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Research article

Normalized solutions for Kirchhoff-Carrier type equation

Jie Yang[1] **and Haibo Chen**[2,*]

[1] School of Mathematics and Computational Science, Huaihua University, Huaihua, Hunan 418008, China

[2] School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, Hunan 410083, China

* **Correspondence:** Email: math_chb@163.com.

Abstract: In this paper, we study the following Kirchhoff-Carrier type equation

$$-\left(a+bM\left(|\nabla u|_2,|u|_\tau\right)\right)\Delta u-\lambda u=|u|^{p-2}u,\qquad \text{in } \mathbb{R}^3,$$

where $a,\ b>0$ are constants, $\lambda\in\mathbb{R},\ p\in(2,6)$. By using a minimax procedure, we obtain infinitely solutions (v_n^b,λ_n) with v_n^b having a prescribed L^2-norm. Moreover, we give a convergence property of v_n^b as $b\to 0^+$.

Keywords: Kirchhoff-Carrier type equation; normalized solution; variational methods
Mathematics Subject Classification: 49J35

1. Introduction and statement of results

In this paper, we study the following Kirchhoff-Carrier type equation

$$-\left(a+bM(|\nabla u|_2,|u|_\tau)\right)\Delta u-\lambda u=|u|^{p-2}u,\qquad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $a,\ b>0$ are constants, $\lambda\in\mathbb{R},\ 2<p<\tau<6$. M satisfies the following condition
(H) $M:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous and

$$0\le M(\xi_1,\xi_2)\le C_0(\xi_1^{d_1}+\xi_2^{d_2}+1), \tag{1.2}$$

for some $C_0,\ d_1,\ d_2>0$. Moreover, for each $\sigma\in[0,S_\tau],\ t\ge 0$,

$$\int_0^t sM(s,\sigma s)ds\ge\frac{2}{3(p-2)}t^2M(t,\sigma t), \tag{1.3}$$

where S_τ is the best constant of $H^1(\mathbb{R}^3)$ embedding into $L^\tau(\mathbb{R}^3)$.

It is well known that problem (1.1) comes from two classes of typical nonlocal problem which are Kirchhoff type and Carrier type problems. In 1883, Kirchhoff [10] firstly proposed the Kirchhoff type problem

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0,$$

which extends the claasical D'Alembert wave equation by considering the effects of changes in the length of the strings during the vibrations. In [12], Lions proposed a functional analysis method to solve the following Kirchhoff problem

$$\begin{cases}-\left(a+b\int_\Omega|\nabla u(x)|^2dx\right)\Delta u=f(x,u), & x\in\Omega,\\ u(x)=0, & x\in\partial\Omega.\end{cases}\tag{1.4}$$

Since then, problem (1.4) has attracted the attention of researchers, see [4, 5, 14, 15, 19, 20] and the references therein.

Carrier in [7] proposed the following problem

$$\frac{\partial^2 u}{\partial t^2}-\left(1+\frac{\alpha^{-2}}{2\pi}\int_0^\pi|u(x,t)|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0\quad in\ (0,\pi)\times(0,+\infty).\tag{1.5}$$

The vibration of elastic string is described by (1.5) when the tension change is not very small. We note that problem (1.5) is similar to the Kirchhoff equation which has many research results. However, there is very little work on the problem (1.5) or its generalization. The appearance of nonlocal terms leads to some difficulties. It lacks a variational structure so we cannot study the problem (1.5) with variational method. Some authors are concerned about the existence of positive solutions or some generalized cases of problem (1.5) by the topological theory and pseudo montone operator theory, see [1, 6]. It is worth mentioning that in [9], Jin and Yan studied the following Carrier type problem

$$\begin{cases}-\left(a+b\int_\Omega|u(x)|^\gamma dx\right)\Delta u=f(u), & x\in\Omega,\\ u(x)=0, & x\in\partial\Omega,\end{cases}\tag{1.6}$$

where $a, b > 0$ and $\Omega\subset\mathbb{R}^N$ is a bounded open set with smooth boundary and $\gamma\geq 1$. They obtained the existence of sign-changing solutions of problem (1.6) by using the fixed point index method. Xu and Qin [21] considered the Kirchhoff-Carrier type equation

$$\begin{cases}-\left(a+b\alpha(\|u\|,|u|_\gamma)\right)\Delta u=f(u), & x\in\Omega,\\ u(x)=0, & x\in\partial\Omega,\end{cases}\tag{1.7}$$

where $a, b > 0$ and $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary and $1<\gamma<2^*$, $\alpha:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$. By applying the mountain pass theorem and the Ekeland theorem, they obtained some existence results for problem (1.7). Let us emphasize that all the previous existence results for Kirchhoff-Carrier type equation are obtained in a bounded domain. Obviously, the approaches adopted in [9, 21] do not work when the Kirchhoff-Carrier type problem defined on the whole space $\mathbb{R}^3$.

When $b = 0$, adding a repulsive nonlocal Coulombic potential, problem (1.1) reduces to the Schrödinger-Poisson-Slater equation

$$\Delta u-\lambda u+(|x|^{-1}*|u|^2)u=|u|^{p-2}u,\quad \text{in }\mathbb{R}^3.\tag{1.8}$$

Luo [13] proved a multiplicity result of solutions for problem (1.8) if $p \in (\frac{10}{3}, 6)$. They were interested in normalized solutions, that is, solutions to (1.8) satisfying

$$\int_{\mathbb{R}^3} |u|^2 dx = c.$$

The normalized solutions associated to Schrödinger or Kirchhoff problems have been extensively studied in recent years, see [11, 17, 22]. Because people are particularly interested in normalized solutions, we can search for normalized solutions of (1.1). Precisely, for given $c > 0$ we look for

$$(u, \lambda) \in H_r^1(\mathbb{R}^3) \times \mathbb{R} \quad \text{with} \quad |u|_2^2 = c.$$

For this reason, we construct the variational structure of problem (1.1). Define $J_b : H_r^1(\mathbb{R}^3) \to \mathbb{R}$ is a functional with its Fréchet derivative operator given by

$$J_b'(u)h = (a + bM(|\nabla u|_2, |u|_\tau)) \int_{\mathbb{R}^3} \nabla u \cdot \nabla h dx - \int_{\mathbb{R}^3} |u|^{p-2} u h dx, \quad \forall u,\ h \in H_r^1(\mathbb{R}^3). \tag{1.9}$$

For any $u \in H_r^1(\mathbb{R}^3) \setminus \{0\}$ set $\beta_u = \frac{u}{|\nabla u|_2}$ and $v(t) = t\beta_u$ for all $t \geq 0$. It follows from (1.9) that

$$\frac{d(J_b(v(s)))}{ds} = J_b'(v(s))v'(s) = s\,(a + bM(s, s|\beta_u|_\tau)) - s^{p-1} \int_{\mathbb{R}^3} |\beta_u|^p dx.$$

Integrating over $[0, |\nabla u|_2]$, using Fubini's theorem and $J_b(0) = 0$ we obtain

$$J_b(u) = \frac{a}{2}|\nabla u|_2^2 + b \int_0^{|\nabla u|_2} sM(s, s|\beta_u|_\tau) ds - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx \tag{1.10}$$

for $u \in H_r^1(\mathbb{R}^3)$. Since M is continuous, $J_b \in C^1(H_r^1(\mathbb{R}^3), \mathbb{R})$ and the couple of weak solution as above can be viewed as a critical point of J_b restricted on the constraint

$$X_r(c) = \left\{ u \in H_r^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} |u_c|^2 dx = c \right\}, \quad c > 0.$$

As far as we know, there is no result on the existence and asymptotic behaviour of normalized solutions of problem (1.1). In this paper, we focus on the existence of high energy normalized solutions to (1.1). Let us state our main result.

Theorem 1.1. *Let* $\max\left\{\frac{2}{3}(d_1 + 2) + 2,\ \frac{2}{3}(d_2 + 2) + 2\right\} < p < 6$. *Then, for any fixed* $c > 0$ *(1.1) has a sequence of couples of weak solutions* $\{(v_n, \lambda_n)\} \subset H_r^1(\mathbb{R}^3) \times \mathbb{R}^-$ *with* $|v_n|_2^2 = c$ *for each* $n \in \mathbb{N}^+$,

$$\|v_n\|^2 \to +\infty, \quad \text{and} \quad J_b(v_n) \to +\infty, \quad \text{as } n \to +\infty.$$

Theorem 1.1 shows (v_n, λ_n) depends on the parameter b. Therefore, we next focus on whether some converence phenomena appear when $b \to 0$.

Theorem 1.2. *Let* $\max\left\{\frac{2}{3}(d_1 + 2) + 2,\ \frac{2}{3}(d_2 + 2) + 2\right\} < p < 6$. $\{(v_n^b, \lambda_n^b)\} \subset X_r(c) \times \mathbb{R}^-$ *are obtained in Theorem 1.1. Then, for any sequence* $\{b_m\} \to 0^+ (m \to +\infty)$ *there exists a subsequence still denoted by* $\{b_m\}$ *such that for any* $n \in \mathbb{N}^+$ $v_n^{b_m} \to v_n^0$ *in* $H_r^1(\mathbb{R}^3)$ *and* $\lambda_n^{b_m} \to \lambda_n^0$ *in* $\mathbb{R}$ *as* $m \to +\infty$ *where* $\{(v_n^0, \lambda_n^0)\} \subset X_r(c) \times \mathbb{R}^-$ *is a sequence of couples of weak solutions to the following equation*

$$-a\Delta u - \lambda u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^3. \tag{1.11}$$

Remark 1.1. *A typical example of M satisfying condition (H) is* $M(s,t) = a + 2bt^2$ *with* $a, b > 0$.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries results. In Section 3, we prove Theorems 1.1 and 1.2.

2. Preliminaries

Throughout the paper, let $\|u\| = \left(\int_{\mathbb{R}^3}(|u|^2 + |\nabla u|^2)dx\right)^{\frac{1}{2}}$ denote the usual norm of $H^1(\mathbb{R}^3)$, $|\cdot|_\tau$ denote the norm of $L^\tau(\mathbb{R}^3)$ for $2 < \tau < 6$. $H^{-1}(\mathbb{R}^3)$ is the dual space of $H^1(\mathbb{R}^3)$ and $H_r^1(\mathbb{R}^3)$ is the subspace of radially symmetric functions in $H^1(\mathbb{R}^3)$.

Proposition 2.1. *[3, Lemma 2.1] Assume that $p \in (2, 6)$. Then, there holds*

$$\mu_n := \inf_{u\in V_{n-1}^{\perp}} \frac{\int_{\mathbb{R}^3}(|\nabla u|^2 + |u|^2)dx}{(\int_{\mathbb{R}^3} |u|^p dx)^{\frac{2}{p}}} \to +\infty, \quad \text{as } n \to +\infty$$

where $\{V_n\} \subset H_r^1(\mathbb{R}^3)$ be a strictly increasing sequence of finite-dimensional linear subspace in $H_r^1(\mathbb{R}^3)$ such that $\cup_n V_n$ is dense in $H_r^1(\mathbb{R}^3)$ and $V_n^\perp$ denotes the orthogonal complementary space of V_n in $H_r^1(\mathbb{R}^3)$.

This lemma is useful to show the compactness of the Palais-Smale sequence.

Lemma 2.2. *[2, Lemma 3] Let F be a C^1 functional on $H^1(\mathbb{R}^3)$, if $\{x_k\} \subset X(c)$ is bounded in $H^1(\mathbb{R}^3)$. Then,*

$$F'|_{X(c)}(x_k) \to 0 \quad \text{in } H^1(\mathbb{R}^3) \Longleftrightarrow F'(x_k) - \langle F'(x_k), x_k\rangle\, x_k \to 0 \quad \text{in } H^1(\mathbb{R}^3) \quad \text{as } k \to \infty,$$

where $X(c) = \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} |u_c|^2 dx = c\right\}, \quad c > 0.$

Set

$$A(u) = \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad B(u) = \int_0^{|\nabla u|_2} sM(s, s|\beta_u|_\tau)ds, \quad C(u) = \int_{\mathbb{R}^3} |u|^p dx, \quad D(u) = \int_{\mathbb{R}^3} |u|^2 dx.$$

Now, we introduce a scaling. Define $u_t(x) = t^{\frac{3}{2}} u(tx)$, for $t > 0,\ u \in X_r(c)$. Then,

$$A(u_t) = t^2 A(u), \quad B(u_t) = \int_0^{t|\nabla u|_2} sM(s, s^{\frac{3}{2}-\frac{3}{\tau}}|\beta_u|_\tau)ds, \quad C(u_t) = t^{\frac{3}{2}(p-2)}C(u) \quad \text{and} \quad D(u_t) = D(u).$$

Lemma 2.3. *Let* $p > \max\left\{\frac{2}{3}(d_1 + 2) + 2, \frac{2}{3}(d_2 + 2) + 2\right\}, c > 0$ *and* $u \in X_r(c)$. *Then*

$$u_t \in X_r(c),\ A(u_t) \to +\infty, \quad \text{and} \quad J_b(u_t) \to -\infty, \quad \text{as} \quad t \to +\infty.$$

Proof. It follows from $D(u_t) = D(u)$ that $u_t \in X_r(c)$ for any $u \in X_r(c)$. By (1.2) and (1.10), we obtain

$$\begin{aligned}
J_b(u_t) &= \frac{at^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + b\int_0^{t|\nabla u|_2} sM(s, s^{\frac{3}{2}-\frac{3}{\tau}}|\beta_u|_\tau)ds - \frac{t^{\frac{3}{2}(p-2)}}{p}\int_{\mathbb{R}^3} |u|^p dx \\
&\le \frac{at^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + b\int_0^{t|\nabla u|_2} C_0(s^{d_1+1} + s^{\left(\frac{3}{2}-\frac{3}{\tau}\right)d_2+1}|\beta_u|_\tau^{d_2+1} + s)ds - \frac{t^{\frac{3}{2}(p-2)}}{p}\int_{\mathbb{R}^3} |u|^p dx \\
&= \frac{at^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + C(t^{d_1+2}|\nabla u|_2^{d_1+2} + t^{\left(\frac{3}{2}-\frac{3}{\tau}\right)d_2+2}|\nabla u|_2^{d_2+2}|\beta_u|_\tau^{d_2+2} + t^2|\nabla u|_2^2) \\
&\quad - \frac{t^{\frac{3}{2}(p-2)}}{p}\int_{\mathbb{R}^3} |u|^p dx,
\end{aligned}$$

which implies that $J_b(u_t) \to -\infty$, as $t \to +\infty$ since $\tau \in (2, 6)$ and $p > \max\left\{\frac{2}{3}(d_1 + 2) + 2, \frac{2}{3}(d_2 + 2) + 2\right\}$. □

For $c > 0$ fixed, $n \in \mathbb{N}^+$ and $n \geq 2$ let

$$\theta_n := L^{-\frac{2}{p-2}} \cdot a^{\frac{2}{p-2}} \cdot \mu_n^{\frac{2}{p-2}}, \quad E_n := \left\{ u \in V_{n-1}^{\perp} \bigcap X_r(c) : |\nabla u|_2^2 = \theta_n \right\}$$

and

$$\sigma_n := \inf_{u \in E_n} J_b(u),$$

where $L = \max_{x>0} \frac{(x^2+c)^{p/2}}{x^p + c^{p/2}}$, μ_n is given in Proposition 2.1.

Lemma 2.4. *For any $p \in (2,6)$, then $\sigma_n \to +\infty$ as $n \to +\infty$. Particularly, we can suppose that $\sigma_n \geq 1$ for every $n \in \mathbb{N}^+$ without any restriction.*

Proof. For any $u \in E_n$, we deduce that

$$\begin{aligned} J_b(u) &= \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + b \int_0^{|\nabla u|_2} sM(s, s|\beta_u|_\tau) ds - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx \\ &\geq \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{p\mu_n} (|\nabla u|_2^2 + c)^{p/2} \\ &\geq \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{L}{p\mu_n} (|\nabla u|_2^p + c^{p/2}) \\ &= \left(\frac{1}{2} - \frac{1}{p} \right) a\theta_n - \frac{L}{p\mu_n} c^{p/2}. \end{aligned}$$

Combining with Proposition 2.1, we obtain that $\sigma_n \to +\infty$ as $n \to +\infty$ since $p > 2$. □

Define the continuous map

$$\eta : \mathbb{R} \times H_r^1(\mathbb{R}^3) \to H_r^1(\mathbb{R}^3), \quad \eta(t,v)(x) = e^{\frac{3}{2}t} v(e^t x), \quad \text{for } t \in \mathbb{R},\ v \in H_r^1(\mathbb{R}^3) \text{ and } x \in \mathbb{R}^3. \tag{2.1}$$

It follows from Lemma 2.3 that $\eta(t,v) \in X_r(c)$ for any $v \in X_r(c)$, $t \in \mathbb{R}$ and

$$\begin{cases} A(\eta(t,v)) \to 0, \quad J_b(\eta(t,v)) \to 0, \quad t \to -\infty, \\ A(\eta(t,v)) \to +\infty, \quad J_b(\eta(t,v)) \to -\infty, \quad t \to +\infty. \end{cases}$$

Recalling that V_n is finite dimensional, we obtain that for any $n \in \mathbb{N}^+$, there exists $t_n > 0$ such that

$$\bar{\gamma}_n : [0,1] \times (X_r(c) \cap V_n) \to X_r(c), \quad \bar{\gamma}_n(s,u) = \eta((2s-1)t_n, u)$$

satisfies

$$\begin{cases} A(\bar{\gamma}_n(0,u)) < \theta_n, \quad A(\bar{\gamma}_n(1,u)) > \theta_n, \\ J_b(\bar{\gamma}_n(0,u)) < \sigma_n, \quad J_b(\bar{\gamma}_n(1,u)) < \sigma_n. \end{cases}$$

Now, define

$$\begin{aligned} \Gamma_n := \{ \gamma : [0,1] \times (X_r(c) \cap V_n) \to X_r(c) | \gamma \text{ is continuous, odd in } u \\ \text{and such that for any } u : \gamma(0,u) = \bar{\gamma}_n(0,u), \gamma(1,u) = \bar{\gamma}_n(1,u) \}. \end{aligned}$$

Obviously, $\bar{\gamma}_n \in \Gamma_n$. According to [3, Lemma 2.3], we obtain the next key intersection result.

Lemma 2.5. *For each $n \in \mathbb{N}^+$,*

$$\kappa_n^b(c) := \inf_{\gamma\in\Gamma_n} \max_{0\le s\le 1, u\in X_r(c)\cap V_n} J_b(\gamma(s,u)) \ge \sigma_n.$$

Proof. The proof of the lemma can be easily obtained by [3, Lemma 2.3]. □

Next, we will show that $\{\kappa_n^b(c)\}$ is indeed a sequence of critical values of J_b restricted on $X_r(c)$. First, we shall prove that there exists a bounded (PS) sequence at each level $\kappa_n^b(c)$. Now, we fix an $n \in \mathbb{N}^+$. We define the following auxiliary functional:

$$\tilde{J}_b : \mathbb{R} \times X_r(c) \to \mathbb{R}, (t, u) \to J_b(\eta(t, u)),$$

where $\eta(t, u)$ is defined in (2.1). Set

$$\begin{aligned}\tilde{\Gamma}_n := \{&\tilde{\gamma} : [0, 1] \times (X_r(c) \cap V_n) \to X_r(c) \times \mathbb{R} | \tilde{\gamma} \text{ is continuous, odd in } u\\ &\text{and such that } \eta \circ \tilde{\gamma} \in \Gamma_n\}.\end{aligned}$$

Obviously, $\tilde{\gamma} := (0, \gamma) \in \tilde{\Gamma}_n$ for any $\gamma \in \Gamma_n$. Define

$$\tilde{\kappa}_n^b(c) := \inf_{\tilde{\gamma}\in\tilde{\Gamma}_n} \max_{0\le s\le 1, u\in X_r(c)\cap V_n} \tilde{J}_b(\tilde{\gamma}(s, u)).$$

Applying the fact that the maps

$$\phi : \Gamma_n \to \tilde{\Gamma}_n, \quad \gamma \to \phi(\gamma) := (0, \gamma)$$

and

$$\varphi : \tilde{\Gamma}_n \to \Gamma_n, \quad \tilde{\gamma} \to \varphi(\tilde{\gamma}) := \eta \circ \tilde{\gamma}$$

satisfy

$$\tilde{J}_b(\phi(\gamma)) = J_b(\gamma), \quad J_b(\varphi(\tilde{\gamma})) = \tilde{J}_b(\tilde{\gamma}),$$

we get $\tilde{\kappa}_n^b(c) = \kappa_n^b(c)$. Let us denote by E the space $\mathbb{R}\times H_r^1(\mathbb{R}^3)$ endowed with the norm $\|\cdot\|_E^2 = |\cdot|_{\mathbb{R}}^2 + \|\cdot\|^2$ and by E^* its dual space. Similar to [8, Lemma 2.3], we get the following result.

Lemma 2.6. *For any $\epsilon > 0$, suppose that $\tilde{\gamma}_0 \in \tilde{\Gamma}_n$ satisfies*

$$\max_{0\le s\le 1, u\in X_r(c)\cap V_n} \tilde{J}_b(\tilde{\gamma}_0(s, u)) \le \tilde{\kappa}_n^b(c) + \epsilon.$$

Then, there exists a pair of $(t_0, u_0) \in \mathbb{R} \times X_r(c)$ such that
(i) $\tilde{J}_b(t_0, u_0) \in [\tilde{\kappa}_n^b(c) - \epsilon, \tilde{\kappa}_n^b(c) + \epsilon]$;
(ii) $\min_{0\le s\le 1, u\in X_r(c)\cap V_n} \|(t_0, u_0) - \tilde{\gamma}_0(s, u)\|_E \le \sqrt{\epsilon}$;
(iii) $\left\|\left(\tilde{J}_b|_{X_r(c)\times\mathbb{R}}\right)' (t_0, u_0)\right\|_{E^*} \le 2\sqrt{\epsilon}$, i.e.,

$$\left|\left\langle \tilde{J}_b(t_0, u_0), z\right\rangle_{E^*\times E}\right| \le 2\sqrt{\epsilon}\|z\|_E$$

holds, for all $z \in \tilde{T}_{(t_0,u_0)} := \{(z_1, z_2) \in E, \langle u_0, z_2\rangle_{L^2} = 0\}$.

Lemma 2.7. *For any $c > 0$ fixed and $n \in \mathbb{N}^+$, there exists a sequence $\{v_k^n \subset X_r(c)\}$ satisfying*

$$J_b(v_k^n) \to \kappa_n^b(c), \quad J'_b|_{X_r(c)}(v_k^n) \to 0, \quad G_b(v_k^n) \to 0, \tag{2.2}$$

where

$$G_b(u) = a\int_{\mathbb{R}^3} |\nabla u|^2 dx + b\int_{\mathbb{R}^3} |\nabla u|^2 dx \cdot M(|\nabla u|_2, |\nabla u|_2|\beta_u|_\tau) - \frac{3(p-2)}{2p}\int_{\mathbb{R}^3} |u|^p dx.$$

Particularly, $\{v_k^n\}$ is bounded in $X_r(c)$.

Proof. By the definition of $\kappa_n^b(c)$ we have that for every $k \in \mathbb{N}^+$, there exists a $\gamma \in \Gamma_n$ such that

$$\max_{0\le s\le 1, u\in X_r(c)\cap V_n} J_b(\gamma_k(s,u)) \le \kappa_n^b(c) + \frac{1}{k}.$$

It follows from $\tilde{\kappa}_n^b(c) = \kappa_n^b(c)$ and $\tilde{\gamma}_k := (0, \gamma_k) \in \tilde{\Gamma}_n$ that

$$\max_{0\le s\le 1, u\in X_r(c)\cap V_n} \tilde{J}_b(\tilde{\gamma}_k(s,u)) \le \tilde{\kappa}_n^b(c) + \frac{1}{k}.$$

Using Lemma 2.6, we infer that there exists a sequence $\{(t_k^n, u_k^n)\} \subset \mathbb{R} \times X_r(c)$ such that
(i) $\tilde{J}_b(t_k^n, u_k^n) \in [\kappa_n^b(c) - \frac{1}{k}, \kappa_n^b(c) + \frac{1}{k}]$;
(ii) $\min_{0\le s\le 1, u\in X_r(c)\cap V_n} \|(t_k^n, u_k^n) - (0, \gamma_k)\|_E \le \sqrt{\frac{1}{k}}$;
(iii) $\left\|\left(\tilde{J}_b|_{X_r(c)\times\mathbb{R}}\right)'(t_k^n, u_k^n)\right\|_{E^*} \le \frac{4}{\sqrt{k}}$, i.e.,

$$\left|\left\langle \tilde{J}_b(t_k^n, u_k^n), z\right\rangle_{E^*\times E}\right| \le \frac{4}{\sqrt{k}}\|z\|_E$$

holds, for all $z \in \tilde{T}_{(t_k^n, u_k^n)} := \{(z_1, z_2) \in E, \langle u_k^n, z_2\rangle_{L^2} = 0\}$.
For every $k \in \mathbb{N}^+$, set $v_k^n = \eta(t_k^n, u_k^n)$. We shall show that $\{v_k^n\} \subset X_r(c)$ satisfies (2.2). It follows from (i) that $J_b(v_k^n) \to \kappa_n^b(c)$ as $k \to \infty$, since $J_b(v_k^n) = J_b(\eta(t_k^n, u_k^n)) = \tilde{J}_b(t_k^n, u_k^n)$. Noting that

$$\begin{aligned}
&\left\langle \tilde{J}'_b(t,u), (r,v)\right\rangle \\
&= are^{2t}\int_{\mathbb{R}^3} |\nabla u|^2 dx + ae^{2t}\int_{\mathbb{R}^3} \nabla u \nabla v dx + bre^{2t}|\nabla u|_2^2 \cdot M\left(e^t|\nabla u|_2, e^{(\frac{3}{2}-\frac{3}{\tau})t}|\nabla u|_2^{\frac{3}{2}-\frac{3}{\tau}}|\beta_u|_\tau\right) \\
&\quad + be^{2t}M\left(e^t|\nabla u|_2, e^{(\frac{3}{2}-\frac{3}{\tau})t}|\nabla u|_2^{\frac{3}{2}-\frac{3}{\tau}}|\beta_u|_\tau\right)\int_{\mathbb{R}^3} \nabla u \nabla v dx \\
&\quad - \frac{3(p-2)}{2p} re^{\frac{3(p-2)t}{2}}\int_{\mathbb{R}^3} |u|^p dx - e^{\frac{3(p-2)t}{2}}\int_{\mathbb{R}^3} |u|^{p-2} uv dx,
\end{aligned}$$

we obtain

$$\begin{aligned}
G_b(v_k^n) &= aA(v_k^n) + b|\nabla v_k^n|_2^2 M(|\nabla v_k^n|_2, |\nabla v_k^n|_2|\beta_{v_k^n}|_\tau) - \frac{3(p-2)}{2p}C(v_k^n) \\
&= ae^{2t_k^n}\int_{\mathbb{R}^3} |\nabla u_k^n|^2 dx + be^{2t_k^n}\int_{\mathbb{R}^3} |\nabla u_k^n|^2 dx \cdot M(e^{t_k^n}|\nabla u_k^n|, e^{(\frac{3}{2}-\frac{3}{\tau})t_k^n}|\nabla u_k^n|_2|\beta_{u_k^n}|_\tau) \\
&\quad - \frac{3(p-2)}{2p} e^{\frac{3(p-2)t_k^n}{2}}\int_{\mathbb{R}^3} |u_k^n|^p dx \\
&= \left\langle \tilde{J}'_b(t_k^n, u_k^n), (1,0)\right\rangle.
\end{aligned}$$

Hence, by (iii), we see that $G_b(v_k^n) \to 0$ as $k \to \infty$ for $(1,0) \in \tilde{T}_{(t_k^n, u_k^n)}$.

Finally, we shall show that

$$J_b'|_{X_r(c)}(v_k^n) \to 0, \quad \text{as } k \to \infty.$$

We claim that for $k \in \mathbb{N}$ large enough

$$|\langle J_b'(v_k^n), w\rangle| \le \frac{4}{\sqrt{k}}\|w\|^2$$

holds for all $w \in T_{v_k^n}$, where $T_{v_k^n} := \{w \in H_r^1(\mathbb{R}), \langle v_k^n, w\rangle_{L^2} = 0\}$. Indeed, for $w \in T_{v_k^n}$, taking $\tilde{w} = \eta(-t_k^n, w)$ we have

$$\begin{aligned}
\langle J_b'(v_k^n), w\rangle &= a\int_{\mathbb{R}^3} \nabla v_k^n \nabla w + bM(|\nabla v_k^n|_2, |\nabla v_k^n|_2 |e_{v_k^n}|_\tau)\int_{\mathbb{R}^3} \nabla v_k^n \nabla w dx - \int_{\mathbb{R}^3} |v_k^n|^{p-2} v_k^n w dx \\
&= ae^{2t_k^n}\int_{\mathbb{R}^3} \nabla u_k^n \nabla \tilde{w} dx + be^{2t_k^n}\int_{\mathbb{R}^3} \nabla u_k^n \nabla \tilde{w} dx \cdot M(e^{t_k^n}|\nabla u_k^n|_2, e^{(\frac{3}{2}-\frac{3}{\tau})t_k^n}|\nabla u_k^n|_2 |\beta_{u_k^n}|_\tau) \\
&\quad - e^{\frac{3(p-2)t_k^n}{2}}\int_{\mathbb{R}^3} |u_k^n|^{p-2} u_k^n \tilde{w} dx \\
&= \langle \tilde{J}_b'(t_k^n, v_k^n), (0, \tilde{w})\rangle.
\end{aligned}$$

From $\int_{\mathbb{R}^3} u_k^n \tilde{w} dx = \int_{\mathbb{R}^3} v_k^n w dx$, we see that $(0, \tilde{w}) \in \tilde{T}_{(t_k^n, u_k^n)}$ is equivalent to show $w \in T_{v_k^n}$. It follows from (ii) that

$$|t_k^n| = |t_k^n - 0| \le \min_{0\le s\le 1, u\in X_r(c)\cap V_n} \|(t_k^n, u_k^n) - (0, \gamma_k(t,u))\|_E \le \frac{1}{\sqrt{k}}.$$

Hence, we conclude that

$$\|(0,\tilde{w})\|_E^2 = \|\tilde{w}\|^2 = \int_{\mathbb{R}^3} |w|^2 dx + e^{-2t_k^n}\int_{\mathbb{R}^3} |\nabla w|^2 dx \le 4\|w\|^2.$$

Therefore,

$$|\langle J_b'(v_k^n), w\rangle| = \left\langle \tilde{J}_b'(t_k^n, v_k^n), (0,\tilde{w})\right\rangle \le \frac{4}{\sqrt{k}}\|(0,\tilde{w})\|_E^2 \le \frac{4}{\sqrt{k}}\|w\|^2,$$

which yields

$$\|J_b'|_{X_r(c)}(v_k^n)\| = \sup_{w\in T_{v_k^n}, \|w\|\le 1} |\langle J_b'(v_k^n), w\rangle| \le \frac{4}{\sqrt{k}} \to 0,$$

as $k \to \infty$. Since (1.3) and $p \in (\frac{10}{3}, 6)$, we obtain

$$\begin{aligned}
&J_b(u) - \frac{2}{3(p-2)}G_b(u) \\
&= \frac{(3p-10)a}{6(p-2)}|\nabla u|_2^2 + b\left[\int_0^{|\nabla u|_2} sM(s, s|\beta_u|_\tau)ds - \frac{2|\nabla u|_2^2}{3(p-2)}M(|\nabla u|_2, |\nabla u|_2|\beta_u|_\tau)\right] \\
&\ge \frac{(3p-10)a}{6(p-2)}|\nabla u|_2^2,
\end{aligned} \tag{2.3}$$

which completes the proof. □

Lemma 2.8. *Let $p \in (2,6)$, $\lambda \in \mathbb{R}$. If $u \in H^1(\mathbb{R}^3)$ is a weak solution of*

$$-(a + bM(|\nabla u|_2, |\nabla u|_2|\beta_u|_\tau))\,\Delta u - |u|^{p-2}u = \lambda u,$$

then $G_b(u) = 0$. Moreover if $\lambda \geq 0$, we obtain $u = 0$.

Proof. If $u \in H^1(\mathbb{R}^3)$ is a weak solution of (1.1) then it satisfies the Pohožaev identity

$$\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{b}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx\cdot M(|\nabla u|_2, |\nabla u|_2|\beta_u|_\tau) - \frac{3}{p}\int_{\mathbb{R}^3}|u|^pdx = \frac{3}{2}\lambda\int_{\mathbb{R}^3}|u|^2dx$$

and

$$a\int_{\mathbb{R}^3}|\nabla u|^2dx + b\int_{\mathbb{R}^3}|\nabla u|^2dx\cdot M(|\nabla u|_2, |\nabla u|_2|\beta_u|_\tau) - \int_{\mathbb{R}^3}|u|^pdx = \lambda\int_{\mathbb{R}^3}|u|^2dx. \tag{2.4}$$

Thus, we obtain

$$G_b(u) = a\int_{\mathbb{R}^3}|\nabla u|^2dx + b\int_{\mathbb{R}^3}|\nabla u|^2dx\cdot M(|\nabla u|_2, |\nabla u|_2|\beta_u|_\tau) - \frac{3(p-2)}{2p}\int_{\mathbb{R}^3}|u|^pdx = 0. \tag{2.5}$$

Combining (2.4) and (2.5), we can see that

$$0 \leq a|\nabla u|_2^2 + b|\nabla u|_2^2M(|\nabla u|_2, |\nabla u|_2|\beta_u|_\tau) = \frac{3(p-2)}{p-6}\lambda\int_{\mathbb{R}^3}|u|^2dx.$$

Since $p \in (2,6)$, if $\lambda > 0$, we obtain $u \equiv 0$ immediately. If $\lambda = 0$, we get $A(u) = 0$. Together with $G_b(u) = 0$ then $u \equiv 0$. □

Lemma 2.9. *Let $v_k^n \subset X_r(c)$ be the Palais-Smale sequence obtained in Lemma 2.7. Then, there exist $v_n \in H_r^1(\mathbb{R}^3)$ and $\{\lambda_k^n\} \subset \mathbb{R}$ such that up to a subsequence as $k \to +\infty$*
(i) $v_k^n \rightharpoonup v_n \neq 0$ in $H_r^1(\mathbb{R}^3)$;
(ii) $\lambda_k^n \to \lambda_n < 0$ in $\mathbb{R}$;
(iii) $-\left(a + bM(|\nabla v_k^n|_2, |\nabla v_k^n|_2|\beta_{v_k^n}|_\tau)\right)\Delta v_k^n - \lambda_k^n v_k^n - |v_k^n|^{p-2}v_k^n \to 0$ in $H_r^1(\mathbb{R}^3)$;
(iv) $-(a + bM(|\nabla v_n|_2, |\nabla v_n|_2|\beta_{v_n}|_\tau))\,\Delta v_n - \lambda_n v_n - |v_n|^{p-2}v_n = 0$ in $H_r^{-1}(\mathbb{R}^3)$.
Moreover, if $\lambda_n < 0$, then we obtain

$$v_k^n \to v_n \quad \text{in } H_r^1(\mathbb{R}^3) \text{ as } k \to \infty.$$

Proof. Since $\{v_k^n\} \subset X_r(c)$ is bounded, we may assume that there exists $v_n \in H_r^1(\mathbb{R}^3)$ such that

$$\begin{cases} v_k^n \rightharpoonup v_n, & \text{in } H_r^1(\mathbb{R}^3),\\ v_k^n \to v_n, & \text{in } L^p(\mathbb{R}^3),\\ v_k^n \to v_n, & \text{a.e. in } \mathbb{R}^3.\end{cases}$$

We claim that $v_n \neq 0$. In fact, we assume by contradiction that $v_n = 0$, then $C(v_n^k) = o(1)$. By $G_b(v_k^n) = o(1)$, we obtain $A(v_k^n) = o(1)$. Consequently, $J_b(v_k^n) = o(1)$ which contradicts Lemma 2.4. Hence, (i) is proved. It follows from Lemma 2.2 that

$$J_b'|_{X_r(c)}(v_k^n) \to 0 \text{ in } H_r^{-1}(\mathbb{R}^3) \Longleftrightarrow J_b'(v_k^n) - \langle J_b'(v_k^n), v_k^n\rangle v_k^n \to 0 \text{ in } H_r^{-1}(\mathbb{R}^3), \text{as } k \to \infty.$$

Note that for any $w \in H_r^1(\mathbb{R}^3)$,

$$\begin{aligned}&\langle J_b'(v_k^n) - \langle J_b'(v_k^n), v_k^n\rangle v_k^n, w\rangle \\ &= \Big(a + bM(|\nabla v_k^n|_2, |\nabla v_k^n|_2|\beta_{v_k^n}|_\tau)\Big)\int_{\mathbb{R}^3} \nabla v_k^n \nabla w dx - \int_{\mathbb{R}^3} |v_k^n|^{p-2} v_k^n w dx \\ &- \lambda_k^n \int_{\mathbb{R}^3} v_k^n w dx,\end{aligned}$$

where

$$\lambda_k^n = \frac{1}{|v_k^n|_2^2}\left[\Big(a + bM(|\nabla v_k^n|_2, |\nabla v_k^n|_2|\beta_{v_k^n}|_\tau)\Big)\int_{\mathbb{R}^3} |\nabla v_k^n|^2 dx - \int_{\mathbb{R}^3} |v_k^n|^p dx\right] + o(1). \tag{2.6}$$

Therefore, (iii) is proved. By (2.6) and the fact $\{v_k^n\} \subset X_r(c)$ is bounded up to a subsequence, there exists $\lambda_n \in \mathbb{R}$ such that $\lambda_k^n \to \lambda_n$ as $k \to +\infty$. Moreover, from $2 < p < 6$ and (2.5), we deduce that

$$\begin{aligned}\lambda_n &= \lim_{k\to\infty} \frac{1}{c}\left[\Big(a + bM(|\nabla v_k^n|_2, |\nabla v_k^n|_2|\beta_{v_k^n}|_\tau)\Big)|\nabla v_k^n|_2^2 - \int_{\mathbb{R}^3} |v_k^n|^p dx\right] \\ &\le \lim_{k\to\infty} \frac{1}{c}\left[\Big(a + bM(|\nabla v_k^n|_2, |\nabla v_k^n|_2|\beta_{v_k^n}|_\tau)\Big)|\nabla v_k^n|_2^2 - \frac{3(p-2)}{2p}\int_{\mathbb{R}^3} |v_k^n|^p dx\right] \\ &= 0,\end{aligned} \tag{2.7}$$

which yields $\lambda_n \le 0$. Consequently, it follows from Lemma 2.8 that $\lambda_n < 0$, that is, (ii) is true.

Assume that $\lim_{k\to\infty}\int_{\mathbb{R}^3} |\nabla v_k^n|^2 dx = l^2 \ge 0$. By (ii) and (iii), we obtain

$$-(a + bM(l, |v_n|_\tau))\,\Delta v_n - \lambda_n v_n - |v_n|^{p-2} v_n = 0 \quad in\ H_r^{-1}(\mathbb{R}^3). \tag{2.8}$$

From (2.8), we get

$$(a + bM(l, |v_n|_\tau))\int_{\mathbb{R}^3} \nabla v_n \nabla(v_k^n - v_n) dx - \lambda_n \int_{\mathbb{R}^3} v_n(v_k^n - v_n) dx = \int_{\mathbb{R}^3} |v_n|^{p-2} v_n (v_k^n - v_n) dx + o(1). \tag{2.9}$$

From (ii) and (iii), we obtain

$$(a + bM(l, |v_n|_\tau))\int_{\mathbb{R}^3} \nabla v_k^n \nabla(v_k^n - v_n) dx - \lambda_n \int_{\mathbb{R}^3} v_k^n(v_k^n - v_n) dx = \int_{\mathbb{R}^3} |v_k^n|^{p-2} v_k^n (v_k^n - v_n) dx + o(1). \tag{2.10}$$

By (2.9) and (2.10), we see that

$$\begin{aligned}&(a + bM(l, |v_n|_\tau))\int_{\mathbb{R}^3} |\nabla v_k^n - \nabla v_n|^2 dx - \lambda_n \int_{\mathbb{R}^3} |v_k^n - v_n|^2 dx + o(1) \\ &= \int_{\mathbb{R}^3} (|v_k^n|^{p-2} v_k^n - |v_n|^{p-2} v_n)(v_k^n - v_n) dx.\end{aligned}$$

Recalling that $v_k^n \to v_n$ in $L^p(\mathbb{R}^3)$, we deduce that

$$(a + bM(l, |v_n|_\tau))\int_{\mathbb{R}^3} |\nabla v_k^n - \nabla v_n|^2 dx - \lambda_n \int_{\mathbb{R}^3} |v_k^n - v_n|^2 dx = o(1),$$

which yields that $(a+bM(l,|v_n|_\tau))\int_{\mathbb{R}^3}|\nabla v_k^n-\nabla v_n|^2dx=o(1)$ for $a,\ b>0, l\geq 0, \lambda_n\leq 0$. Hence,

$$\int_{\mathbb{R}^3}|\nabla v_k^n|^2dx\to\int_{\mathbb{R}^3}|\nabla v_n|^2dx.$$

Together with (ii) and (iii), (iv) is easily obtained. Then, we obtain that $\int_{\mathbb{R}^3}|v_k^n-v_n|^2dx=o(1)$. Thus, we obtain

$$v_k^n\to v_n \quad in\ H_r^1(\mathbb{R}^3)\ as\ k\to\infty.$$

□

3. Proofs of the main results

Proof of Theorem 1.1. From Lemma 2.9, we obtain that for any fixed $c>0$, $\{(v_n,\lambda_n)\}\subset H_r^1(\mathbb{R}^3)\times\mathbb{R}^-$ with $|v_n|_2^2=c$ is a sequence of couples of weak solutions to (1.1) for each $n\in\mathbb{N}^+$. Since $G_b(v_n)=0$, (1.2) and (2.3) we obtain

$$\begin{aligned}\kappa_n^b(c)&=J_b(v_n)-\frac{2}{3(p-2)}G_b(v_n)\\&=\frac{(3p-10)a}{6(p-2)}|\nabla v_n|_2^2+b\left[\int_0^{|\nabla v_n|_2}sM(s,s|\beta_{v_n}|_\tau)ds-\frac{2|\nabla v_n|_2^2}{3(p-2)}M(|\nabla v_n|_2,|\nabla v_n|_2|\beta_{v_n}|_\tau)\right]\\&\leq\frac{(3p-10)a}{6(p-2)}\|v_n\|^2+C(\|v_n\|^{d_1+2}+\|v_n\|^{d_2+2}|\beta_{v_n}|_\tau^{d_2+2}+\|v_n\|),\end{aligned}$$

which implies that $\{v_n\}$ is unbounded due to Lemmas 2.4 and 2.5. □

Proof of Theorem 1.2. For every $b>0$, by Theorem 1.1 there exists a sequence of couples of weak solutions $\{(v_n^b,\lambda_n^b)\}\subset H_r^1(\mathbb{R}^3)\times\mathbb{R}^-$ to (1.1). We show that for any sequence $b_m\to 0^+$ as $m\to+\infty$, $\{v_n^{b_m}\}_{m\in\mathbb{N}^+}$ is bounded in $H_r^1(\mathbb{R}^3)$. It follows from V_n is finite-dimensional that for each $n\in\mathbb{N}^+$,

$$\kappa_n^{b_m}(c):=\inf_{\gamma\in\Gamma_n}\max_{0\leq s\leq 1,u\in S_r(c)\cap V_n}J_{b_m}(\gamma(s,u))\leq\inf_{\gamma\in\Gamma_n}\max_{0\leq s\leq 1,u\in S_r(c)\cap V_n}J_1(\gamma(s,u)):=\alpha_n<+\infty.$$

Since $\{(v_n^{b_m},\lambda_n^{b_m})\}\subset H_r^1(\mathbb{R}^3)\times\mathbb{R}^-$ is a sequence of couples of weak solutions to (1.1) with $b=b_m$ and

$$\lambda_n^{b_m}=\frac{1}{c}\left[\left(a+bM(|\nabla v_n^{b_m}|_2,|\beta_{v_n^{b_m}}|_\tau)\right)A(v_n^{b_m})-C(v_n^{b_m})\right],$$

then by Lemma 2.8, we conclude that $G_{b_m}(v_n^{b_m})=0$. Together with (1.3) and (2.3), we deduce that

$$\kappa_n^{b_m}(c)=J_{b_m}(v_n^{b_m})-\frac{2}{3(p-2)}G_{b_m}(v_n^{b_m})\geq\frac{(3p-10)a}{6(p-2)}|\nabla v_n^{b_m}|_2^2,$$

which implies that $\{A(v_n^{b_m})\}_{m\in\mathbb{N}^+}$ is bounded in $\mathbb{R}$, $\{v_n^{b_m}\}_{m\in\mathbb{N}^+}$ is bounded in $H_r^1(\mathbb{R}^3)$ and $\{\lambda_n^{b_m}\}_{m\in\mathbb{N}^+}$ is bounded in $\mathbb{R}$. Therefore, there exist a subsequence of $\{b_m\}$ (still denoted by $\{b_m\}$) and $\lambda_n^0\leq 0$ such that $\lambda_n^{b_m}\to\lambda_n^0$ as $m\to+\infty$. Meanwhile,

$$\begin{cases}v_n^{b_m}\rightharpoonup v_n^0 & \text{in } H_r^1(\mathbb{R}^3),\\ v_n^{b_m}\to v_n^0 & \text{in } L^p(\mathbb{R}^3),\\ v_n^{b_m}\to v_n^0 & \text{a.e. in } \mathbb{R}^3.\end{cases}\tag{3.1}$$

Thus, for every $n \in \mathbb{N}^+$ (v_n^0, λ_n^0) is a couple of weak solution to (1.1) with $b = 0$, that is, for any $w \in H_r^1(\mathbb{R}^3)$,

$$a\int_{\mathbb{R}^3} \nabla v_n^0 \nabla w dx - \lambda_n^0 \int_{\mathbb{R}^3} v_n^0 w dx = \int_{\mathbb{R}^3} |v_n^0|^{p-2} v_n^0 w dx. \tag{3.2}$$

Taking $w = v_n^{b_m} - v_n^0$ in (3), we obtain

$$a\int_{\mathbb{R}^3} \nabla v_n^0 \nabla (v_n^{b_m} - v_n^0) dx - \lambda_n^0 \int_{\mathbb{R}^3} v_n^0 (v_n^{b_m} - v_n^0) dx = \int_{\mathbb{R}^3} |v_n^0|^{p-2} v_n^0 (v_n^{b_m} - v_n^0) dx. \tag{3.3}$$

By $\{(v_n^{b_m}, \lambda_n^{b_m})\} \subset H_r^1(\mathbb{R}^3) \times \mathbb{R}^-$ is a sequence of couples of weak solutions to (1.1) with $b = b_m$, $\lambda_n^{b_m} \to \lambda_n^0$ and $\{v_n^{b_m}\}$ is bounded in $H_r^1(\mathbb{R}^3)$, we obtain

$$a\int_{\mathbb{R}^3} \nabla v_n^{b_m} \nabla (v_n^{b_m} - v_n^0) dx - \lambda_n^0 \int_{\mathbb{R}^3} v_n^{b_m} (v_n^{b_m} - v_n^0) dx = \int_{\mathbb{R}^3} |v_n^{b_m}|^{p-2} v_n^0 (v_n^{b_m} - v_n^0) dx + o(1). \tag{3.4}$$

Combining (3.1), (3.3) and (3.4) we can see that

$$aA(v_n^{b_m} - v_n^0) - \lambda_n^0 D(v_n^{b_m} - v_n^0) = o(1). \tag{3.5}$$

According to $\lambda_n^0 \le 0$, $A(v_n^{b_m} - v_n^0) \to 0$ as $m \to +\infty$. If $\lambda_n^0 = 0$, from (3.2) it follows that $v_n^0 \in H_r^1(\mathbb{R}^3)$ is a weak solution to $-a\Delta v = |v|^{p-2}v$. This yields $v_n^0 = 0$. On the other hand, by Lemma 2.5 we obtain that

$$1 \le \sigma_n \le \kappa_n^b(c) := J_b(v_n^{b_m}) \to 0,$$

which yields a contradiction. Hence, $\lambda_n^0 < 0$ and $D(v_n^{b_m} - v_n^0) \to 0$, as $m \to +\infty$, due to (3.5). Thus, $\{(v_n^0, \lambda_n^0)\} \subset H_r^1(\mathbb{R}^3) \times \mathbb{R}^-$ is a sequence of couples of weak solutions to (1.11). □

4. Conclusions

The main purpose of this paper is to study the existence of high energy normalized solutions for the Kirchhoff-Carrier type equation. To prove Theorem 1.1, we first show that

$$u_t \in X_r(c),\ A(u_t) \to +\infty, \quad and \quad J_b(u_t) \to -\infty, \quad as \quad t \to +\infty.$$

Combining with Proposition 2.1, we obtain that $\sigma_n \to +\infty$ as $n \to +\infty$. Similar to [8, Lemma 2.3], we get Lemma 2.6. We use Lemma 2.6 to obtain a Palais-Smale sequence $\{v_k^n \subset X_r(c)\}$ of functional J_b satisfying $G_b(v_k^n) \to 0$ where $G_b(u)$ is given in Lemma 2.7. Finally, we conclude that

$$v_k^n \to v_n \quad in\ H_r^1(\mathbb{R}^3)\ as\ k \to \infty$$

by using Lemma 2.2. Meanwhile, we find some converence phenomena appear when $b \to 0$.

In the proof, $H_r^1(\mathbb{R}^3)$ is the subspace of radially symmetric functions in $H^1(\mathbb{R}^3)$ is very crucial, we do not know whether the solution can still exist in the more general space $H^1(\mathbb{R}^3)$. This is a question that we need to further consider.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

References

1. C. O. Alves, D. P. Covei, Existence of solution for a class of nonlocall elliptic problem via sub-supersolution method, *Nonlinear Anal.-Real*, **23** (2015), 1–8. https://doi.org/10.1016/j.nonrwa.2014.11.003
2. H. Berestycki, P. L. Lions, Nonlinear scalar field equations, II existence of infinitely many solutions, *Arch. Ration. Mech. An.*, **82** (1983), 347–375. Available from: `https://link.springer.com/article/10.1007/BF00250556`.
3. T. Bartsch, S. De Valeriola, Normalized solutions of nonlinear Schrödinger equations, *Arch. Math.*, **100** (2013), 75–83. Available from: `https://arxiv.org/abs/1209.0950v1`.
4. G. Che, H. Chen, Existence and concentration result for Kirchhoff equations with critical exponent and hartree nonlinearity, *J. Appl. Anal. Comput.*, **10** (2020), 2121–2144. https://doi.org/10.11948/20190338
5. G. Che, H. Chen, Existence and multiplicity of systems of Kirchhoff-type equations with general potentials, *Math. Method. Appl. Sci.*, **40** (2019), 775–785. https://doi.org/10.1002/mma.4007
6. M. Chipot, B. Lovat, Some remarks on nonlocal elliptic and parabolic problems, *Nonlinear Anal.-Theor.*, **30** (1997), 4619–4627. https://doi.org/10.1016/S0362-546X(97)00169-7
7. G. F. Carrier, On the nonlinear vibration problem of the elastic string, *Q. J. Appl. Math.*, **3** (1945), 157–165.
8. L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal.*, **28** (1997), 1633–1659.
9. F. Jin, B. Yan, The sign-changing solutions for nonlinear elliptic problem with Carrier type, *J. Math. Anal. Appl.*, **487** (2020), 124002. https://doi.org/10.1016/j.jmaa.2020.124002
10. G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.
11. L. Kong, H. Chen, Normalized solutions for nonlinear Kirchhoff type equations in high dimensions, *Electron. Res. Arch.*, **30** (2022), 1282–1295. https://doi.org/10.3934/era.2022067
12. J. L. Lions, On some questions in boundary value problems of mathematical physics, *North-Holland Math. Stud.*, **30** (1978), 284–346. https://doi.org/10.1016/S0304-0208(08)70870-3

13. T. Luo, Multiplicity of normalized solutions for a class of nonlinear Schrödinger-Poisson-Slater equations, *J. Math. Anal. Appl.*, **416** (2014), 195–204. https://doi.org/10.1016/j.jmaa.2014.02.038

14. Z. Liu, Y. Lou, J. Zhang, A perturbation approach to studying sign-changing solutions of Kirchhoff equations with a general nonlinearity, *Ann. Mat. Pur. Appl.*, **201** (2022), 1229–1255. https://doi.org/10.1007/s10231-021-011

15. Z. Liu, H. Luo, J. Zhang, Existence and multiplicity of bound state solutions to a Kirchhoff type equation with a general nonlinearity, *J. Geom. Anal.,* **32** (2022), 125. https://doi.org/10.1007/s12220-021-00849-0

16. X. Luo, Q. Wang, Existence and asymptotic behavior of high energy normalized solutions for the Kirchhoff type equations in $\mathbb{R}^3$, *Nonlinear Anal.-Real*, **33** (2017), 19–32. https://doi.org/10.1016/j.nonrwa.2016.06.001

17. S. Yao, H. Chen, V. Radulescu, J. Sun, Normalized solutions for lower critical Choquard equations with critical sobolev perturbation, *Siam. J. Math. Anal.*, **54** (2022), 3696–3723. https://doi.org/10.1137/21M1463136

18. Y. Su, Z. Feng, Fractional sobolev embedding with radial potential, *J. Differ. Equations*, **340** (2022), 1–44. https://doi.org/10.1016/j.jde.2022.08.030

19. Y. Su, S. Liu, Nehari-Pohozaev-type ground state solutions of Kirchhoff-type equation with singular potential and critical exponent, *Can. Math. Bull.*, **65** (2022), 473–495. https://doi.org/10.4153/S0008439521000436

20. J. Sun, K. Wang, T. Wu, On indefinite Kirchhoff-type equations under the combined effect of linear and superlinear terms, *J. Math. Phys.*, **62** (2021), 031505. https://doi.org/10.1063/5.0030427

21. X. Xu, B. Qin, A variational approach for Kirchhoff-Carrier type non-local equation boundary value problems, *J. Math. Anal. Appl.*, **508** (2022), 125885. https://doi.org/10.1016/j.jmaa.2021.125885

22. W. Xie, H. Chen, Existence and multiplicity of normalized solutions for a class of Schrodinger-Poisson equations with general nonlinearities, *Math. Method. Appl. Sci.*, **43** (2020), 3602–3616. https://doi.org/10.1002/mma.6140