+1)x}) + (2n - 4\gamma)x - 4(n+1)x^2}{(n+\delta)^2(n-1)}\right]. \end{aligned}$$

Proof. By Lemma 2.1, we have

$$S_{n,\lambda}^{\gamma,\delta}(e_0, x) = \sum_{i=0}^{\infty} s_{n,i}^{(\lambda)}(x) = S_{n,\lambda}(e_0, x) = 1.$$

$$\begin{aligned} S_{n,\lambda}^{\gamma,\delta}(e_1, x) &= \sum_{i=0}^{\infty} \frac{i + \gamma}{n + \delta} s_{n,i}^{(\lambda)}(x) = \frac{n}{n+\delta} S_{n,\lambda}(e_1, x) + \frac{\gamma}{n+\delta} S_{n,\lambda}(e_0, x) \\ &= x + \frac{\gamma - \delta x}{n + \delta} + \lambda\left[\frac{1 - e^{-(n+1)x} - 2x}{(n+\delta)(n-1)}\right]. \end{aligned}$$

$$
\begin{aligned}
S_{n,\lambda}^{\gamma,\delta}(e_2,x) &= \sum_{i=0}^{\infty}\left(\frac{i+\gamma}{n+\delta}\right)^2 s_{n,i}^{(\lambda)}(x)\\
&= \frac{n^2}{(n+\delta)^2}S_{n,\lambda}(e_2,x)+\frac{2n\gamma}{(n+\delta)^2}S_{n,\lambda}(e_1,x)+\frac{\gamma^2}{(n+\delta)^2}S_{n,\lambda}(e_0,x)\\
&= x^2+\frac{\gamma^2+n(2\gamma+1)x-(2n\delta+\delta^2)x^2}{(n+\delta)^2}\\
&+ \lambda\left[\frac{(2\gamma-1)(1-e^{-(n+1)x})+(2n-4\gamma)x-4(n+1)x^2}{(n+\delta)^2(n-1)}\right].
\end{aligned}
$$

Using a completely similar derivation method we can get the expression of $S_{n,\lambda}^{\gamma,\delta}(e_3,x)$ and $S_{n,\lambda}^{\gamma,\delta}(e_4,x)$. Here, we omit it.

From Lemma 2.2 and simple calculation, we can get the following.

Lemma 2.3. For $x\in[0,\infty)$, we have

$$
S_{n,\lambda}^{\gamma,\delta}(t-x,x)=\frac{\gamma-\delta x}{n+\delta}+\lambda\left[\frac{1-e^{-(n+1)x}-2x}{(n+\delta)(n-1)}\right], \tag{2.1}
$$

$$
S_{n,\lambda}^{\gamma,\delta}((t-x)^2,x)=\xi_{n,\gamma,\delta,\lambda}(x), \tag{2.2}
$$

$$
S_{n,\lambda}^{\gamma,\delta}((t-x)^4,x)=O(n^{-2}), \tag{2.3}
$$

$$
\lim_{n\to\infty} nS_{n,\lambda}^{\gamma,\delta}(t-x,x)=\gamma-\delta x, \tag{2.4}
$$

$$
\lim_{n\to\infty} nS_{n,\lambda}^{\gamma,\delta}((t-x)^2,x)=x \tag{2.5}
$$

where $\xi_{n,\gamma,\delta,\lambda}(x)$ is defined as follows

$$
\begin{aligned}
\xi_{n,\gamma,\delta,\lambda}(x) &= \frac{\gamma^2+(n-2\gamma\delta)x+\delta^2x^2}{(n+\delta)^2}\\
&+ \lambda\left[\frac{(2\gamma-1-2x(n+\delta))(1-e^{-(n+1)x})+(2n-4\gamma)x-4nx^2}{(n+\delta)^2(n-1)}\right].
\end{aligned}
$$

Lemma 2.4. For $x\in[0,\infty)$, we have

$$
S_{n,\lambda}^{\gamma,\delta}(|t-x|,x)\le\sqrt{\xi_{n,\gamma,\delta,\lambda}(x)}. \tag{2.6}
$$

Proof. Because of $S_{n,\lambda}^{\gamma,\delta}(1,x)=1$, by Cauchy-Schwarz inequality and (2.2) we get

$$
S_{n,\lambda}^{\gamma,\delta}(|t-x|,x)\le\sqrt{S_{n,\lambda}^{\gamma,\delta}((t-x)^2,x)}\cdot\sqrt{S_{n,\lambda}^{\gamma,\delta}(1,x)}=\sqrt{\xi_{n,\gamma,\delta,\lambda}(x)}.
$$

3. Results

Let $C_B[0,\infty)$ be defined as the space of bounded and uniformly continuous functions σ on $[0,\infty)$, endowed with the norm $\|\sigma\|=sup_{x\in[0,\infty)}|\sigma|$.

Theorem 3.1. For $\sigma\in C_B[0,\infty)$, $x\in[0,\infty)$, the following inequality holds

$$
\|S_{n,\lambda}^{\gamma,\delta}(\sigma,x)\|\le\|\sigma\|. \tag{3.1}
$$

Proof. Since $S_{n,\lambda}^{\gamma,\delta}(1,x) = 1$, we get

$$\|S_{n,\lambda}^{\gamma,\delta}(\sigma,x)\| \leq S_{n,\lambda}^{\gamma,\delta}(1,x)\cdot\|\sigma\| = \|\sigma\|.$$

Theorem 3.2. For $\sigma \in C_B[0,\infty)$, $x \in [0,\infty)$, the following equality holds

$$\lim_{n\to\infty} S_{n,\lambda}^{\gamma,\delta}(\sigma,x) = \sigma(x). \tag{3.2}$$

Proof. By Lemma 2.2, we get

$$\lim_{n\to\infty} S_{n,\lambda}^{\gamma,\delta}(e_k,x) = x^k, k = 0,1,2.$$

The Korovkin theorem [24] is applied to obtain the conclusion.

Theorem 3.3. Let $\sigma \in C_B[0,\infty)$, $\tau > 0$ and

$$\omega(\sigma,\tau) = \sup_{0<\epsilon\leq\tau}\ \sup_{x,x+\epsilon\in[0,\infty)} |\sigma(x+\epsilon)-\sigma(x)|.$$

When n sufficiently large we have

$$\left|S_{n,\lambda}^{\gamma,\delta}(\sigma,x)-\sigma(x)\right| \leq 2\omega\left(\sigma,\ \sqrt{\xi_{n,\gamma,\delta,\lambda}(x)}\right).$$

Proof. For $\nu > 0$ and $\rho > 0$, it is widely known that $\omega(\sigma,\nu\rho) \leq (\rho+1)\omega(\sigma,\nu)$. So, we get

$$\begin{aligned}
\left|S_{n,\lambda}^{\gamma,\delta}(\sigma,x)-\sigma(x)\right| &\leq \left|S_{n,\lambda}^{\gamma,\delta}(|\sigma(t)-\sigma(x)|,x)\right| \leq \left|S_{n,\lambda}^{\gamma,\delta}(\omega(\sigma,|t-x|),x)\right| \\
&= \left|S_{n,\lambda}^{\gamma,\delta}\left(\omega(\sigma,\frac{|t-x|}{\nu}\cdot\nu),x\right)\right| \leq \left|S_{n,\lambda}^{\gamma,\delta}\left(\omega(\sigma,\nu)(1+\frac{|t-x|}{\nu}),x\right)\right| \\
&= \omega(\sigma,\nu)\left(1+\frac{1}{\nu}S_{n,\lambda}^{\gamma,\delta}(|t-x|,x)\right) \leq \omega(\sigma,\nu)\left(1+\frac{1}{\nu}\sqrt{\xi_{n,\gamma,\delta,\lambda}(x)}\right).
\end{aligned}$$

The last inequality is obtained from (2.6). Let $\nu = \sqrt{\xi_{n,\gamma,\delta,\lambda}(x)}$, we get Theorem 3.3 immediately.

For $\tau > 0$ and $W^2[0,\infty) = \{g|g, g'' \in C_B[0,\infty)\}$, the appropriate Peetre's K-functional is defined by

$$K_2(\sigma,\tau) = \inf_{g\in W^2[0,\infty)}\{\|\sigma-g\|+\tau\|g''\|\}.$$

Let

$$\omega_2(\sigma,\tau) = \sup_{0<|h|\leq\tau}\ \sup_{x,x+h,x+2h\in[0,\infty)} |\sigma(x+2h)-2\sigma(x+h)+\sigma(x)|$$

where ω_2 is the second order modulus of continuity of $\sigma \in C_B[0,\infty)$.

From [25], there exists an absolute constant $D > 0$ such that

$$K_2(\sigma,\tau) \leq D\cdot\omega_2(\sigma,\ \sqrt{\tau}). \tag{3.3}$$

Theorem 3.4. For $\sigma \in C_B[0,\infty)$ there exists an absolute constant $D > 0$ such that

$$\begin{aligned}
|S_{n,\lambda}^{\gamma,\delta}(\sigma,x)-\sigma(x)| \leq D\cdot\omega_2&\left(\sigma,\frac{1}{2}\sqrt{\xi_{n,\gamma,\delta,\lambda}(x)+\left(\varpi_{n,\gamma,\delta,\lambda}(x)-x\right)^2}\right) \\
&+\omega\left(\sigma,\varpi_{n,\gamma,\delta,\lambda}(x)-x\right).
\end{aligned} \tag{3.4}$$

Proof. We introduce the auxiliary operators

$$\widetilde{S}^{\gamma,\delta}_{n,\lambda}(\sigma, x) = S^{\gamma,\delta}_{n,\lambda}(\sigma, x) + \sigma(x) - \sigma(\varpi_{n,\gamma,\delta,\lambda}(x)). \tag{3.5}$$

By Lemma 2.2, we get

$$\widetilde{S}^{\gamma,\delta}_{n,\lambda}(1, x) = S^{\gamma,\delta}_{n,\lambda}(1, x) = 1, \tag{3.6}$$

$$\widetilde{S}^{\gamma,\delta}_{n,\lambda}(t, x) = S^{\gamma,\delta}_{n,\lambda}(t, x) + x - \varpi_{n,\gamma,\delta,\lambda}(x) = x. \tag{3.7}$$

Let $g \in W^2$. By Taylor's expansion, we get

$$g(t) = g(x) + g'(x)(t - x) + \int_x^t (t - u)g''(u)du, x, t \in [0, \infty).$$

Apply the operators $\widetilde{S}^{\gamma,\delta}_{n,\lambda}$ to the above equality and note that (3.5)–(3.7), we have

$$\begin{aligned}
\widetilde{S}^{\gamma,\delta}_{n,\lambda}(g, x) &= g(x) + \widetilde{S}^{\gamma,\delta}_{n,\lambda}\Big(\int_x^t (t - u)g''(u)du, x\Big) \\
&= g(x) + S^{\gamma,\delta}_{n,\lambda}\Big(\int_x^t (t - u)g''(u)du, x\Big) - \int_x^{\varpi_{n,\gamma,\delta,\lambda}(x)} \left(\varpi_{n,\gamma,\delta,\lambda}(x) - u\right) g''(u)du.
\end{aligned}$$

So,

$$\begin{aligned}
|\widetilde{S}^{\gamma,\delta}_{n,\lambda}(g, x) - g(x)| &\leq S^{\gamma,\delta}_{n,\lambda}\left(\int_x^t |t - u||g''(u)|du, x\right) \\
&+ \left|\int_x^{\varpi_{n,\gamma,\delta,\lambda}(x)} \left|\varpi_{n,\gamma,\delta,\lambda}(x) - u\right| |g''(u)|du\right| \\
&\leq \|g''\| \left\{S^{\gamma,\delta}_{n,\lambda}((t - x)^2, x) + \left(\varpi_{n,\gamma,\delta,\lambda}(x) - x\right)^2\right\} \\
&= \|g''\| \left\{\xi_{n,\gamma,\delta,\lambda}(x) + \left(\varpi_{n,\gamma,\delta,\lambda}(x) - x\right)^2\right\}.
\end{aligned}$$

By (3.5) and Theorem 3.1, we get

$$|\widetilde{S}^{\gamma,\delta}_{n,\lambda}(\sigma, x)| \leq |S^{\gamma,\delta}_{n,\lambda}(\sigma, x)| + |\sigma(x)| + |\sigma(\varpi_{n,\gamma,\delta,\lambda}(x))| \leq 3\|\sigma\|.$$

So,

$$\begin{aligned}
|S^{\gamma,\delta}_{n,\lambda}(\sigma, x) - \sigma(x)| &\leq \left|\widetilde{S}^{\gamma,\delta}_{n,\lambda}(\sigma, x) - \sigma(x) + \sigma(x) - \sigma\left(\varpi_{n,\gamma,\delta,\lambda}(x)\right)\right| \\
&\leq |\widetilde{S}^{\gamma,\delta}_{n,\lambda}(\sigma - g, x)| + |\widetilde{S}^{\gamma,\delta}_{n,\lambda}(g, x) - g(x)| \\
&+ |\sigma(x) - g(x)| + \left|\sigma(x) - \sigma\left(\varpi_{n,\gamma,\delta,\lambda}(x)\right)\right| \\
&\leq 4\|\sigma - g\| + \|g''\| \left\{\xi_{n,\gamma,\delta,\lambda}(x) + \left(\varpi_{n,\gamma,\delta,\lambda}(x) - x\right)^2\right\} \\
&+ \omega\left(\sigma, \varpi_{n,\gamma,\delta,\lambda}(x) - x\right).
\end{aligned}$$

Taking the infimum on the right hand side over all $g \in W^2$, we obtain

$$\begin{aligned}
|S^{\gamma,\delta}_{n,\lambda}(\sigma, x) - \sigma(x)| &\leq 4K_2\left(\sigma, \frac{1}{4}\left\{\xi_{n,\gamma,\delta,\lambda}(x) + \left(\varpi_{n,\gamma,\delta,\lambda}(x) - x\right)^2\right\}\right) \\
&+ \omega\left(\sigma, \varpi_{n,\gamma,\delta,\lambda}(x) - x\right).
\end{aligned}$$

By the inequality of (3.3), we get Theorem 3.4 immediately.

Remark 3.1. When $\gamma = \delta = 0$, Theorem 3.4 is the form of the Theorem 3.2 of Qi [21].

Let $\phi(x) = \sqrt{x}$ and $\sigma \in C_B[0,\infty)$. The first order Ditzian-Totik modulus of smoothness and corresponding K-functional are given by

$$\omega_\phi(\sigma,\tau) = \sup_{0<h\leq\tau}\left|\sigma(x+\frac{h\phi(x)}{2}) - \sigma(x-\frac{h\phi(x)}{2})\right|, x \pm \frac{h\phi(x)}{2} \in [0,\infty)$$

and

$$K_\phi(\sigma,\tau) = \inf_{g\in W_\phi[0,\infty)}\{\|\sigma - g\| + \tau\|\phi g'\|\}(\tau > 0),$$

respectively. Here, $W_\phi[0,\infty) = \{g|g \in AC[0,\infty), \|\phi g'\| < \infty\}$ means that g is differentiable and absolutely continuous on every compact subset of $[0,\infty)$. By [26], there exists a constant $E > 0$ such that

$$K_\phi(\sigma,\tau) \leq E \cdot \omega_\phi(\sigma,\tau). \tag{3.8}$$

Theorem 3.5. For $\sigma \in C_B[0,\infty)$ there exists an absolute constant $E > 0$ such that

$$\left|S^{\gamma,\delta}_{n,\lambda}(\sigma,x) - \sigma(x)\right| \leq E \cdot \omega_\phi\left(\sigma, \frac{\sqrt{\xi_{n,\gamma,\delta,\lambda}(x)}}{\sqrt{x}}\right). \tag{3.9}$$

Proof. Applying the operators $S^{\gamma,\delta}_{n,\lambda}(\cdot,x)$ to the representation

$$g(t) = g(x) + \int_x^t g'(u)du,$$

we have

$$S^{\gamma,\delta}_{n,\lambda}(g,x) = g(x) + S^{\gamma,\delta}_{n,\lambda}\left(\int_x^t g'(u)du, x\right).$$

For any $x,t \in (0,\infty)$, we can get

$$\left|\int_x^t g'(u)du\right| = \left|\int_x^t \frac{g'(u)\phi(u)}{\phi(u)}du\right| \leq \|\phi g'\|\left|\int_x^t \frac{1}{\phi(u)}du\right| \leq 2\|\phi g'\|\frac{|t-x|}{\phi(x)}.$$

By (2.6), we have

$$\begin{aligned}|S^{\gamma,\delta}_{n,\lambda}(g,x) - g(x)| &\leq 2\|\phi g'\|\phi^{-1}(x)S^{\gamma,\delta}_{n,\lambda}(|t-x|,x)\\ &\leq 2\|\phi g'\|\phi^{-1}(x)\cdot\sqrt{\xi_{n,\gamma,\delta,\lambda}(x)}.\end{aligned}$$

Thus,

$$\begin{aligned}|S^{\gamma,\delta}_{n,\lambda}(\sigma,x) - \sigma(x)| &\leq |S^{\gamma,\delta}_{n,\lambda}(\sigma - g,x)| + |\sigma - g| + |S^{\gamma,\delta}_{n,\lambda}(g,x) - g(x)|\\ &\leq 2\|\sigma - g\| + 2\|\phi g'\|\cdot\phi^{-1}(x)\cdot\sqrt{\xi_{n,\gamma,\delta,\lambda}(x)}.\end{aligned}$$

Taking the infimum on the right hand side over all $g \in W_\phi(0,\infty)$, we can get

$$|S^{\gamma,\delta}_{n,\lambda}(\sigma,x) - f(x)| \leq 2K_\phi\left(\sigma, \phi^{-1}(x)\cdot\sqrt{\xi_{n,\gamma,\delta,\lambda}(x)}\right).$$

By (3.8) and the above inequality, we get (3.9) immediately. This completes the proof of Theorem 3.5.

Now we compute the rate of convergence of the operators $S_{n,\lambda}^{\gamma,\delta}(\sigma, x)$ for the Lipschitz class $Lip_M(\kappa)(0 < \kappa \leq 1, M > 0)$. As usual, we say that a function σ belongs to $Lip_M(\kappa)$ if the inequality

$$|\sigma(t) - \sigma(x)| \leq M|t - x|^{\kappa}$$

holds for all $t, x \in R$.

Theorem 3.6. For $\sigma \in Lip_M(\kappa) \bigcap C_B[0, \infty)$ and $x \in [0, \infty)$, we have

$$\left|S_{n,\lambda}^{\gamma,\delta}(\sigma, x) - \sigma(x)\right| \leq M\left[\xi_{n,\gamma,\delta,\lambda}(x)\right]^{\kappa/2}. \tag{3.10}$$

Proof. Let $e_1 = \frac{2}{\kappa}, e_2 = \frac{2}{2-\kappa}$. Then, $\frac{1}{e_1} + \frac{1}{e_2} = 1$. By the Hölder inequality, we get

$$\begin{aligned} |S_{n,\lambda}^{\gamma,\delta}(\sigma, x) - \sigma(x)| &\leq S_{n,\lambda}^{\gamma,\delta}(|\sigma(t) - \sigma(x)|, x) \\ &\leq M \cdot S_{n,\lambda}^{\gamma,\delta}(|t - x|^{\kappa}, x) = M \cdot S_{n,\lambda}^{\gamma,\delta}(|t - x|^{\kappa} \cdot 1, x) \\ &\leq M\left(S_{n,\lambda}^{\gamma,\delta}((t - x)^{\kappa e_1}, x)\right)^{1/e_1} \cdot \left(S_{n,\lambda}^{\gamma,\delta}(1^{e_2}, x)\right)^{1/e_2} \\ &= M\left(S_{n,\lambda}^{\gamma,\delta}((t - x)^2, x)\right)^{1/e_1} \cdot 1 = M\left[\xi_{n,\gamma,\delta,\lambda}(x)\right]^{\kappa/2}. \end{aligned}$$

Remark 3.2. When $\gamma = \delta = 0$, Theorem 3.6 is the form of the Theorem 3.4 of Qi [21].

Lastly, we will consider the Voronvskaya type asymptotic expansion of the operators $S_{n,\lambda}^{\gamma,\delta}(\sigma, x)$.

Theorem 3.7. Let $\sigma', \sigma'' \in C_B[0, \infty)$, we have

$$\lim_{n\to\infty} n[S_{n,\lambda}^{\gamma,\delta}(\sigma, x) - \sigma(x)] = (\gamma - \delta x)\sigma'(x) + \frac{x}{2}\sigma''(x).$$

Proof. In view of Taylor's expansion formula, we have

$$\sigma(t) = \sigma(x) + \sigma'(x)(t - x) + \frac{1}{2}\sigma''(x)(t - x)^2 + \eta(t; x)(t - x)^2$$

where $\eta(t; x)$ ia a Peano of the rest term, $\eta(t; x) \in C[0, \infty)$ and $\lim_{t\to x} \eta(t; x) = 0$.

So,

$$\begin{aligned} n[S_{n,\lambda}^{\gamma,\delta}(\sigma, x) - \sigma(x)] &= n\sigma'(x)S_{n,\lambda}^{\gamma,\delta}(t - x, x) + \frac{n}{2}\sigma''(x)S_{n,\lambda}^{\gamma,\delta}((t - x)^2, x) \\ &\quad + nS_{n,\lambda}^{\gamma,\delta}(\eta(t; x)(t - x)^2, x). \end{aligned} \tag{3.11}$$

By Cauchy-Schwarz inequality, we have

$$S_{n,\lambda}^{\gamma,\delta}(\eta(t; x)(t - x)^2, x) \leq \sqrt{S_{n,\lambda}^{\gamma,\delta}(\eta^2(t; x), x)} \cdot \sqrt{S_{n,\lambda}^{\gamma,\delta}((t - x)^4, x)}. \tag{3.12}$$

Noting $\eta^2(x; x) = 0$, $\eta^2(t; x) \in C[0, \infty)$ and Theorem 3.2, we get

$$\lim_{n\to\infty} S_{n,\lambda}^{\gamma,\delta}(\eta^2(t; x), x) = \eta^2(x; x) = 0. \tag{3.13}$$

By (2.3), (3.12) and (3.13), we get

$$\lim_{n\to\infty} nS_{n,\lambda}^{\gamma,\delta}(\eta(t; x)(t - x)^2, x) = 0. \tag{3.14}$$

Theorem 3.7 is obtained by (2.4), (2.5), (3.11) and (3.14).

4. Graphical analysis

In this section, we show several graphics to present the convergence of operators (1.3) to certain functions with different values of γ, δ, n and λ.

In Figure 1, we choose the function $\sigma(x) = e^x$ (black), $\lambda = 0.9, \gamma = 1, \delta = 2$, $n = 10$ (red), $n = 30$ (blue) and $n = 50$ (green).

In Figure 2, we choose the function $\sigma(x) = cos(3\pi x)$ (black), $\lambda = 0.5, \gamma = 1, \delta = 2$, $n = 10$ (red), $n = 30$ (blue) and $n = 50$ (green).

In Figure 3, we choose the function $\sigma(x) = 3\pi x \cdot sin(3\pi x)$ (black), $\lambda = -0.9, \gamma = 2, \delta = 3$, $n = 10$ (red), $n = 50$ (blue) and $n = 90$ (green).

It is clear from Figures 1–3 that for the different values of λ as the values of n increases the convergence of operators (1.3) to the functions $\sigma(x)$ becomes better.

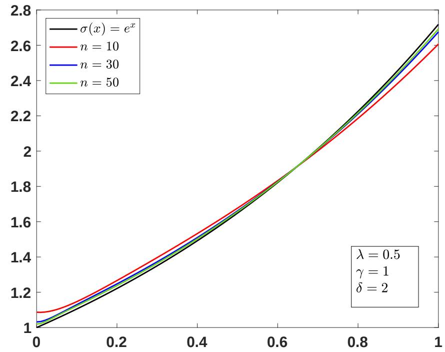

Figure 1. The convergence of $S^{\gamma,\delta}_{n,\lambda}(\sigma, x)$ to $\sigma(x) = e^x$ for $\lambda = 0.9, \gamma = 1, \delta = 2$.

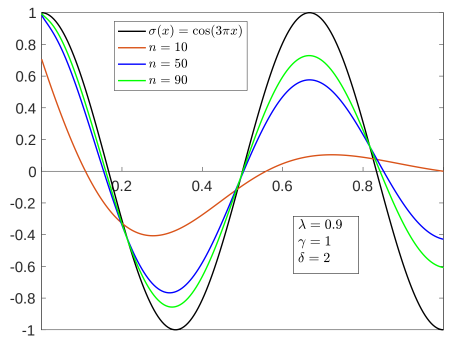

Figure 2. The convergence of $S^{\gamma,\delta}_{n,\lambda}(\sigma, x)$ to $\sigma(x) = cos(3\pi x)$ for $\lambda = 0.5, \gamma = 1, \delta = 2$.

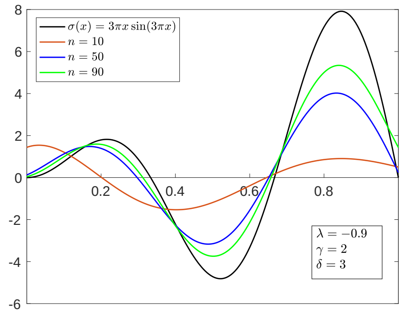

Figure 3. The convergence of $S_{n,\lambda}^{\gamma,\delta}(\sigma, x)$ to $\sigma(x) = 3\pi x \cdot sin(3\pi x)$ for $\lambda = -0.9, \gamma = 2, \delta = 3$.

5. Conclusions

In this paper, we introduce a class of Stancu type Szász Mirakjan operators and discuss their approximation properties using tools such as modulus of continuity, modulus of smoothness and K-functional. In addition, the estimation of Lipschitz function classes by the operators is also studied. Later, the Voronvskaya type asymptotic expansion of the operators is established. Finally, we give the comparison of the convergence of operators (1.3) to certain functions with some graphics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

References

1. O. Szász, Generalization of the Bernstein polynomials to the infinite interval, *J. Res. Nat. Bur. Stand.*, **45** (1950), 239–245.
2. F. Cheng, On the rate of convergence of the Szász-Mirakyan operator for functions of bounded variation, *J. Approx. Theory*, **40** (1984), 226–241. https://doi.org/10.1016/0021-9045(84)90064-9

3. D. X. Zhou, Weighted approximation by Szász-Mirakjan operators, *J. Approx. Theory*, **76** (1994), 393–402. https://doi.org/10.1006/jath.1994.1025

4. V. Gupta, The bézier variant of Kantorovitch operators, *Comput. Math. Appl.*, **47** (2004), 227–232. https://doi.org/10.1016/S0898-1221(04)90019-3

5. V. Gupta, D. Soybaş, Approximation by complex genuine hybrid operators, *Appl. Math. Comput.*, **244** (2014), 526–532. https://doi.org/10.1016/j.amc.2014.07.025

6. M. Heilmann, I. Rasa, A nice representation for a link between Baskakov- and Szász-Mirakjan-Durrmeyer operators and their Kantorovich variants, *Results Math.*, **74** (2019). https://doi.org/10.48550/arXiv.1809.05661

7. A. M. Acu, G. Tachev, Yet another new variant of Szász-Mirakyan operator, *Symmetry*, **13** (2021), 2018. https://doi.org/10.3390/sym13112018

8. L. S. Xie, S. L. Wang, Strong converse inequality for linear combinations of Szász-Mirakjan operators, *J. Approx. Theory*, **273** (2022). https://doi.org/10.1016/j.jat2021.105651

9. D. D. Stancu, Asupra unei generalizari a polinoamelor lui Bernstein, *Studia Univ. Babes-Bolyai*, **14** (1969), 31–45.

10. O. Doğru, C. Muraru, Statistical approximation by Stancu type bivariate generalization of Meyer-König and Zeller type operators, *Math. Comput. Model.*, **48** (2008), 961–968. https://doi.org/10.1016/j.mcm.2007.12.005

11. A. Aral, V. Gupta, On the q analogue of Stancu-Beta operators, *Appl. Math. Lett.*, **25** (2012), 67–71. https://doi.org/10.1016/j.aml.2011.07.009

12. A. Kumar, A new kind of variant of the Kantorovich type modification operators introduced by D. D. Stancu, *Results Appl. Math.*, **11** (2021). https://doi.org/10.1016/j.rinam.2021.100158

13. Z. Ye, X. Long, X. M. Zeng, *Adjustment algorithms for Bézier curve and surface*, In: International Conference on Computer Science and Education, 2010.

14. Q. B. Cai, B. Y. Lian, G. Zhou, Approximation properties of λ-Bernstein operators, *J. Inequal. Appl.*, **61** (2018). https://doi.org/10.1186/s13660-018-1653-7

15. Q. B. Cai, W. T. Cheng, Convergence of λ-Bernstein operators based on (p, q)-integers, *J. Inequal. Appl.*, **35** (2020). https://doi.org/10.1186/s13660-020-2309-y

16. Q. B. Cai, R. Aslan, On a new construction of generalized q-Bernstein polynomials based on shape parameter λ, *Symmetry*, **13** (2021), 1919. https://doi.org/10.3390/sym13101919

17. P. N. Agrawal, B. Baxhaku, R. Shukla, A new kind of Bi-variate λ-Bernstein-Kantorovich type operator with shifted knots and its associated GBS form, *Math. Fdn. Comput.*, **5** (2022), 157–172. https://doi.org/10.3934/mfc.2021025

18. P. N. Agrawal, V. A. Radu, J. K. Singh, Better numerical approximation by λ-Durrmeyer-Bernstein type operators, *Filomat*, **35** (2021), 1405–1419. https://doi.org/10.2298/FIL2104405R

19. M. Mursaleen, A. A. H. A. Abied, M. A. Salman, Chlodowsky type λ, q-Bernstein-Stancu operators, *Azerbaijan J. Math.*, **10** (2020), 97–110. https://doi.org/75–101. 10.1515/jaa-2020-2009

20. R. Aslan, M. Mursaleen, Some approximation results on a class of new type λ-Bernstein polynomials, *J. Math. Inequal.*, **16** (2022), 445–462. https://doi.org/10.7153/jmi-2022-16-32

21. Q. Qi, D. Guo, G. Yang, Approximation properties of λ-Szász-Mirakian operators, *Int. J. Eng. Res.*, **12** (2019), 662–669.
22. R. Aslan, Some approximation results on λ-Szász-Mirakjan-Kantorivich operators, *Fund. J. Math. Appl.*, **4** (2021), 150–158. https://doi.org/10.33401/fujma.903140
23. R. Aslan, Approximation by Szász-Mirakjan-Durrmeyer operators based on shape parameter λ, *Commun. Fac. Sci. Univ.*, **71** (2022), 407–421. https://doi.org/10.31801/cfsuasmas.941919
24. P. P. Korovkin, *Linear operators and approximation theory*, Delhi: Hindustan Pub Corp, 1960.
25. R. A. Devore, G. G. Lorentz, *Construtive approximation*, Berlin: Springer-Verlag, 1993.
26. Z. Ditzian, V. Totik, *Moduli of smoothness*, New York: Springer, 1987.