Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 21270–21297.
DOI: 10.3934/math.20231084
Received: 07 April 2023
Revised: 09 June 2023
Accepted: 25 June 2023
Published: 03 July 2023

Research article

A posteriori error estimates of mixed discontinuous Galerkin method for a class of Stokes eigenvalue problems

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Abstract: For a class of Stokes eigenvalue problems including the classical Stokes eigenvalue problem and the magnetohydrodynamic Stokes eigenvalue problem a residual type a posteriori error estimate of the mixed discontinuous Galerkin finite element method using $\mathbb{P}_k - \mathbb{P}_{k-1}$ element ($k \geq 1$) is studied in this paper. The a posteriori error estimators for approximate eigenpairs are given. The reliability and efficiency of the posteriori error estimator for the eigenfunction are proved and the reliability of the estimator for the eigenvalue is also analyzed. The numerical results are provided to confirm the theoretical predictions and indicate that the method considered in this paper can reach the optimal convergence order $O(dof^{\frac{-2k}{d}})$.

Keywords: the classical Stokes eigenvalue problem; the magnetohydrodynamic Stokes eigenvalue problem; discontinuous Galerkin method; a posteriori error estimates of residual type; adaptive algorithm
Mathematics Subject Classification: 65N25, 65N30

1. Introduction

Stokes eigenvalue problem is of great significance because of its role in the stability analysis of fluid mechanics. Therefore, it is of great interest to study efficient numerical methods for solving this problem (e.g. see [1–8]).

The a posteriori error estimates of the classical Stokes eigenvalue problem based on the velocity-pressure formulation received much attention from scholars. For example, the authors in [9–11] studied the low-order conforming mixed method and Jia et al. [12] and Sun et al. [6] explored the low-order non-conforming finite elements. Since the accuracy of low-order elements is not high Gedicke et al. [1] used the Arnold-Winther hybrid finite element method to analyze the a posteriori error estimation based on the stress-velocity formulation in $\mathbb{R}^2$ and Gedicke et al. [2] adopted the divergence-conforming discontinuous Galerkin finite element method (DGFEM for short) to discuss the a posteriori error estimate for velocity-pressure formulation on shape-regular rectangular meshes. Lepe et al. [13] proposed a mixed numerical method to study the error estimates for a vorticity-based velocity-stress formulation of the Stokes eigenvalue problem.

In this paper, for a class of Stokes eigenvalue problems (see (2.1)), including the classical Stokes eigenvalue problem in $\mathbb{R}^d$ ($d = 2, 3$) and the magnetohydrodynamic (MHD) Stokes eigenvalue problem et al. based on the velocity-pressure formulation we study the residual type a posteriori error estimates of the mixed DGFEM using $\mathbb{P}_k - \mathbb{P}_{k-1}$ ($k \geq 1$) element on shape-regular simplex meshes. For the Stokes equations the DGFEM was researched by [14–19] which laid a foundation for us to study further the Stokes eigenvalue problem. Among them, Badia et al. [14] proved the well-posedness of discrete DG formulation and studied the a priori error estimate of DGFEM with $\mathbb{P}_k - \mathbb{P}_{k-1}$ element without using the discrete inf-sup condition. Our main work is as follows:

(1) We present the a posteriori error estimators for approximate eigenpairs. Referring to [20, 21], using the enriching operator (see [22,23]) and the lifting operator (see [24,25]) we prove the reliability and efficiency of the estimator for eigenfunctions. We establish an identity (see Lemma 3.8) and use it to analyze the reliability of the estimator for eigenvalues. The characteristic of the adaptive DGFEM discussed in this paper is that for the Stokes eigenvalue problem in two and three-dimensional domains due to the usage of high-order elements it can capture smooth solutions and achieve the optimal convergence order for local less smooth solutions (eigenfunctions that have local singularity or local low smoothness) on adaptive locally refined graded meshes.

(2) We implement adaptive computing and the numerical results confirm our theoretical predictions and show that our method is stable, efficient and can obtain high-accuracy approximate eigenvalues. In the existing literature on the classical Stokes eigenvalue problem, Gedicke et al. [1, 2] presented the approximate eigenvalues of 11, 10 and 9 significant digits in the unit square, the L-shaped domain and the slit domain, respectively, which are the most accurate approximations in the existing literature. In this paper, we obtain approximate eigenvalues that have the same accuracy as those in [1, 2].

Note that C in different positions in this article represents different positive constants which is independent of mesh size h. We use $a \lesssim b$ to represent $a \leq Cb$ and $a \simeq b$ to represent $a \lesssim b$ and $b \lesssim a$.

2. Preliminaries

Consider the following class of Stokes eigenvalue problems:

$$\begin{cases} -\mu\Delta\mathbf{u} + A\mathbf{u} + \nabla p = \lambda\mathbf{u}, & in\ \ \Omega, \\ div\mathbf{u} = 0, & in\ \ \Omega, \\ \mathbf{u} = 0, & on\ \ \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is a bounded polyhedral domain, $\mathbf{u} = (u_1, ..., u_d)^T$ is the velocity of the flow, p is the pressure, $\mu > 0$ is the kinematic viscosity parameter of the fluid, λ is the eigenvalue of the problem (2.1) and A is a $d \times d$ symmetric semi-definite matrix whose elements belong to $L^\infty(\Omega)$.

The problem (2.1) includes the classical Stokes eigenvalue problem in R^d ($d = 2, 3$) and the MHD Stokes eigenvalue problem et al. When A is a zero matrix (2.1) is the classical Stokes eigenvalue problem. In the case of $d = 2$ when

$$A=(Ha)^2\begin{pmatrix} H_0^2 & 0 \\ 0 & 0 \end{pmatrix},$$

the problem (2.1) is the MHD Stokes eigenvalue problem (see [7, 8]) where H_0 is the intensity of the externally applied magnetic field on the vertical direction, i.e., magnetic field $\mathbf{H} = (0, H_0, 0)$, and Ha is the Hartmann number (see [7, 8]) and when

$$A=(Ha)^2\begin{pmatrix} 0 & 0 \\ 0 & H_0^2 \end{pmatrix},$$

the problem (2.1) is the MHD Stokes eigenvalue problem while the magnetic field is applied horizontally.

For the sake of narrative simplicity we take $\mu = 1$ in this paper.

Let $H^\rho(\Omega)$ be the Sobolev space on Ω of order $\rho \geq 0$ equipped with the norm $\|\cdot\|_{\rho,\Omega}$ (denoted by $\|\cdot\|_\rho$ for simplicity). $H_0^1(\Omega) = \{z \in H^1(\Omega), z|_{\partial\Omega} = 0\}$. We denote $\|\mathbf{z}\|_\rho = \sum\limits_{i=1}^{d} \|z_i\|_\rho$ for $\mathbf{z} = (z_1, \cdots, z_d) \in H^\rho(\Omega)^d$. We denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)^d$ which is given by $(u, z) = \int_\Omega uzdx$ $(d = 1)$ and $(\mathbf{u}, \mathbf{z}) = \int_\Omega \mathbf{u}\cdot\mathbf{z}dx$ $(d = 2, 3)$. Define $\mathbb{X} = H_0^1(\Omega)^d$ with the norm $\|\mathbf{z}\|_{\mathbb{X}} = (\nabla\mathbf{z}, \nabla\mathbf{z})^{\frac{1}{2}}$ and define $\mathbb{W} = L_0^2(\Omega) = \{\varrho \in L^2(\Omega) : (\varrho, 1) = 0\}$.

The weak formulation of (2.1) is given by: Find $(\lambda, \mathbf{u}, p) \in \mathbb{R}\times\mathbb{X}\times\mathbb{W}$, $\|\mathbf{u}\|_0 = 1$ such that

$$\mathbb{A}(\mathbf{u}, \mathbf{z}) + \mathbb{B}(\mathbf{z}, p) = \lambda(\mathbf{u}, \mathbf{z}), \quad \forall \mathbf{z} \in \mathbb{X}, \tag{2.2}$$

$$\mathbb{B}(\mathbf{u}, \varrho) = 0, \quad \forall \varrho \in \mathbb{W}, \tag{2.3}$$

where

$$\mathbb{A}(\mathbf{u}, \mathbf{z}) = (\nabla\mathbf{u}, \nabla\mathbf{z}) + (A\mathbf{u}, \mathbf{z}),$$
$$\mathbb{B}(\mathbf{z}, \varrho) = -(div\mathbf{z}, \varrho).$$

Let $\mathcal{T}_h = \{\tau\}$ be a regular simplex partition of Ω with the mesh diameter $h = \max\limits_{\tau\in\mathcal{T}_h} h_\tau$ where h_τ is the diameter of element τ. We use $\mathcal{E}_h^i$ and $\mathcal{E}_h^b$ to denote the set of interior faces (edges) and the set of faces (edges) on $\partial\Omega$, respectively. $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$. We use h_F to denote the measure of $F \in \mathcal{E}_h$. We denote by $(\cdot,\cdot)_\tau$ and $(\cdot,\cdot)_F$ the inner product in $L^2(\tau)$ and $L^2(F)$, respectively. We use $\omega(\tau)$ to represent the union of all elements which share at least one edge (face) with τ and use $\omega(F)$ to represent the union of the elements having in common with F.

The broken Sobolev space is defined by

$$H^1(\Omega, \mathcal{T}_h) = \{z \in L^2(\Omega) : z|_\tau \in H^1(\tau),\ \forall\tau \in \mathcal{T}_h\}.$$

For any $F \in \mathcal{E}_h^i$, there are two simplices τ^+ and τ^- such that $F = \tau^+ \cap \tau^-$ (e.g. see [14]). Let $\mathbf{n}^+$ be the unit normal of F pointing from τ^+ to τ^- and let $\mathbf{n}^- = -\mathbf{n}^+$.

For any $\varphi \in H^1(\Omega, \mathcal{T}_h)$ we denote its jump and mean on $F \in \mathcal{E}_h^i$ by $[\![\varphi]\!] = \varphi^+\mathbf{n}^+ + \varphi^-\mathbf{n}^-$ and $\{\varphi\} = \frac{1}{2}(\varphi^+ + \varphi^-)$, respectively, where $\varphi^\pm = \varphi|_{\tau^\pm}$. For $\boldsymbol{\varphi} \in H^1(\Omega, \mathcal{T}_h)^d$ we denote by $[\![\boldsymbol{\varphi}]\!] = \boldsymbol{\varphi}^+\cdot\mathbf{n}^+ + \boldsymbol{\varphi}^-\cdot\mathbf{n}^-$

the jump and $\{\boldsymbol{\varphi}\} = \frac{1}{2}(\boldsymbol{\varphi}^+ + \boldsymbol{\varphi}^-)$ the mean of $\boldsymbol{\varphi}$ on $F \in \mathcal{E}_h^i$. We also denote by $[\![\underline{\boldsymbol{\varphi}}]\!] = \boldsymbol{\varphi}^+ \otimes \mathbf{n}^+ + \boldsymbol{\varphi}^- \otimes \mathbf{n}^-$ the full jump of $\boldsymbol{\varphi}$ on $F \in \mathcal{E}_h^i$, where $\boldsymbol{\varphi} \otimes \mathbf{n} = [\varphi_i n_j]_{1 \le i,j \le d}$, $\boldsymbol{\varphi} = (\varphi_i)$, $\mathbf{n} = (n_j)$. For tensors $\boldsymbol{\chi} \in H^1(\Omega, \mathcal{T}_h)^{d \times d}$ we denote by $[\![\boldsymbol{\chi}]\!] = \boldsymbol{\chi}^+ \mathbf{n}^+ + \boldsymbol{\chi}^- \mathbf{n}^-$ the jump and $\{\boldsymbol{\chi}\} = \frac{1}{2}(\boldsymbol{\chi}^+ + \boldsymbol{\chi}^-)$ the mean on $F \in \mathcal{E}_h^i$.

For the sake of simplicity, when $F \in \mathcal{E}_h^b$ by modifying the above definitions appropriately, we obtain the jump and the mean on $\partial\Omega$. That is to say, we modify $\varphi^- = 0$ (similarly, $\boldsymbol{\varphi}^- = 0$ and $\boldsymbol{\chi}^- = 0$) to obtain the definition of jump on $\partial\Omega$ and modify $\varphi^- = \varphi^+$(similarly, $\boldsymbol{\varphi}^- = \boldsymbol{\varphi}^+$ and $\boldsymbol{\chi}^- = \boldsymbol{\chi}^+$) to obtain the definition of mean on $\partial\Omega$.

The discrete velocity and pressure spaces are defined as follows (see [7]):

$$\mathbb{X}_h = \{\mathbf{z}_h \in L^2(\Omega)^d : \mathbf{z}_h|_\tau \in \mathbb{P}_k(\tau)^d, \ \forall \tau \in \mathcal{T}_h\},$$
$$\mathbb{W}_h = \{\varrho_h \in \mathbb{W} : \varrho_h|_\tau \in \mathbb{P}_{k-1}(\tau), \ \forall \tau \in \mathcal{T}_h\},$$

where $\mathbb{P}_k(\tau)$ is the space of polynomials of degree less than or equal to $k \ge 1$ on τ.

The DGFEM for the problem (2.1) is to find $(\lambda_h, \mathbf{u}_h, p_h) \in \mathbb{R} \times \mathbb{X}_h \times \mathbb{W}_h$, $\|\mathbf{u}_h\|_0 = 1$ such that

$$\mathbb{A}_h(\mathbf{u}_h, \mathbf{z}_h) + \mathbb{B}_h(\mathbf{z}_h, p_h) = \lambda_h(\mathbf{u}_h, \mathbf{z}_h), \ \ \forall \mathbf{z}_h \in \mathbb{X}_h, \tag{2.4}$$
$$\mathbb{B}_h(\mathbf{u}_h, \varrho_h) = 0, \quad \forall \varrho_h \in \mathbb{W}_h, \tag{2.5}$$

where

$$\begin{aligned}\mathbb{A}_h(\mathbf{u}_h, \mathbf{z}_h) = &\sum_{\tau \in \mathcal{T}_h} \int_\tau \nabla \mathbf{u}_h : \nabla \mathbf{z}_h dx + \sum_{\tau \in \mathcal{T}_h} \int_\tau A\mathbf{u}_h \cdot \mathbf{z}_h dx - \sum_{F \in \mathcal{E}_h} \int_F \{\nabla \mathbf{u}_h\} : [\![\underline{\mathbf{z}_h}]\!] ds \\ &- \sum_{F \in \mathcal{E}_h} \int_F \{\nabla \mathbf{z}_h\} : [\![\underline{\mathbf{u}_h}]\!] ds + \sum_{F \in \mathcal{E}_h} \int_F \frac{\gamma}{h_F} [\![\underline{\mathbf{u}_h}]\!] : [\![\underline{\mathbf{z}_h}]\!] ds, \end{aligned} \tag{2.6}$$

$$\mathbb{B}_h(\mathbf{z}_h, \varrho_h) = -\sum_{\tau \in \mathcal{T}_h} \int_\tau \varrho_h div \mathbf{z}_h dx + \sum_{F \in \mathcal{E}_h} \int_F \{\varrho_h\} [\![\mathbf{z}_h]\!] ds. \tag{2.7}$$

Here, γ/h_F is the interior penalty parameter. We select $\gamma = C_1 k^2$ with $C_1 = 10$ in the actual numerical implementations from Remark 2.1 in [26].

Define the DG-norm as follows:

$$\|\mathbf{z}_h\|_h^2 = \sum_{\tau \in \mathcal{T}_h} \|\mathbf{z}_h\|_{1,\tau}^2 + \sum_{F \in \mathcal{E}_h} \int_F \frac{\gamma}{h_F} [\![\underline{\mathbf{z}_h}]\!]^2 ds, \ \ on \ \ \mathbb{X}_h + \mathbb{X}; \tag{2.8}$$

$$|||\mathbf{z_h}|||^2 = \|\mathbf{z}_h\|_h^2 + \sum_{F \in \mathcal{E}_h} \int_F \frac{h_F}{\gamma} |\nabla \mathbf{z}_h|^2 ds, \ \ on \ \ \mathbb{X}_h + H^{1+s}(\Omega)^d \ (s > \frac{1}{2}). \tag{2.9}$$

Note that $\|\cdot\|_h$ is equivalent to $|||\cdot|||$ on $\mathbb{X}_h$.

From [27] we know that the continuity and coercivity properties hold:

$$|\mathbb{A}_h(\mathbf{u}_h, \mathbf{z}_h)| \lesssim |||\mathbf{u}_h||| \ |||\mathbf{z}_h|||, \quad \forall \ \mathbf{u}_h, \mathbf{z}_h \in \mathbb{X}_h + H^{1+s}(\Omega)^d \ (s > \frac{1}{2}),$$
$$\|\mathbf{u}_h\|_h^2 \lesssim \mathbb{A}_h(\mathbf{u}_h, \mathbf{u}_h), \quad \forall \mathbf{u}_h \in \mathbb{X}_h.$$

We consider the boundary problem : Given $\boldsymbol{g} \in (L^2(\Omega))^d$,

$$\begin{cases} -\Delta \mathbf{u}^g + A\mathbf{u}^g + \nabla p^g = \boldsymbol{g}, \quad in \ \ \Omega, \\ div \mathbf{u}^g = 0, \quad in \ \ \Omega, \\ \mathbf{u}^g = 0, \quad on \ \ \partial\Omega. \end{cases} \tag{2.10}$$

From the Lax-Milgram theorem we have the existence and uniqueness of the velocity $\mathbf{u}$ in the space $\mathbf{Z} = \{\mathbf{z} \in \mathbb{X} : b(\mathbf{z}, \varrho) = 0, \forall \varrho \in \mathbb{W}\}$. From the well-known inf-sup condition (see [28]):

$$\beta\|\varrho\|_{L^2(\Omega)} \leq \sup_{\mathbf{z}\in\mathbb{X},\mathbf{z}\neq 0} \frac{\mathbb{B}(\mathbf{z}, \varrho)}{\|\mathbf{z}\|_{\mathbb{X}}},\ \forall\ \varrho \in \mathbb{W},$$

the stability of the pressure holds.

The weak formulation of (2.10) reads: Find $(\mathbf{u}^g, p^g) \in \mathbb{X} \times \mathbb{W}$ such that

$$\mathbb{A}(\mathbf{u}^g, \mathbf{z}) + \mathbb{B}(\mathbf{z}, p^g) = (\boldsymbol{g}, \mathbf{z}), \quad \forall \mathbf{z} \in \mathbb{X}, \tag{2.11}$$

$$\mathbb{B}(\mathbf{u}^g, \varrho) = 0, \quad \forall \varrho \in \mathbb{W}, \tag{2.12}$$

and its DGFEM form reads: Find $(\mathbf{u}_h^g, p_h^g) \in \mathbb{X}_h \times \mathbb{W}_h$ such that

$$\mathbb{A}_h(\mathbf{u}_h^g, \mathbf{z}_h) + \mathbb{B}_h(\mathbf{z}_h, p_h^g) = (\boldsymbol{g}, \mathbf{z}_h), \quad \forall \mathbf{z}_h \in \mathbb{X}_h, \tag{2.13}$$

$$\mathbb{B}_h(\mathbf{u}_h^g, \varrho_h) = 0, \quad \forall \varrho_h \in \mathbb{W}_h. \tag{2.14}$$

We assume that the following regularity is valid: For any $\boldsymbol{g} \in (L^2(\Omega))^d$ $(d = 2, 3)$ there exists $(\mathbf{u}^g, p^g) \in (H^{1+r}(\Omega)^d \times H^r(\Omega)) \cap (W^{2,p}(\Omega)^d \times W^{1,p}(\Omega))$ $(\frac{1}{2} < r \leq 1,\ p > \frac{2d}{d+1})$ satisfying (2.10) and

$$\|\mathbf{u}^g\|_{1+r} + \|p^g\|_r \leq C\|\boldsymbol{g}\|_0, \tag{2.15}$$

where C is a positive constant independent of $\boldsymbol{g}$.

From Lemma 6.5 in [27] we can obtain the consistency of the DGFEM that is to say when $(\mathbf{u}^g, p^g)$ is the solution of the boundary problem (2.10), there hold the following equations:

$$\mathbb{A}_h(\mathbf{u}^g, \mathbf{z}_h) + \mathbb{B}_h(\mathbf{z}_h, p^g) = (\boldsymbol{g}, \mathbf{z}_h), \quad \forall \mathbf{z}_h \in \mathbb{X}_h, \tag{2.16}$$

$$\mathbb{B}_h(\mathbf{u}^g, \varrho_h) = 0, \quad \forall \varrho_h \in \mathbb{W}_h. \tag{2.17}$$

From (2.13), (2.14), (2.16) and (2.17) we have

$$\mathbb{A}_h(\mathbf{u}^g - \mathbf{u}_h^g, \mathbf{z}_h) + \mathbb{B}_h(\mathbf{z}_h, p^g - p_h^g) = 0, \quad \forall\ \mathbf{z}_h \in \mathbb{X}_h, \tag{2.18}$$

$$\mathbb{B}_h(\mathbf{u}^g - \mathbf{u}_h^g, \varrho_h) = 0, \quad \forall\ \varrho_h \in \mathbb{W}_h. \tag{2.19}$$

Badia et al. [14], Cockburn et al. [29], Hansbo et al. [30] and Schötzau et al. [25] proved that (2.13) and (2.14) are well defined and gave the a priori error estimate. From [14] we obtain the following lemma.

Lemma 2.1. Assume that $(\mathbf{u}^g, p^g) \in H^{1+s}(\Omega)^d \times H^s(\Omega)$ $(r \leq s \leq k)$ with $\boldsymbol{g} \in H^l(\Omega)^d$ $(0 \leq l \leq k+1)$. Then,

$$\|\mathbf{u}^g - \mathbf{u}_h^g\|_h + \|p^g - p_h^g\|_0 \lesssim h^s(\|\mathbf{u}^g\|_{1+s} + \|p^g\|_s) + h^{1+l}\|\boldsymbol{g}\|_l. \tag{2.20}$$

Proof. Since the bilinear form $\mathbb{A}(\cdot, \cdot)$ is coercive on $\mathbb{X}$, $\mathbb{A}_h(\cdot, \cdot)$ is also coercive on $\mathbb{X}_h$. Using the proof method of Theorem 4.1 in [14] we can obtain (2.20).

Let $I_h : \mathbb{X} \cap C^0(\overline{\Omega})^d \to \mathbb{X}_h \cap \mathbb{X}$ be the conforming element interpolation operator and let $\vartheta_h : H^s(\Omega) \to \mathbb{W}_h$ be the local L^2 projection operator satisfying $\vartheta_h p|_\tau \in \mathbb{P}_{k-1}(\tau)$ and

$$\int_\tau (p - \vartheta_h p) z dx = 0, \quad \forall z \in \mathbb{P}_{k-1}(\tau), \quad \forall \tau \in \mathcal{T}_h.$$

We introduce the following auxiliary problem before estimating the error of velocity in the sense of L^2 norm:

$$\mathbb{A}(\boldsymbol{\omega}, \mathbf{z}) + \mathbb{B}(\mathbf{z}, \varrho) = (\mathbf{u}^g - \mathbf{u}_h^g, \mathbf{z}), \quad \forall \mathbf{z} \in \mathbb{X}, \tag{2.21}$$

$$\mathbb{B}(\boldsymbol{\omega}, v) = 0, \quad \forall v \in \mathbb{W}. \tag{2.22}$$

Using (2.15), we have

$$\|\boldsymbol{\omega}\|_{1+r} + \|\varrho\|_r \lesssim \|\mathbf{u}^g - \mathbf{u}_h^g\|_0. \tag{2.23}$$

From Theorem 6.12 in [27] using the Nitsche's technique can prove the following lemma.

Lemma 2.2. Suppose that the conditions of Lemma 2.1 and (2.15) hold. Then,

$$\|\mathbf{u}^g - \mathbf{u}_h^g\|_0 \lesssim h^r(|||\mathbf{u}^g - \mathbf{u}_h^g||| + \|p^g - p_h^g\|_0). \tag{2.24}$$

Proof. From (2.18) and (2.19) we can derive

$$\begin{aligned}
\|\mathbf{u}^g - \mathbf{u}_h^g\|_0^2 &= \mathbb{A}_h(\boldsymbol{\omega}, \mathbf{u}^g - \mathbf{u}_h^g) + \mathbb{B}_h(\mathbf{u}^g - \mathbf{u}_h^g, \varrho)\\
&= \mathbb{A}_h(\mathbf{u}^g - \mathbf{u}_h^g, \boldsymbol{\omega} - I_h\boldsymbol{\omega}) - \mathbb{A}_h(\mathbf{u}^g - \mathbf{u}_h^g, I_h\boldsymbol{\omega}) + \mathbb{B}_h(\mathbf{u}^g - \mathbf{u}_h^g, \varrho - \vartheta_h\varrho) + \mathbb{B}_h(\mathbf{u}^g - \mathbf{u}_h^g, \vartheta_h\varrho)\\
&= \mathbb{A}_h(\mathbf{u}^g - \mathbf{u}_h^g, \boldsymbol{\omega} - I_h\boldsymbol{\omega}) - \mathbb{B}_h(I_h\boldsymbol{\omega}, p^g - p_h^g) + \mathbb{B}_h(\mathbf{u}^g - \mathbf{u}_h^g, \varrho - \vartheta_h\varrho)\\
&\equiv E_1 + E_2 + E_3.
\end{aligned} \tag{2.25}$$

Using the continuity of $\mathbb{A}_h(\cdot,\cdot)$ and the approximation property of $I_h\boldsymbol{\omega}$ we obtain

$$|E_1| \lesssim |||\mathbf{u}^g - \mathbf{u}_h^g|||\ |||\boldsymbol{\omega} - I_h\boldsymbol{\omega}||| \lesssim h^r|||\mathbf{u}^g - \mathbf{u}_h^g|||\ \|\boldsymbol{\omega}\|_{1+r}.$$

Since $div\boldsymbol{\omega} = 0$, $\boldsymbol{\omega} \in [H_0^1(\Omega)]^d$, $[\boldsymbol{\omega}] = 0$ and $[I_h\boldsymbol{\omega}] = 0, \forall F \in \mathcal{E}_h$, we have

$$\begin{aligned}
|E_2| &= | - \mathbb{B}_h(I_h\boldsymbol{\omega}, p^g - p_h^g)| = |\mathbb{B}_h(\boldsymbol{\omega} - I_h\boldsymbol{\omega}, p^g - p_h^g)|\\
&= | - (p^g - p_h^g, div(\boldsymbol{\omega} - I_h\boldsymbol{\omega})) + \sum_{F\in\mathcal{E}_h}\int_F \{p^g - p_h^g\}[\![\boldsymbol{\omega} - I_h\boldsymbol{\omega}]\!]ds|\\
&\lesssim h^r\|p^g - p_h^g\|_0\|\boldsymbol{\omega}\|_{1+r}.
\end{aligned}$$

Using the approximation property of $\vartheta_h\varrho$ we can get

$$\begin{aligned}
|E_3| &\le \|\varrho - \vartheta_h\varrho\|_0\|div(\mathbf{u}^g - \mathbf{u}_h^g)\|_0 + \left(\sum_{F\in\mathcal{E}_h}\int_F \frac{h_F}{\gamma}|\{\varrho - \vartheta_h\varrho\}|^2 ds\right)^{\frac{1}{2}} \|\mathbf{u}^g - \mathbf{u}_h^g\|_h\\
&\lesssim h^r\|\mathbf{u}^g - \mathbf{u}_h^g\|_h\|\varrho\|_r.
\end{aligned}$$

Substituting E_1, E_2, E_3 into (2.25) and using (2.23) we obtain (2.24). The proof is completed.

From the inf-sup condition and [14] we know that (2.11) and (2.12) are uniquely solvable and stable. Then, we define

$$\begin{aligned}
\mathbb{T} : L^2(\Omega)^d \to \mathbb{X}, \qquad \mathbb{T}\boldsymbol{g} = \mathbf{u}^g,\\
\mathbb{S} : L^2(\Omega)^d \to \mathbb{W}, \qquad \mathbb{S}\boldsymbol{g} = p^g,
\end{aligned}$$

and it is valid that

$$\|\mathbb{T}\boldsymbol{g}\|_1 + \|\mathbb{S}\boldsymbol{g}\|_0 \lesssim \|\boldsymbol{g}\|_0. \tag{2.26}$$

From [14] we also know that (2.13) and (2.14) are uniquely solvable and stable and we define

$$\mathbb{T}_h : L^2(\Omega)^d \to \mathbb{X}_h, \quad \mathbb{T}_h\boldsymbol{g} = \mathbf{u}_h^g,$$
$$\mathbb{S}_h : L^2(\Omega)^d \to \mathbb{W}_h, \quad \mathbb{S}_h\boldsymbol{g} = p_h^g.$$

Hence,

$$\|\mathbb{T}_h\boldsymbol{g}\|_h + \|\mathbb{S}_h\boldsymbol{g}\|_0 \lesssim \|\boldsymbol{g}\|_0. \tag{2.27}$$

Thus, (2.2), (2.3) and (2.4), (2.5) have the following equivalent operator forms, respectively:

$$\lambda\mathbb{T}\mathbf{u} = \mathbf{u}, \qquad \mathbb{S}(\lambda\mathbf{u}) = p, \tag{2.28}$$
$$\lambda_h\mathbb{T}_h\mathbf{u}_h = \mathbf{u}_h, \quad \mathbb{S}_h(\lambda_h\mathbf{u}_h) = p_h. \tag{2.29}$$

Next, we will derive the error estimates for the eigenvalue problem.

From (2.24), (2.20) and (2.15) we have

$$\|\mathbb{T}_h - \mathbb{T}\|_0 \to 0, \ \ (h \to 0). \tag{2.30}$$

Thus, we can obtain the following Lemma 2.3 (see Lemma 2.3 in [31]) from the Babuška-Osborn spectral approximation theory [32, 33].

From (2.8) and (2.9) we know that $|||\cdot|||$ is a norm stronger than $\|\cdot\|_h$, i.e., $\|z\|_h \lesssim |||z|||$. Additionally, we have

$$|||\mathbf{u} - \mathbf{u}_h|||^2 \lesssim \|\mathbf{u} - \mathbf{u}_h\|_h^2 + \sum_{\tau\in\mathcal{T}_h} h_\tau^{2r}|\mathbf{u} - I_h\mathbf{u}|_{1+r,\tau}^2. \tag{2.31}$$

To show the validity of (2.31), using the trace theorem on the reference element and the scaling argument we have for any $\tau \in \mathcal{T}_h$ that

$$\|w\|_{0,\partial\tau} \lesssim h_\tau^{-\frac{1}{2}}\|w\|_{0,\tau} + h_\tau^{r-\frac{1}{2}}|w|_{r,\tau}, \ \ \forall w \in H^r(\tau), \ r \in (\frac{1}{2}, 1], \tag{2.32}$$

and from the inverse inequality and the interpolation estimate and taking $w = \nabla(\mathbf{u} - I_h\mathbf{u})$ in (2.32) we deduce

$$\begin{aligned}\sum_{F\in\mathcal{E}_h} h_F\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{0,F}^2 &\lesssim \sum_{F\in\mathcal{E}_h} h_F\|\nabla(I_h\mathbf{u} - \mathbf{u}_h)\|_{0,F}^2 + \sum_{F\in\mathcal{E}_h} h_F\|\nabla(\mathbf{u} - I_h\mathbf{u})\|_{0,F}^2\\ &\lesssim \sum_{\tau\in\mathcal{T}_h}\|\nabla(I_h\mathbf{u} - \mathbf{u}_h)\|_{0,\tau}^2 + \sum_{\tau\in\mathcal{T}_h}(\|\nabla(\mathbf{u} - I_h\mathbf{u})\|_{0,\tau}^2 + h_\tau^{2r}|\mathbf{u} - I_h\mathbf{u}|_{1+r,\tau}^2)\\ &\lesssim \|\mathbf{u} - \mathbf{u}_h\|_h^2 + \sum_{\tau\in\mathcal{T}_h} h_\tau^{2r}|\mathbf{u} - I_h\mathbf{u}|_{1+r,\tau}^2.\end{aligned}$$

By the above inequality and (2.9) we obtain (2.31).

Theorem 2.1. Let $(\lambda, \mathbf{u}, p)$ and $(\lambda_h, \mathbf{u}_h, p_h)$ be the jth eigenpair of (2.2), (2.3) and (2.4), (2.5), respectively. Assume that the regularity estimate (2.15) is valid and $(\mathbf{u}, p) \in H^{1+s}(\Omega)^d \times H^s(\Omega)$ for some $s \in [r, k]$. Then,

$$\|\mathbf{u}_h - \mathbf{u}\|_0 \lesssim h^r(\|\mathbf{u} - \mathbf{u}_h\|_h + \|p - p_h\|_0), \tag{2.33}$$

$$|\lambda_h - \lambda| \lesssim h^{2s}, \tag{2.34}$$

$$\|\mathbf{u} - \mathbf{u}_h\|_h + \|p - p_h\|_0 \lesssim h^s(\|\mathbf{u}\|_{1+s} + \|p\|_s). \tag{2.35}$$

Proof. In (2.11)–(2.14), we take $\boldsymbol{g} = \lambda\mathbf{u}$ then we obtain $\mathbf{u}^g = \lambda\mathbb{T}\mathbf{u}$, $\mathbf{u}_h^g = \lambda\mathbb{T}_h\mathbf{u}$, $p^g = \lambda\mathbb{S}\mathbf{u}$ and $p_h^g = \lambda\mathbb{S}_h\mathbf{u}$. Hence, using (2.20) we deduce

$$\|\lambda\mathbb{T}\mathbf{u} - \lambda\mathbb{T}_h\mathbf{u}\|_h + \|\lambda\mathbb{S}\mathbf{u} - \lambda\mathbb{S}_h\mathbf{u}\|_0 \lesssim h^s(\|\mathbf{u}\|_{1+s} + \|p\|_s). \tag{2.36}$$

By using (2.16), (2.18), (2.19) and (2.36) we obtain

$$\begin{aligned}
((\mathbb{T} - \mathbb{T}_h)\mathbf{u}, \mathbf{u}) &= \mathbb{A}_h((\mathbb{T} - \mathbb{T}_h)\mathbf{u}, \mathbb{T}\mathbf{u}) + \mathbb{B}_h((\mathbb{T} - \mathbb{T}_h)\mathbf{u}, \mathbb{S}\mathbf{u}) \\
&= \mathbb{A}_h(\mathbb{T}\mathbf{u} - \mathbb{T}_h\mathbf{u}, \mathbb{T}\mathbf{u} - \mathbb{T}_h\mathbf{u}) + \mathbb{A}_h(\mathbb{T}\mathbf{u} - \mathbb{T}_h\mathbf{u}, \mathbb{T}_h\mathbf{u}) \\
&\quad + 2\mathbb{B}_h(\mathbb{T}\mathbf{u} - \mathbb{T}_h\mathbf{u}, \mathbb{S}\mathbf{u} - \mathbb{S}_h\mathbf{u}) + \mathbb{B}_h(\mathbb{T}\mathbf{u} - \mathbb{T}_h\mathbf{u}, 2\mathbb{S}_h\mathbf{u}) - \mathbb{B}_h(\mathbb{T}\mathbf{u} - \mathbb{T}_h\mathbf{u}, \mathbb{S}\mathbf{u}) \\
&= \mathbb{A}_h(\mathbb{T}\mathbf{u} - \mathbb{T}_h\mathbf{u}, \mathbb{T}\mathbf{u} - \mathbb{T}_h\mathbf{u}) + 2\mathbb{B}_h(\mathbb{T}\mathbf{u} - \mathbb{T}_h\mathbf{u}, \mathbb{S}\mathbf{u} - \mathbb{S}_h\mathbf{u}) \\
&\quad + (\mathbb{A}_h(\mathbb{T}\mathbf{u} - \mathbb{T}_h\mathbf{u}, \mathbb{T}_h\mathbf{u}) + \mathbb{B}_h(\mathbb{T}_h\mathbf{u}, \mathbb{S}\mathbf{u} - \mathbb{S}_h\mathbf{u})) \\
&\quad + \mathbb{B}_h(2\mathbb{T}\mathbf{u} - \mathbb{T}_h\mathbf{u}, \mathbb{S}_h\mathbf{u}) - \mathbb{B}_h(\mathbb{T}\mathbf{u}, \mathbb{S}\mathbf{u}) \\
&= \mathbb{A}_h(\mathbb{T}\mathbf{u} - \mathbb{T}_h\mathbf{u}, \mathbb{T}\mathbf{u} - \mathbb{T}_h\mathbf{u}) + 2\mathbb{B}_h(\mathbb{T}\mathbf{u} - \mathbb{T}_h\mathbf{u}, \mathbb{S}\mathbf{u} - \mathbb{S}_h\mathbf{u}) \\
&\lesssim h^{2s}(\|\mathbf{u}\|_{1+s} + \|p\|_s)^2.
\end{aligned} \tag{2.37}$$

From Lemma 2.3 in [31] we know that

$$\|\mathbf{u}_h - \mathbf{u}\|_0 \lesssim \|(\mathbb{T} - \mathbb{T}_h)\mathbf{u}\|_0, \tag{2.38}$$

$$|\lambda_h - \lambda| \lesssim \lambda^2((\mathbb{T} - \mathbb{T}_h)\mathbf{u}, \mathbf{u}) + \|(\mathbb{T} - \mathbb{T}_h)\mathbf{u}\|_0^2. \tag{2.39}$$

Substituting (2.37) and (2.24) into (2.39) we obtain (2.34).

Applying the triangle inequality and (2.27) we get

$$|\ \|\mathbf{u} - \mathbf{u}_h\|_h - \|\lambda\mathbb{T}\mathbf{u} - \lambda\mathbb{T}_h\mathbf{u}\|_h\ | \le \|\lambda_h\mathbb{T}_h\mathbf{u}_h - \lambda\mathbb{T}_h\mathbf{u}\|_h \lesssim \|\lambda_h\mathbf{u}_h - \lambda\mathbf{u}\|_0, \tag{2.40}$$

$$|\ \|p - p_h\|_0 - \|\lambda\mathbb{S}\mathbf{u} - \lambda\mathbb{S}_h\mathbf{u}\|_0\ | \le \|\lambda_h\mathbb{S}_h\mathbf{u}_h - \lambda\mathbb{S}_h\mathbf{u}\|_0 \lesssim \|\lambda_h\mathbf{u}_h - \lambda\mathbf{u}\|_0. \tag{2.41}$$

From (2.24), (2.38) and (2.39) we deduce

$$\begin{aligned}
\|\lambda_h\mathbf{u}_h - \lambda\mathbf{u}\|_0 \lesssim |\lambda_h - \lambda| + \|\mathbf{u}_h - \mathbf{u}\|_0 &\lesssim \|\lambda\mathbb{T}\mathbf{u} - \lambda\mathbb{T}_h\mathbf{u}\|_0 \\
&\lesssim h^r(\|\lambda\mathbb{T}\mathbf{u} - \lambda\mathbb{T}_h\mathbf{u}\|_h + \|\lambda\mathbb{S}\mathbf{u} - \lambda\mathbb{S}_h\mathbf{u}\|_0).
\end{aligned} \tag{2.42}$$

Then, from (2.40)–(2.42) we obtain

$$\|\mathbf{u} - \mathbf{u}_h\|_h + \|p - p_h\|_0 \simeq \|\lambda\mathbb{T}\mathbf{u} - \lambda\mathbb{T}_h\mathbf{u}\|_h + \|\lambda\mathbb{S}\mathbf{u} - \lambda\mathbb{S}_h\mathbf{u}\|_0. \tag{2.43}$$

Thus, we get (2.33).

Combining (2.43) with (2.36) we get (2.35).

3. The a posteriori error estimates

Let $(\lambda_h, \mathbf{u}_h, p_h) \in \mathbb{R}^+ \times \mathbb{X}_h \times \mathbb{W}_h$ be an approximate eigenpair. First, for each element $\tau \in \mathcal{T}_h$ we introduce the residuals:

$$\eta_{R_\tau}^2 = h_\tau^2 \|\lambda_h \mathbf{u}_h + \Delta \mathbf{u}_h - A\mathbf{u}_h - \nabla p_h\|_{0,\tau}^2 + \|div \mathbf{u}_h\|_{0,\tau}^2,$$
$$\eta_{F_\tau}^2 = \frac{1}{2} \sum_{F \subset \partial\tau \backslash \partial\Omega} h_F \|[\![(p_h \mathbf{I} - \nabla \mathbf{u}_h)]\!]\|_{0,F}^2,$$

where $\mathbf{I}$ denotes the $d \times d$ $(d = 2, 3)$ identity matrix. Next, we introduce the following estimator η_{J_τ} to measure the jump of the approximate solution $\mathbf{u}_h$:

$$\eta_{J_\tau}^2 = \sum_{F \subset \partial\tau, F \in \mathcal{E}_h^i} \gamma h_F^{-1} \|[\![\underline{\mathbf{u}_h}]\!]\|_{0,F}^2 + \sum_{F \subset \partial\tau, F \in \mathcal{E}_h^b} \gamma h_F^{-1} \|\mathbf{u}_h \otimes \mathbf{n}\|_{0,F}^2.$$

The local error indictor is defined as

$$\eta_\tau^2 = \eta_{R_\tau}^2 + \eta_{F_\tau}^2 + \eta_{J_\tau}^2.$$

Then, the global a posteriori error estimator is defined as

$$\eta_h = (\sum_{\tau \in \mathcal{T}_h} \eta_\tau^2)^{\frac{1}{2}}.$$

We denote θ_τ =int$\{\bigcup_{\bar{\tau}_i \cap \bar{\tau} \neq \emptyset} \bar{\tau}_i, \tau_i \in \mathcal{T}_h\}$ for $\tau \in \mathcal{T}_h$ and use θ_F to represent the set of all elements which share at least one node with face F. We denote by $\mathbf{z}^I$ the Scott-Zhang interpolation function [34], then $\mathbf{z}^I \in \mathbb{X} \cap \mathbb{X}_h$ and

$$\|\mathbf{z} - \mathbf{z}^I\|_{0,\tau} + h_\tau \|\nabla(\mathbf{z} - \mathbf{z}^I)\|_{0,\tau} \lesssim h_\tau |\mathbf{z}|_{1,\theta_\tau}, \qquad \forall \tau \in \mathcal{T}_h, \tag{3.1}$$
$$\|\mathbf{z} - \mathbf{z}^I\|_{0,F} \lesssim h_F^{\frac{1}{2}} |\mathbf{z}|_{1,\theta_F}, \qquad \forall F \subset \partial\tau. \tag{3.2}$$

Denote

$$\underline{\sum}_h = \{\underline{\boldsymbol{o}} \in L^2(\Omega)^{d\times d} : \underline{\boldsymbol{o}}|_\tau \in \mathbb{P}_k(\tau)^{d\times d}, \tau \in \mathcal{T}_h\}.$$

The lifting operator $\mathcal{L} : \mathbb{X}(h) \to \underline{\sum}_h$ is defined by

$$\int_\Omega \mathcal{L}(\mathbf{z}) : \underline{\boldsymbol{o}} dx = \sum_{F \in \mathcal{E}_h^i} \int_F [\![\underline{\mathbf{z}}]\!] : \{\underline{\boldsymbol{o}}\} ds, \ \ \forall \ \underline{\boldsymbol{o}} \in \underline{\sum}_h, \tag{3.3}$$

and has the following property (see [24, 25]):

$$\|\mathcal{L}(\mathbf{z})\|_0^2 \lesssim \sum_{F \in \mathcal{E}_h^i} \|h_F^{-\frac{1}{2}} [\![\underline{\mathbf{z}}]\!]\|_{0,F}^2, \quad \forall \ \mathbf{z} \in \mathbb{X} + \mathbb{X}_h. \tag{3.4}$$

Using the lifting operator, we define the following form:

$$\widetilde{\mathbb{A}}_h(\cdot,\cdot) : (\mathbb{X}+\mathbb{X}_h)\times(\mathbb{X}+\mathbb{X}_h)\to\mathbb{R} \tag{3.5}$$

by

$$\begin{aligned}\widetilde{\mathbb{A}}_h(\mathbf{w},\mathbf{z}) =& \sum_{\tau\in\mathcal{T}_h}\int_\tau \nabla\mathbf{w}:\nabla\mathbf{z}dx + \sum_{\tau\in\mathcal{T}_h}\int_\tau A\mathbf{w}\cdot\mathbf{z}dx - \sum_{\tau\in\mathcal{T}_h}\int_\tau \mathcal{L}(\mathbf{z}):\nabla\mathbf{w}dx\\ &- \sum_{\tau\in\mathcal{T}_h}\int_\tau \mathcal{L}(\mathbf{w}):\nabla\mathbf{z}dx + \sum_{F\in\mathcal{E}_h}\int_F \frac{\gamma}{h_F}[\![\underline{\mathbf{w}}]\!]:[\![\underline{\mathbf{z}}]\!]ds, \quad \forall\, \mathbf{z}\in\mathbb{X}+\mathbb{X}_h. \end{aligned}\tag{3.6}$$

Note that $\widetilde{\mathbb{A}}_h = \mathbb{A}_h$ on $\mathbb{X}_h\times\mathbb{X}_h$ and $\mathbb{A} = \widetilde{\mathbb{A}}_h$ on $\mathbb{X}\times\mathbb{X}$. The DGFEM presented in (2.4) and (2.5) is equivalent to finding $(\lambda_h, \mathbf{u}_h, p_h)\in\mathbb{R}^+\times\mathbb{X}_h\times\mathbb{W}_h$ and satisfying

$$\begin{aligned}\widetilde{\mathbb{A}}_h(\mathbf{u}_h,\mathbf{z}_h) + \mathbb{B}_h(\mathbf{z}_h,p_h) &= \lambda_h(\mathbf{u}_h,\mathbf{z}_h), \quad \forall \mathbf{z}_h\in\mathbb{X}_h,\\ \mathbb{B}_h(\mathbf{u}_h,\varrho_h) &= 0, \quad \forall\varrho_h\in\mathbb{W}_h.\end{aligned}\tag{3.7}$$

Lemma 3.1. Let $(\mathbf{u}^g, p^g)$ and $(\mathbf{u}_h^g, p_h^g)$ be the solutions of (2.11), (2.12) and (2.13), (2.14), respectively. Then,

$$\|\mathbf{u}^g-\mathbf{u}_h^g\|_h + \|p^g-p_h^g\|_0 \backsimeq \sup_{0\neq\mathbf{z}\in\mathbb{X}}\frac{|(\boldsymbol{g},\mathbf{z}) - \widetilde{\mathbb{A}}_h(\mathbf{u}_h^g,\mathbf{z}) - \mathbb{B}_h(\mathbf{z},p_h^g)|}{\|\mathbf{z}\|_h} + \inf_{\mathbf{z}\in\mathbb{X}}\|\mathbf{u}_h^g-\mathbf{z}\|_h. \tag{3.8}$$

Proof. For $\forall\bar{\mathbf{u}}\in\mathbb{X}$, from (2.11) we have

$$\begin{aligned}\widetilde{\mathbb{A}}_h(\mathbf{u}^g-\bar{\mathbf{u}},\mathbf{u}^g-\bar{\mathbf{u}}) &= \widetilde{\mathbb{A}}_h(\mathbf{u}^g,\mathbf{u}^g-\bar{\mathbf{u}}) - \widetilde{\mathbb{A}}_h(\bar{\mathbf{u}},\mathbf{u}^g-\bar{\mathbf{u}})\\ &= (\boldsymbol{g},\mathbf{u}^g-\bar{\mathbf{u}}) - \mathbb{B}(\mathbf{u}^g-\bar{\mathbf{u}},p^g) - \widetilde{\mathbb{A}}_h(\mathbf{u}_h^g,\mathbf{u}^g-\bar{\mathbf{u}}) + \widetilde{\mathbb{A}}_h(\mathbf{u}_h^g-\bar{\mathbf{u}},\mathbf{u}^g-\bar{\mathbf{u}}).\end{aligned}$$

For $\forall\bar{\mathbf{u}}\in\mathbb{X}$, $\overline{p}\in\mathbb{W}$ we have

$$\mathbb{B}_h(\mathbf{u}^g-\bar{\mathbf{u}},p^g-\bar{p}) = \mathbb{B}_h(\mathbf{u}^g-\bar{\mathbf{u}},p^g) - \mathbb{B}_h(\mathbf{u}^g-\bar{\mathbf{u}},p_h^g) - \mathbb{B}_h(\mathbf{u}^g-\bar{\mathbf{u}},\bar{p}-p_h^g).$$

Combining the above two equations and taking $\mathbf{z} = \mathbf{u}^g-\bar{\mathbf{u}}$ we obtain

$$\begin{aligned}&\|\mathbf{u}^g-\bar{\mathbf{u}}\|_h\|\mathbf{z}\|_h + \mathbb{B}_h(\mathbf{z},p^g-\bar{p})\\ =&(\boldsymbol{g},\mathbf{z}) - \mathbb{B}_h(\mathbf{z},p^g) - \widetilde{\mathbb{A}}_h(\mathbf{u}_h^g,\mathbf{z}) + \widetilde{\mathbb{A}}_h(\mathbf{u}_h^g-\bar{\mathbf{u}},\mathbf{z}) + \mathbb{B}_h(\mathbf{z},p^g) - \mathbb{B}_h(\mathbf{z},p_h^g) - \mathbb{B}_h(\mathbf{z},\bar{p}-p_h^g)\\ =&(\boldsymbol{g},\mathbf{z}) - \widetilde{\mathbb{A}}_h(\mathbf{u}_h^g,\mathbf{z}) - \mathbb{B}_h(\mathbf{z},p_h^g) + \widetilde{\mathbb{A}}_h(\mathbf{u}_h^g-\bar{\mathbf{u}},\mathbf{z}) - \mathbb{B}_h(\mathbf{z},\bar{p}-p_h^g).\end{aligned}\tag{3.9}$$

From the well-known inf-sup condition we obtain

$$\sup_{\mathbf{z}\in\mathbb{X}}\frac{\mathbb{B}_h(\mathbf{z},p^g-\bar{p})}{\|\mathbf{z}\|_h} \gtrsim \|p^g-\bar{p}\|_0.$$

Dividing both sides of (3.9) by $\|\mathbf{z}\|_h$ and taking supremum for $\mathbf{z}\in\mathbb{X}$ we get

$$\|\mathbf{u}^g-\bar{\mathbf{u}}\|_h + \|p^g-\bar{p}\|_0 \lesssim \sup_{\mathbf{z}\in\mathbb{X}}\frac{(\boldsymbol{g},\mathbf{z}) - \widetilde{\mathbb{A}}_h(\mathbf{u}_h^g,\mathbf{z}) - \mathbb{B}_h(\mathbf{z},p_h^g)}{\|\mathbf{z}\|_h} + \|\mathbf{u}_h^g-\bar{\mathbf{u}}\|_h + \|\bar{p}-p_h^g\|_0, \ \forall(\bar{\mathbf{u}},\bar{p})\in\mathbb{X}\times\mathbb{W}. \tag{3.10}$$

Using the triangle inequality we obtain

$$\begin{aligned}&\|\mathbf{u}^g-\mathbf{u}_h^g\|_h+\|p^g-p_h^g\|_0\\ \lesssim\ &\sup_{\mathbf{z}\in\mathbb{X}}\frac{(\boldsymbol{g},\mathbf{z})-\widetilde{\mathbb{A}}_h(\mathbf{u}_h^g,\mathbf{z})-\mathbb{B}_h(\mathbf{z},p_h^g)}{\|\mathbf{z}\|_h}+\|\mathbf{u}_h^g-\bar{\mathbf{u}}\|_h+\|\bar{p}-p_h^g\|_0,\quad\forall(\bar{\mathbf{u}},\bar{p})\in\mathbb{X}\times\mathbb{W}.\end{aligned}\tag{3.11}$$

Since $(\bar{\mathbf{u}},\bar{p})$ is arbitrary and $\inf_{\bar{p}\in\mathbb{W}}\|\bar{p}-p_h^g\|_0=0$, the part $\lesssim$ in (3.8) is valid. The other part $\gtrsim$ in (3.8) is obvious.

Lemma 3.1 can be extended to the eigenvalue problem.

Theorem 3.1. Let $(\lambda,\mathbf{u},p)$ and $(\lambda_h,\mathbf{u}_h,p_h)$ be the jth eigenpair of (2.2), (2.3) and (2.4), (2.5), respectively. Then,

$$\|\mathbf{u}-\mathbf{u}_h\|_h+\|p-p_h\|_0\simeq\sup_{0\neq\mathbf{z}\in\mathbb{X}}\frac{|\widetilde{\mathbb{A}}_h(\mathbf{u}-\mathbf{u}_h,\mathbf{z})+\mathbb{B}_h(\mathbf{z},p-p_h)|}{\|\mathbf{z}\|_h}+\inf_{\mathbf{z}\in\mathbb{X}}\|\mathbf{u}_h-\mathbf{z}\|_h+\|\lambda_h\mathbf{u}_h-\lambda\mathbf{u}\|_0.\tag{3.12}$$

Proof. Using (2.26) and (2.27) we can obtain

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}_h\|_h+\|p-p_h\|_0\\ =\ &\|\lambda\mathbb{T}\mathbf{u}-\lambda_h\mathbb{T}\mathbf{u}_h+\lambda_h\mathbb{T}\mathbf{u}_h-\lambda_h\mathbb{T}_h\mathbf{u}_h\|_h+\|\lambda\mathbb{S}\mathbf{u}-\lambda_h\mathbb{S}\mathbf{u}_h+\lambda_h\mathbb{S}\mathbf{u}_h-\lambda_h\mathbb{S}_h\mathbf{u}_h\|_h\\ \le\ &\|\lambda_h\mathbb{T}\mathbf{u}_h-\lambda_h\mathbb{T}_h\mathbf{u}_h\|_h+\|\lambda_h\mathbb{S}\mathbf{u}_h-\lambda_h\mathbb{S}_h\mathbf{u}_h\|_h+\|\lambda\mathbf{u}-\lambda_h\mathbf{u}_h\|_0.\end{aligned}\tag{3.13}$$

In (2.11)–(2.14) we take $\boldsymbol{g}=\lambda_h\mathbf{u}_h$ and obtain $\mathbf{u}^g=\lambda_h\mathbb{T}\mathbf{u}_h$, $\mathbf{u}_h^g=\lambda_h\mathbb{T}_h\mathbf{u}_h$, $p^g=\lambda_h\mathbb{S}\mathbf{u}_h$ and $p_h^g=\lambda_h\mathbb{S}_h\mathbf{u}_h$. Therefore, from (3.8) we have

$$\begin{aligned}&\|\lambda_h\mathbb{T}\mathbf{u}_h-\lambda_h\mathbb{T}_h\mathbf{u}_h\|_h+\|\lambda_h\mathbb{S}\mathbf{u}_h-\lambda_h\mathbb{S}_h\mathbf{u}_h\|_0\\ \lesssim\ &\sup_{0\neq\mathbf{z}\in\mathbb{X}}\frac{|(\lambda_\mathbf{h}\mathbf{u}_h,\mathbf{z})-\widetilde{\mathbb{A}}_h(\lambda_h\mathbb{T}_h\mathbf{u}_h,\mathbf{z})-\mathbb{B}_h(\mathbf{z},\mathbb{S}_h(\lambda_h\mathbf{u}_h))|}{\|\mathbf{z}\|_h}+\inf_{\mathbf{z}\in\mathbb{X}}\|\mathbf{u}_h-\mathbf{z}\|_h.\end{aligned}\tag{3.14}$$

From (2.11) with $\boldsymbol{g}=\lambda_h\mathbf{u}_h$, (2.26) and (2.27) we deduce

$$\begin{aligned}&|(\lambda_\mathbf{h}\mathbf{u}_\mathbf{h},\mathbf{z})-\widetilde{\mathbb{A}}_h(\lambda_h\mathbb{T}_h\mathbf{u}_h,\mathbf{z})-\mathbb{B}_h(\mathbf{z},\mathbb{S}_h(\lambda_h\mathbf{u}_h))|\\ &=|\widetilde{\mathbb{A}}_h(\lambda_h\mathbb{T}\mathbf{u}_h,\mathbf{z})+\mathbb{B}_h(\mathbf{z},\mathbb{S}(\lambda_h\mathbf{u}_h))-\widetilde{\mathbb{A}}_h(\lambda_h\mathbb{T}_h\mathbf{u}_h,\mathbf{z})-\mathbb{B}_h(\mathbf{z},\mathbb{S}_h(\lambda_h\mathbf{u}_h))|\\ &=|\widetilde{\mathbb{A}}_h(\lambda_h\mathbb{T}\mathbf{u}_h-\lambda_h\mathbb{T}_h\mathbf{u}_h,\mathbf{z})+\mathbb{B}_h(\mathbf{z},\mathbb{S}(\lambda_h\mathbf{u}_h)-\mathbb{S}_h(\lambda_h\mathbf{u}_h))|\\ &=|\widetilde{\mathbb{A}}_h(\lambda_h\mathbb{T}\mathbf{u}_h-\lambda\mathbb{T}\mathbf{u}+\mathbf{u}-\mathbf{u}_h,\mathbf{z})+\mathbb{B}_h(\mathbf{z},\mathbb{S}(\lambda_h\mathbf{u}_h)-\mathbb{S}(\lambda\mathbf{u})+p-p_h)|\\ &\le|\widetilde{\mathbb{A}}_h(\mathbf{u}-\mathbf{u}_h,\mathbf{z})+\mathbb{B}_h(\mathbf{z},p-p_h))|+C\|\lambda_h\mathbf{u}_h-\lambda\mathbf{u}\|_0\|\mathbf{z}\|_h.\end{aligned}\tag{3.15}$$

Substituting (3.15) into (3.14) gives us

$$\begin{aligned}&\|\lambda_h\mathbb{T}\mathbf{u}_h-\lambda_h\mathbb{T}_h\mathbf{u}_h\|_h+\|\lambda_h\mathbb{S}\mathbf{u}_h-\lambda_h\mathbb{S}_h\mathbf{u}_h\|_0\\ \lesssim\ &\sup_{0\neq\mathbf{z}\in\mathbb{X}}\frac{|\widetilde{\mathbb{A}}_h(\mathbf{u}-\mathbf{u}_h,\mathbf{z})+\mathbb{B}_h(\mathbf{z},p-p_h)|}{\|\mathbf{z}\|_h}+C(\|\lambda_h\mathbf{u}_h-\lambda\mathbf{u}\|_0+\inf_{\mathbf{z}\in\mathbb{X}}\|\mathbf{u}_h-\mathbf{z}\|_h).\end{aligned}\tag{3.16}$$

Theorem 2.1 indicates that $\|\lambda_h\mathbf{u}_h-\lambda\mathbf{u}\|_0$ is a small quantity of higher order compared with $\|\mathbf{u}-\mathbf{u}_h\|_h+\|p-p_h\|_0$. From (3.16) and (3.14) the side $\lesssim$ in (3.12) is true. The other side $\gtrsim$ in (3.12) is obvious.

Lemma 3.2. Under the conditions of Theorem 2.1,

$$\widetilde{\mathbb{A}}_h(\mathbf{u}-\mathbf{u}_h,\mathbf{z})+\mathbb{B}_h(\mathbf{z},p-p_h)\lesssim\sum_{\tau\in\mathcal{T}_h}(\eta_{R_\tau}+\eta_{F_\tau}+\eta_{J_\tau})\,\|\mathbf{z}\|_h+\|\lambda\mathbf{u}-\lambda_h\mathbf{u}_h\|_0\|\mathbf{z}\|_h,\quad\forall z\in\mathbb{X}.\tag{3.17}$$

Proof. Using (2.7), (3.6), (3.7) and the Green's formula we deduce that

$$\begin{aligned}
&\widetilde{\mathbb{A}}_h(\mathbf{u}-\mathbf{u}_h,\mathbf{z})+\mathbb{B}_h(\mathbf{z},p-p_h))\\
=&\widetilde{\mathbb{A}}_h(\mathbf{u},\mathbf{z})-\widetilde{\mathbb{A}}_h(\mathbf{u}_h,\mathbf{z})+\mathbb{B}_h(\mathbf{z},p)-\mathbb{B}_h(\mathbf{z},p_h)\\
=&\lambda(\mathbf{u},\mathbf{z})-\widetilde{\mathbb{A}}_h(\mathbf{u}_h,\mathbf{z})-\mathbb{B}_h(\mathbf{z},p_h)\\
=&\lambda\sum_{\tau\in\mathcal{T}_h}\int_\tau\mathbf{u}\cdot\mathbf{z}dx-\sum_{\tau\in\mathcal{T}_h}\int_\tau\nabla\mathbf{u}_h:\nabla\mathbf{z}dx-\sum_{\tau\in\mathcal{T}_h}\int_\tau A\mathbf{u}_h\cdot\mathbf{z}dx+\sum_{\tau\in\mathcal{T}_h}\int_\tau\mathcal{L}(\mathbf{z}):\nabla\mathbf{u}_hdx\\
&+\sum_{\tau\in\mathcal{T}_h}\int_\tau\mathcal{L}(\mathbf{u}_h):\nabla\mathbf{z}dx+\sum_{F\in\mathcal{E}_h}\int_F\frac{\gamma}{h_F}[\![\underline{\mathbf{u}_h}]\!]:[\![\underline{\mathbf{z}}]\!]ds+\sum_{\tau\in\mathcal{T}_h}\int_\tau div\mathbf{z}p_hdx\\
=&\lambda\sum_{\tau\in\mathcal{T}_h}\int_\tau\mathbf{u}\cdot\mathbf{z}dx+\sum_{\tau\in\mathcal{T}_h}\int_\tau\Delta\mathbf{u}_h\cdot\mathbf{z}dx-\sum_{\tau\in\mathcal{T}_h}\sum_{F\subset\partial\tau}\int_F\frac{\partial\mathbf{u}_h}{\partial\mathbf{n}}\cdot\mathbf{z}ds-\sum_{\tau\in\mathcal{T}_h}\int_\tau A\mathbf{u}_h\cdot\mathbf{z}dx\\
&+\sum_{\tau\in\mathcal{T}_h}\int_\tau\mathcal{L}(\mathbf{z}):\nabla\mathbf{u}_hdx+\sum_{\tau\in\mathcal{T}_h}\int_\tau\mathcal{L}(\mathbf{u}_h):\nabla\mathbf{z}dx-\sum_{\tau\in\mathcal{T}_h}\int_\tau\nabla p_h\cdot\mathbf{z}dx\\
&+\sum_{\tau\in\mathcal{T}_h}\sum_{F\subset\partial\tau}\int_F p_h\mathbf{z}\cdot\mathbf{n}ds.
\end{aligned}\tag{3.18}$$

By $\mathbf{z}^I\in\mathbb{X}\cap\mathbb{X}_h$ and (2.2)–(2.5) we obtain

$$\widetilde{\mathbb{A}}_h(\mathbf{u}-\mathbf{u}_h,\mathbf{z})+\mathbb{B}_h(\mathbf{z},p-p_h)=\widetilde{\mathbb{A}}_h(\mathbf{u}-\mathbf{u}_h,\mathbf{z}-\mathbf{z}^I)+\mathbb{B}_h(\mathbf{z}-\mathbf{z}^I,p-p_h)+\sum_{\tau\in\mathcal{T}_h}\int_\tau(\lambda\mathbf{u}-\lambda_h\mathbf{u}_h)\cdot\mathbf{z}^Idx.$$

Using (3.6), the Cauchy-Schwartz inequality, (3.1) and (3.2), (3.18) can be written as follows:

$$\begin{aligned}
&\widetilde{\mathbb{A}}_h(\mathbf{u}-\mathbf{u}_h,\mathbf{z})+\mathbb{B}_h(\mathbf{z},p-p_h)\\
=&\ \lambda\sum_{\tau\in\mathcal{T}_h}\int_\tau\mathbf{u}\cdot(\mathbf{z}-\mathbf{z}^I)dx+\sum_{\tau\in\mathcal{T}_h}\int_\tau\Delta\mathbf{u}_h\cdot(\mathbf{z}-\mathbf{z}^I)dx-\sum_{\tau\in\mathcal{T}_h}\sum_{F\subset\partial\tau}\int_F\frac{\partial\mathbf{u}_h}{\partial\mathbf{n}}\cdot(\mathbf{z}-\mathbf{z}^I)ds\\
&-\sum_{\tau\in\mathcal{T}_h}\int_\tau A\mathbf{u}_h\cdot(\mathbf{z}-\mathbf{z}^I)dx+\sum_{\tau\in\mathcal{T}_h}\int_\tau\mathcal{L}(\mathbf{z}-\mathbf{z}^I):\nabla\mathbf{u}_hdx+\sum_{\tau\in\mathcal{T}_h}\int_\tau\mathcal{L}(\mathbf{u}_h):\nabla(\mathbf{z}-\mathbf{z}^I)dx\\
&-\sum_{\tau\in\mathcal{T}_h}\int_\tau\nabla p_h\cdot(\mathbf{z}-\mathbf{z}^I)dx+\sum_{\tau\in\mathcal{T}_h}\sum_{F\subset\partial\tau}\int_F p_h(\mathbf{z}-\mathbf{z}^I)\cdot\mathbf{n}ds+\sum_{\tau\in\mathcal{T}_h}\int_\tau(\lambda\mathbf{u}-\lambda_h\mathbf{u}_h)\cdot\mathbf{z}^Idx\\
=&\sum_{\tau\in\mathcal{T}_h}\int_\tau(\Delta\mathbf{u}_h+\lambda_h\mathbf{u}_h-A\mathbf{u}_h-\nabla p_h)\cdot(\mathbf{z}-\mathbf{z}^I)dx-\sum_{\tau\in\mathcal{T}_h}\sum_{F\subset\partial\tau}\int_F\frac{\partial\mathbf{u}_h}{\partial\mathbf{n}}\cdot(\mathbf{z}-\mathbf{z}^I)ds\\
&+\sum_{\tau\in\mathcal{T}_h}\int_\tau\mathcal{L}(\mathbf{z}-\mathbf{z}^I):\nabla\mathbf{u}_hdx+\sum_{\tau\in\mathcal{T}_h}\int_\tau\mathcal{L}(\mathbf{u}_h):\nabla(\mathbf{z}-\mathbf{z}^I)dx\\
&+\sum_{\tau\in\mathcal{T}_h}\sum_{F\subset\partial\tau}\int_F p_h(\mathbf{z}-\mathbf{z}^I)\cdot\mathbf{n}ds+\sum_{\tau\in\mathcal{T}_h}\int_\tau(\lambda\mathbf{u}-\lambda_h\mathbf{u}_h)\cdot\mathbf{z}dx
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\tau\in\mathcal{T}_h}\int_\tau(\Delta\mathbf{u}_h+\lambda_h\mathbf{u}_h-A\mathbf{u}_h-\nabla p_h)\cdot(\mathbf{z}-\mathbf{z}^I)dx+\sum_{\tau\in\mathcal{T}_h}\sum_{F\subset\partial\tau\backslash\partial\Omega}\int_F(p_h\mathbf{I}-\nabla\mathbf{u}_h)\mathbf{n}\cdot(\mathbf{z}-\mathbf{z}^I)ds\\
&+\sum_{\tau\in\mathcal{T}_h}\int_\tau\mathcal{L}(\mathbf{z}-\mathbf{z}^I):\nabla\mathbf{u}_h dx+\sum_{\tau\in\mathcal{T}_h}\int_\tau\mathcal{L}(\mathbf{u}_h):\nabla(\mathbf{z}-\mathbf{z}^I)dx+\sum_{\tau\in\mathcal{T}_h}\int_\tau(\lambda\mathbf{u}-\lambda_h\mathbf{u}_h)\cdot\mathbf{z}dx\\
&\equiv B_1+B_2+B_3+B_4+B_5. \qquad (3.19)
\end{aligned}
$$

Next, we will analyze each item on the right-hand side of (3.19). Using the Cauchy-Schwartz inequality and the approximation property (3.1) and (3.2), we have

$$
\begin{aligned}
|B_1| &\le \sum_{\tau\in\mathcal{T}_h}\|\Delta\mathbf{u}_h+\lambda_h\mathbf{u}_h-A\mathbf{u}_h-\nabla p_h\|_{0,\tau}\|\mathbf{z}-\mathbf{z}^I\|_{0,\tau}\\
&\lesssim \sum_{\tau\in\mathcal{T}_h}h_\tau\|\Delta\mathbf{u}_h+\lambda_h\mathbf{u}_h-A\mathbf{u}_h-\nabla p_h\|_{0,\tau}\|\mathbf{z}\|^2_{1,\theta_\tau}\\
&\lesssim \left(\sum_{\tau\in\mathcal{T}_h}h_\tau^2\|\Delta\mathbf{u}_h+\lambda_h\mathbf{u}_h-A\mathbf{u}_h-\nabla p_h\|^2_{0,\tau}\right)^{\frac{1}{2}}\|\mathbf{z}\|_h.
\end{aligned}
$$

From (3.2) we deduce

$$
\begin{aligned}
|B_2| &= |\frac{1}{2}\sum_{\tau\in\mathcal{T}_h}\sum_{F\subset\partial\tau\backslash\partial\Omega}\int_F[\![p_h\mathbf{I}-\nabla\mathbf{u}_h]\!]\cdot(\mathbf{z}-\mathbf{z}^I)ds|\\
&\le \sum_{\tau\in\mathcal{T}_h}\sum_{F\subset\partial\tau\backslash\partial\Omega}\|[\![p_h\mathbf{I}-\nabla\mathbf{u}_h]\!]\|_{0,F}Ch_F^{\frac{1}{2}}\|\mathbf{z}\|_{1,\theta_F}\\
&\lesssim \left(\sum_{\tau\in\mathcal{T}_h}\sum_{F\subset\partial\tau\backslash\partial\Omega}(h_F^{\frac{1}{2}}\|[\![p_h\mathbf{I}-\nabla\mathbf{u}_h]\!]\|_{0,F})^2\right)^{\frac{1}{2}}\|\mathbf{z}\|_h.
\end{aligned}
$$

For the third term, by the properties of the interpolation function $\mathbf{z}^I$ we know $[\![\mathbf{z}-\mathbf{z}^I]\!]=0$. Therefore, from the definition of lifting operation $\mathcal{L}$ we have

$$
B_3=\sum_{\tau\in\mathcal{T}_h}\int_\tau\mathcal{L}(\mathbf{z}-\mathbf{z}^I):\nabla\mathbf{u}_h ds=\sum_{F\in\mathcal{E}_h}\int_F\{\underline{\nabla\mathbf{u}_h}\}:[\![\underline{\mathbf{z}-\mathbf{z}^I}]\!]ds=0.
$$

By the Cauchy-Schwartz inequality, (3.4) and (3.1) we obtain

$$
\begin{aligned}
|B_4| &\le \left(\sum_{\tau\in\mathcal{T}_h}\|\mathcal{L}(\mathbf{u}_h)\|^2_{0,\tau}\right)^{\frac{1}{2}}\left(\sum_{\tau\in\mathcal{T}_h}\|\nabla(\mathbf{z}-\mathbf{z}^I)\|^2_{0,\tau}\right)^{\frac{1}{2}}\\
&\lesssim \left(\sum_{F\in\mathcal{E}_h^i}\|h_F^{-\frac{1}{2}}[\![\underline{\mathbf{u}_h}]\!]\|^2_{0,F}\right)^{\frac{1}{2}}\left(\sum_{\tau\in\mathcal{T}_h}\|\nabla(\mathbf{z}-\mathbf{z}^I)\|^2_{0,\tau}\right)^{\frac{1}{2}}\\
&\lesssim \left(\sum_{F\in\mathcal{E}_h^i}\|h_F^{-\frac{1}{2}}[\![\underline{\mathbf{u}_h}]\!]\|^2_{0,F}\right)^{\frac{1}{2}}\|\mathbf{z}\|_h.
\end{aligned}
$$

For the last term of (3.19) we get

$$B_5 = \sum_{\tau\in\mathcal{T}_h}\int_\tau(\lambda\mathbf{u}-\lambda_h\mathbf{u}_h)\cdot\mathbf{z}dx \le \|\lambda\mathbf{u}-\lambda_h\mathbf{u}_h\|_0\|\mathbf{z}\|_0.$$

Substituting B_1–B_5 into (3.19) results in (3.17).

In [22, 23], the authors constructed the enriching operator $E_h : \mathbb{X}_h \to \mathbb{X}_h \cap \mathbb{X}$ by averaging and proved the following lemma.

Lemma 3.3. It is valid the following estimate:

$$\|\mathbf{u}_h - E_h\mathbf{u}_h\|_h \lesssim \sum_{F\in\mathcal{E}_h^i}\gamma h_F^{-1}\|[\![\underline{\mathbf{u}_h}]\!]\|_{0,F}^2 + \sum_{F\in\mathcal{E}_h^b}\gamma h_F^{-1}|\mathbf{u}_h\otimes\mathbf{n}|_{0,F}^2. \tag{3.20}$$

Theorem 3.2. Suppose that the conditions of Theorem 2.1 hold. Then,

$$\|\mathbf{u}-\mathbf{u}_h\|_h + \|p-p_h\|_0 \lesssim \eta_h + \|\lambda_h\mathbf{u}_h - \lambda\mathbf{u}\|_0. \tag{3.21}$$

Proof. Substituting (3.17) and (3.20) into (3.12), we obtain (3.21).

Let b_τ and b_F be the standard bubble function on element τ and face F ($d = 3$) or edge F ($d = 2$) of τ, respectively. Then, from [20, 21, 35] we obtain the following lemma.

Lemma 3.4. For any vector-valued polynomial function $\mathbf{z}_h$ on τ,

$$\|\mathbf{z}_h\|_{0,\tau} \lesssim \|b_\tau^{1/2}\mathbf{z}_h\|_{0,\tau}, \tag{3.22}$$

$$\|b_\tau\mathbf{z}_h\|_{0,\tau} \lesssim \|\mathbf{z}_h\|_{0,\tau}, \tag{3.23}$$

$$\|\nabla(b_\tau\mathbf{z}_h)\|_{0,\tau} \lesssim h_\tau^{-1}\|\mathbf{z}_h\|_{0,\tau}. \tag{3.24}$$

For any vector-valued polynomial function σ on F it is valid that

$$\|b_E\sigma\|_{0,F} \lesssim \|\sigma\|_{0,F}, \tag{3.25}$$

$$\|\sigma\|_{0,F} \lesssim \|b_F^{1/2}\sigma\|_{0,F}. \tag{3.26}$$

Furthermore, for each $b_F\sigma$ there exists an extension $\sigma_b \in H_0^1(\omega(F))$ satisfying $\sigma_b|_F = b_F\sigma$ and

$$\|\sigma_b\|_{0,\tau} \lesssim h_F^{1/2}\|\sigma\|_{0,F}, \quad \forall\tau\in\omega(F), \tag{3.27}$$

$$\|\nabla\sigma_b\|_{0,\tau} \lesssim h_F^{-1/2}\|\sigma\|_{0,F}, \quad \forall\tau\in\omega(F). \tag{3.28}$$

Using the standard arguments (see, e.g., Lemma 3.13 in [36]) and Lemmas 7 and 8 in [2], we can deduce the following local bounds.

Lemma 3.5. Under the conditions of Theorem 2.1,

$$\eta_{R_\tau} \lesssim \|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\tau} + \|p-p_h\|_{0,\tau} + h_\tau\|\lambda_h\mathbf{u}_h - \lambda\mathbf{u}\|_{0,\tau}, \tag{3.29}$$

$$\eta_{F_\tau} \lesssim \|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\omega(\tau)} + \|p-p_h\|_{0,\omega(\tau)} + \left(\sum_{\tau\in\omega(\tau)}h_\tau^2\|\lambda\mathbf{u}-\lambda_h\mathbf{u}_h\|_{0,\tau}^2\right)^{\frac{1}{2}}, \tag{3.30}$$

$$\eta_{J_\tau}^2 = \sum_{F\subset\partial\tau, F\in\mathcal{E}_h^i}\gamma h_F^{-1}\|[\![\underline{\mathbf{u}_h-\mathbf{u}}]\!]\|_{0,F}^2 + \sum_{F\subset\partial\tau, F\in\mathcal{E}_h^b}\gamma h_F^{-1}|(\mathbf{u}_h-\mathbf{u})\otimes\mathbf{n}|_{0,F}^2. \tag{3.31}$$

Proof. For any $\tau \in \mathcal{T}_h$ define the function R and K locally by

$$R|_\tau = \lambda_h \mathbf{u}_h + \triangle \mathbf{u}_h - A\mathbf{u}_h - \nabla p_h \quad \text{and} \quad \mathrm{K}|_\tau = \mathrm{h}_\tau^2 \mathrm{Rb}_\tau.$$

From (3.22) and using $\lambda \mathbf{u} + \Delta \mathbf{u} - A\mathbf{u} - \nabla p = 0$, we have

$$\begin{aligned} h_\tau^2 \|R\|_{0,\tau}^2 &\lesssim \int_\tau R \cdot (h_\tau^2 R b_\tau) dx \\ &= \int_\tau (\lambda_h \mathbf{u}_h + \Delta \mathbf{u}_h - A\mathbf{u}_h - \nabla p_h) \cdot K dx \\ &= \int_\tau (\lambda_h \mathbf{u}_h + \Delta \mathbf{u}_h - A\mathbf{u}_h - \nabla p_h - (\lambda \mathbf{u} + \Delta \mathbf{u} - A\mathbf{u} - \nabla p)) \cdot K dx \\ &= \int_\tau \Delta(\mathbf{u}_h - \mathbf{u}) \cdot K dx - \int_\tau \nabla(p_h - p) \cdot K dx + \int_\tau (\lambda_h \mathbf{u}_h - \lambda \mathbf{u}) \cdot K dx - \int_\tau A(\mathbf{u}_h - \mathbf{u}) \cdot K dx. \end{aligned}$$

Using integration by parts and $K|_{\partial\tau} = 0$, we obtain

$$h_\tau^2 \|R\|_{0,\tau}^2 \lesssim \int_\tau \nabla(\mathbf{u} - \mathbf{u}_h) \cdot \nabla K dx + \int_\tau (p_h - p) div K dx + \int_\tau (\lambda_h \mathbf{u}_h - \lambda \mathbf{u}) \cdot K dx + \int_\tau A(\mathbf{u}_h - \mathbf{u}) \cdot K dx.$$

Applying the Cauchy-Schwartz inequality yields

$$h_\tau^2 \|R\|_{0,\tau}^2 \lesssim (\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{0,\tau} + \|p - p_h\|_{0,\tau} + h_\tau \|\lambda_h \mathbf{u}_h - \lambda \mathbf{u}\|_{0,\tau} + h_\tau \|A\mathbf{u}_h - A\mathbf{u}\|_{0,\tau}) \left(\|\nabla K\|_{0,\tau} + h_\tau^{-1} \|K\|_{0,\tau}\right). \tag{3.32}$$

From (3.23) and (3.24) we get

$$\|\nabla K\|_{0,\tau} + h_\tau^{-1} \|K\|_{0,\tau} \lesssim h_\tau \|R\|_{0,\tau}.$$

Dividing (3.32) by $h_\tau \|R\|_{0,\tau}$ and noting $\|\nabla \cdot \mathbf{u}_h\|_0 = \|\nabla \cdot (\mathbf{u}_h - \mathbf{u})\|_0$, we obtain (3.29).

For any interior edge $F \in \mathcal{E}_h^i$ let the functions R and Θ be such that

$$R|_F = [\![p_h \mathbf{I} - \nabla \mathbf{u}_h]\!]|_F \quad \text{and} \quad \Theta = h_F R b_F.$$

Using (3.26) and $[\![p\mathbf{I} - \nabla \mathbf{u}]\!]|_F = 0$ we get

$$h_F \|R\|_{0,F}^2 \lesssim \int_F R \cdot (h_F R b_F) ds = \int_F [\![(p_h - p)\mathbf{I} - \nabla(\mathbf{u}_h - \mathbf{u})]\!] \cdot \Theta ds.$$

Applying the Green's formula over each element of $\omega(F)$ we derive

$$\begin{aligned} h_E \|R\|_{0,F}^2 &\lesssim \int_F [\![((p_h - p)\mathbf{I} - \nabla(\mathbf{u}_h - \mathbf{u}))]\!] \cdot \Theta ds \\ &= C\Big(\sum_{\tau \in \omega(F)} \int_\tau (-\Delta(\mathbf{u} - \mathbf{u}_h) + \nabla(p - p_h)) \cdot \Theta dx - \sum_{\tau \in \omega(F)} \int_\tau (\nabla(\mathbf{u} - \mathbf{u}_h) - (p - p_h)\mathbf{I}) : \nabla \Theta dx\Big). \end{aligned}$$

Using $\lambda \mathbf{u} + \Delta \mathbf{u} - A\mathbf{u} - \nabla p = 0$ we deduce

$$h_F\|R\|_{0,F}^2 \lesssim \sum_{\tau\in\omega(F)}\int_\tau(\lambda_h\mathbf{u}_h+\Delta\mathbf{u}_h-A\mathbf{u}_h-\nabla p_h)\cdot\Theta dx+\sum_{\tau\in\omega(F)}\int_\tau(\lambda\mathbf{u}-\lambda_h\mathbf{u}_h)\cdot\Theta dx$$
$$+\sum_{\tau\in\omega(F)}\int_\tau(-\nabla(u-u_h)+(p-p_h)\mathbf{I}):\nabla\Theta dx+\sum_{\tau\in\omega(F)}\int_\tau(A\mathbf{u}-A\mathbf{u}_h)\cdot\Theta dx$$
$$\equiv T_1+T_2+T_3+T_4. \tag{3.33}$$

Using the Cauchy-Schwartz inequality, (3.27) and (3.28) yieids

$$T_1 \lesssim \left(\sum_{\tau\in\omega(F)}\eta_{R_\tau}^2\right)^{1/2}\left(\sum_{\tau\in\omega(F)}h_\tau^{-2}\|\Theta\|_{0,\tau}^2\right)^{1/2} \lesssim \left(\sum_{\tau\in\omega(F)}\eta_{R_\tau}^2\right)^{1/2} h_F^{1/2}\|R\|_{0,F},$$
$$T_2 \lesssim \left(\sum_{\tau\in\omega(F)}\left(h_\tau^2\|\lambda\mathbf{u}-\lambda_h\mathbf{u}_h\|_{0,\tau}^2\right)\right)^{1/2} h_F^{1/2}\|R\|_{0,F},$$
$$T_3 \lesssim \left(\sum_{\tau\in\omega(F)}(\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\tau}^2+\|p-p_h\|_{0,\tau}^2)\right)^{1/2} h_F^{1/2}\|R\|_{0,F},$$
$$T_4 \lesssim \left(\sum_{\tau\in\omega(F)}\left(h_\tau^2\|A\mathbf{u}-A\mathbf{u}_h\|_{0,\tau}^2\right)\right)^{1/2} h_F^{1/2}\|R\|_{0,F}.$$

Combing the above estimates of T_1, T_2, T_3 and T_4, dividing (3.33) by $h_F^{1/2}\|R\|_{0,F}$ and summing over all interior edges of τ gives us (3.30).

For any $F\in\mathcal{E}_h^i(\Omega)$, $[\![\underline{\mathbf{u}}]\!]=0$ and for any $F\in\mathcal{E}_h\cap\partial\Omega$, $\mathbf{u}\otimes\mathbf{n}=0$. Therefore, we obtain (3.31) and finish the proof.

Theorem 3.3. Suppose that the conditions of Theorem 2.1 hold. Then, the a posteriori error estimator η_h is efficient:

$$\eta_\tau^2 \lesssim \sum_{\tau\in\omega(\tau)}(\|\mathbf{u}-\mathbf{u}_h\|_{0,\tau}^2+\|p-p_h\|_{0,\tau}^2+h_\tau^2\|\lambda\mathbf{u}-\lambda_h\mathbf{u}_h\|_{0,\tau}^2), \tag{3.34}$$
$$\eta_h^2 \lesssim \|\mathbf{u}-\mathbf{u}_h\|_h^2+\|p-p_h\|^2+\sum_{\tau\in\mathcal{T}_h}h_\tau^2\|\lambda\mathbf{u}-\lambda_h\mathbf{u}_h\|_{0,\tau}^2. \tag{3.35}$$

Lemma 3.6. Let $(\lambda,\mathbf{u},p)$ and $(\lambda_h,\mathbf{u}_h,p_h)$ be the eigenpairs of (2.2), (2.3) and (2.4), (2.5), respectively. Then,

$$\lambda_h-\lambda=\mathbb{A}_h(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h)+2\mathbb{B}_h(\mathbf{u}-\mathbf{u}_h,p-p_h)-\lambda(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h). \tag{3.36}$$

Proof. By using (2.16) and (2.17) we get

$$\mathbb{A}_h(\mathbf{u},\mathbf{z}_h)+\mathbb{B}_h(\mathbf{z}_h,p)=\lambda(\mathbf{u},\mathbf{z}_h),\quad \forall\mathbf{z}_h\in\mathbb{X}_h, \tag{3.37}$$
$$\mathbb{B}_h(\mathbf{u},\varrho_h)=0,\quad \forall\varrho_h\in\mathbb{W}_h. \tag{3.38}$$

From (2.2) and (2.3) with $(\mathbf{z},\varrho) = (\mathbf{u}, p)$, (2.4) and (2.5) with $(\mathbf{z}_h,\varrho_h) = (\mathbf{u_h}, p_h)$ and (3.37), (3.38) we deduce

$$\begin{aligned} & \mathbb{A}_h(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h) + 2\mathbb{B}_h(\mathbf{u}-\mathbf{u}_h, p-p_h) - \lambda(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h) \\ = \quad & \mathbb{A}_h(\mathbf{u},\mathbf{u}) - 2\mathbb{B}_h(\mathbf{u},\mathbf{u}_h) + \mathbb{A}_h(\mathbf{u}_h,\mathbf{u}_h) + 2\mathbb{B}_h(\mathbf{u}, p) - 2\mathbb{B}_h(\mathbf{u}_h, p) \\ & -2\mathbb{B}_h(\mathbf{u}, p_h) + 2\mathbb{B}_h(\mathbf{u}_h, p_h) - \lambda(\mathbf{u},\mathbf{u}) + 2\lambda(\mathbf{u},\mathbf{u}_h) - \lambda(\mathbf{u}_h,\mathbf{u}_h) \\ = \quad & \lambda_h(\mathbf{u}_h,\mathbf{u}_h) - \lambda(\mathbf{u}_h,\mathbf{u}_h) = \lambda_h - \lambda. \end{aligned}$$

We complete the proof.

Theorem 3.4. Under the conditions of Theorem 2.1,

$$|\lambda-\lambda_h| \lesssim \eta_h^2 + \sum_{\tau\in\mathcal{T}_h} h_\tau^{2r}(|\mathbf{u}-I_h\mathbf{u}|^2_{1+r,\tau} + \|p-\vartheta_h p\|_r^2). \tag{3.39}$$

Proof. Theorem 2.1 indicates that $\|\mathbf{u}-\mathbf{u}_h\|_0$ is a term of higher order than $|||\mathbf{u}-\mathbf{u}_h||| + \|p-p_h\|_0$. Hence, from (3.36) and (3.21), we obtain

$$|\lambda-\lambda_h| \lesssim |||\mathbf{u}-\mathbf{u}_h|||^2 + \|p-p_h\|_0^2 + \sum_{F\in\mathcal{E}_h} h_F\|p-p_h\|^2_{0,F}.$$

Thus, from (2.39) and (3.21) we obtain (3.39).

Remark 3.1. Theorem 2.1 indicates that $\|\lambda_h\mathbf{u}_h - \lambda\mathbf{u}\|_0$ is a small quantity of higher order than $\|\mathbf{u} - \mathbf{u}_h\|_h + \|p-p_h\|_0$. Theorems 3.2 and 3.3 show that the estimator η_h for the eigenfunction error $\|\mathbf{u}-\mathbf{u}_h\|_h + \|p-p_h\|_0$ is reliable and efficient up to data oscillation. Therefore, a good graded mesh is generated by the adaptive algorithm for the estimator, which makes the eigenfunction error $\|\mathbf{u}-\mathbf{u}_h\|_h + \|p-p_h\|_0$ reach the optimal convergence rate $O(dof^{-\frac{2k}{d}})$. Hence, from [37–39] we can look forward to getting $\sum_{\tau\in\mathcal{T}_h} h_\tau^{2r}(|\mathbf{u}-I_h\mathbf{u}|^2_{1+r,\tau} + \|p-\vartheta_h p\|_r^2) \lesssim dof^{-\frac{2k}{d}}$. Thereby from (3.39) we have $|\lambda-\lambda_h| \lesssim dof^{-\frac{2k}{d}}$. Therefore, we think η_h^2 can be regarded as the error estimator of λ_h.

Remark 3.2. Based on [17], for the problem (2.1) all analysis and conclusions in this paper are valid for the mixed DGFEM using the $\mathbb{Q}_k - \mathbb{Q}_{k-1}$ element.

4. Numerical experiments

When λ is a multiple eigenvalue the exact eigenfunction approximated by the discrete eigenfunction will change with the change of mesh diameter. In order to implement the adaptive algorithm better, we will conduct our numerical experiments on simple eigenpairs (multiplicity 1).

Based on [40–42], we design an adaptive DGFEM algorithm (ADGFEM) by adopting the standard adaptive loop with the steps *solve, estimate, mark* and *refine* with the a posteriori error estimator given in Section 3. We compile our program with the help of the iFEM package [43] and solve the matrix eigenvalue problem by means of the command $'eigs'$ in MATLAB.

We adapt the following symbols in our tables:

l: the lth iteration.

λ_{k,h_l}: the kth approximate eigenvalue at the lth iteration.

dof: the degrees of freedom at the lth iteration.

4.1. The numerical results for the classical Stokes eigenvalue problem

4.1.1. The results for two-dimensional domains

The experiment is conducted in three two-dimensional domains the slit domain $\Omega_{slit} = (-1,1)^2 \setminus \{0 \le x \le 1, y = 0\}$, the L-shaped domain $\Omega_L = (-1,1)^2 \setminus [0,1] \times [-1,0]$ and the unit square domain $\Omega_{square} = (0,1)^2$. In the step *mark* we select the parameter $\theta = 0.5$, and the initial mesh π_{h_0} with $h_0 = \frac{\sqrt{2}}{16}$ for the above three two-dimensional domains.

The reference values for the first eigenvalue of the classical Stokes eigenvalue problem are $\lambda_{1,slit} = 29.9168629$, $\lambda_{1,L} = 32.13269465$ and $\lambda_{1,square} = 52.344691168$ for Ω_{slit}, Ω_L and Ω_{square}, respectively (see [1, 2]) and the reference value for the fourth eigenvalue reads $\lambda_{4,square} = 128.209584313$ in Ω_{square} (see [2]). We choose the values $\lambda_{4,slit} = 40.1527333966$ and $\lambda_{4,L} = 48.9835839778$ as the reference values for the Ω_{slit} and Ω_L respectively, which are obtained by adaptive procedure using $\mathbb{P}_3 - \mathbb{P}_2$ element with as much degrees of freedom as possible.

The error curves for the first eigenvalue of the classical Stokes eigenvalue problem are shown in Figures 1–3 and the fourth eigenvalue are shown in Figures 4–6. The adaptive refined meshes for the first eigenvalue of the classical Stokes eigenvalue problem by the ADGFEM are shown in Figure 7.

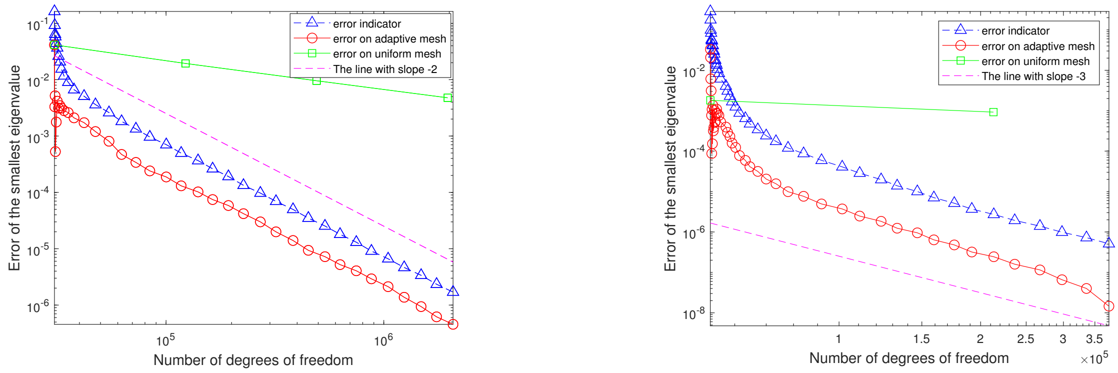


Figure 1. The error curves of the first eigenvalue by the ADGFEM using $\mathbb{P}_2-\mathbb{P}_1$ element (left) and $\mathbb{P}_3 - \mathbb{P}_2$ element (right) for the classical Stokes eigenvalue problem in Ω_{slit}.

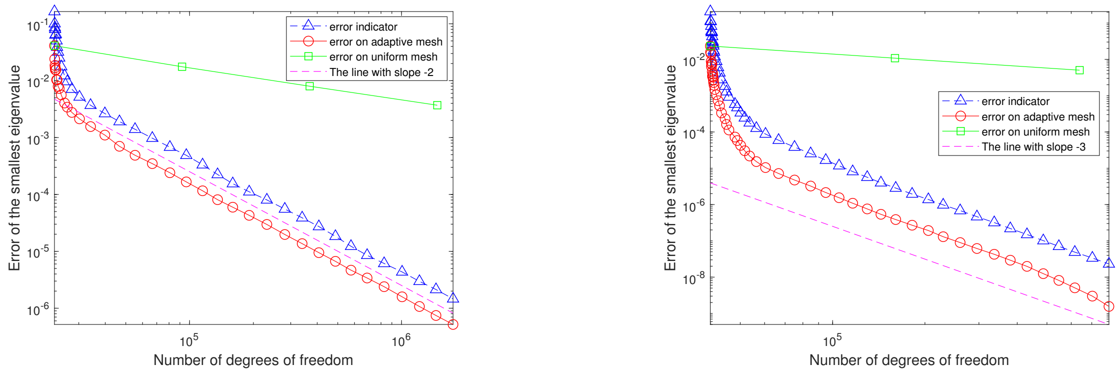


Figure 2. The error curves of the first eigenvalue by the ADGFEM using $\mathbb{P}_2-\mathbb{P}_1$ element (left) and $\mathbb{P}_3 - \mathbb{P}_2$ element (right) for the classical Stokes eigenvalue problem in Ω_L.

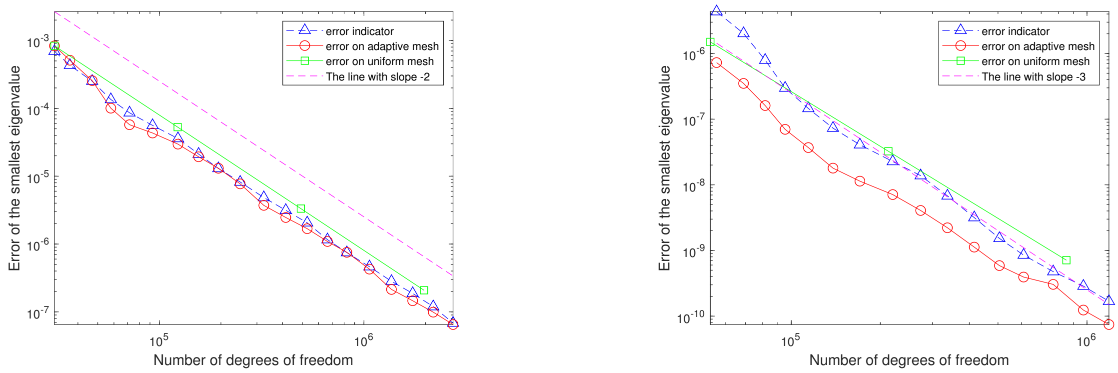


Figure 3. The error curves of the first eigenvalue by the ADGFEM using $\mathbb{P}_2-\mathbb{P}_1$ element (left) and $\mathbb{P}_3-\mathbb{P}_2$ element (right) for the classical Stokes eigenvalue problem in Ω_{square}.

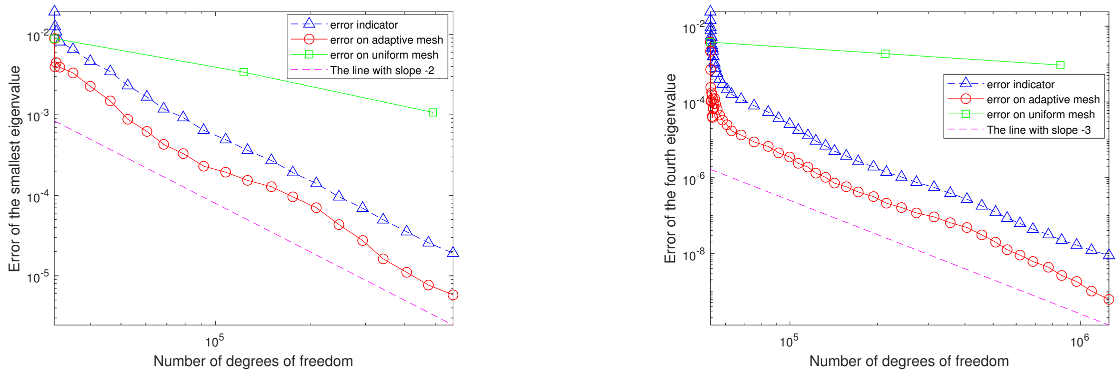


Figure 4. The error curves of the fourth eigenvalue by the ADGFEM using $\mathbb{P}_2-\mathbb{P}_1$ element (left) and $\mathbb{P}_3-\mathbb{P}_2$ element (right) for the classical Stokes eigenvalue problem in Ω_{slit}.

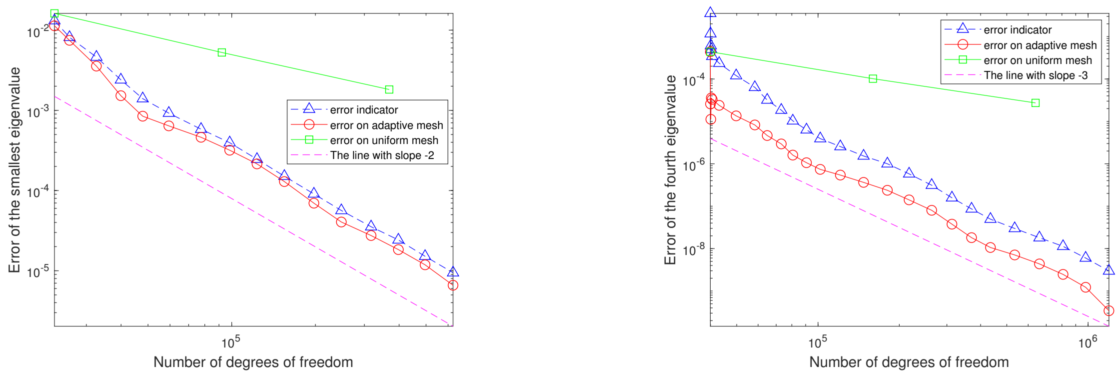


Figure 5. The error curves of the fourth eigenvalue by the ADGFEM using $\mathbb{P}_2-\mathbb{P}_1$ element (left) and $\mathbb{P}_3-\mathbb{P}_2$ element (right) for the classical Stokes eigenvalue problem in Ω_L.

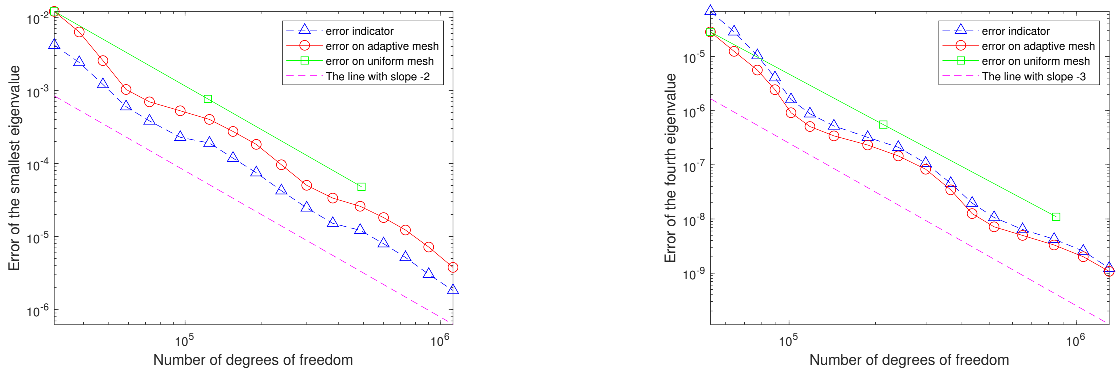


Figure 6. The error curves of the fourth eigenvalue by the ADGFEM using $\mathbb{P}_2 - \mathbb{P}_1$ element (left) and $\mathbb{P}_3 - \mathbb{P}_2$ element (right) for the classical Stokes eigenvalue problem in Ω_{square}.

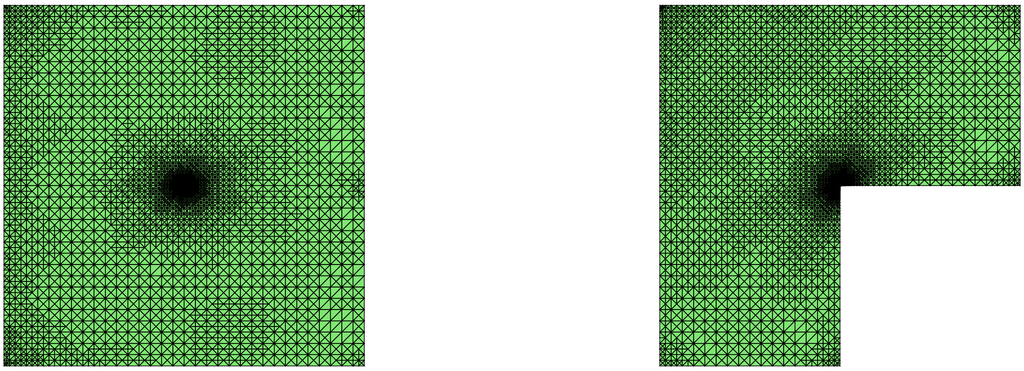


Figure 7. The adaptive meshes for the first eigenvalue of the classical Stokes eigenvalue problem by the ADGFEM at $l = 25$ refinement times using $\mathbb{P}_3 - \mathbb{P}_2$ element in Ω_{slit} (left) and Ω_L (right).

We observe from Figures 1–6 that the error curves and error estimators curves for ADGFEM are both almost parallel to the straight line with a slope of $-k$ which indicates that the error estimators are reliable and efficient and the adaptive algorithm can achieve the optimal convergence order. This is consistent with our theoretical results. We also observe from the error curves that under the same dof the approximations obtained by the ADGFEM are more accurate than those computed on uniform meshes.

The approximations of the first eigenvalue obtained by $\mathbb{P}_3 - \mathbb{P}_2$ element in Ω_{slit}, Ω_L and Ω_{square} are listed in Tables 1–3, respectively. These eigenvalues have the same accuracy as those in [1, 2] which achieve 9, 10 and 11 significant digits in Ω_{slit}, Ω_L and Ω_{square}, respectively. Furthermore, it shows that our method is effective. The approximations of the fourth eigenvalue obtained by $\mathbb{P}_3 - \mathbb{P}_2$ element in Ω_{slit}, Ω_L and Ω_{square} are listed in Tables 4–6, respectively.

Table 1. The approximation of the first eigenvalue of the classical Stokes eigenvalue problem in Ω_{slit} obtained by the ADGFEM using $\mathbb{P}_3 - \mathbb{P}_2$ element.

l	dof	λ_{1,h_l}	l	dof	λ_{1,h_l}
1	53248	29.950023991	26	63206	29.916921865
5	53560	29.917626784	30	73424	29.916878484
10	54028	29.917180037	35	110630	29.916865373
15	54756	29.917731006	40	175812	29.916863378
25	61412	29.916940636	50	537862	29.916862882

Table 2. The approximation of the first eigenvalue of the classical Stokes eigenvalue problem in Ω_L obtained by the ADGFEM using $\mathbb{P}_3 - \mathbb{P}_2$ element.

l	dof	λ_{1,h_l}	l	dof	λ_{1,h_l}
1	39936	32.155997914	27	53612	32.132716405
5	40248	32.139031080	31	75140	32.132699385
15	41288	32.134171324	41	229424	32.132694780
24	48880	32.132752576	50	703092	32.132694653
25	50128	32.132737367	51	796276	32.132694652
26	51688	32.132725042			

Table 3. The approximation of the first eigenvalue of the classical Stokes eigenvalue problem in Ω_{square} obtained by the ADGFEM using $\mathbb{P}_3 - \mathbb{P}_2$ element.

l	dof	λ_{1,h_l}	l	dof	λ_{1,h_l}
1	53248	52.3446926681	10	273780	52.3446911721
5	95316	52.3446912380	14	610376	52.3446911684
8	170612	52.3446911794	17	1186328	52.3446911679
9	220324	52.3446911751			

Table 4. The approximation of the fourth eigenvalue of the classical Stokes eigenvalue problem in Ω_{slit} obtained by the ADGFEM using $\mathbb{P}_3 - \mathbb{P}_2$ element.

l	dof	λ_{4,h_l}	l	dof	λ_{4,h_l}
1	53248	40.1565119894	27	106860	40.1527357945
15	55068	40.1528559112	41	457002	40.1527334275
25	91416	40.1527379227	51	1421836	40.1527333968
26	99788	40.1527369150	52	1618838	40.1527333966

Table 5. The approximation of the fourth eigenvalue of the classical Stokes eigenvalue problem in Ω_L obtained by the ADGFEM using $\mathbb{P}_3 - \mathbb{P}_2$ element.

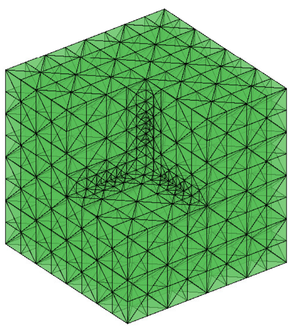
l	dof	λ_{4,h_l}	l	dof	λ_{4,h_l}
1	39936	48.9840225306	15	147472	48.9835843422
2	39988	48.9836097839	16	180648	48.9835842150
5	40560	48.9836170562	19	313248	48.9835840155
10	73112	48.9835869187	24	807768	48.9835839802
13	101920	48.9835847163	27	1414244	48.9835839778
14	120952	48.9835845197			

Table 6. The approximation of the fourth eigenvalue of the classical Stokes eigenvalue problem in Ω_{square} obtained by the ADGFEM using $\mathbb{P}_3 - \mathbb{P}_2$ element.

l	dof	λ_{4,h_l}	l	dof	λ_{4,h_l}
1	53248	128.2096127378	10	299052	128.2095843960
2	64376	128.2095967670	11	365664	128.2095843475
7	143312	128.2095846555	16	1057888	128.2095843150
8	187824	128.2095845453	17	1302184	128.2095843141
9	240084	128.2095844591			

4.1.2. The results for three-dimensional domains

The experiment is conducted in two three-dimensional domains: $\Omega_1 = (0,1)^3 \setminus \{0 \le x \le 0.5, 0 \le y \le 0.5, 0.5 \le z \le 1\}$ and $\Omega_2 = (0,1)^3$. In computation we select the initial mesh π_{h_0} with $h_0 = \frac{\sqrt{3}}{8}$ and $\theta = 0.25$.



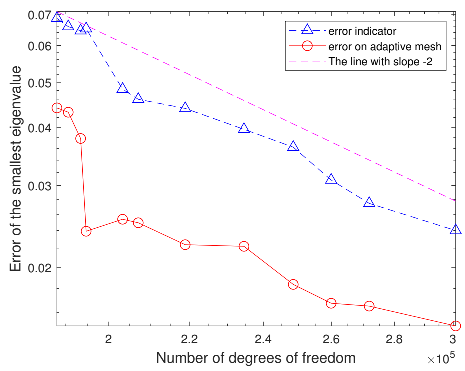


Figure 8. Adaptive mesh after l=12 refinement times (left) and the error curves (right) of the first eigenvalue by the ADGFEM using $\mathbb{P}_3 - \mathbb{P}_2$ element for the classical Stokes eigenvalue problem in Ω_1.

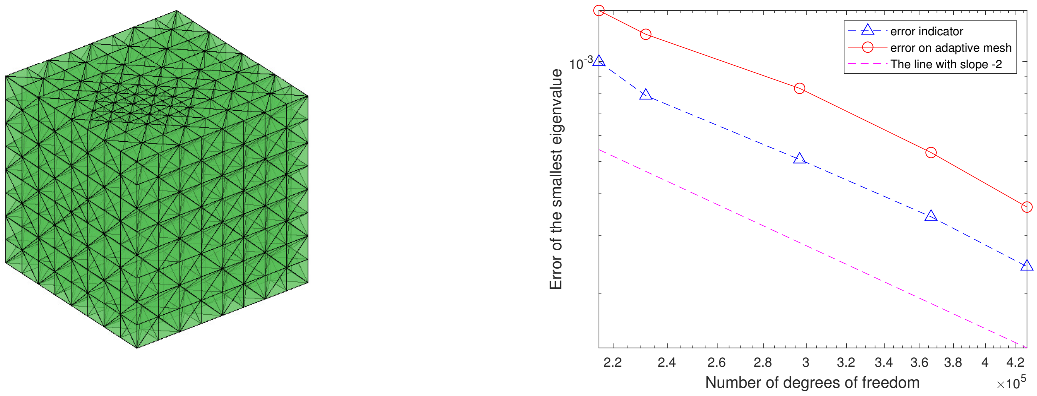


Figure 9. Adaptive mesh after l=5 refinement times (left) and the error curves (right) of the first eigenvalue by the ADGFEM using $\mathbb{P}_3 - \mathbb{P}_2$ element for the classical Stokes eigenvalue problem in Ω_2.

The reference values for the first eigenvalue of the classical Stokes eigenvalue problem are $\lambda_{\Omega_1} = 70.98560$ and $\lambda_{\Omega_2} = 62.17341$ for the domains Ω_1 and Ω_2 respectively, which are calculated by adaptive procedure with as much degrees of freedom as possible.

The adaptive refined meshes and the error curves are shown in Figures 8 and 9. We observe from Figures 8 and 9 that the error estimators are reliable and efficient and the adaptive algorithm achieve the optimal convergence order.

4.2. The numerical results for the MHD Stokes eigenvalue problem

We conduct experiments in Ω_L and Ω_{square}. We select $\theta = 0.5$ and the initial mesh π_{h_0} with $h_0 = \frac{\sqrt{2}}{16}$ for the above two two-dimensional domains.

For the MHD Stokes eigenvalue problem with $Ha = 5$, we choose the values $\lambda_{1,square} = 64.68920947$ and $\lambda_{1,L} = 40.2764915$ as the reference values for Ω_{square} and Ω_L, respectively, and while $Ha = 30$, we choose the values $\lambda_{1,square} = 234.34458093$ and $\lambda_{1,L} = 125.24247135$ as the reference values for Ω_{square} and Ω_L respectively. These reference values are obtained by the ADGFEM using $\mathbb{P}_3 - \mathbb{P}_2$ element with as much degrees of freedom as possible.

The error curves for the first eigenvalue are shown in Figures 10 and 11 and the adaptive refined meshes for the first eigenvalue of the MHD Stokes eigenvalue problem by the ADGFEM are shown in Figure 12. We observe from Figures 10 and 11 that the error curves and error estimators curves are both approximately parallel to the line with slope $-k$, which indicates that the error estimators are reliable and efficient and the adaptive algorithm can achieve the optimal convergence order.

The approximations of the first eigenvalue for the MHD Stokes eigenvalue problem in Ω_{square} using $\mathbb{P}_3 - \mathbb{P}_2$ element are listed in Tables 7 and 8, from which we can see that the approximate eigenvalues also has high accuracy.

We also use the ADGFEM with $\mathbb{P}_k - \mathbb{P}_{k-1} (k = 1, 2)$ element to calculate the classical Stokes eigenvalue problem and the MHD Stokes eigenvalue problem. The numerical results indicate that the discrete formulations are stable and effective. Due to article length limitations, these results are not listed in the paper.

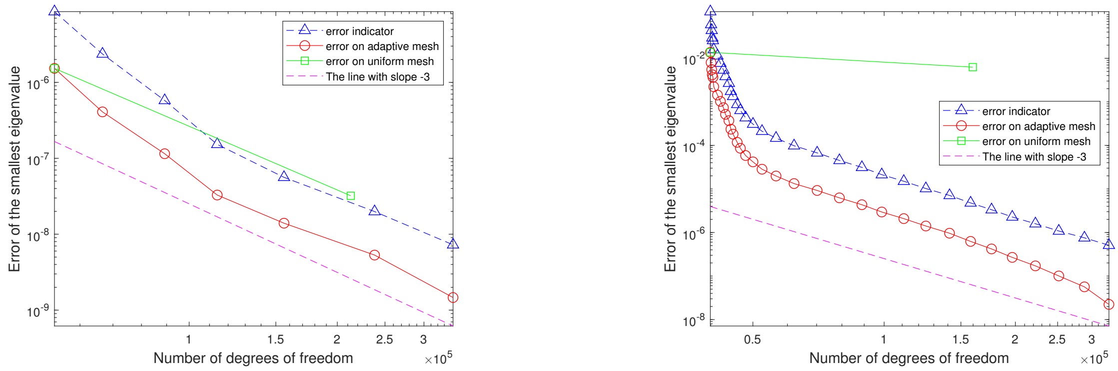


Figure 10. The error curves of the first eigenvalue by the ADGFEM using $\mathbb{P}_3 - \mathbb{P}_2$ element for the MHD Stokes eigenvalue problem with $Ha = 5$ in Ω_{square} (left) and Ω_L (right).

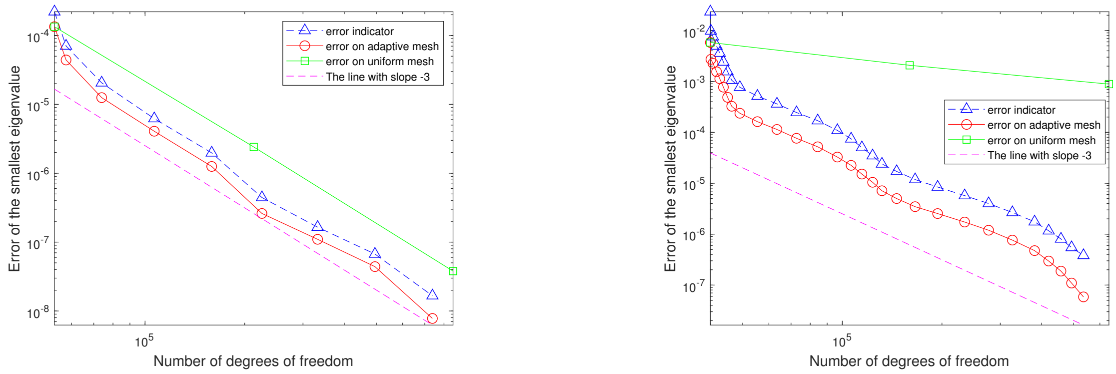


Figure 11. The error curves of the first eigenvalue by the ADGFEM using $\mathbb{P}_3 - \mathbb{P}_2$ element for the MHD Stokes eigenvalue problem with $Ha = 30$ in Ω_{square} (left) and Ω_L (right).

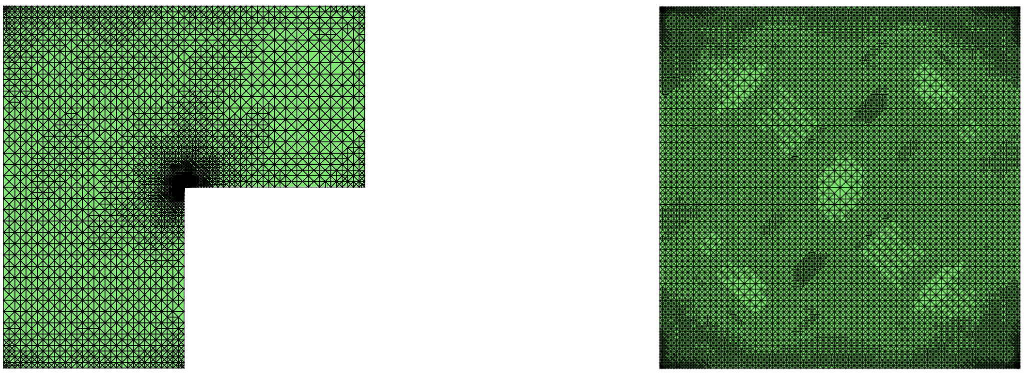


Figure 12. The adaptive meshes for the first eigenvalue of the MHD Stokes eigenvalue problem when $Ha = 5$ by the ADGFEM at $l = 25$ refinement times using $\mathbb{P}_3 - \mathbb{P}_2$ element in Ω_L (left) and at $l = 8$ refinement times using $\mathbb{P}_3 - \mathbb{P}_2$ element in Ω_{square} (right).

Table 7. The approximation of the first eigenvalue of the MHD Stokes eigenvalue problem with $Ha = 5$ in Ω_{square} obtained by the ADGFEM using $\mathbb{P}_3 - \mathbb{P}_2$ element.

l	dof	λ_{1,h_l}	l	dof	λ_{1,h_l}
1	53248	64.689210998	5	155844	64.689209484
2	66664	64.689209879	6	238108	64.689209476
3	89180	64.689209585	7	343720	64.689209472
4	114036	64.689209503	8	482144	64.689209470

Table 8. The approximation of the first eigenvalue of the MHD Stokes eigenvalue problem with $Ha = 30$ in Ω_{square} obtained by the ADGFEM using $\mathbb{P}_3 - \mathbb{P}_2$ element.

l	dof	λ_{1,h_l}	l	dof	λ_{1,h_l}
1	53248	234.34471492	6	225472	234.34458119
2	57720	234.34462502	7	332124	234.34458104
3	73944	234.34459345	8	494832	234.34458097
4	106600	234.34458500	9	738192	234.34458094
5	158964	234.34458219	10	994656	234.34458093

5. Conclusions

In this paper, for a class of Stokes eigenvalue problems including the classical Stokes eigenvalue problem in R^d $(d = 2, 3)$ and the MHD Stokes eigenvalue problem et al, based on the velocity-pressure formulation we studied the residual type a posteriori error estimates of the mixed DGFEM using $\mathbb{P}_k - \mathbb{P}_{k-1}$ $(k \geq 1)$ element on shape-regular simplex meshes. We proposed the a posteriori error estimator for approximate eigenpairs and proved the reliability and efficiency of the estimator for eigenfunctions and also analyzed their reliability for eigenvalues. The characteristic of the adaptive DGFEM is that it can use high-order elements and capture local low smooth solutions and can achieve the optimal convergence order $O(dof^{-\frac{2k}{d}})$ in two and three-dimensional domains. Our method is easy to implement on existing software packages. The numerical results confirmed our theoretical predictions and showed that our method is stable, efficient and can obtain high-accuracy approximate eigenvalues.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Nos. 11561014, 11761022), and Science and Technology Planning Project of Guizhou Province (Guizhou Kehe fundamental research-ZK[2022] No.324).

Conflict of interest

The authors declare that this work does not have any conflicts of interest.

References

1. J. Gedicke, A. Khan, Arnold-Winther mixed finite elements for Stokes eigenvalue problems, *SIAM J. Sci. Comput.*, **40** (2018), A3449–A3469. https://doi.org/10.1137/17M1162032
2. J. Gedicke, A. Khan, Divergence-conforming discontinuous Galerkin finite elements for Stokes eigenvalue problems, *Numer. Math.*, **144** (2020), 585–614. https://doi.org/10.1007/s00211-019-01095-x
3. F. Lepea, D. Mora, Symmetric and nonsymmetric discontinuous Galerkin methods for a pseudostress formulation of the Stokes spectral problem, *SIAM J. Sci. Comput.*, **42** (2020), A698–A722. https://doi.org/10.1137/19M1259535
4. F. Lepe, G. Rivera, A virtual element approximation for the pseudostress formulation of the Stokes eigenvalue problem, *Comput. Methods Appl. Mech. Eng.*, **379** (2021), 113753. https://doi.org/10.1016/j.cma.2021.113753
5. B. Mercier, J. Osborn, J. Rappaz, P. A. Raviart, Eigenvalue approximation by mixed and hybrid methods, *Math. Comp.*, **36** (1981), 427–453. https://doi.org/10.2307/2007651
6. L. L. Sun, Y. D. Yang, The a posteriori error estimates and adaptive computation of nonconforming mixed finite elements for the Stokes eigenvalue problem, *Appl. Math. Comput.*, **421** (2022), 126951. https://doi.org/10.1016/j.amc.2022.126951
7. L. L. Sun, H. Bi, Y. D. Yang, The discontinuous Galerkin and the nonconforming ECR element approximations for an MHD Stokes eigenvalue problem, *Math. Methods Appl. Sci.*, **46** (2022), 6154–6176. https://doi.org/10.1002/mma.8897
8. Ö. Türk, An MHD Stokes eigenvalue problem and its approximation by a spectral collocation method, *Comput. Math. Appl.*, **80** (2020), 2045–2056. https://doi.org/10.1016/j.camwa.2020.09.002
9. M. G. Armentano, V. Moreno, A posteriori error estimates of stabilized low-order mixed finite elements for the Stokes eigenvalue problem, *J. Comput. Appl. Math.*, **269** (2014), 132–149. https://doi.org/10.1016/j.cam.2014.03.027
10. H. P. Liu, W. Gong, S. H. Wang, N. N. Yan, Superconvergence and a posteriori error estimates for the Stokes eigenvalue problems, *BIT Numer. Math.*, **53** (2013), 665–687. https://doi.org/10.1007/s10543-013-0422-8
11. C. Lovadina, M. Lyly, R. Stenberg, A posteriori estimates for the Stokes eigenvalue problem, *Numer. Methods Partial Differ. Equ.*, **25** (2009), 244–257. https://doi.org/10.1002/num.20342
12. S. H. Jia, F. S. Lu, H. H. Xie, A Posterior error analysis for the nonconforming discretization of Stokes eigenvalue problem, *Acta Math. Sin. Engl. Ser.*, **30** (2014), 949–967. https://doi.org/10.1007/s10114-014-3121-8
13. F. Lepe, G. Rivera, J. Vellojin, Error estimates for a vorticity-based velocity-stress formulation of the Stokes eigenvalue problem, *J. Comput. Appl. Math.*, **420** (2023), 114798. https://doi.org/10.1016/j.cam.2022.114798

14. S. Badia, R. Codina, T. Gudi, J. Guzmán, Error analysis of discontinuous Galerkin methods for the Stokes problem under minimal regularity, *IMA. J. Numer. Anal.*, **34** (2014), 800–819. https://doi.org/10.1093/imanum/drt022
15. B. Cockburn, G. E. Karniadakis, C. W. Shu, *Discontinuous Galerkin methods: theory, computation and applications*, Berlin, Heidelberg: Springer, 2000. https://doi.org/10.1007/978-3-642-59721-3
16. A. Ern, J. Guermond, *Finite elements II: Galerkin approximation, elliptic and mixed PDEs*, Cham: Springer, 2021. https://doi.org/10.1007/978-3-030-56923-5
17. P. Houston, D. Schötzau, T. P. Wihler, Energy norm a posteriori error estimation for mixed discontinuous Galerkin approximations of the Stokes problem, *J. Sci. Comput.*, **22** (2005), 347–370. https://doi.org/10.1007/s10915-004-4143-7
18. P. Houston, D. Schötzau, T. P. Wihler, *hp*-adaptive discontinuous Galerkin finite element methods for the Stokes problem, *European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS)*, 2004, 1–20.
19. D. A. Di Pietro, A. Ern, Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier-Stokes equations, *Math. Comp.*, **79** (2010), 1303–1330. https://doi.org/10.1090/s0025-5718-10-02333-1
20. M. Ainsworth, J. T. Oden, *A posteriori error estimation in finite element analysis*, New York: Wiley, 2011.
21. R. Verfürth, *A posteriori error estimation techniques*, New York: Oxford University Press, 1996.
22. S. C. Brenner, Poincaré-Friedrichs inequalities for piecewise H^1 functions, *SIAM J. Numer. Anal.*, **41** (2003), 306–324. https://doi.org/10.1137/S0036142902401311
23. O. A. Karakashian, F. Pascal, A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems, *SIAM J. Numer. Anal.*, **41** (2003), 2374–2399. https://doi.org/10.1137/s0036142902405217
24. I. Perugia, D. Schötzau, The *hp*-local discontinuous Galerkin method for low-frequency time-harmonic Maxwell equations, *Math. Comp.*, **72** (2003), 1179–1214. https://doi.org/10.1090/s0025-5718-02-01471-0
25. D. Schötzau, C. Schwab, A. Toselli, Mixed *hp*-DGFEM for incompressible flows, *SIAM J. Numer. Anal.*, **40** (2002), 2171–2194. https://doi.org/10.1137/s0036142901399124
26. P. Houston, I. Perugia, D. Schötzau, An a posteriori error indicator for discontinuous Galerkin discretizations of H(curl)-elliptic partial differential equations, *IMA J. Numer. Anal.*, **27** (2007), 122–150. https://doi.org/10.1093/imanum/drl012
27. B. Rivière, *Discontinuous Galerkin methods for solving elliptic and parabolic equations: theory and implementation*, SIAM, 2008.
28. V. Girault, P. A. Raviart, *Finite element approximation of the Navier-Stokes equations*, Berlin, Heidelberg: Springer, 1979.
29. B. Cockburn, G. Kanschat, D. Schötzau, C. Schwab, Local discontinuous Galerkin methods for the Stokes system, *SIAM J. Numer. Anal.*, **40** (2002), 319–343. https://doi.org/10.1137/S0036142900380121

30. P. Hansbo, M. G. Larson, Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method, *Comput. Methods Appl. Mech. Eng.*, **191** (2002), 1895–1908. https://doi.org/10.1016/S0045-7825(01)00358-9
31. Y. D. Yang, Z. M. Zhang, F. B. Lin, Eigenvalue approximation from below using non-conforming finite elements, *Sci. China Ser. A Math.*, **53** (2010), 137–150. https://doi.org/10.1007/s11425-009-0198-0
32. I. Babuska, J. Osborn, Eigenvalue problems, In: *Handbook of numerical analysis*, Elsevier, 1991, 641–787. https://doi.org/10.1016/S1570-8659(05)80042-0
33. D. Boffi, Finite element approximation of eigenvalue problems, *Acta Numer.*, **19** (2010), 1–120. https://doi.org/10.1017/S0962492910000012
34. L. R. Scott, S. Y. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions, *Math. Comp.*, **54** (1990), 483–493. https://doi.org/10.1090/S0025-5718-1990-1011446-7
35. G. Kanschat, D. Schötzau, Energy norm a posteriori error estimation for divergence-free discontinuous Galerkin approximations of the Navier-Stokes equations, *Int. J. Numer. Methods Fluids*, **57** (2008), 1093–1113. https://doi.org/10.1002/fld.1795
36. Y. P. Zeng, F. Wang, A posteriori error estimates for a discontinuous Galerkin approximation of Steklov eigenvalue problems, *Appl. Math.*, **62** (2017), 243–267. https://doi.org/10.21136/AM.2017.0115-16
37. H. J. Wu, Z. M. Zhang, Can we have superconvergent gradient recovery under adaptive meshes? *SIAM J. Numer. Anal.*, **45** (2007), 1701–1722. https://doi.org/10.1137/060661430
38. Y. D. Yang, Y. Zhang, H. Bi, A type of adaptive C^0 non-conforming finite element method for the Helmholtz transmission eigenvalue problem, *Comput. Methods Appl. Mech. Eng.*, **360** (2020), 112697. https://doi.org/10.1016/j.cma.2019.112697
39. Y. J. Li, H. Bi, Y. D. Yang, The a priori and a posteriori error estimates of DG method for the Steklov eigenvalue problem in inverse scattering, *J. Sci. Comput.*, **91** (2022), 20. https://doi.org/10.1007/s10915-022-01787-x
40. X. Y. Dai, J. C. Xu, A. H. Zhou, Convergence and optimal complexity of adaptive finite element eigenvalue computations, *Numer. Math.*, **110** (2008), 313–355. https://doi.org/10.1007/s00211-008-0169-3
41. W. Dörfler, A convergent adaptive algorithm for Poisson's equation, *SIAM J. Numer. Anal.*, **33** (1996), 1106–1124. https://doi.org/10.1137/0733054
42. P. Morin, R. H. Nochetto, K. G. Siebert, Convergence of adaptive finite element methods, *SIAM Rev.*, **44** (2002), 631–658. https://doi.org/10.1137/s0036144502409093
43. L. Chen, iFEM: An integrated finite element method package in Matlab, Technical Report, University of California at Irvine, 2009.