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Research article

On the k-th power mean of one kind generalized cubic Gauss sums

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Abstract: The main purpose of this paper is using the elementary methods and properties of the recurrence sequence to study the calculating problem of the k-th power mean of one kind generalized cubic Gauss sums, and give an exact calculating formula for it.

Keywords: the generalized Gauss sums; k-th power mean; elementary method; recurrence sequence; calculating formula
Mathematics Subject Classification: 11L03, 11L05

1. Introduction

Let p be an odd prime, χ denotes a Dirichlet character modulo p. For any integers $k > h \geq 1$ and integers m and n, the generalized two-term exponential sums $S(m, n, k, h, \chi; p)$ is defined as follows:

$$S(m, n, k, h, \chi; p) = \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k + na^h}{p}\right),$$

where $e(y) = e^{2\pi i y}$ and i is the imaginary unit, i.e., $i^2 = -1$.

This sum plays a very important role in the study of analytic number theory and additive number theory, many important problems in number theory are closely related to it, such as prime distribution and Waring's problems. And because of that, many number theorists and scholars had studied the various properties of $S(m, n, k, h, \chi; p)$, and obtained a series of meaningful research results. For example, R. Duan and W. P. Zhang [1] proved that for any prime p with $3 \nmid (p-1)$, and any Dirichlet character λ mod p, one has the identity

$$\sum_{m=1}^{p-1} \left| \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{ma^3 + na}{p}\right) \right|^4 = \begin{cases} 3p^3 - 8p^2 & \text{if } \lambda = \left(\frac{*}{p}\right), \\ 2p^3 - 7p^2 & \text{if } \lambda \neq \chi_0, \left(\frac{*}{p}\right), \\ 2p^3 - 3p^2 - 3p - 1 & \text{if } \lambda = \chi_0, \end{cases}$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre symbol, χ_0 is the principal character modulo p.

L. Chen and X. Wang [2] studied the fourth power mean of $S(m, 1, 4, 1, \chi_0; p)$, and proved the identities

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^4 = \begin{cases} 2p^2(p-2) & \text{if } p \equiv 7 \bmod 12, \\ 2p^3 & \text{if } p \equiv 11 \bmod 12, \\ 2p(p^2-10p-2\alpha^2) & \text{if } p \equiv 1 \bmod 24, \\ 2p(p^2-4p-2\alpha^2) & \text{if } p \equiv 5 \bmod 24, \\ 2p(p^2-6p-2\alpha^2) & \text{if } p \equiv 13 \bmod 24, \\ 2p(p^2-8p-2\alpha^2) & \text{if } p \equiv 17 \bmod 24, \end{cases}$$

where $\alpha = \alpha(p) = \sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\overline{a}}{p}\right)$ is an integer, $a\cdot\overline{a} \equiv 1 \bmod p$, α satisfies the following identity (see Theorems 4–11 in [3]):

$$p = \alpha^2+\beta^2 = \left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\overline{a}}{p}\right)\right)^2 + \left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+r\overline{a}}{p}\right)\right)^2,$$

and r is any quadratic non-residue modulo p. That is, $\left(\frac{r}{p}\right) = -1$.

Recently, W. P. Zhang and Y. Y. Meng [4] studied the sixth power mean of $S(m, n, 3, 1, \chi_0; p)$, and proved that for any odd prime p and integer n with $(n, p) = 1$, we have the identities

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^6 = \begin{cases} 5p^3\cdot(p-1) & \text{if } p \equiv 5 \bmod 6; \\ p^2\cdot\left(5p^2-23p-d^2\right) & \text{if } p \equiv 1 \bmod 6, \end{cases}$$

where $4p = d^2+27\cdot b^2$, b is an integer and d is uniquely determined by $d \equiv 1 \bmod 3$.

X. Y. Liu and W. P. Zhang [5] proved the following conclusion: For any odd prime p with $3 \nmid (p-1)$, one has the identity

$$\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^3+a}{p}\right)\right|^6 = p(p-1)\left(6p^3-28p^2+39p+5\right).$$

On the other hand, if $p \mid n$, then $S(m, n, k, h, \chi; p)$ becomes the generalized k-th Gauss sums. W. P. Zhang and H. N. Liu [6] proved that for any prime p with $3 \mid (p-1)$, one has the identity

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{a^3}{p}\right)\right|^4 = 5p^3-18p^2+20p+1+\frac{U^5}{p}+5pU-5U^3-4U^2+4U,$$

where U is the cubic Gauss sums. That is, $U = \sum_{a=0}^{p-1} e\left(\frac{a^3}{p}\right)$.

For more works towards this direction, see references [7–12], to save space, we will not list them all here.

In this paper, we use the elementary methods and the properties of the recurrence sequence to study the calculating problem of the k-th power mean

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left(\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3}{p} \right) \right)^k, \tag{1.1}$$

and give an exact calculating formula for (1.1) with the restrictive condition $p \equiv 1 \bmod 3$ and $k \nmid (p-1)$. In fact, when k is relatively large, the problem of calculating (1.1) is very difficult. However, we find that as long as $(k, p-1) = 1$, we can transform the studied problem into an interesting recursion relation. Thus, we have completely solved the problem of calculating the k-th power mean (1.1). That is, we will prove the following two conclusions:

Theorem 1.1. Let p be a prime with $p \equiv 1 \bmod 3$, and

$$S_k(p) = \sum_{m=1}^{p-1} (A(m) - 1)^k \, e\left(\frac{m}{p} \right).$$

Then for any positive integer $k \geq 3$, we have the recurrence formula

$$S_k(p) = -3S_{k-1}(p) + 3(p-1)S_{k-2}(p) + (dp + 3p - 1)S_{k-3}(p),$$

where the initial values $S_0(p) = -1$, $S_1(p) = 2p + 1$ and $S_2(p) = dp - 6p - 1$, $A(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p} \right)$ is the cubic Gauss sums, $4p = d^2 + 27 \cdot b^2$, and d is uniquely determined by $d \equiv 1 \bmod 3$.

As an application of Theorem 1.1, we immediately give an efficient method for calculating (1.1). That is, we have the following:

Theorem 1.2. Let p be a prime with $p \equiv 1 \bmod 3$. Then for any positive integer k with $(k, p-1) = 1$, we have the identity

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left(\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3}{p} \right) \right)^k = (p-1)^k + (p-1) \sum_{m=1}^{p-1} (A(m) - 1)^{k-1} \, e\left(\frac{m}{p} \right).$$

It is clear that combining Theorem 1.1 and Theorem 1.2 we immediately solved the problem of calculating k-th power mean (1.1). Especially take $k = 5$, 7 and 11, we can deduce the following three corollaries:

Corollary 1.1. Let p be a prime with $p \equiv 1 \bmod 3$. If $5 \nmid (p-1)$, then we have the identity

$$\frac{1}{p(p-1)} \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left(\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3}{p} \right) \right)^5 = p^3 - 4p^2 - 24p - 24 + 5dp + 10d.$$

Corollary 1.2. Let p be a prime with $p \equiv 1 \bmod 3$. If $7 \nmid (p-1)$, then we have the identity

$$\frac{1}{p(p-1)} \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left(\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3}{p} \right) \right)^7$$

$$= \quad p^5 - 6p^4 + 15p^3 - 146p^2 - 195p - 48 + 21dp^2 + 105dp - 7d^2p + 35d.$$

Corollary 1.3. Let p be a prime with $p \equiv 1 \bmod 3$. If $11 \nmid (p-1)$, then we have the identity

$$\frac{1}{p(p-1)} \sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left(\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3}{p}\right) \right)^{11} = p^9 - 10p^8 + 45p^7 - 120p^6 + 210p^5$$
$$-2034p^4 - 8700p^3 - 8436p^2 - 1935p - 120 + 11d^3p^3 + 55d^3p^2 - 495d^2p^3$$
$$-1320d^2p^2 - 462d^2p + 297dp^4 + 4455dp^3 + 6930dp^2 + 2310dp + 165d.$$

Remarks: Firstly in Theorem 1.2, we only considered the case $p \equiv 1 \bmod 3$. In fact, if $3 \nmid (p-1)$, then for any non-principal character χ modulo p and any integer m with $(m, p) = 1$, we have

$$\left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3}{p}\right) \right| = \sqrt{p}.$$

So in this time, for any real number $k \geq 0$, we have the identity

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^3}{p}\right) \right|^{2k} = (p-1)^{2k} + (p-1) + (p-2)(p-1) \cdot p^k.$$

Secondly, there is no restriction for positive integer k in Theorem 1.1. So for any positive integer k with $(k, p-1) = 1$, combining Theorems 1.1 and 1.2 we can get the exact values of the k-th power mean (1.1).

Thirdly, with the mathematical software such as Matlab (See Appendix program A), we can use computers to calculate the exact value of Theorem 1.2. Thus, computers can be truly realized in solving theoretical mathematical problems.

Finally, if $k \geq 5$ and $(k, p-1) > 1$, whether there exists an exact calculating formula for (1.1) is an open problem. This will be further studied by us.

2. The direct proofs of the theorems

To complete the proof of Theorem 1.1, we need following simple lemma. Of course, the proof of this lemma need some knowledge of elementary or analytic number theory, all them can be found in references [13, 14] or [3], we will not repeat them here. First we have the following:

Lemma 2.1. Let p be a prime with $p \equiv 1 \bmod 3$. Then for any third-order primitive character λ modulo p, we have the identity

$$\tau^3(\lambda) + \tau^3\left(\overline{\lambda}\right) = dp,$$

where $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a}{p}\right)$ denotes the classical Gauss sums, $4p = d^2 + 27 \cdot b^2$, b is an integer and d is uniquely determined by $d \equiv 1 \bmod 3$.

Proof. See [15] or [16].

Now we using this Lemma 2.1 to complete the proof of our Theorem 1.1. Let λ be any third-order primitive character modulo p. For any integer m, let

$$A(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right).$$

Then for integer m with $(m, p) = 1$, from Lemma 2.1, the properties of the third-order character modulo p and Gauss sums we have

$$\begin{aligned} A(m) &= 1 + \sum_{a=1}^{p-1} e\left(\frac{ma^3}{p}\right) = 1 + \sum_{a=1}^{p-1} \left(1 + \lambda(a) + \overline{\lambda}(a)\right) e\left(\frac{ma}{p}\right) \\ &= \lambda(m)\tau\left(\overline{\lambda}\right) + \overline{\lambda}(m)\tau\left(\lambda\right). \end{aligned}$$

Note that $\tau(\lambda) \cdot \tau\left(\overline{\lambda}\right) = p$, $\lambda^2 = \overline{\lambda}$ and $\lambda^3(m) = 1$ we also have

$$A^3(m) = \tau^3(\lambda) + \tau^3\left(\overline{\lambda}\right) + 3p \cdot A(m) = dp + 3p + 3p \cdot (A(m) - 1). \tag{2.1}$$

It is clear that

$$S_0(p) = \sum_{m=1}^{p-1} (A(m) - 1)^0 e\left(\frac{m}{p}\right) = \sum_{m=1}^{p-1} e\left(\frac{m}{p}\right) = -1. \tag{2.2}$$

$$\begin{aligned} S_1(p) &= \sum_{m=1}^{p-1} (A(m) - 1) e\left(\frac{m}{p}\right) = \sum_{m=1}^{p-1} \left(\lambda(m)\tau\left(\overline{\lambda}\right) + \overline{\lambda}(m)\tau\left(\lambda\right) - 1\right) e\left(\frac{m}{p}\right) \\ &= \tau(\lambda)\tau\left(\overline{\lambda}\right) + \tau\left(\overline{\lambda}\right)\tau(\lambda) + 1 = 2p + 1. \end{aligned} \tag{2.3}$$

$$\begin{aligned} S_2(p) &= \sum_{m=1}^{p-1} (A(m) - 1)^2 e\left(\frac{m}{p}\right) = \sum_{m=1}^{p-1} \left(\lambda(m)\tau^2\left(\lambda\right) + \overline{\lambda}(m)\tau^2\left(\overline{\lambda}\right)\right) e\left(\frac{m}{p}\right) \\ &\quad -2\sum_{m=1}^{p-1} \left(\lambda(m)\tau\left(\overline{\lambda}\right) + \overline{\lambda}(m)\tau\left(\lambda\right)\right) e\left(\frac{m}{p}\right) + (2p+1)\sum_{m=1}^{p-1} e\left(\frac{m}{p}\right) \\ &= \tau^3(\lambda) + \tau^3\left(\overline{\lambda}\right) - 4p - (2p + 1) = dp - 6p - 1. \end{aligned} \tag{2.4}$$

From (2.1) we also have

$$\begin{aligned} &(A(m) - 1)^3 = A^3(m) - 3\left(A(m) - 1\right)^2 - 3\left(A(m) - 1\right) - 1 \\ = \ & pd + 3p - 1 - 3\left(A(m) - 1\right)^2 + 3(p - 1)\left(A(m) - 1\right). \end{aligned} \tag{2.5}$$

If $k \geq 3$, then from (2.5) we have the recursive formula

$$S_k(p) = \sum_{m=1}^{p-1} (A(m) - 1)^k e\left(\frac{m}{p}\right) = \sum_{m=1}^{p-1} (A(m) - 1)^{k-3} \left(A(m) - 1\right)^3 e\left(\frac{m}{p}\right)$$

$$
\begin{aligned}
&= \sum_{m=1}^{p-1}(A(m)-1)^{k-2}\left(3(p-1)-3\left(A(m)-1\right)\right)e\left(\frac{m}{p}\right)\\
&\quad +(pd+3p-1)\sum_{m=1}^{p-1}(A(m)-1)^{k-3}\,e\left(\frac{m}{p}\right)\\
&= -3S_{k-1}(p)+3(p-1)S_{k-2}(p)+(dp+3p-1)S_{k-3}(p). \qquad (2.6)
\end{aligned}
$$

Now Theorem 1.1 follows from (2.2)–(2.4) and (2.6).

Now we prove Theorem 1.2. From the properties of the reduced residue system, the orthogonality of the characters modulo p and the trigonometrical identity

$$
\sum_{a=0}^{p-1} e\left(\frac{na}{p}\right)=\begin{cases} p & \text{if } p \mid n;\\ 0 & \text{if } p \nmid n\end{cases}
$$

we have

$$
\begin{aligned}
\frac{1}{p(p-1)}\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left(\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^3}{p}\right)\right)^k &= \sum_{\substack{a_1=1\\ a_1^3+a_2^3+\cdots+a_k^3\equiv 0 \bmod p\\ a_1a_2\cdots a_k\equiv 1 \bmod p}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1} 1\\
= \sum_{\substack{a_1=1\\ a_1^3+a_2^3+\cdots+a_{k-1}^3+1\equiv 0 \bmod p\\ a_1a_2\cdots a_{k-1}a_k^k\equiv 1 \bmod p}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1} 1 &= \sum_{\substack{a_1=1\\ a_1^3+a_2^3+\cdots+a_{k-1}^3+1\equiv 0 \bmod p\\ a_1a_2\cdots a_{k-1}\equiv a_k^k \bmod p}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1} 1. \qquad (2.7)
\end{aligned}
$$

Since $(k, p-1) = 1$, when a_k passes through a reduced residue system modulo p, then a_k^k also passes through a reduced residue system modulo p. From (2.7) we have

$$
\begin{aligned}
\frac{1}{p(p-1)}\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left(\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^3}{p}\right)\right)^k &= \sum_{\substack{a_1=1\\ a_1^3+a_2^3+\cdots+a_{k-1}^3+1\equiv 0 \bmod p\\ a_1a_2\cdots a_{k-1}\equiv a_k \bmod p}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1} 1\\
= \sum_{\substack{a_1=1\\ a_1^3+a_2^3+\cdots+a_{k-1}^3+1\equiv 0 \bmod p}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_{k-1}=1}^{p-1} 1 &= \frac{1}{p}\sum_{m=0}^{p-1}\left(\sum_{a=1}^{p-1}e\left(\frac{ma^3}{p}\right)\right)^{k-1} e\left(\frac{m}{p}\right)\\
&= \frac{(p-1)^{k-1}}{p}+\frac{1}{p}\sum_{m=1}^{p-1}(A(m)-1)^{k-1}\,e\left(\frac{m}{p}\right).
\end{aligned}
$$

This completes the proof of Theorem 1.2.

3. Conclusions

The main results of this paper is to give two theorems. Theorem 1.1 proved that one kind exponential sums related to cubic Gauss sums satisfies a third-order linear recursion property. Theorem 1.2 gave the transformation form of k-th power mean of the generalized cubic Gauss sums, which is closely related to Theorem 1.1. Finally, the calculation of Theorem 1.2 can be achieved by Theorem 1.1 and computer programs such as Matlab.

Of course, our results also provides some new and effective methods for the calculating problem of the k-th power mean of the generalized cubic Gauss sums. We have reason to believe that these works will play a positive role in promoting the study of relevant problems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Appendix program A

Program 1:
function[s_n] = RECURRENCE($d, p, s_{n1}, s_{n2}, s_{n3}$)
$s_n = -3 * s_{n1} + 3 * (p - 1) * s_{n2} + (d * p + 3 * p - 1) * s_{n3}$;
Program 2 :
clc
clear all
syms p;
syms d;
syms $s0$;
syms $s1$;
syms $s2$;
syms $s3$;
syms s_{n3};
syms s_{n2};
syms s_{n1};
syms s_n;
$s0 = -1$
$s1 = 2 * p + 1$
$s2 = d * p - 6 * p - 1$
$s_{n1} = s2$;
$s_{n2} = s1$;
$s_{n3} = s0$;
for $n = 1 : 8$
$n + 2$
s_n = RECURRENCE($d, p, s_{n1}, s_{n2}, s_{n3}$);
expand(s_n)

$s_{n3} = s_{n2}$;
$s_{n2} = s_{n1}$;
$s_{n1} = s_n$;
end

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