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Research article

Results on a neutrosophic sub-rings

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Abstract: The goal of this paper is to create an algebraic structure based on single-valued neutrosophic sets. We present a novel approach to the neutrosophic sub-ring and ideal by combining the classical ring with neutrosophic sets. We also introduce and investigate some of the fundamental properties of the concepts. Finally, we show how to use a neutrosophic ideal to make a decision.

Keywords: neutrosophic set; neutrosophic sub-ring; neutrosophic ideal
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1. Introduction

Many types of uncertainty are found in everyday human life. Zadeh [22] first provided the definition of a fuzzy set since the classical set is not able to handle the outlined uncertainties. This definition states that a fuzzy set is a function that can be represented by a membership value graded over a unit interval. However, it has since been determined that this definition is inadequate when both membership and non-membership degrees are considered. To deal with the stated ambiguity, Atanassov [1] created the intuitionistic fuzzy theory which is a generalization of the fuzzy set. Because this set has several application-related issues, Smarandache [17] proposed a neutrosophic set to address the problems with ambiguous and inconsistent information.

In recent years, researchers from diverse fields have taken a keen interest in this topic. For instance, [4, 7, 8, 10, 11, 14, 15, 18] explores the topic in the context of algebraic structures, while [5, 13] discusses its relevance to analysis and [16] to graphs theory. Additionally, the papers highlights various practical applications of the topic, as outlined in [2, 3]. The authors provide a comprehensive overview of the latest developments in this area, bringing together insights from different disciplines to offer a holistic view of the field.

The operators used in the neutrosophic set (logic) are approximations rather than exact outcomes, as they deal with partial truths (memberships) unlike the classical fuzzy set (logic). As a result, choosing the right operator in fuzzy logic depends on the situation and the user's knowledge. Different operators can give different levels of accuracy. So, it is important to pick the one that works best. This requires experience to make good choices.

Later on, researchers explored the fundamental algebraic operations of neutrosophic sets from three distinct perspectives (see, e.g., [18, 21, 23]). Furthermore, Vildan and Halis developed a strategy for the neutrosophic sub-ring in [6] that was based on the second viewpoint. In addition, Elrawy et al. [9] recently investigated a neutrosophic group and level sub-groups of a neutrosophic sub-group based on the second viewpoint.

In this paper, motivated by some of these aforementioned works, we introduce and study a new approach to neutrosophic sub-ring, ideal, level sub-ring and ideal based on the first viewpoint. Also, we establish an application of neutrosophic ideal in decision making.

The present study has been formulated as follows: Section 2 provides an introduction to fundamental concepts and terminology. The new approaches of neutrosophic sub-ring, ideal, level sub-ring, and ideal are introduced and examined, and we have set up some examples to explain these concepts in Section 3. Finally, in Section 4, we present some observations and conclusions from the work.

2. Basic concepts

Here, we go through some of the concepts and results that we use in the following section.

Definition 2.1. [18] Presume $\mathbb{N}$ is an universe set. Then a neutrosophic set $\mathcal{N}$ on $\mathbb{N}$ is defined by the following:

$$\mathcal{N} = \{< s, \mu(s), \gamma(s), \zeta(s) >: s \in \mathbb{N}\},$$

with $\mu, \gamma, \zeta : \mathbb{N} \to [0, 1]$.

Definition 2.2. [17, 19, 20] Assume that $\mathcal{N}_1 = \{< s, \mu_1(s), \gamma_1(s), \zeta_1(s) >: s \in \mathbb{N}\}$ and $\mathcal{N}_2 = \{< s, \mu_2(s), \gamma_2(s), \zeta_2(s) >: s \in \mathbb{N}\}$ are two neutrosophic sets on $\mathbb{N}$. Then,

1) $\mathcal{N}_1 \subset_1 \mathcal{N}_2 = \{< s, \mu_1(s) \leq \mu_2(s), \gamma_1(s) \geq \gamma_2(s), \zeta_1(s) \geq \zeta_2(s) >: s \in \mathbb{N}\}$,
2) $\mathcal{N}_1 \cup_1 \mathcal{N}_2 = \{< s, \mu_1(s) \vee \mu_2(s), \gamma_1(s) \wedge \gamma_2(s), \zeta_1(s) \wedge \zeta_2(s) >: s \in \mathbb{N}\}$,
3) $\mathcal{N}_1 \cap_1 \mathcal{N}_2 = \{< s, \mu_1(s) \wedge \mu_2(s), \gamma_1(s) \vee \gamma_2(s), \zeta_1(s) \vee \zeta_2(s) >: s \in \mathbb{N}\}$.

Definition 2.3. [9] Assume that $\mathcal{D}$ is a neutrosophic subset of $\mathcal{N}$. For $\alpha \in [0, 1]$, the set

$$\mathcal{D}_\alpha = \{< s, \mu(s), \gamma(s), \zeta(s) >, s \in \mathcal{R} : \mu(s) \geq \alpha, \gamma(s) \leq \alpha, \zeta(s) \leq \alpha\}$$

is a level subset of $\mathcal{D}$.

Obviously, $\mathcal{D}_{\alpha_1} \subset \mathcal{D}_{\alpha_2}$, whenever $\alpha_1 > \alpha_2$.

3. Main result

This section is divided into four subsections. In the first subsection, we introduce and study the new approach of neutrosophic sub-rings of a classical ring in a way similar to the fuzzy situation and give an example. In the second subsection, we investigate the definition of a neutrosophic sub-ring to define the concept of the neutrosophic ideal and give its properties and examples. The level sub-ring and ideal are defined and studied in the third subsection. In the last subsection, we explain an application of the neutrosophic ideal in decision-making.

In what follows, we assume that $(\mathcal{R}, +, .)$ is a ring.

3.1. Neutrosophic sub-ring

Definition 3.1. A neutrosophic subset $\mathcal{S} = \{< s, \mu(s), \gamma(s), \zeta(s) >: s \in \mathcal{R}\}$ of $\mathcal{R}$ is called a neutrosophic sub-ring of $\mathcal{R}$ if the next axioms are satisfied:
(i) $\mu(s-t) \geqslant \min(\mu(s), \mu(t))$,
(ii) $\mu(st) \geqslant \min(\mu(s), \mu(t))$,
(iii) $\gamma(s-t) \leqslant \max(\gamma(s), \gamma(t))$,
(iv) $\gamma(st) \leqslant \max(\gamma(s), \gamma(t))$,
(v) $\zeta(s-t) \leqslant \max(\zeta(s), \zeta(t))$,
(vi) $\zeta(st) \leqslant \max(\zeta(s), \zeta(t))$,
where $s, t \in \mathcal{R}$.

Example 3.2. Consider $\mathcal{R} = Z_3$ is a classical ring. Define a neutrosophic set $\mathcal{S}$ on $\mathcal{R}$ as follows $\mathcal{S} = \{< 0, 0.7, 0.4, 0.5 >, < 1, 0.7, 0.3, 0.4 >, < 2, 0.6, 0.1, 0.3 >\}$. It is easy to show that $\mathcal{S}$ be a neutrosophic sub-ring of $\mathcal{R}$.

Example 3.3. Presume $(\mathbb{R}, +, .)$ is a ring of real number. Consider a neutrosophic subset $\mathcal{S} = \{< s, \mu(s), \gamma(s), \zeta(s) >: s \in \mathbb{R}\}$ define as follows:

$$\mu(s) = \begin{cases} 0.7 & \text{if } s = 0. \\ 0.8 & \text{if } s \neq 0. \end{cases}$$

$$\gamma(s) = \begin{cases} 0.4 & \text{if } s \neq 0. \\ 0.2 & \text{if } s = 0. \end{cases}$$

$$\zeta(s) = \begin{cases} 0.6 & \text{if } s \neq 0. \\ 0.3 & \text{if } s = 0. \end{cases}$$

All axioms of Definition 3.1 are satisfied. Therefore, $\mathcal{S}$ is neutrosophic sub-ring.

Proposition 3.4. *The intersection of a finite set of neutrosophic sub-rings is a neutrosophic sub-ring.*

Proof. We only verify the (iii) and (iv) axioms in Definition 3.1 as the other axioms are well-known.

$$(iii) \quad [\cap_1 \gamma_i](s-t) = \sup[\gamma_i(s-t)] \leq \sup[max(\gamma_i(s), \gamma_i(t))]$$
$$= max(\sup \gamma_i(s), \sup \gamma_i(t))$$
$$= max([\cap_1 \gamma_i](s), [\cap_1 \gamma_i](t)),$$
$$(iv) \quad [\cap_1 \gamma_i](st) = \sup[\gamma_i(st)] \leq \sup[\max(\gamma_i(s), \gamma_i(t))]$$
$$= \max(\cap_1 \gamma_i(s), \cap_1 \gamma_i(t)),$$

where $i = 1, 2, ..., n$. Consequently, the proposition is desired. □

Proposition 3.5. *Let $\mathcal{S}$ be a neutrosophic sub-ring of $\mathcal{R}$, then $\mathcal{S}' = \{s \in \mathcal{R} : \mu(s) = \mu(0), \gamma(s) = \gamma(0), \zeta(s) = \zeta(0)\}$ is a neutrosophic sub-ring.*

Proof. Suppose that $s, t \in \mathcal{S}'$. Then, we have

$$\begin{aligned}\mu(s-t) &\geq min(\mu(s), \mu(t)) \\ &= min(\mu(0), \mu(0)) \\ &= \mu(0).\end{aligned}$$

Again,

$$\begin{aligned}\mu(0) &= \mu([s-t]-[s-t]) \\ &\geq min(\mu(s-t), \mu(s-t)) \\ &= \mu(s-t).\end{aligned}$$

Thus, $s-t \in \mathcal{S}'$.

$$\begin{aligned}\mu(st) &\geq min(\mu(s), \mu(t)) \\ &= min(\mu(0), \mu(0)) \\ &= \mu(0).\end{aligned}$$

Again,

$$\begin{aligned}\mu(0) &= \mu([st]-[st]) \\ &\geq min(\mu(st), \mu(st)) \\ &= \mu(st).\end{aligned}$$

Thus, $st \in \mathcal{S}'$.
Similarly, in case γ and ζ we can show $s-t, st \in \mathcal{S}'$. We only prove in case γ as follows

$$\begin{aligned}\gamma(s-t) &\leq max(\gamma(s), \gamma(t)) \\ &= max(\gamma(0), \gamma(0)) \\ &= \gamma(0).\end{aligned}$$

Again,

$$\begin{aligned}\gamma(0) &= \gamma([s-t]-[s-t]) \\ &\leq max(\gamma(s-t), \gamma(s-t)) \\ &= \gamma(s-t).\end{aligned}$$

Thus, $s-t \in \mathcal{S}'$.

$$\begin{aligned}\gamma(st) &\leq max(\gamma(s), \gamma(t)) \\ &= max(\gamma(0), \gamma(0)) \\ &= \gamma(0).\end{aligned}$$

Again,

$$\begin{aligned}\gamma(0) &= \gamma([st]-[st]) \\ &\leq max(\gamma(st), \gamma(st)) \\ &= \gamma(st).\end{aligned}$$

Thus, $st \in \mathcal{S}'$. □

Proposition 3.6. *Let $\mathcal{S} = \{< s, \mu(s), \gamma(s), \zeta(s) >: s \in \mathcal{R}\}$ be a neutrosophic sub-ring of $\mathcal{R}$. Then, the next axioms are held*

(i) $\mu(s+t) = min(\mu(s), \mu(t))\ \forall s, t \in \mathcal{R}$ *with* $\mu(s) \neq \mu(t)$,
(ii) $\gamma(s+t) = max(\gamma(s), \gamma(t))\ \forall s, t \in \mathcal{R}$ *with* $\gamma(s) \neq \gamma(t)$,
(iii) $\zeta(s+t) = max(\zeta(s), \zeta(t))\ \forall s, t \in \mathcal{R}$ *with* $\zeta(s) \neq \zeta(t)$.

Proof. Assume that $\mathcal{S}$ is a neutrosophic sub-ring of $\mathcal{R}$ and $s, t \in \mathcal{R}$ with $\mu(s) \neq \mu(t)$, $\gamma(s) \neq \gamma(t)$ and $\zeta(s) \neq \zeta(t)$. Then, we have the following:

(i) Assume that $\mu(s+t) > min(\mu(s), \mu(t))$ and take $\mu(s) < \mu(t)$. Thus, $\mu(s+t) > \mu(s)$. Again, since $\mu(s) \geqslant min(\mu(s+t), \mu(-t))$. Thus, $\mu(s) \geqslant \mu(s+t)$ which is a contradiction. Therefore, $\mu(s+t) \leqslant min(\mu(s), \mu(t))$. Hence, $\mu(s+t) = min(\mu(s), \mu(t))$.
(ii) Assume that $\gamma(s+t) < max(\gamma(s), \gamma(t))$ and take $\gamma(s) > \gamma(t)$. Thus, $\gamma(s+t) < \gamma(s)$. Again, since $\gamma(s) \leqslant max(\gamma(s+t), \gamma(-t))$. Thus, $\gamma(s) \leqslant \gamma(s+t)$ which is a contradiction. Therefore, $\gamma(s+t) \leqslant max(\gamma(s), \gamma(t))$. Hence, $\gamma(s+t) = max(\gamma(s), \gamma(t))$.
(iii) Similar to (ii).

□

3.2. Neutrosophic ideal

Definition 3.7. Presume $\mathcal{I}$ is a neutrosophic sub-ring. Then, we called $\mathcal{I}$ a neutrosophic left ideal if the following is satisfied
(i) $\mu(st) \geqslant \mu(t)$,
(ii) $\gamma(st) \leqslant \gamma(t)$,
(iii) $\zeta(st) \leqslant \zeta(t)$.
Also, it is called $\mathcal{I}$ is a neutrosophic right ideal if
(i) $\mu(st) \geqslant \mu(s)$,
(ii) $\gamma(st) \leqslant \gamma(s)$,
(iii) $\zeta(st) \leqslant \zeta(s)$.
Again, a neutrosophic ideal, if it is a neutrosophic left and right ideal.

Proposition 3.8. *$\mathcal{I}$ is a neutrosophic ideal of $\mathcal{R}$ iff for all $s, t \in \mathcal{R}$ the following axioms are true:*
(i) $\mu(s-t) \geqslant \min(\mu(s), \mu(t))$,
(ii) $\mu(st) \geqslant \max(\mu(s), \mu(t))$,
(iii) $\gamma(s-t) \leqslant \max(\gamma(s), \gamma(t))$,
(iv) $\gamma(st) \leqslant \min(\gamma(s), \gamma(t))$,
(v) $\zeta(s-t) \leqslant \max(\zeta(s), \zeta(t))$,
(vi) $\zeta(st) \leqslant \min(\zeta(s), \zeta(t))$.

Proof. It is stratified for the definition of a neutrosophic ideal. □

Example 3.9. Let $(Z_8, \oplus_8, \otimes_8)$ be a ring. Consider a neutrosophic subset $\mathcal{S} = \{< s, \mu(s), \gamma(s), \zeta(s) >: s \in Z_8\}$ define as follows:

$$\mu(s) = \begin{cases} 0.7 & \text{if } s = 0. \\ 0.5 & \text{if } s \in \{2, 4, 6\}. \\ 0.4 & \text{otherwise.} \end{cases}$$

$$\gamma(s)=\begin{cases}0.5 & \text{if } s=0.\\ 0.7 & \text{if } s\in\{2,4,6\}.\\ 0.8 & \text{otherwise.}\end{cases}$$

$$\zeta(s)=\begin{cases}0.4 & \text{if } s=0.\\ 0.6 & \text{if } s\in\{2,4,6\}.\\ 0.7 & \text{otherwise.}\end{cases}$$

All axioms of Proposition 3.8 are satisfied. Therefore, $\mathcal{S}$ is neutrosophic ideal.

Now, we assume that $(\mathcal{D},+,.)$ is a division ring and o, ε are a unit of $\mathcal{D}$ for + and ., respectively.

Proposition 3.10. *$\mathcal{I}$ is a neutrosophic ideal of $\mathcal{D}$ if and only if for all $s\neq o$ in $\mathcal{D}$ the next axioms are held*

(*i*) $\mu(s)=\mu(\varepsilon)\leqslant\mu(o)$,
(*ii*) $\gamma(s)=\gamma(\varepsilon)\geqslant\gamma(o)$,
(*iii*) $\zeta(s)=\zeta(\varepsilon)\geqslant\zeta(o)$.

Proof. Assume that $\mathcal{I}$ is a neutrosophic ideal of $\mathcal{D}$, then we explain (*ii*), (*iii*) similarly and (*i*) see [12]. Since

$$\gamma(o)=\gamma(\varepsilon-\varepsilon)\leqslant\max(\gamma(\varepsilon),\gamma(\varepsilon))=\gamma(\varepsilon).$$

Again, we assume that $s\in\mathcal{D}$ with $s\neq o$

$$\gamma(s)=\gamma(s.\varepsilon)\leqslant\min(\gamma(s),\gamma(\varepsilon))=\gamma(\varepsilon).$$

Also,

$$\gamma(\varepsilon)=\gamma(ss^{-1})\leqslant\gamma(s).$$

Therefore, $\gamma(\varepsilon)=\gamma(s)\geqslant\gamma(o)$. Conversely, we assume that $s,t\in\mathcal{D}$. Then, we explain the only (*iii*) and (*iv*) axioms in Proposition 3.8. Now, to prove (*iii*) we have the two cases:
Case 1. If $s\neq t$, then we find

$$\begin{aligned}\gamma(s-t)&=\gamma(\varepsilon)\\&\leqslant\max(\gamma(s),\gamma(t)).\end{aligned}$$

Case 2. If $s=t$, then we get

$$\begin{aligned}\gamma(s-t)&=\gamma(o)\\&\leqslant\max(\gamma(s),\gamma(t)).\end{aligned}$$

Again, to prove (*iv*) we have the two cases:
Case 1. If $s\neq o$ or $t\neq o$, then we obtain

$$\begin{aligned}\gamma(st)&=\gamma(\varepsilon)\\&\leqslant\min(\gamma(s),\gamma(t)).\end{aligned}$$

Case 2. If $s=o$ or $t=o$, then we arrive at

$$\gamma(s-t)\leqslant\min(\gamma(s),\gamma(t)).$$

□

Now, we assume that $\mathcal{R}'$ is a commutative ring with identity ε.

Proposition 3.11. *Let $\mathcal{I}$ be a neutrosophic ideal of $\mathcal{R}'$ with $\mu(l) = \mu(\varepsilon) \leqslant \mu(o)$, $\gamma(l) = \gamma(\varepsilon) \geqslant \gamma(o)$ and $\zeta(l) = \zeta(\varepsilon) \geqslant \zeta(o)$, $l \in \mathcal{R}'$, $l \neq o$. Then, $\mathcal{R}'$ is a field.*

Proof. Presume that $\mathcal{I}$ is a neutrosophic ideal of $\mathcal{R}'$, then there exist $r \in \mathcal{R}'$ and $r \neq \mathcal{I}$ with $\mu(r) = \gamma(r) = \zeta(r) = o$ and $\mu(l) = \gamma(l) = \zeta(l) = o$, $\forall l \in \mathcal{R}'$ with $l \neq o$. Therefore, $\mathcal{I} = o$ and $\mathcal{R}'$ is a field. □

Proposition 3.12. *Let $\mathcal{S}$ be a neutrosophic sub-ring and $\mathcal{I}$ be a neutrosophic ideal of $\mathcal{R}$. Then, $\mathcal{S} \cap_1 \mathcal{I}$ is a neutrosophic ideal of the sub-ring $\mathcal{S}' = \{s \in \mathcal{R} : \mu(s) = \mu(0), \gamma(s) = \gamma(0), \zeta(s) = \zeta(0)\}$.*

Proof. To show $\mathcal{S} \cap_1 \mathcal{I}$ is a neutrosophic ideal of the sub-ring $\mathcal{S}' = \{s \in \mathcal{R} : \mu(s) = \mu(0), \gamma(s) = \gamma(0), \zeta(s) = \zeta(0)\}$ it is enough check all axioms of Proposition 3.8. Suppose that $\mathcal{S} = \{< s, \mu_1(s), \gamma_1(s), \zeta_1(s) >: s \in \mathcal{R}\}$ and $\mathcal{I} = \{< s, \mu_2(s), \gamma_2(s), \zeta_2(s) >: s \in \mathcal{R}\}$. Then,

$$(i) \quad \begin{aligned}[][\mu_1 \cap_1 \mu_2](s-t) &= \sup[(\mu_1 \wedge \mu_2)(s-t)] \\ &= \sup[\mu_1(s-t) \wedge \mu_2(s-t)] \\ &\geq \sup[min(\mu_1(s), \mu_1(t)) \wedge min(\mu_2(s), \mu_2(t))] \\ &= min[sup[(\mu_1 \wedge \mu_2)(s), (\mu_1 \wedge \mu_2)(t)]] \\ &= min[sup[(\mu_1 \cap_1 \mu_2)(s), (\mu_1 \cap_1 \mu_2)(t)]].\end{aligned}$$

$$(ii) \quad \begin{aligned}[][\mu_1 \cap_1 \mu_2](st) &= \sup[(\mu_1 \wedge \mu_2)(st)] \\ &= \sup[\mu_1(st) \wedge \mu_2(st)] \\ &\geq \sup[max(\mu_1(s), \mu_1(t)) \wedge max(\mu_2(s), \mu_2(t))] \\ &= max[\sup[(\mu_1 \wedge \mu_2)(s), (\mu_1 \wedge \mu_2)(t)]] \\ &= max[\sup[(\mu_1 \cap_1 \mu_2)(s), (\mu_1 \cap_1 \mu_2)(t)]].\end{aligned}$$

$$(iii) \quad \begin{aligned}[][\gamma_1 \cap_1 \gamma_2](s-t) &= \sup[(\gamma_1 \wedge \gamma_2)(s-t)] \\ &= \sup[\gamma_1(s-t) \wedge \gamma_2(s-t)] \\ &\leq \sup[max(\gamma_1(s), \gamma_1(t)) \wedge max(\gamma_2(s), \gamma_2(t))] \\ &= max[\sup[(\gamma_1 \wedge \gamma_2)(s), (\gamma_1 \wedge \gamma_2)(t)]] \\ &= max[\sup[(\gamma_1 \cap_1 \gamma_2)(s), (\gamma_1 \cap_1 \gamma_2)(t)]].\end{aligned}$$

$$(iv) \quad \begin{aligned}[][\gamma_1 \cap_1 \gamma_2](st) &= \sup[(\gamma_1 \wedge \gamma_2)(st)] \\ &= \sup[\gamma_1(st) \wedge \gamma_2(st)] \\ &\leq \sup[min(\gamma_1(s), \gamma_1(t)) \wedge min(\gamma_2(s), \gamma_2(t))] \\ &= min[\sup[(\gamma_1 \wedge \gamma_2)(s), (\gamma_1 \wedge \gamma_2)(t)]] \\ &= min[\sup[(\gamma_1 \cap_1 \gamma_2)(s), (\gamma_1 \cap_1 \gamma_2)(t)]].\end{aligned}$$

Similarly, (v) and (vi). Therefore, $\mathcal{S} \cap_1 \mathcal{I}$ is a neutrosophic ideal of the sub-ring $\mathcal{S}' = \{s \in \mathcal{R} : \mu(s) = \mu(0), \gamma(s) = \gamma(0), \zeta(s) = \zeta(0)\}$. □

Definition 3.13. Presume that $\mathcal{I}_1$ and $\mathcal{I}_2$ are two neutrosophic ideals of $\mathcal{R}$. Then, we define a product of $\mathcal{I}_1$ and $\mathcal{I}_2$ as follows

$$\begin{aligned}(\mu_1 \bullet \mu_2)(s) &= \sup_{s=\sum_i u_i v_i} (min_i(min(\mu_1(u_i), \mu_2(v_i)))), \\ (\gamma_1 \bullet \gamma_2)(s) &= \inf_{s=\sum_i u_i v_i} (max_i(max(\gamma_1(u_i), \gamma_2(v_i)))), \\ (\zeta_1 \bullet \zeta_2)(s) &= \inf_{s=\sum_i u_i v_i} (max_i(max(\zeta_1(u_i), \zeta_2(v_i)))),\end{aligned}$$

where $s, u_i, v_i \in \mathcal{R}$.

Proposition 3.14. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two neutrosophic ideal of $\mathcal{R}$. Then $\mathcal{I}_1 \cap_1 \mathcal{I}_2$ is also neutrosophic ideal. Moreover, $\mathcal{I}_1 \bullet \mathcal{I}_2$ is a neutrosophic ideal.*

Proof. Since $\mathcal{I}_1$ and $\mathcal{I}_2$ are two neutrosophic ideals of $\mathcal{R}$, $\mathcal{I}_1 \cap_1 \mathcal{I}_2$ is also sub-ring. Now, it is enough to check axioms (*ii*), (*iv*) and (*vi*).

$$
\begin{aligned}
(ii)\quad (\mu_1 \cap_1 \mu_2)(st) &= \sup[\mu_1(st), \mu_2(st)] \\
&\leqslant \sup[max(\mu_1(s), \mu_1(t)), max(\mu_2(s), \mu_2(t))] \\
&= max[(\mu_1 \cap_1 \mu_2)(s), (\mu_1 \cap_1 \mu_2)(t)].
\end{aligned}
$$

$$
\begin{aligned}
(iv)\quad (\gamma_1 \cap_1 \gamma_2)(st) &= \sup[\gamma_1(st), \gamma_2(st)] \\
&\geqslant \sup[min(\gamma_1(s), \gamma_1(t)), min(\gamma_2(s), \gamma_2(t))] \\
&= min[(\gamma_1 \cap_1 \gamma_2)(s), (\gamma_1 \cap_1 \gamma_2)(t)].
\end{aligned}
$$

$$
\begin{aligned}
(iv)\quad (\zeta_1 \cap_1 \zeta_2)(st) &= \sup[\zeta_1(st), \zeta_2(st)] \\
&\geqslant \sup[min(\zeta_1(s), \zeta_1(t)), min(\zeta_2(s), \zeta_2(t))] \\
&= min[(\zeta_1 \cap_1 \zeta_2)(s), (\zeta_1 \cap_1 \zeta_2)(t)].
\end{aligned}
$$

Therefore, $\mathcal{I}_1 \cap_1 \mathcal{I}_2$ is neutrosophic ideal. Next, we show $\mathcal{I}_1 \bullet \mathcal{I}_2$ is neutrosophic ideal. Consider $s, t \in \mathcal{R}$. Then,

(*i*) $(\mu_1 \bullet \mu_2)(s-t) = \sup_{s-t=\sum_i (u_i v_i - w_i z_i)} (min_i(min(\mu_1(u_i - w_i), \mu_2(v_i - z_i)))$,
since

$$
\begin{aligned}
min[(\mu_1 \bullet \mu_2)(s), (\mu_1 \bullet \mu_2)(t)] &= min[\sup_{s=\sum_i u_i v_i} (min_i(\mu_1(u_i), \mu_2(v_i))), \sup_{t=\sum_i w_i z_i} (min_i(\mu_1(w_i), \mu_2(z_i)))] \\
&= \sup_{\substack{s=\sum_i u_i v_i \\ t=\sum_i w_i z_i}} [min_i(min(\mu_1(u_i), \mu_1(w_i), \mu_2(v_i), \mu_2(z_i)))] \\
&\leq \sup_{s-t=\sum_i (u_i v_i - w_i z_i)} [min_i(min(\mu_1(u_i - w_i), \mu_2(v_i - z_i)))] \\
&= (\mu_1 \bullet \mu_2)(s-t).
\end{aligned}
$$

(*ii*) $(\mu_1 \bullet \mu_2)(st) = \sup_{st=\sum_i u_i w_i v_i z_i} [min_i(min(\mu_1(u_i w_i), \mu_2(v_i z_i)))]$.
Also,

$$
\begin{aligned}
(\mu_1 \bullet \mu_2)(s) &= \sup_{s=\sum_i u_i v_i} [min_i(min(\mu_1(u_i), \mu_2(v_i)))] \\
&\leq \sup_{st=\sum_i u_i w_i v_i z_i} [min_i(min(\mu_1(u_i w_i), \mu_2(v_i z_i)))] \\
&= (\mu_1 \bullet \mu_2)(st).
\end{aligned}
$$

The same direct $(\mu_1 \bullet \mu_2)(st) \geq (\mu_1 \bullet \mu_2)(t)$.

(*iii*) $(\gamma_1 \bullet \gamma_2)(s-t) = \inf_{s-t=\sum_i (u_i v_i - w_i z_i)} (max_i(max(\gamma_1(u_i - w_i), \gamma_2(v_i - z_i)))$,
since

$$
\begin{aligned}
max[(\gamma_1 \bullet \gamma_2)(s), (\gamma_1 \bullet \gamma_2)(t)] &= max[\inf_{s=\sum_i u_i v_i} (max_i(\gamma_1(u_i), \gamma_2(v_i))), \inf_{t=\sum_i w_i z_i} (max_i(\gamma_1(w_i), \gamma_2(z_i)))] \\
&= \inf_{\substack{s=\sum_i u_i v_i \\ t=\sum_i w_i z_i}} [max_i(max(\gamma_1(u_i), \gamma_1(w_i), \gamma_2(v_i), \gamma_2(z_i)))] \\
&\geq \inf_{s-t=\sum_i (u_i v_i - w_i z_i)} [max_i(max(\gamma_1(u_i - w_i), \gamma_2(v_i - z_i)))] \\
&= (\gamma_1 \bullet \gamma_2)(s-t).
\end{aligned}
$$

(iv) $(\gamma_1 \bullet \gamma_2)(st) = \inf_{st=\sum_i u_i w_i v_i z_i}[max_i(max(\gamma_1(u_i w_i), \gamma_2(v_i z_i)))]$.

Also,

$$\begin{aligned}(\gamma_1 \bullet \gamma_2)(s) &= \inf_{s=\sum_i u_i v_i}[max_i(max(\gamma_1(u_i), \gamma_2(v_i)))]\\ &\geq \inf_{st=\sum_i u_i w_i v_i z_i}[max_i(max(\gamma_1(u_i w_i), \gamma_2(v_i z_i)))]\\ &= (\gamma_1 \bullet \gamma_2)(st).\end{aligned}$$

The same direct $(\gamma_1 \bullet \gamma_2)(st) \leq (\gamma_1 \bullet \gamma_2)(t)$.

(v) $(\zeta_1 \bullet \zeta_2)(s-t) = \inf_{s-t=\sum_i(u_i v_i - w_i z_i)}(max_i(max(\zeta_1(u_i - w_i), \zeta_2(v_i - z_i)))$,

since

$$\begin{aligned}max[(\zeta_1 \bullet \zeta_2)(s), (\zeta_1 \bullet \zeta_2)(t)] &= max[\inf_{s=\sum_i u_i v_i}(max_i(\zeta_1(u_i), \zeta_2(v_i))), \inf_{t=\sum_i w_i z_i}(max_i(\zeta_1(w_i), \zeta_2(z_i)))]\\ &= \inf_{\substack{s=\sum_i(u_i v_i)\\ t=\sum_i(w_i z_i)}}[max_i(max(\zeta_1(u_i), \zeta_1(w_i), \zeta_2(v_i), \zeta_2(z_i)))]\\ &\geq \inf_{s-t=\sum_i(u_i v_i - w_i z_i)}[max_i(max(\zeta_1(u_i - w_i), \zeta_2(v_i - z_i)))]\\ &= (\zeta_1 \bullet \zeta_2)(s-t).\end{aligned}$$

(vi) $(\zeta_1 \bullet \zeta_2)(st) = \inf_{st=\sum_i u_i w_i v_i z_i}[max_i(max(\zeta_1(u_i w_i), \zeta_2(v_i z_i)))]$.

Also,

$$\begin{aligned}(\zeta_1 \bullet \zeta_2)(s) &= \inf_{s=\sum_i u_i v_i}[max_i(max(\zeta_1(u_i), \zeta_2(v_i)))]\\ &\geq \inf_{st=\sum_i u_i w_i v_i z_i}[max_i(max(\zeta_1(u_i w_i), \zeta_2(v_i z_i)))]\\ &= (\zeta_1 \bullet \zeta_2)(st).\end{aligned}$$

The same direct $(\zeta_1 \bullet \zeta_2)(st) \leq (\zeta_1 \bullet \zeta_2)(t)$.

□

3.3. Level sub-ring (ideal)

Definition 3.15. Presume $\mathcal{S}$ is a neutrosophic sub-ring (ideal) of a ring $\mathcal{R}$ with $0 \leq \alpha \leq \mu(0)$, and $0 \leq \gamma(0), \zeta(0) \leq \alpha$. The sub-ring (ideal) $\mathcal{S}_\alpha$ is called a level sub-ring (level ideal) of $\mathcal{S}$.

Now, we consider the family of level sub-ring (ideal) of a neutrosophic sub-ring (ideal) $\mathcal{S}$ of a ring $\mathcal{R}$ as follows

$$\mathcal{S}_\varrho = \{\mathcal{S}_{\alpha_i} : \alpha_i \in A\},$$

where $A = \{\alpha_0, \alpha_1, \alpha_2, ..., \alpha_n\}$, and when $\alpha_0 > \alpha_1 > \alpha_2 > ... > \alpha_n$. Then, we have the following chain

$$\mathcal{S}_{\alpha_0} \subset \mathcal{S}_{\alpha_1} \subset \mathcal{S}_{\alpha_2} \subset ... \subset \mathcal{S}_{\alpha_n} = \mathcal{R}.$$

This result as parallel to the corresponding results on neutrosophic sub-groups [9].

Theorem 3.16. *Let $\mathcal{S}$ be a neutrosophic sub-ring (ideal) of $\mathcal{R}$ if and only if the level subsets $\mathcal{S}_\varrho$ are sub-rings (ideals) of $\mathcal{R}$.*

Proof. Obviously, $\mathcal{S}_\varrho$ is nonempty. Assume that $s, t \in \mathcal{S}_\varrho$. Then, we have $\alpha_i \leqslant \mu(0)$, $\alpha_i \geqslant \gamma(0)$ and $\alpha_i \geqslant \zeta(0)$. Since $\mathcal{S}$ is a neutrosophic sub-ring (ideal) of $\mathcal{R}$ by Definition 3.1 (3.7) we have $\alpha_i \leqslant \mu(s-t)$, $\alpha_i \geqslant \gamma(s-t)$ and $\alpha_i \geqslant \zeta(s-t)$. Thus, $< s-t, \mu(s-t), \gamma(s-t), \zeta(s-t) > \in \mathcal{S}_\varrho$. Again, we have $< st, \mu(st), \gamma(st), \zeta(st) > \in \mathcal{S}_\varrho$. The other direction is the same routine. □

Proposition 3.17. *Presume $\mathcal{I}$ is a neutrosophic left (right) ideal of $\mathcal{R}$. When $0 \leqslant \alpha \leqslant \mu(0)$ and $0 \leqslant \gamma(0), \zeta(0) \leqslant \alpha$, then $\mathcal{I}_\alpha = \{< s, \mu_\alpha(s), \gamma_\alpha(s), \zeta_\alpha(s) >: s \in \mathbb{R}\}$ is a neutrosophic left (right) ideal of $\mathcal{R}$.*

Proof. Since $0 \leqslant \alpha \leqslant \mu(0)$ and $0 \leqslant \gamma(0), \zeta(0) \leqslant \alpha$, then $\mathcal{I}_\alpha \neq \phi$. Suppose that $s, t \in \mathcal{I}_\alpha$ and $r \in \mathcal{R}$. Thus,

(*i*) $\mu(s-t) \geqslant min(\mu(s), \mu(t)) \geqslant \alpha$, so $s - t \in \mu_\alpha$.
(*ii*) $\mu(rs) \geqslant \mu(s) \geqslant \alpha$, so $rs \in \mu_\alpha$, and $\mu(sr) \geqslant \mu(s) \geqslant \alpha$, so $sr \in \mu_\alpha$.
(*iii*) $\gamma(s-t) \leqslant max(\gamma(s), \gamma(t)) \leqslant \alpha$, so $s - t \in \gamma_\alpha$.
(*iv*) $\gamma(rs) \leqslant \gamma(s) \leqslant \alpha$, so $rs \in \gamma_\alpha$, and $\gamma(sr) \leqslant \gamma(s) \leqslant \alpha$, so $sr \in \gamma_\alpha$.
(*v*) $\zeta(s-t) \leqslant max(\zeta(s), \zeta(t)) \leqslant \alpha$, so $s - t \in \zeta_\alpha$.
(*vi*) $\zeta(rs) \leqslant \zeta(s) \leqslant \alpha$, so $rs \in \zeta_\alpha$, and $\zeta(sr) \leqslant \zeta(s) \leqslant \alpha$, so $sr \in \zeta_\alpha$.

Therefore, $\mathcal{I}_\alpha$ is a neutrosophic left (right) ideal of $\mathcal{R}$. □

Proposition 3.18. *Presume $\mathcal{I}$ is a neutrosophic subset of $\mathcal{R}$. For any $\alpha \in A$, if $\mathcal{I}_\alpha$ is a neutrosophic left (right) ideal, then $\mathcal{I}$ is also a neutrosophic left (right) ideal.*

Proof. Suppose that $\mathcal{I}_\alpha$ is a neutrosophic left ideal for any $\alpha \in A$. Then, $0 \in \mathcal{I}_\alpha \forall \alpha \in A$. Thus, $\mu(0) \geq \alpha$ and $\gamma(0), \zeta(0) \leq \alpha$. Now, assume that $s, t \in \mathbb{R}$ with $\mathcal{I}(s) = \alpha_1$ and $\mathcal{I}(t) = \alpha_2$ for any $\alpha_1, \alpha_2 \in A$ with $\alpha_1 \geq \alpha_2$. Thus, $s, t, s-t, st \in \mathcal{I}_\alpha$. Since $\mathcal{I}_\alpha$ is a neutrosophic ideal we have the following:

$$\mu(s-t) \geq \alpha_2 = min(\mu(t), \mu(s)), \quad \mu(st) \geq s = \mu(\alpha_1).$$

$$\gamma(s-t) \leq \alpha_1 = max(\gamma(t), \gamma(s)), \quad \gamma(st) \leq t = \gamma(\alpha_2).$$

$$\zeta(s-t) \leq \alpha_1 = max(\zeta(t), \zeta(s)), \quad \zeta(st) \leq t = \zeta(\alpha_2).$$

Therefore, $\mathcal{I}$ is a neutrosophic left ideal and similarly $\mathcal{I}$ is a neutrosophic right ideal. □

3.4. An application of neutrosophic ideal in decision making

The neutrosophic ideal algebraic structure can be used to model complex decision making processes with involving uncertain, incomplete, or inconsistent information. It provides a flexible framework for representing and analyzing decision making problems in a variety of real-world contexts.
Now, we give an example of using the algebraic structure of the neutrosophic ideal to make a decision based on a set of data presented in a neutrosophic form:
Suppose we have the following data (as shown in Table 1) representing the satisfaction level of customers in a restaurant:

Table 1. Data on the satisfaction level of customers in a restaurant.

customers	Food quality	Service quality	Ambience quality
customer 1	$< 0.7, 0.5, 0.4 >$	$< 0.4, 0.6, 0.7 >$	$< 0.5, 0.7, 0.6 >$
customer 2	$< 0.5, 0.7, 0.6 >$	$< 0.7, 0.5, 0.4 >$	$< 0.4, 0.6, 0.7 >$
customer 3	$< 0.4, 0.6, 0.7 >$	$< 0.5, 0.7, 0.6 >$	$< 0.7, 0.5, 0.4 >$

Here, each entry represents a neutrosophic set with three values indicating, respectively, the degree of truth, indeterminacy, and falsity. For example, the first entry $< 0.7, 0.5, 0.4 >$ means that the food quality is 70% true, 50% indeterminate, and 40% false.

To make a decision about the overall satisfaction level of customers at the restaurant, we can use the algebraic structure of the neutrosophic ideal. Also, it is easy to show that the set of data in Table 1 represents a neutrosophic ideal. Then, we use the neutrosophic ideal to select the most satisfactory option. To do this, we find the ideal elements in each set by using the axioms of a neutrosophic ideal. This gives us the following Table 2.

Table 2. The ideal elements.

	Food quality	Service quality	Ambience quality
ideal elements	$< 0.4, 0.7, 0.7 >$	$< 0.4, 0.7, 0.7 >$	$< 0.4, 0.7, 0.7 >$

Next, we take the intersection of these ideal elements, which gives us the following neutrosophic set:

$$< 0.4, 0.7, 0.7 > .$$

This set represents the ideal combination of food quality, service quality, and ambiance quality that maximizes customer satisfaction. We can then use this set to make a decision about how to improve the restaurant's performance. For example, we might focus on improving the quality of the food to achieve a higher level of overall customer satisfaction.

4. Conclusions

In this work, we presented a study of neutrosophic sub-ring and ideals, which are mathematical structures that add indeterminacy and ambiguity to the classical concepts of sub-ring and ideals. We introduced the notion of a neutrosophic sub-ring and showed that it has several interesting properties. We have also introduced the concept of a neutrosophic ideal and established some basic results.

In the next work, we will shown the applicability of neutrosophic sub-ring and ideals in several areas, such as computer science, image processing, and control theory. Our results provide a new perspective on sub-ring and ideals in neutrosophic algebra and pave the way for further research in this area. In particular, the study of neutrosophic sub-ring and ideals opens new avenues for the study of the algebraic structures of neutrosophic rings and their applications in real-world problems. Furthermore, as a possible area for future research, the authors could investigate neutro rings, which are part of the neutro algebra structures that have partially true axioms. This could involve exploring the properties and behavior of neutro rings, as well as their potential applications in areas such as computer science, cryptography and coding theory. Further study of neutro rings could contribute to the development of new mathematical tools and techniques for dealing with uncertainty and incomplete information.

Use of AI tools declaration

We declare that we have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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