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Research article

On the series solution of the stochastic Newell Whitehead Segel equation

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Abstract: The purpose of this paper is to present a two-step approach for finding the series solution of the stochastic Newell-Whitehead-Segel (NWS) equation. The proposed two-step approach starts with the use of the Wiener-Hermite expansion (WHE) technique, which allows the conversion of the stochastic problem into a set of coupled deterministic partial differential equations (PDEs) by components. The deterministic kernels of the WHE serve as the solution to the stochastic NWS equation by decomposing the stochastic process. The second step involves solving these PDEs using the reduced differential transform (RDT) algorithm, which enables the determination of the deterministic kernels. The final step involves plugging these kernels back into the WHE to derive the series solution of the stochastic NWS equation. The expectation and variance of the solution are calculated and graphically displayed to provide a clear visual representation of the results. We believe that this two-step technique for computing the series solution process can be used to a great extent for stochastic PDEs arising in a variety of sciences.

Keywords: amplitude equations; Wiener-Hermite expansion; differential transform; series solution
Mathematics Subject Classification: 58J35, 60H15, 60H35

1. Introduction

This paper focuses on a specific version of the initial-boundary value problem, known as the Newell-Whitehead-Segel (NWS) equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u - \beta u^3 + \sigma \dot{\mathscr{W}}(x), \tag{1.1}$$

subject to the following initial and boundary conditions:

$$u = 0, \ on \, \Gamma = \partial\Omega,$$
$$u(0) = u_0(x), \ for \, x \in \Omega.$$

Here, $\Gamma = \partial\Omega$ represents a bounded area with a smooth boundary, $u_0 \in \mathscr{H}_0^1(\Omega)$ represents the initial condition, and $\dot{\mathscr{W}}(x)$ represents spatial white noise. One possible interpretation of the variable $u(x,t)$ could be that it is a function with randomness, which can represent the random temperature distribution in an infinitely long, thin rod, or the flow velocity of a fluid through an infinitely long, narrow pipe. In Eq (1.1), the first term on the left-hand side, $\frac{\partial u}{\partial t}$, expresses the time variations of $u(x,t)$ at a fixed location; the first term on the right-hand side, $\frac{\partial^2 u}{\partial x^2}$, represents the spatial variations of $u(x,t)$ at a specific time, and $\alpha u - \beta u^3$ is the source term. Moreover, $\alpha > 0$ and $\beta > 0$ can be interpreted as strength factors of nonlinearity. Many physical phenomena are stochastic or random when observed at the microscale and magnified. Thus, it is natural to consider differential equations that involve some randomness. If the randomness is confined to the solution of the differential equations, such problems are known as stochastic differential equations. This type of equation is related to the family of amplitude equations, which are used to study the dynamics of weakly nonlinear systems such as stripe patterns in various physical phenomena like zebra skin, human fingerprints, and ripples in sand [1]. Amplitude equations are a class of mathematical equations that describe the behavior of weakly nonlinear systems, assuming that the system's variables are small deviations from their equilibrium values. The resulting equations are typically a set of ordinary differential equations that describe the evolution of the amplitude and phase of the oscillations. The most fundamental equations in this family of amplitude equations are the NWS equation [2, 3] and the Swift-Hohenberg equation [4]. The problem (1.1) of our interest also belongs to the family of amplitude equations and is closely related to the NWS equation. Some more recent and relevant references are [5–8].

The NWS equation is a nonlinear partial differential equation used to model chemical and biological systems involving substance movement through a medium. This equation was first derived by mathematicians John G. Newell, Richard A. Whitehead and Joel Segel, and it has been applied to various phenomena, such as chemical spill diffusion and pattern formation in reaction-diffusion systems. The NWS equation has been extensively studied and applied in diverse contexts, including astrophysics [9], plasma physics [10], and Bernard-Rayleigh convection of a fluid mixture around a bifurcation point [11].

Several studies have investigated the NWS equation, with studies including analyses of Lie and discrete symmetry transformations [12], comparative analyses of different solution methods [13], and numerical computation lifting schemes [14]. Various analytical and numerical methods have been used to solve the NWS equation, such as the extended direct algebraic method [15], exponential finite difference methods [16], and fractional variational iteration [17]. Furthermore, researchers have explored the NWS equation's applications in biology [18], rheological models [19], and fractional order time and space [20, 21].

Studies on the NWS equation have also examined exact solutions obtained using methods such as the first integral method [22] and the conformable Laplace decomposition method [23]. Other studies include the classification of the reduction operators and exact solutions of the variable coefficient NWS equation [24], the periodic solution of wave-type NWS equations [25], the numerical solution via the exponential B-spline collocation method [26], and the fractional NWS based on a residual power series algorithm [27].

Gaussian white noise-driven nonlinear stochastic systems play a crucial role in a range of engineering and scientific fields. To examine such equations, a group of techniques called spectral decomposition techniques are utilized. These techniques, including Wiener-Hermite expansion (WHE) technique, introduced by Norbert Wiener [28], have been employed to solve nonlinear stochastic differential equations (NSDEs). The WHE technique has been applied in various areas, including turbulence solutions of the Burger equation by Meecham and Siegel [29] and perturbed NSDEs; see [30–32]. The Wiener-Chaos expansion (WCE) technique, introduced by Cameron and Martin in [33], discretizes the Gaussian white noise through Fourier expansion to solve stochastic differential equations [34]. It produces a system of delay differential equations, known as propagators, which can be resolved using a simple deterministic numerical method like the 5th order Runge-Kutta [35]. Researchers have used the WHE technique to examine different aspects of NSDEs, such as the flow turbulence [29], random vibration of nonlinear oscillators [36], stochastic nonlinear diffusion problems [37], and the stochastic Navier-Stokes equations [38]. Additionally, the WHEP technique, which uses WHE with perturbation, has been used in various studies, including the stochastic heat equation with nonlinear losses and the stochastic quadratic nonlinear oscillatory equation [39].

Some of the recent progress in the spectral decomposition techniques, including WHEP/WHE is as follows. In [40], the authors carried out a numerical study of the stochastic Duffing-van der Pol equation under both multiplicative and additive random forcing using the WCE technique and the WHE technique. In [41], El-Beltagy and co-authors have employed the WHEP technique to study the stochastic models driven by fractional Brownian motion. In [42], the authors proposed two semi-analytical techniques i.e., the variation of parameters method with an auxiliary parameter and the Temimi and Ansari method, for solving a class of stochastic ordinary differential equations. In [43] El-Beltagy and collegues, studied the stochastic point nuclear-reactor using WHEs. Hamed and co-authors in [44] used the WHEP technique to study the general form of stochastic van der pol equation under an external excitation, described by Gaussian white noise.

The novelty of this paper lies in proposing a two-step method for solving the stochastic NWS equation series solution. The method involves using the WHE technique and the reduced differential transform (RDT) algorithm to obtain deterministic kernels, which are then used to derive the series solution of the stochastic NWS Eq (1.1). Similarly, the approach has been previously used in [45, 46], where the WHEP technique is coupled with the homotopy perturbation method and multistep transformed method, respectively, to study nonlinear stochastic partial differential equations (SPDEs). The RDT is a useful analytical and numerical technique first introduced by Keskin and Oturanc [47,48] for the study of wave equations. The method is simple to use, efficient, and provides high-precision numerical solutions with rapid convergence and reduced computational work. It can handle both linear and nonlinear equations, and it is more straightforward to use than other methods such as the homotopy perturbation method, differential transform method, variational iteration method (VIM), and Adomian decomposition method. The RDT method has been applied to various types of differential equations, including ordinary differential equations, partial differential equations, fractional differential equations, Volterra integral equations, and integro-differential equations. Its versatility and effectiveness have made it a popular choice among researchers for solving a wide range of mathematical problems.

The paper is structured as follows. Section 1 serves as an introduction to the topic. To make the paper more accessible to the reader, Section 2 is dedicated to preliminaries about the WHE and the RDT algorithm. In Section 3, we present our two-step technique and use it to obtain an analytic random solution process for problem (1.1). Section 4 is dedicated to computing the expectation and variance of the solution process and presenting plots of the results. Finally, Section 5 provides the conclusion of the paper.

2. Theoretical background: Description of WHE technique and RDT algorithm

The widely used WHE method is essential to solve stochastic partial differential equations (SPDEs) based on white noise. It is especially significant for mechanical and structural engineering, as it enables the study of linear and nonlinear systems' response to random influences. By transforming the SPDE into a system of deterministic equations, the WHE method facilitates the use of conventional numerical methods, thereby eliminating the need for random numbers and repeated solving of the SPDEs. Although the truncated series solution obtained through the WHE method comprises a Gaussian part (incorporating the first two terms) and a non-Gaussian part, nonlinear cases may pose difficulties in solving a system of deterministic integro-differential equations obtained after applying a set of comprehensive averages to the stochastic integro-differential equation. Various solutions have been proposed to overcome these challenges, including the WHEP technique, which utilizes perturbation methods to solve perturbed nonlinear problems.

2.1. Basic introduction to WHE technique

The WHE has been effectively used to solve linear and nonlinear problems in dynamics, as shown in works such as [31, 32, 49]. It makes use of Wiener-Hermite polynomials, which form a complete set of statistically orthogonal random functions. The Wiener-Hermite polynomial $\mathscr{H}^{(i)}(t_1, t_2, \ldots, t_i)$ follows the recurrence relation given blow:

$$\mathscr{H}^{(0)} = 1,$$
$$\mathscr{H}^{(1)}(t) = \dot{\mathscr{W}}(t),$$
$$\mathscr{H}^{(i)}(t_1, t_2, \ldots, t_i) = \mathscr{H}^{(i-1)}(t_1, t_2, \ldots, t_{i-1})\mathscr{H}^{(1)}(t_i) - \sum_{m=1}^{i-1} \mathscr{H}^{(i-2)}(t_1, t_2, \ldots, t_{i-2})\delta(t_{i-m} - t_i), i \geq 2,$$

where $\dot{\mathscr{W}}(t)$ is white noise with the following statistical properties:

$$\mathbb{E}\left(\dot{\mathscr{W}}(t)\right) = 0, \tag{2.1}$$

$$\mathbb{E}\left(\dot{\mathscr{W}}(t_1)\dot{\mathscr{W}}(t_2)\right) = \delta(t_1 - t_2), \tag{2.2}$$

in which $\delta(t)$ is the Dirac delta function and $\mathbb{E}$ denotes the ensemble average operator. The Wiener-Hermite set is an orthogonal set, i.e.,

$$\mathbb{E}\left[\mathscr{H}^{(i)}\mathscr{H}^{(j)}\right] = 0, \forall i \neq j.$$

The average of almost all $\mathscr{H}$ functions vanishes, particularly,

$$\mathbb{E}\left[\mathscr{H}^{(i)}\right] = 0, \forall i \geq 1.$$

The Wiener-Hermite polynomial set is complete, thus any stochastic process can be represented by an expansion of Wiener-Hermite polynomials that converges to the original process with a probability of one. The solution function can also be represented in terms of Wiener-Hermite functionals,

$$u(t;w) = u^0(t;w) + \int_{-\infty}^{\infty} \mathscr{U}^{(1)}(t;t_1)\mathscr{H}^{(1)}(t_1;w), dt_1 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^{(2)}(t;t_1,t_2)\mathscr{H}^{(2)}(t_1,t_2;w), dt_1dt_2 + \dots$$

Or, after eliminating the parameters for the sake of brevity, we get:

$$u(t;w) = u^0(t) + \sum_{k=1}^{\infty} \int_{\mathbb{R}^k} u^{(k)}\mathscr{H}^{(k)} d\tau_k.$$

Here, $d\tau_k = dt_1dt_2 \dots dt_k$, and $\int_{\mathbb{R}^k}$ is a k-dimensional integral over the variables $t_1, t_2, \dots, t_k$.

The WHE starts with the ensemble mean, or non-random component, of the function. The next two terms reflect the Gaussian part of the solution. Further terms in the expansion are increasingly different from the Gaussian form. The functions $u^{(j)}(t;t_1,t_2,\dots,t_k)$ in the expansion are referred to as deterministic kernels. The variable w represents a random output from the triple probability space (Ω, B, P), where Ω is the sample space, B is the σ-algebra of Ω, and P is a probability measure. For clarity, the variable w is later omitted. The n^{th}-order Wiener-Hermite time-independent functional, $\mathscr{H}^{(n)}(t_1,t_2,\dots,t_n)$, is also part of the expansion. The Wiener-Hermite functionals are a complete set, with $\mathscr{H}^{(0)} = 1$ and $\mathscr{H}^{(1)}(t_1)\dot{\mathscr{W}}(t_1)$ being the white noise. These functions are symmetrical in their arguments and statistically orthonormal;

$$\mathbb{E}\left[\mathscr{H}^{(i)}\mathscr{H}^{(j)}\right] = 0, \forall i \neq j.$$

The average of almost all Wiener-Hermite functionals vanishes, particularly,

$$\mathbb{E}\left[\mathscr{H}^{(i)}\right] = 0, \forall i \geq 1.$$

The expectation and variance of the solution will be,

$$\mathbb{E}[u(t)] = u^{(0)} \quad \text{and} \quad \text{var}[u(t)] = \sum_{k=1}^{m} k! \int_{\mathbb{R}^k} (u_i^{(k)})^2 d\tau_k.$$

The methodology for solving SPDEs using WHE is as follows. First, the solution process and the stochastic input processes are expressed in terms of Wiener-Hermite polynomials. By substituting these expansions into the original equation, a real or complex integro-differential stochastic equation is obtained. However, by taking the ensemble averages of this equation and utilizing the statistical properties of the Wiener-Hermite polynomials, the equation can be simplified into a set of basic deterministic equations that can be solved using the deterministic kernels of the solution expansion.

2.2. *Description of RDT algorithm*

Let $u(x,t)$ be a function of two variables that can be decomposed as the product of two single-variable functions $f(x)g(t)$. Utilizing the properties of one-dimensional differential transforms, we can write $u(x,t)$ in the following form:

$$u(x,t) = \left(\sum_{i=0}^{\infty} \mathscr{F}(i)x^i\right)\left(\sum_{i=0}^{\infty} \mathscr{G}(j)t^j\right) = \sum_{k=0}^{\infty} \mathscr{U}k(x)t^k, \tag{2.3}$$

where $\mathscr{U}_k(x)$ is referred to as the t-dimensional spectrum function of $u(x,t)$. The key aspects of the RDT algorithm are outlined as follows:

Definition 2.1. *The function $u(x,t)$ is considered to be analytic and continuously differentiable with respect to both x and t in the area of investigation. The transformed function $\mathscr{U}k(x)$ is defined as,*

$$\mathscr{U}k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]t = 0. \tag{2.4}$$

Here, k is a positive integer representing the order of differentiation. The original function is denoted by the lowercase letter $u(x,t)$, while the transformed function is denoted by the scripted letter $\mathscr{U}_k(x)$.

Definition 2.2. *The inverse transform of $\mathscr{U}_k(x)$ is defined as*

$$u(x,t) = \sum_{k=0}^{\infty} \mathscr{U}_k(x)t^k. \tag{2.5}$$

By combining Eqs (2.4) and (2.5), the expression for $u(x,t)$ can be written as:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k. \tag{2.6}$$

The underlying principle of the RDT Algorithm is based on the power series expansion of a function.

The RDT algorithm is a method used to approximate solutions to nonlinear partial differential equations. The algorithm uses an iterative formula to approximate the solution, and the resulting approximation can be improved by increasing the order of the approximation. The steps for using the RDT algorithm are as follows:

(1) Start with the nonlinear partial differential equation expressed in operator form:

$$\begin{aligned}(k+1)\mathscr{U}_{k+1}(x) &= \mathscr{L}\mathscr{U}_k(x) + \mathscr{R}\mathscr{U}_k(x) + \mathscr{N}_k(g(x,t)), \qquad k = 1,2,3,...\\ u_0(x) &= f(x).\end{aligned} \tag{2.7}$$

In Eq (2.7), $\mathscr{L}$ is a linear operator that takes the partial derivative with respect to time t, $\mathscr{R}$ is a linear operator that includes partial derivatives, $\mathscr{N}u(x,t)$ is a nonlinear operator, and $g(x,t)$ is an inhomogeneous term.

(2) Apply the initial condition $u(x,0) = f(x)$ to write $U_0(x) = f(x)$.

(3) Substitute $U_0(x) = f(x)$ into Eq (2.7) and perform iterative calculations to find the values of $\mathscr{U}_k(x)$.

(4) Use these values to perform the inverse transformation given by Eq (2.5) and obtain an approximate solution of order n.

(5) The exact solution is obtained by taking the limit as n approaches infinity, i.e. $u(x,t) = \lim_{n\to\infty} \widetilde{u}_n(x,t)$.

The iterative formula in Eq (2.7) provides an approximate solution at each step of the iteration, and the inverse transformation in Eq (2.5) is used to obtain an approximation solution of order n. By increasing the order of the approximation, the resulting solution becomes more accurate.

Table 1. Reduced differential transformation.

Functional form	Transformed form
$u(x,t)$	$\mathscr{U}_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]\_t = 0$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = \mathscr{U}_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,t)$	$W_k(x) = \alpha\mathscr{U}\_k(x)$, α is a constant
$w(x,t) = x^m t^n$	$W_k(x) = x^m\delta(k-n)$, the Kronecker delta
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U\_k - n(x)$, when $k \geq n$ else 0
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum\_r = 0^k V\_r(x)U_{k-r}(x) = \sum\_r = 0^k U\_r(x)V\_k - r(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r}u(x,t)$	$W\_k(x) = (k+1)\ldots(k+r)U\_k + r(x) = \frac{(k+r)!}{k!}U\_k + r(x)$
$w(x,t) = \frac{\partial}{\partial x}u(x,t)$	$W\_k(x) = \frac{\partial}{\partial x}\mathscr{U}\_k(x)$

3. Application and results for stochastic NWS equation

By applying the WHE, two sets of equations are obtained:
The First equation is given as

$$\begin{aligned} \frac{\partial}{\partial t}u^0(x,t) &= \frac{\partial^2}{\partial x^2}u^0(x,t) + \alpha u^0(x,t) - \beta\left(u^0(x,t)^3 - 3\beta u^0(x,t)\int_0^L u^1(x,t)^2 dx_1\right), \\ u^0(0,t) &= u^0(L,t) = 0, \quad u^0(x,0) = u_0(x). \end{aligned} \tag{3.1}$$

The Second equation can be written as:

$$\begin{aligned} \frac{\partial}{\partial t}u^1(x,t,x_1) &= \frac{\partial^2}{\partial x^2}u^1(x,t,x_1) - 3\beta u^0(x,t,x_1)^2 u^1(x,t,x_1) - \beta u^1(x,t,x_1)^3 + \sigma\delta(x-x_1), \\ u^1(0,t,x_1) &= u^1(L,t,x_1) = 0, \quad u^1(x,0,x_1) = 0. \end{aligned} \tag{3.2}$$

Upon applying the RDT algorithm, using the properties of Table 1, to Eq (3.1), we obtain the following two equations:

$$(k+1)\mathscr{U}^{(0)}_{k+1}(x) = \frac{\partial^2\mathscr{U}^{(0)}_k(x))}{\partial x^2} + \alpha\mathscr{U}^{(0)}_k(x) - \beta\mathscr{A}^{(0)}_k(x) - 3\beta\mathscr{C}^{(0)}k(x), \tag{3.3}$$

$$(k+1)\mathscr{U}^{(1)}k+1(x) = \frac{\partial^2\mathscr{U}^{(1)}_k(x))}{\partial x^2} - 3\beta\mathscr{D}^{(1)}_k(x) - \beta\mathscr{E}^{(1)}_k(x), \tag{3.4}$$

where $\mathscr{A}^{(0)}_k(x)$, $\mathscr{C}^{(0)}_k(x)$, $\mathscr{D}^{(1)}_k(x)$ and $\mathscr{E}^{(1)}_k(x)$ are given as follows.

$$\begin{aligned} \mathscr{A}^{(0)}_0 &= \left(\mathscr{U}^{(0)}_0\right)^3, \\ \mathscr{A}^{(0)}_1 &= 3\left(\mathscr{U}^{(0)}_0\right)^2\mathscr{U}^{(0)}_1, \\ \mathscr{A}^{(0)}_2 &= 3\left(\mathscr{U}^{(0)}_0\right)^2\mathscr{U}^{(0)}_2 + 3\left(\mathscr{U}^{(0)}_1\right)^2\mathscr{U}^{(0)}_0, \\ \mathscr{A}^{(0)}_3 &= 3\left(\mathscr{U}^{(0)}_0\right)^2 U_3 + 6\mathscr{U}^{(0)}_1\mathscr{U}^{(0)}_2\mathscr{U}^{(0)}_0 + \left(\mathscr{U}^{(0)}_1\right)^3, \\ &\vdots \end{aligned}$$

$$
\begin{aligned}
\mathscr{C}_0^{(0)} &= 0\\
\mathscr{C}_1^{(0)} &= \mathscr{U}_0^{(0)}\left(\mathscr{U}_0^{(1)}\right)^2,\\
\mathscr{C}_2^{(0)} &= 2\mathscr{U}_0^{(0)}\mathscr{U}_0^{(1)}\mathscr{U}_1^{(1)},\\
\mathscr{C}_3^{(0)} &= 2\mathscr{U}_0^{(0)}\mathscr{U}_0^{(1)}\mathscr{U}_2^{(1)} + \mathscr{U}_0^{(0)}\mathscr{U}_1^{(1)}\mathscr{U}_1^{(1)} + 2\mathscr{U}_1^{(0)}\mathscr{U}_0^{(1)}\mathscr{U}_1^{(1)} + \mathscr{U}_2^{(0)}\left(\mathscr{U}_0^{(1)}\right)^2,\\
&\vdots\\
\mathscr{D}_0^{(1)} &= \left(\mathscr{U}_0^{(0)}\right)^2\mathscr{U}_0^{(1)},\\
D_1 &= 2\mathscr{U}_0^{(0)}\mathscr{U}_1^{(0)}\mathscr{U}_0^{(1)} + \left(\mathscr{U}_0^{(0)}\right)^2\mathscr{U}_1^{(1)},\\
\mathscr{D}_2^{(1)} &= 2\mathscr{U}_0^{(0)}\mathscr{U}_2^{(0)}\mathscr{U}_0^{(1)} + \left(\mathscr{U}_1^{(0)}\right)^2\mathscr{U}_0^{(1)} + \left(\mathscr{U}_0^{(0)}\right)^2\mathscr{U}_2^{(1)} + 2\mathscr{U}_0^{(0)}\mathscr{U}_1^{(0)}\mathscr{U}_1^{(1)},\\
\mathscr{D}_3^{(1)} &= 2\mathscr{U}_0^{(0)}U_3\mathscr{U}_0^{(1)} + 2\mathscr{U}_1^{(0)}\mathscr{U}_2^{(0)}\mathscr{U}_0^{(1)} + 2\left(\mathscr{U}_1^{(0)}\right)^2\mathscr{U}_1^{(1)} + 2\left(\mathscr{U}_0^{(0)}\right)^2\mathscr{U}_3^{(1)} + 2\mathscr{U}_0^{(0)}\mathscr{U}_1^{(0)}\mathscr{U}_2^{(1)},\\
&\vdots\\
\mathscr{E}_0^{(1)} &= \left(\mathscr{U}_0^{(1)}\right)^3,\\
\mathscr{E}_1^{(1)} &= 3\left(\mathscr{U}_0^{(1)}\right)^2\mathscr{U}_1^{(1)},\\
\mathscr{E}_2^{(1)} &= 3\left(\mathscr{U}_0^{(1)}\right)^2\mathscr{U}_2^{(1)} + 3\left(\mathscr{U}_1^{(1)}\right)^2\mathscr{U}_0^{(1)}.\\
&\vdots
\end{aligned}
\tag{3.5}
$$

Now by taking $\mathscr{U}_0^{(0)} = e^x$ we have $\mathscr{U}_k^{(0)}$ as

$$
\begin{aligned}
\mathscr{U}_0^{(0)} &= e^x,\\
\mathscr{U}_1^{(0)} &= (1+\alpha)e^x - \beta e^{2x},\\
\mathscr{U}_2^{(0)} &= \left(\frac{(1+\alpha)^2 - 3\beta c^2}{2}\right)e^x + 2(\alpha+5)\beta e^{3x} + 3\beta^2 ce^{4x},\\
\mathscr{U}_3^{(0)} &= ((1+\alpha)(1+2\alpha) - \beta c(c+6))\,\frac{e^x}{3} - (1+\alpha)\beta(9+4\alpha+3\alpha(1+\alpha))\frac{e^{3x}}{3},\\
&\quad + 3\beta((33+9\alpha+2\alpha\beta)(1+\alpha)+\alpha\beta)\frac{e^{4x}}{3} + 3\beta(16(1+\alpha)-\alpha\beta^2)\frac{e^{5x}}{3},\\
&\vdots
\end{aligned}
\tag{3.6}
$$

By setting $\mathscr{U}_0^{(1)} = c$, the expression for $\mathscr{U}_k^{(1)}$ is given by the following equation:

$$
\begin{aligned}
\mathscr{U}^{(1)}0 &= c,\\
\mathscr{U}^{(1)}1 &= -3c\beta e^{2x} - \beta\mathscr{C}^3,\\
\mathscr{U}^{(1)}2 &= -\left(12\beta c + 6(1+\alpha)\beta c + 3\beta^2\mathscr{C}^3\right)\frac{e^{2x}}{2} - 3\beta^2 c\frac{e^{2x}}{2},\\
3\mathscr{U}^{(1)}3 &= -48\beta ce^{2x} - 144\beta^2 ce^{4x} - 6\beta c[4e^{2x}(1+\alpha-\beta e^{2x}) - 20\beta e^{4x}] - 3\beta\mathscr{D}_2^{(1)}(x) - \beta\mathscr{E}_2^{(1)}(x),\\
&\vdots
\end{aligned}
\tag{3.7}
$$

The expression for $u^{(0)}(x,t)$ can be simplified to:

$$u^{(0)}(x,t) = \lim_{n\to\infty} \bar{u}^{(0)}n(x,t) = \lim n \to \infty \sum_{k=0}^{n} \mathscr{U}_k^{(0)}(x)\, t^k = \mathscr{U}_0^{(0)} + \mathscr{U}_1^{(0)} t + \mathscr{U}_2^{(0)} t^2 + \mathscr{U}_3^{(0)} t^3 + \ldots . \quad (3.8)$$

Similarly, $\mathscr{U}^{(1)}(x,t,x_1)$ can be represented as:

$$\mathscr{U}^{(1)}(x,t,x_1) = \lim_{n\to\infty} \bar{u}^{(1)}n(x,t,x_1) = \sum_{k=0}^{n} \mathscr{U}_k^{(1)}(x)\, t^k = \mathscr{U}_0^{(1)} + \mathscr{U}_1^{(1)} t + \mathscr{U}_2^{(1)} t^2 + \mathscr{U}_3^{(1)} t^3 + \cdots \quad (3.9)$$

The approximated solution up to two terms is then given by:

$$u(x,t;w) = u^0(x,t) + \int_0^L \mathscr{U}^{(1)}(x,t;x_1)\mathscr{H}^{(1)}(x_1), dx_1$$

$$u(x,t;w) = (\mathscr{U}_0^{(0)} + \mathscr{U}_1^{(0)} t + \mathscr{U}_2^{(0)} t^2 + \mathscr{U}_3^{(0)} t^3 + \ldots) \quad (3.10)$$

$$+ \int_0^L (\mathscr{U}_0^{(1)} + \mathscr{U}_1^{(1)} t + \mathscr{U}_2^{(1)} t^2 + \mathscr{U}_3^{(1)} t^3 + \ldots)\mathscr{H}^{(1)}(x_1)\, dx_1. \quad (3.11)$$

Finally, substituting the computed $\mathscr{U}_0^{(0)}, \mathscr{U}_1^{(0)}, \mathscr{U}_2^{(0)}, \mathscr{U}_3^{(0)}$ and $\mathscr{U}_0^{(1)} + \mathscr{U}_1^{(1)} + \mathscr{U}_2^{(1)} + \mathscr{U}_3^{(1)}$, we get the approximated stochastic series solution of the main problem (1.1) for the particular choice of initial conditions;

$$\begin{aligned}
u(x,t) &\approx e^x + ((1+\alpha)e^x - \beta e^{2x})t + \left(\left(\frac{(1+\alpha)^2 - 3\beta c^2}{2}\right)e^x + 2(\alpha+5)\beta e^{3x} + 3\beta^2 c e^{4x}\right)t^2 \\
&\left(((1+\alpha)(1+2\alpha) - \beta c(c+6))\,\frac{e^x}{3} - (1+\alpha)\beta(9+4\alpha+3\alpha(1+\alpha))\frac{e^{3x}}{3}\right)t^3 \\
&+ \left(3\beta((33+9\alpha+2\alpha\beta)(1+\alpha)+\alpha\beta)\frac{e^{4x}}{3} + 3\beta(16(1+\alpha)-\alpha\beta^2)\frac{e^{5x}}{3}\right)t^3 + ... \\
&+ \int_0^L \left(c + (-3c\beta e^{2x} - \beta\mathscr{C}^3)t + \left(\left(12\beta c + 6(1+\alpha)\beta c + 3\beta^2\mathscr{C}^3\right)\frac{e^{2x}}{2} - 3\beta^2 c\frac{e^{2x}}{2}\right)t^2 + ...\right)\mathscr{H}^{(1)}(x_1)\, dx_1.
\end{aligned} \quad (3.12)$$

4. Mean and variance

We know that the mean and variance of the given SPDE are respectively,

$$\begin{aligned}
\mathbb{E}[u(x,t)] &= u^{(0)} = \sum_{k=0}^{\infty} U_k(x)\, t^k \\
&= e^x + ((1+\alpha)e^x - \beta e^{2x})t + \left(\left(\frac{(1+\alpha)^2 - 3\beta c^2}{2}\right)e^x + 2(\alpha+5)\beta e^{3x} + 3\beta^2 c e^{4x}\right)t^2 \\
&\left(((1+\alpha)(1+2\alpha) - \beta c(c+6))\,\frac{e^x}{3} - (1+\alpha)\beta(9+4\alpha+3\alpha(1+\alpha))\frac{e^{3x}}{3}\right)t^3 \\
&+ \left(3\beta((33+9\alpha+2\alpha\beta)(1+\alpha)+\alpha\beta)\frac{e^{4x}}{3} + 3\beta(16(1+\alpha)-\alpha\beta^2)\frac{e^{5x}}{3}\right)t^3 + ... \\
\operatorname{var}[u(x,t)] &= \sum_{k=1}^{m}(k!)\int_{\mathbb{R}^k} (u_i^{(k)})^2 d\tau_k = \int_0^L (u^1(x,t,x_1))^2 dx_1 \\
&= \int_0^L \left(c + (-3c\beta e^{2x} - \beta\mathscr{C}^3)t + \left(\left(12\beta c + 6(1+\alpha)\beta c + 3\beta^2\mathscr{C}^3\right)\frac{e^{2x}}{2} - 3\beta^2 c\frac{e^{2x}}{2}\right)t^2 + ...\right)\mathscr{H}^{(1)}(x_1)\, dx_1.
\end{aligned}$$

5. Graphs and discussion

In this section, we present some numerical simulations of solution processes and first and the second kernels and variance of the solutions. For this, we have made the following choice of parameters, $\alpha = 1$, nonlinearity strength $\beta = 0.5$ and fluctuation parameter $\sigma = 0.02$. For the SPDE (1.1) the graph of the mean (i.e., first deterministic kernel) of solution (3.12) is as follows.

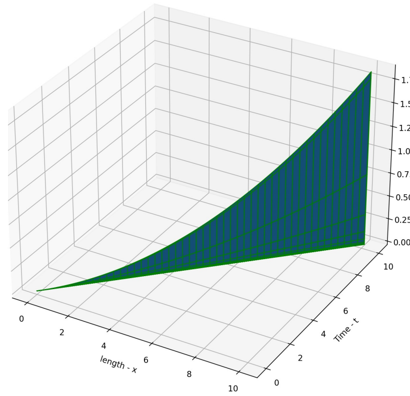

Figure 1. First deterministic kernel

Figure 1 depicts a 3D graph illustrating the evolution of the expectation, where the nonlinearity strength β is set to 0.5. The graph indicates that the mean solution gradually grows over time until it reaches a stable value, which is contingent upon the nonlinearity strength.

For the SPDE, (1.1) the graph of the second deterministic kernel of solution (3.12) is following.

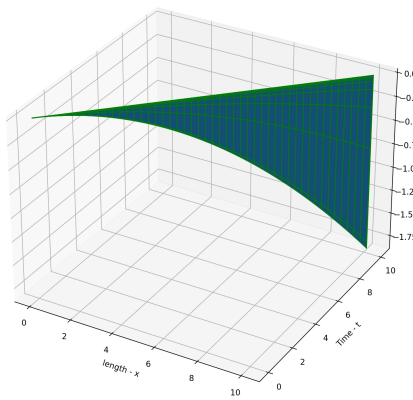

Figure 2. Second deterministic kernel

For the SPDE (1.1), the graph of the variance of solution (3.12) is as follows.

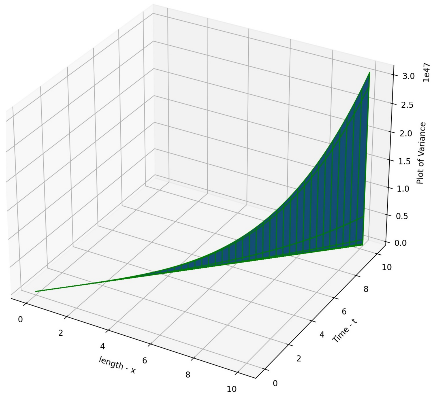

Figure 3. Approximated variance for SPDE

Figures 2 and 3 each showcase a 3D plot that portrays the progression of the second kernel, denoted as u_1, and the variance of the solution process, denoted as u. The variance of the solution process is essentially the square integral of u_1. In this case, the nonlinearity strength, β, is set to 0.5. The graph demonstrates that the second integral and variance exhibit opposite behavior. Specifically, u_1 becomes stable over time, while the solution gradually increases until it attains a stable value that is dependent on the non-linearity strength.

Moreover, $u(x, t)$ is a function that characterizes the temperature distribution in a rod that is infinitely thin and long. As time progresses from zero to one and the spatial variable varies from zero to one, the solution $u(x, t)$ increases and then decreases. The impact of noise is noticeable in the function, but it is not chaotic; instead, it is a genuine reflection of the underlying physical phenomenon.

6. Conclusions

The aim of this study was to propose a novel approach for finding the series solution of the stochastic NWS equation. To achieve this goal, we combined the WHE technique with the RDT algorithm in a two-step approach. The efficacy of the proposed approach was demonstrated through 3D graphical representations that highlight the similarity between the numerical solution of the stochastic NWS and the solution of the original equation. Our approach is not only efficient and accurate, but it also has broad applicability for solving stochastic partial differential equations in various scientific contexts. Thus, it can serve as a valuable tool for researchers and practitioners in the field of applied mathematics. Additionally, this study opens up avenues for future research in the area of solving SPDEs. We plan to extend our work to include numerically solving stochastic reaction-diffusion models, stochastic models appearing in chemistry and biology and stochastic fractional nonlinear models.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflict of interest.

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