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Research article

Some new applications of the quantum-difference operator on subclasses of multivalent q-starlike and q-convex functions associated with the Cardioid domain

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Abstract: In this study, we consider the quantum difference operator to define new subclasses of multivalent q-starlike and q-convex functions associated with the cardioid domain. We investigate a number of interesting problems for functions that belong to these newly defined classes, such as bounds for the first two Taylor-Maclaurin coefficients, estimates for the Fekete-Szegö type functional, and coefficient inequalities. The important point of this article is that all the bounds that we have investigated are sharp. Many well-known corollaries are also presented to demonstrate the relationship between prior studies and the results of this article.

Keywords: analytic functions; univalent functions; quantum difference operator; q-starlike functions and q-convex functions; Fibonacci numbers; multivalent functions; cardioid domain
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1. Introduction and definitions

Assume that $\mathcal{A}$ represents the family of analytic functions in the open unit disc

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

For $f_1, f_2 \in \mathcal{A}$, we say that f_1 subordinate to f_2 in $\mathcal{U}$, indicated by

$$f_1(z) \prec f_2(z), \quad z \in \mathcal{U},$$

if there exists a Schwarz function w, defined by

$$w \in \mathcal{B} = \{w : w \in \mathcal{A}, \ |w(z)| < 1 \text{ and } w(0) = 0, \ z \in \mathcal{U}\},$$

that satisfies the condition

$$f_1(z) = f_2(w(z)), \quad z \in \mathcal{U}.$$

Indeed, it is known that

$$f_1(z) \prec f_2(z) \Longrightarrow f_1(0) = f_2(0) \text{ and } f_1(\mathcal{U}) \subset f_2(\mathcal{U}).$$

Moreover, if the function f_2 is univalent in $\mathcal{U}$ then

$$f_1(z) \prec f_2(z) \Leftrightarrow f_1(0) = f_2(0) \text{ and } f_1(\mathcal{U}) \subset f_2(\mathcal{U}).$$

Let the class $\mathcal{P}$ be defined by

$$\mathcal{P} = \{h \in \mathcal{A} : h(0) = 1 \text{ and } \mathrm{Re}(h(z) > 0\}.$$

The class of all functions in the normalized analytic function class $\mathcal{A}$ that are univalent in $\mathcal{U}$ is also denoted by the symbol $\mathcal{S}$. The maximization of the non-linear functional $\left|a_3 - \mu a_2^2\right|$ or other classes and subclasses of univalent functions has been the subject of a number of established results and these results are known as Fekete-Szegö problems, see [1]. If $f \in \mathcal{S}$ and of the form (1.1), then

$$\left|a_3 - \mu a_2^2\right| \leq \left\{ \begin{array}{ll} 3 - 4\mu, & \text{if } \mu \leq 0, \\ 1 + 2\exp\left(\frac{2\mu}{\mu - 1}\right), & \text{if } 0 \leq \mu < 1, \\ 4\mu - 3, & \text{if } \mu \geq 1, \end{array} \right\}$$

and the result $\left|a_3 - \mu a_2^2\right|$ are sharp (see [1]). The Fekete-Szegö problems have a rich history in literature and for complex number μ.

In the area of geometric function theory (GFT), the q-calculus and fractional q-calculus have been extensively employed by scholars who have developed and investigated a number of novel subclasses of analytic, univalent and bi-univalent functions. Jackson [2, 3] first proposed the concept of the q-calculus operator and gave the definition of the q-difference operator D_q in 1909. In instance, Ismail et al. were the first to define a class of q-starlike functions in open unit disc $\mathcal{U}$ using D_q in [4]. The most significant usages of q-calculus in the perspective of GFT was basically furnished and the basic (or q-) hypergeometric functions were first used in GFT in a book chapter by Srivastava (see, for details, [5]). See the following articles [6–10] for more information about q-calculus operator theory in GFT.

Now we review some fundamental definitions and ideas of the q-calculus, we utilize them to create some new subclasses in this paper.

For a non-negative integer l, the q-number $[l]_q$, $(0 < q < 1)$, is defined by

$$[l]_q = \frac{1 - ql}{1 - q} \text{ and } [0] = 0,$$

and the q-number shift factorial is given by

$$\begin{aligned} [l]_q! &= [1]_q[2]_q[3]_q \cdots [l]_q, \\ [0]_q! &= 1. \end{aligned}$$

For $q \to 1^-$, then $[l]!$ reduces to $l!$.

The q-generalized Pochhammer symbol is defined by

$$[l]_k = \frac{\Gamma_q(l+k)}{\Gamma_q(l)}, \quad k \in \mathbb{N},\ l \in \mathbb{C}.$$

The q-gamma function Γ_q is defined by

$$\Gamma_q(l) = (1-q)^l \prod_{j=0}^{\infty} \frac{1-q^{j+1}}{1-q^{j+l}}.$$

The q-generalized Pochhammer symbol is defined by

$$[l]_k = \frac{\Gamma_q(l+k)}{\Gamma_q(l)}, \quad k \in \mathbb{N},\ l \in \mathbb{C}.$$

Remark 1. *For $q \to 1^-$, then $[l]_{q,k}$ reduces to $(l)_k = \frac{\Gamma(l+k)}{\Gamma(l)}$.*

Definition 1. Jackson [3] defined the q-integral of function $f(z)$ as follows:

$$\int f(z) d_q(z) = \sum_{n=0}^{\infty} z(1-q) f(q^n(z)) q^n.$$

Jackson [2] introduced the q-difference operator for analytic functions as follows:

Definition 2. [2] For $f \in \mathcal{A}$, the q-difference operator is defined as:

$$D_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \quad z \in \mathcal{U}.$$

Note that, for $n \in \mathbb{N}$, $z \in \mathcal{U}$ and

$$D_q(z^n) = [n]_q z^{n-1}, \quad D_q\left(\sum_{n=1}^{\infty} a_n z^n\right) = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}.$$

Let $\mathcal{A}_p$ stand for the class of analytic functions with the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \quad p \in \mathbb{N}, z \in \mathcal{U} \tag{1.1}$$

in the open unit disk $\mathcal{U}$. More specifically, $\mathcal{A}_1 = \mathcal{A}$ and

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1}, \quad z \in \mathcal{U}. \tag{1.2}$$

Consider the q-difference operator for $f \in \mathcal{A}_p$ as follows:

Definition 3. [11] For $f \in \mathcal{A}_p$, the q-difference operator is defined as:

$$D_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \quad z \in \mathcal{U}.$$

Note that, for $n \in \mathbb{N}$, $z \in \mathcal{U}$ and

$$D_q(z^{n+p}) = [n+p]_q z^{n+p-1}, \quad D_q\left(\sum_{n=1}^{\infty} a_{n+p} z^{n+p}\right) = \sum_{n=1}^{\infty} [n+p]_q a_{n+p} z^{n+p-1}.$$

Let $\mathcal{S}^*(p)$ represents the class of p-valent starlike functions and every $f \in \mathcal{S}^*(p)$, if

$$Re\left(\frac{zf'(z)}{pf(z)}\right) > 0, \quad z \in \mathcal{U},$$

and $\mathcal{K}(p)$ represents the class of p-valent convex functions and every $f \in \mathcal{K}(p)$, if

$$\frac{1}{p}\left(1 + Re\left(\frac{zf''(z)}{f'(z)}\right)\right) > 0, \quad z \in \mathcal{U}.$$

These conditions are equivalent in terms of subordination as follows:

$$\mathcal{S}^*(p) = \left\{f \in \mathcal{A}_p : \frac{zf'(z)}{pf(z)} \prec \frac{1+z}{1-z}\right\}$$

and

$$\mathcal{K}(p) = \left\{f \in \mathcal{A}_p : \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{1+z}{1-z}\right\}.$$

The aforementioned two classes can be generalized as follows:

$$\mathcal{S}^*(p,\varphi) = \left\{f \in \mathcal{A}_p : \frac{zf'(z)}{pf(z)} \prec \varphi(z)\right\}$$

and

$$\mathcal{K}(p,\varphi) = \left\{f \in \mathcal{A}_p : \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z)\right\},$$

where $\varphi(z)$ is a real part function that is positive and is normalized by the rule

$$\varphi(0) = 1 \text{ and } \varphi'(0) > 0,$$

and φ maps $\mathcal{U}$ onto a space that is symmetric with regard to the real axis and starlike with respect to 1. If $p = 1$, then

$$\mathcal{S}^*(p,\varphi) = \mathcal{S}^*(\varphi)$$

and

$$\mathcal{K}(p,\varphi) = \mathcal{K}(\varphi).$$

These two classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ defined by Ma [12].

A function $f \in \mathcal{A}_p$, is called p-valently starlike of order α $(0 \leq \alpha < 1)$ with complex order $b \in \mathbb{C}\backslash\{0\}$, if it satisfies the inequality

$$Re\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{pf(z)} - 1\right) > \alpha, \quad z \in \mathcal{U}\right\}.$$

The class $\mathcal{S}_p^*(\alpha, b)$ denotes the collection of all $f \in \mathcal{A}_p$ functions that satisfy the aforementioned condition.

A function $f \in \mathcal{A}_p$, is called p-valently convex function of order α $(0 \leq \alpha < 1)$ with complex order $b \in \mathbb{C}\backslash\{0\}$, if it satisfies the inequality

$$Re\left\{1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf^{''}(z)}{f^{'}(z)}\right) > \alpha, \;\; z \in \mathcal{U}\right\}.$$

The class $\mathcal{K}_p(\alpha, b)$ denotes the collection of all functions $f \in \mathcal{A}_p$ that satisfy the aforementioned condition.

Note that

$$f \in \mathcal{K}_p(\alpha, b) \Leftrightarrow \frac{1}{p}zf^{'} \in \mathcal{S}^*_p(\alpha, b).$$

Kargar et al. [13] investigated the classes $\mathcal{S}_p(\alpha,\beta)$ for $f \in \mathcal{A}_p$ and defined as follows:

$$f \in \mathcal{S}_p(\alpha,\beta) \Leftrightarrow \alpha < Re\left(\frac{1}{p}\frac{zf^{'}(z)}{f(z)}\right) < \beta, \quad (0 \leq \alpha < 1 < \beta, \; z \in \mathcal{U}).$$

For $0 \leq \alpha < 1 < \beta$ and $b \in \mathbb{C}\backslash\{0\}$, then the function $f \in \mathcal{A}_p$ belongs to the class $\mathcal{K}_{b,p}(\alpha,\beta)$ if it satisfies the inequality

$$\alpha < Re\left(1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf^{''}(z)}{f^{'}(z)}\right)\right) < \beta, \quad (0 \leq \alpha < 1 < \beta, \; z \in \mathcal{U}).$$

If $p = 1$, then $\mathcal{K}_{b,p}(\alpha,\beta) = \mathcal{K}_b(\alpha,\beta)$, studied by Kargar et al. in [13] and if $\beta \to \infty$ in above definition, $\mathcal{K}_{b,p}(\alpha,\beta) = \mathcal{K}_{b,p}(\alpha, b)$.

Recently, Bult [14] used the definition of subordination and defined new subclasses of p-valent starlike and convex functions associated with vertical strip domain as follows:

$$\mathcal{S}^*_{p,b}(\alpha,\beta) = \left\{f \in \mathcal{A}_p : 1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf^{'}(z)}{f(z)} - 1\right) \prec f(\alpha,\beta; z)\right\}$$

and

$$\mathcal{K}_{p,b}(\alpha,\beta) = \left\{f \in \mathcal{A}_p : 1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf^{''}(z)}{f^{'}(z)}\right) \prec f(\alpha,\beta; z)\right\},$$

where

$$f(\alpha,\beta; z) = 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z}\right)$$

and

$$0 \leq \alpha < 1 < \beta, \; b \in \mathbb{C}\backslash\{0\}, \; z \in \mathcal{U}.$$

Bult [14] determined the coefficient bounds for functions belonging to these new classes.

On the basis of the geometrical interpretation of their image domains, numerous subclasses of analytic functions have established using the concept of subordination. Right half plane [15], circular disc [16], oval and petal type domains [17], conic domain [18, 19], leaf-like domain [20], generalized conic domains [21], and the most important one is shell-like curve [22–25] are some fascinating geometrical classes we obtain with this domain. The function

$$h(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \tag{1.3}$$

is essential for the shell-like shape, where

$$\tau = \frac{1-\sqrt{5}}{2}.$$

The image of unit circle under the function h gives the conchoid of Maclaurin's, due to the function

$$h\left(e^{i\varphi}\right) = \frac{\sqrt{5}}{2\,(3-2\cos\varphi)} + i\frac{\sin\varphi\,(4\cos\varphi-1)}{2\,(3-2\cos\varphi)\,(1+\cos\varphi)}, \quad 0 \le \varphi < 2\pi.$$

The function given in (1.3) has the following series representation:

$$h(z) = 1 + \sum_{n=1}^{\infty} (u_{n-1} + u_{n+1})\,\tau^n z^n,$$

where

$$u_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}},$$

and u_n produces a Fibonacci series of coefficient constants that are more closely related to the Fibonacci numbers.

Taking motivation from the idea of circular disc and shell-like curves, Malik et al. [26] defined new domain for analytic functions which is named as cardioid domain. A new class of analytic functions is defined associated with cardioid domain, for more detail, see [26].

Definition 4. [26] Assume that $\mathcal{CP}\,(L,N)$ represents the class of functions p that are defined as

$$p(z) \prec \overline{p}\,(L,N,z)\,,$$

where $\overline{p}\,(L,N,z)$ is defined by

$$\overline{p}\,(L,N,z) = \frac{2L\tau^2 z^2 + (L-1)\,\tau z + 2}{2N\tau^2 z^2 + (N-1)\,\tau z + 2} \tag{1.4}$$

with $-1 < N < L \le 1$, $\tau = \frac{1-\sqrt{5}}{2}$ and $z \in \mathcal{U}$.

To understand the class $\mathcal{CP}\,(L,N)$, an explanation of the function $\overline{p}\,(L,N,z)$ in geometric terms might be helpful in this instance. If we denote

$$R_{\overline{p}}\left(L,N;e^{i\theta}\right) = u$$

and

$$I_{\overline{p}}\left(L,N;e^{i\theta}\right) = v,$$

then the image $\overline{p}\left(L, N, e^{i\theta}\right)$ of the unit circle is a cardioid like curve defined by

$$\left\{\begin{array}{l} u = \frac{4+(L-1)(N-1)\tau^2+4LN\tau^4+2\lambda\cos\theta+4(L+N)\tau^2\cos 2\theta}{4+(N-1)^2\tau^2+4N^2\tau^4+4(N-1)\left(\tau+N\tau^3\right)\cos\theta+8N\tau^2\cos 2\theta}, \\ v = \frac{(L-N)\left(\tau-\tau^3\right)\sin\theta+2\tau^2\sin 2\theta}{4+(N-1)^2\tau^2+4N^2\tau^4+4(N-1)\left(\tau+N\tau^3\right)\cos\theta+8N\tau^2\cos 2\theta}, \end{array}\right\} \tag{1.5}$$

where

$$\lambda = (L+N-2)\tau + (2LN-L-N)\tau^3, \ -1 < N < L \leq 1, \ \tau = \frac{1-\sqrt{5}}{2},$$

and

$$0 \leq \theta \leq 2\pi.$$

Moreover, we observe that

$$\overline{p}\,(L, N, 0) = 1,$$

and

$$\overline{p}\,(L, N, 1) = \frac{LN + 9\,(L+N) + 1 + 4\,(N-L)\,\sqrt{5}}{N^2 + 18N + 1}.$$

According to (1.5), the cusp of the cardioid-like curve is provided by

$$\gamma\,(L, N) = \overline{p}\left(L, N; e^{\pm i\arccos\left(\frac{1}{4}\right)}\right) = \frac{2LN - 3\,(L+N) + 2 + (L-N)\,\sqrt{5}}{2\,(N^2 - 3N + 1)}.$$

The image of each inner circle is a nested cardioid-like curve if the open unit disc $\mathcal{U}$ is considered a collection of concentric circles with origin at the center. As a result, the open unit disc $\mathcal{U}$ is mapped onto a cardioid region by the function $\overline{p}(L, N, z)$. This means that $\overline{p}\,(L, N; \mathcal{U})$ is a cardioid domain. The above discussed cardioid like curve with different values of parameters can be seen in Figures 1 and 2.

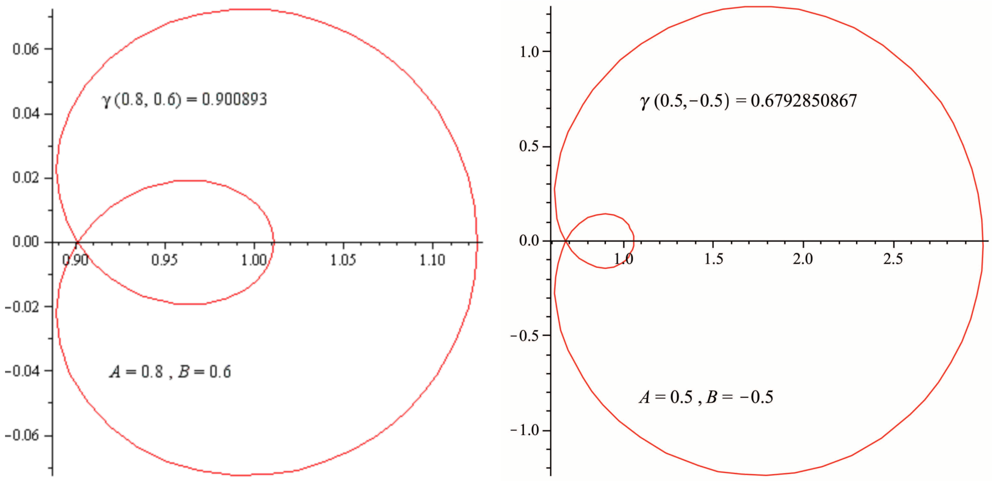

Figure 1. The curve (1.5) with L = 0.8, N = 0.6 and the curve (1.5) with L = 0.5, N=-0.5.

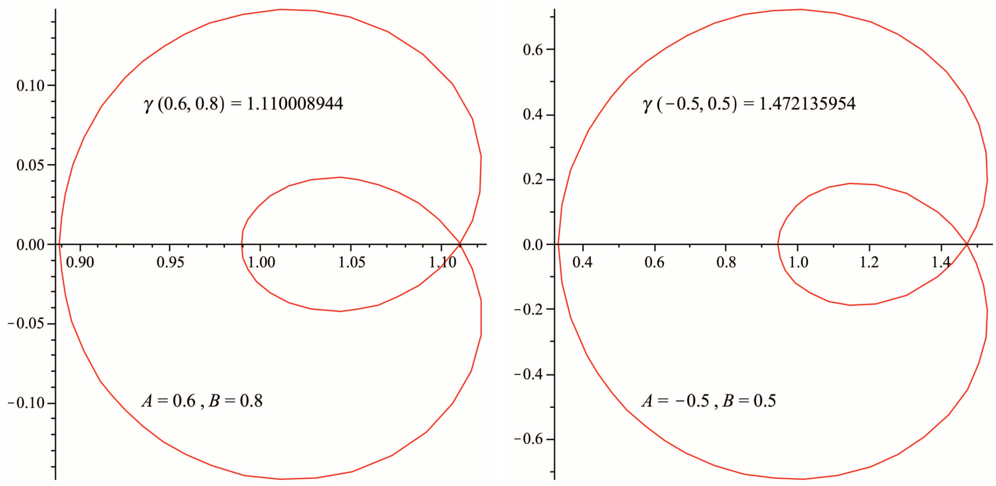

Figure 2. The curve (1.5) with L = 0.6, N = 0.8 and the curve (1.5) with L =-0.5, N = 0.5.

The relationship $N < L$ links the parameters L and N. The cardioid-like curve is flipped by its voilation, as seen in the figures below.

See Figure 3, if collection of concentric circles having origin as center. Thus, the function $\overline{p}(L, N, z)$ maps the open unit disk $\mathcal{U}$ onto a cardioid region. See [26] for more details about cardioid region.

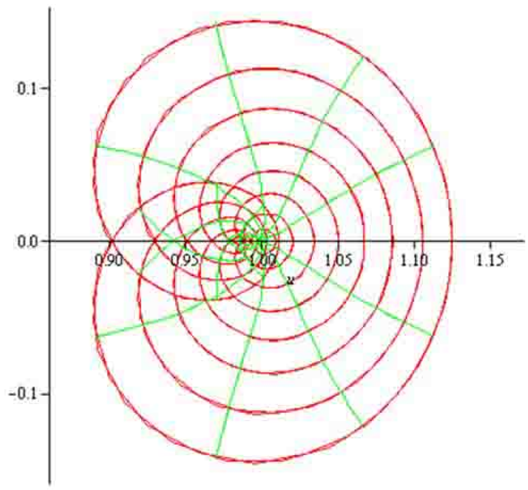

Figure 3. The open unit disk $\mathcal{U}$.

The operator theory of quantum calculus is the primary result of this research. Using standard uses in quantum calculus operator theory and the q-difference operator, we develop numerous novel q-analogous of the differential and integral operators. We construct a large number of new classes of q-starlike and q-convex functions using these operators and study some interesting characteristics of the corresponding analytic functions. In this paper, we gain inspiration from recent research by [14, 26, 27] and define two new classes of p-valent starlike, convex functions connected with the cardioid domain using the q-difference operator.

Influenced by recent studies [14, 26, 27], we defined two new classes of p-valent starlike, convex functions related with cardioid domain.

Definition 5. The function f of the form (1.1) related with cardioid domain, represented by $\mathcal{S}_p^*(L, N, q, b)$, is defined to be the functions f such that

$$1 + \frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_q f(z)}{f(z)} - 1\right) \prec \overline{p}(L, N; z),$$

where, $b \in \mathbb{C}\backslash\{0\}$ and $\overline{p}(L, N; z)$ is given by (1.4).

Definition 6. The function f of the form (1.1) related with cardioid domain, represented by $\mathcal{K}_p(L, N, q, b)$, is defined to be the functions f such that

$$1 - \frac{1}{b} + \frac{1}{b[p]_q}\left(1 + \frac{zD_q^2 f(z)}{D_q f(z)}\right) \prec \overline{p}(L, N; z),$$

where $b \in \mathbb{C}\backslash\{0\}$ and $\overline{p}(L, N; z)$ is given by (1.4) and $\mathcal{K}_p(L, N, q, b)$ is the class of convex functions of order b related with cardioid domain.

Special cases:

(i) For $q \to 1^-$, $b = 1$ and $p = 1$, in Definition 5, we have known class $\mathcal{S}^*(L, N)$ of starlike functions associated with cardioid domain proved by Zainab et al. in [27].

(ii) For $q \to 1^-$, $L = 1$, $N = -1$, $b = 1$ and $p = 1$ in Definition 5, then class $\mathcal{S}_p^*(L, N, q, b) = S\mathcal{L}$ and this class is defined on starlike functions associated with Fibonacci numbers, introduced and studied by Sokół in [28].

(iii) For $q \to 1^-$, $L = 1$, $N = -1$, $b = 1$ and $p = 1$ in Definition 6 then class $\mathcal{K}_p(L, N, q, b) = \mathcal{K}$, and this family is referred to as a class of convex functions connected with Fibonacci numbers.

There are four parts to this article. In Section 1, we briefly reviewed some basic concepts from geometric function theory, quantum calculus, and cardioid domain, studied the q-difference operator, and finally discussed this operator to define two new subclasses of multivalent q-starlike and q-convex functions. The established lemmas are presented in Section 2. Our main results and some known corollaries will be presented in Section 3, then some concluding remarks in Section 4.

2. A set of lemmas

By utilizing the following lemmas, we will determine our main results.

Lemma 1. *[26] Let the function $\overline{p}(L, N; z)$, defined by (1.4). Then,*

(i) For the disc $|z| < \tau^2$, the function $\overline{p}(L, N; z)$ is univalent.

(ii) If $h(z) \prec \overline{p}(L, N; z)$, then $Reh(z) > \alpha$, where

$$\alpha = \frac{2(L + N - 2)\tau + 2(2LN - L - N)\tau^3 + 16(L + N)\tau^2\eta}{4(N - 1)(\tau + N\tau^3) + 32N\tau^2\eta},$$

where

$$\eta = \frac{4 + \tau^2 - N^2\tau^2 - 4N^2\tau^4 - \left(1 - N\tau^2\right)\chi(N)}{4\tau(1 + N^2\tau^2)},$$

$$\chi(N) = \sqrt{5\left(2N\tau^2-(N-1)\tau+2\right)\left(2N\tau^2+(N-1)\tau+2\right)},$$
$$-1<N<L\leq 1$$

and

$$\tau=\frac{1-\sqrt{5}}{2}.$$

(iii) If

$$\overline{p}\,(L,N;z)=1+\sum_{n=1}^{\infty}\overline{Q}_n z^n,$$

then

$$\overline{Q}_n=\begin{cases}(L-N)\frac{\tau}{2}, & \text{for } n=1,\\ (L-N)(5-N)\frac{\tau^2}{2^2}, & \text{for } n=2,\\ \frac{1-N}{2}\tau p_{n-1}-N\tau^2 p_{n-2}, & \text{for } n=3,4,5,\cdots,\end{cases}\tag{2.1}$$

where

$$-1<N<L\leq 1.$$

(iv) Let $h(z)\prec\overline{p}\,(L,N;z)$ and of the form $h(z)=1+\sum\limits_{n=1}^{\infty}h_n z^n$. Then

$$\left|h_2-vh_1^2\right|\leq\frac{(L-N)\,|\tau|}{4}\max\left\{2,\left|\tau\left(v\,(L-N)+N-5\right)\right|\right\},\quad v\in\mathbb{C}.$$

Lemma 2. *[29] Let $h\in\mathcal{P}$, such that $h(z)=1+\sum\limits_{n=1}^{\infty}c_n z^n$. Then*

$$\left|c_2-\frac{v}{2}c_1^2\right|\leq\max\left\{2,2\,|v-1|\right\}=\left\{\begin{array}{cc}2, & \text{if } 0\leq v\leq 2,\\ 2\,|v-1|, & \text{elsewhere},\end{array}\right\}\tag{2.2}$$

and

$$|c_n|\leq 2 \;\; for \;\; n\geq 1.\tag{2.3}$$

Lemma 3. *[30] Let $h\in\mathcal{P}$, such that*

$$h(z)=1+\sum_{n=1}^{\infty}c_n z^n.$$

Then for any complex number v

$$\left|c_2-vc_1^2\right|\leq 2\max\left\{1,|2v-1|\right\}$$

and the result is sharp for

$$h(z)=\frac{1+z^2}{1-z^2}\;\; and \;\; h(z)=\frac{1+z}{1-z}.$$

Lemma 4. *[31] Let the function g given by*

$$g(z)=\sum_{k=1}^{\infty}b_k z^k$$

be convex in $\mathcal{U}$. Also let the function f given by

$$f(z)=\sum_{k=1}^{\infty}a_k z^k$$

be analytic in $\mathcal{U}$. If

$$f(z)\prec g(z),$$

then

$$|a_k|<|b_1|,\quad k=1,2,3,\cdots.$$

For the recently described classes of multivalent q-starlike ($\mathcal{S}_p^*(L,N,q,b)$) and multivalent q-convex ($\mathcal{K}_p(L,N,q,b)$) functions, we get sharp estimates for the coefficients of Taylor series, Fekete-Szegő problems and coefficient inequalities.

3. Main results

In the following theorems, we investigate the functions $f(z)$ which can be used to find the sharpness of the results of this article.

Theorem 5. *A function $f\in\mathcal{A}_p$ given by (1.1) is in the class $\mathcal{S}_p^*(L,N,q,b)$ if and only if there exists an analytic function S,*

$$S(z)\prec\overline{p}(L,N,z)=\frac{2L\tau^2z^2+(L-1)\tau z+2}{2N\tau^2z^2+(N-1)\tau z+2},$$

where, $\overline{p}(L,N,0)=1$, such that

$$f(z)=z^p\exp\left\{b[p]_q\int_0^z\frac{S(t)-1}{t}dt\right\},\quad z\in\mathcal{U}. \tag{3.1}$$

Proof. Let $f\in\mathcal{S}_p^*(L,N,q,b)$ and

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_qf(z)}{f(z)}-1\right)=h(z)\prec\overline{p}(L,N,z).$$

Then by integrating this equation we obtain (3.1). Conversely, if given by (3.1) with an analytic function $S(z)$ such that $S(z)\prec\overline{p}(L,N,z)$, then by logarithmic differentiation of (3.1) we obtain

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_qf(z)}{f(z)}-1\right)=S(z).$$

Therefore we have

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_qf(z)}{f(z)}-1\right)\prec\overline{p}(L,N,z)$$

and $f\in\mathcal{S}_p^*(L,N,q,b)$. □

The initial coefficient bounds $|a_{p+1}|$ and $|a_{p+2}|$ for the functions $f\in\mathcal{S}_p^*(L,N,b)$ are investigated in Theorem 6 using the Lemma 2.

Theorem 6. *Let $f \in \mathcal{S}_p^*(L,N,q,b)$ be given by (1.1), $-1 \leq N < L \leq 1$. Then*

$$\begin{aligned}
\left|a_{p+1}\right| &\leq \frac{[p]_q\,|b|\,(L-N)\,\tau}{2},\\
\left|a_{p+2}\right| &\leq \frac{[p]_q\,|b|\,(L-N)\,|\tau|^2}{8}\left(5-N+[p]_q\,b(L-N)\right).
\end{aligned}$$

These bounds are sharp.

Proof. Let $f \in \mathcal{S}_p^*(L,N,q,b)$, and of the form (1.1). Then

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_qf(z)}{f(z)}-1\right) \prec \overline{p}\,(L,N;z), \tag{3.2}$$

where

$$\overline{p}\,(L,N,z)=\frac{2L\tau^2z^2+(L-1)\,\tau z+2}{2N\tau^2z^2+(N-1)\,\tau z+2}.$$

By applying the concept of subordination, there exists a function w with

$$w(0)=0 \text{ and } |w(z)|<1,$$

such that

$$1+\frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_qf(z)}{f(z)}-1\right)=\overline{p}\,(L,N;w(z)). \tag{3.3}$$

Let

$$\begin{aligned}
w\,(z) &= \frac{h(z)-1}{h(z)+1}\\
&= \frac{c_1z+c_2z^2+c_3z^3+\cdots}{2+c_1z+c_2z^2+\cdots}\\
&= \frac{1}{2}c_1z+\frac{1}{2}\left(c_2-\frac{1}{2}c_1^2\right)z^2+\frac{1}{2}\left(c_3-c_1c_2+\frac{1}{4}c_1^3\right)z^3+\cdots.
\end{aligned}\tag{3.4}$$

Since

$$\overline{p}\,(L,N;z)=1+\sum_{n=1}^{\infty}\overline{Q}_nz^n,$$

then

$$\begin{aligned}
\overline{p}\,(L,N;w(z)) &= 1+\overline{Q}_1\left\{\frac{1}{2}c_1z+\frac{1}{2}\left(c_2-\frac{1}{2}c_1^2\right)z^2\cdots\right\}+\overline{Q}_2\left\{\frac{1}{2}c_1z+\frac{1}{2}\left(c_2-\frac{1}{2}c_1^2\right)z^2\cdots\right\}^2+\cdots\\
&= 1+\frac{\overline{Q}_1c_1}{2}z+\left(\frac{1}{2}\left(c_2-\frac{1}{2}c_1^2\right)\overline{Q}_1+\frac{\overline{Q}_2c_1^2}{4}\right)z^2+\cdots.
\end{aligned}\tag{3.5}$$

Also consider the function

$$\overline{p}\,(L,N;z)=\frac{2L\tau^2z^2+(L-1)\,\tau z+2}{2N\tau^2z^2+(N-1)\,\tau z+2}.$$

Let $\tau z = \alpha_0$, then

$$\begin{aligned}\overline{p}(L, N, z) &= \frac{2L\alpha_0^2 + (L-1)\alpha_0 + 2}{2N\alpha_0^2 + (N-1)\alpha_0 + 2} \\ &= \frac{L\alpha_0^2 + \frac{(L-1)}{2}\alpha_0 + 1}{N\alpha_0^2 + \frac{(N-1)}{2}\alpha_0 + 1} \\ &= \left(L\alpha_0^2 + \frac{(L-1)}{2}\alpha_0 + 1\right)\left[1 + \frac{1}{2}(1-N)\alpha_0 + \left(\frac{N^2 - 6N + 1}{4}\right)\alpha_0^2 + \cdots\right] \\ &= 1 + \frac{1}{2}(L-N)\alpha_0 + \frac{1}{4}(L-N)(5-N)\alpha_0^2 + \cdots .\end{aligned}$$

This implies that

$$\overline{p}(L, N; z) = 1 + \frac{1}{2}(L-N)\tau z + \frac{1}{4}(L-N)(5-N)\tau^2 z^2 + \cdots . \tag{3.6}$$

It is simple to observe from (3.5) that

$$\overline{p}(L, N; w(z)) = 1 + \frac{1}{4}(L-N)\tau c_1 z + \left(\frac{1}{4}(L-N)\tau\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{(L-N)(5-N)\tau^2 c_1^2}{16}\right)z^2 + \cdots . \tag{3.7}$$

Since $f \in \mathcal{S}_p^*(L, N, b)$, then

$$1 + \frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_q f(z)}{f(z)} - 1\right) = 1 + \frac{1}{b[p]_q}a_{p+1}z + \frac{1}{b[p]_q}\left(2a_{p+2} - a_{p+1}^2\right)z^2 + \cdots . \tag{3.8}$$

It is simple to show that by utilizing (3.3) and comparing the coefficients from (3.7) and (3.8), we get

$$a_{p+1} = \frac{b[p]_q (L-N)\tau c_1}{4}. \tag{3.9}$$

Applying modulus on both side, we have

$$\left|a_{p+1}\right| \leq \frac{[p]_q |b| (L-N)\tau}{2}.$$

Now again comparing the coefficients from (3.7) and (3.8), we have

$$\frac{2}{b[p]_q}a_{p+2} = \frac{1}{4}(L-N)\tau\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{(L-N)(5-N)\tau^2 c_1^2}{16} + \frac{1}{[p]_q b}a_{p+1}^2,$$

$$\left|a_{p+2}\right| = \frac{b[p]_q (L-N)\tau}{8}\left|c_2 - \frac{v}{2}c_1^2\right|, \tag{3.10}$$

where

$$v = 1 - \frac{\tau}{2}\left(5 - N + [p]_q b(L-N)\right).$$

It shows that $v > 2$ which is satisfied by the relation $L > N$. Hence, by applying Lemma 2, we obtain the required result.

Result is sharp for the function

$$\begin{aligned} f_*(z) &= z^p \exp\left(b[p]_q \int_0^z \frac{\overline{p}(L,N,t)-1}{t} dt \right) \\ &= z^p + \frac{b[p]_q (L-N)\tau}{2} z^{p+1} + \frac{b[p]_q (L-N)(5-N)\tau^2}{8} z^{p+2} + \cdots, \end{aligned} \tag{3.11}$$

where $\overline{p}(L,N,)$ defined in (1.4). □

Letting $q \to 1^-$, $b = 1$ and $p = 1$ in Theorem 6, we get the known corollary proved in [32] for starlike functions connected with cardioid domain.

Corollary 1. *[32] Let $f \in \mathcal{S}^*(L,N)$ be given by (1.2), $-1 \leq N < L \leq 1$. Then*

$$\begin{aligned} |a_2| &\leq \frac{(L-N)|\tau|}{2}, \\ |a_3| &\leq \frac{(L-N)|\tau|^2}{8} \{L-2N+5\}. \end{aligned}$$

Fekete-Szegö problem $\left|a_{p+2} - \mu a_{p+1}^2\right|$ for the functions $f \in \mathcal{S}_p^*(L,N,b)$ are investigated in Theorem 7.

Theorem 7. *Let $f \in \mathcal{S}_p^*(L,N,q,b)$ and of the form (1.1). Then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{[p]_q |b| (L-N)|\tau|}{8} \max\left\{2, \left|\tau\left(-(L-N)[p]_q b + N - 5 + 2[p]_q b (L-N)\mu\right)\right|\right\}.$$

This result is sharp.

Proof. Since $f \in \mathcal{S}_p^*(L,N,q,b)$, we have

$$1 + \frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_q f(z)}{f(z)} - 1\right) = \overline{p}(L,N;w(z)), \quad z \in \mathcal{U},$$

where w is Schwarz function such that $w(0)$ and $|w(z)| < 1$ in $\mathcal{U}$. Therefore

$$\begin{aligned} 1 + \frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_q f(z)}{f(z)} - 1\right) &= h(z), \\ 1 + \frac{1}{[p]_q b}\frac{zD_q f(z)}{f(z)} &= \frac{1}{b} + \left(1 + h_1 z + h_2 z^2 + \cdots\right), \\ zD_q f(z) &= [p]_q b f(z)\left(\frac{1}{b} + h_1 z + h_2 z^2 + \cdots\right), \end{aligned}$$

and after some simple calculation, we have

$$\begin{aligned} [p]_q z^p &+ [p+1]_q a_{p+1} z^{p+1} + [p+2]_q a_{p+2} z^{p+2} + \cdots \\ &= [p]_q b \left\{z^p + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + \cdots\right\}\left(\frac{1}{b} + h_1 z + h_2 z^2 + \cdots\right) \end{aligned}$$

$$= \quad [p]_q \left\{z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots\right\}\left(1 + bh_1z + bh_2z^2 + \cdots\right).$$

Comparing the coefficients of both sides, we get

$$a_{p+1} = [p]_q bh_1, \quad 2a_{p+2} = [p]_q b\left(h_1 a_{p+1} + h_2\right).$$

This implies that

$$\begin{aligned}\left|a_{p+2} - \mu a_{p+1}^2\right| &= \frac{[p]_q\,|b|}{2}\left|h_2 + (1-2\mu)\,[p]_q bh_1^2\right| \\ &= \frac{[p]_q\,|b|}{2}\left|h_2 - vh_1^2\right|,\end{aligned}$$

where

$$v = (2\mu - 1)\,[p]_q\, b.$$

By using (iv) of Lemma 1 for

$$v = (2\mu - 1)\,[p]_q b,$$

we have the required result. The equality

$$\left|a_{p+2} - \mu a_{p+1}^2\right| = \frac{[p]_q\,|b|\,(L-N)\,|\tau|^2}{8}\left|(L-N)\,[p]_q b - N + 5 - 2[p]_q b\,(L-N)\,\mu\right|$$

holds for f_* given in (3.11). Consider f_0: $\mathcal{U} \to \mathbb{C}$ defined as:

$$f_0(z) = z^p \exp\left([p]_q b \int_0^z \frac{\overline{p}\left(L, N; t^2\right) - 1}{t} dt\right) = z^p + \frac{[p]_q b\tau}{2}\,(L-N)\,z^{p+2} + \cdots, \tag{3.12}$$

where, $\overline{p}\,(L, N; z)$ is defined in (1.4). Hence

$$1 + \frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_q f(z)}{f(z)} - 1\right) = \overline{p}\left(L, N; z^2\right).$$

This demonstrates $f_0 \in \mathcal{S}_p^*\,(L, N, q, b)$. Hence the equality

$$\left|a_{p+2} - \mu a_{p+1}^2\right| = \frac{[p]_q\,|b|\,(L-N)\,|\tau|}{2}$$

holds for the function f_0 given in (3.12). □

Letting $q \to 1^-$, $b = 1$ and $p = 1$ in Theorem 7, we get the known corollary proved in [32] for starlike functions associated with cardioid domain.

Corollary 2. *[32] Let $f \in \mathcal{S}^*\,(L, N)$ and of the form (1.2). Then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(L-N)\,|\tau|}{8}\max\left\{2, |\tau\,(-\,(L - 2N + 5) + 2\,(L-N)\,\mu)|\right\}.$$

This result is sharp.

Coefficient inequality for the class $\mathcal{S}_p^*(L,N,b)$:

Theorem 8. *For function $f\in\mathcal{A}_p$, given by (1.1), if $f\in\mathcal{S}_p^*(L,N,q,b)$, then*

$$\left|a_{p+n}\right|\leq\frac{\prod\limits_{k=2}^{n+1}\left([k-2]_q+\frac{[p]_q}{q^p}\left|b\left((L-N)\frac{\tau}{2}\right)\right|\right)}{[n]_q!},\quad (p,n\in\mathbb{N}).$$

Proof. Suppose $f\in\mathcal{S}_p^*(L,N,q,b)$ and the function $S(z)$ define by

$$S(z)=1+\frac{1}{b}\left(\frac{1}{[p]_q}\frac{zD_qf(z)}{f(z)}-1\right).\tag{3.13}$$

Then by Definition 5, we have

$$S(z)\prec\overline{p}(L,N;z),$$

where, $b\in\mathbb{C}\backslash\{0\}$ and $\overline{p}(L,N;z)$ is given by (1.4). Hence, applying the Lemma 4, we get

$$\left|\frac{S^{(m)}(0)}{m!}\right|=|c_m|\leq\left|\overline{Q}_1\right|,\quad m\in\mathbb{N},\tag{3.14}$$

where

$$S(z)=1+c_1z+c_2z^2+\cdots,$$

and by (2.1), we have

$$\left|\overline{Q}_1\right|=\left|(L-N)\frac{\tau}{2}\right|.\tag{3.15}$$

Also from (3.13), we find

$$zD_qf(z)=[p]_q\{b[q(z)-1]+1\}f(z).\tag{3.16}$$

Since $a_p=1$, in view of (3.16), we obtain

$$\begin{aligned}[n+p]_q-[p]_q\,a_{p+n}&=[p]_qb\left\{c_n+c_{n-1}a_{p+1}+\cdots+c_1a_{p+n-1}\right\}\\&=b[p]_q\sum_{i=1}^{n}c_ia_{p+n-i}.\end{aligned}\tag{3.17}$$

Applying (3.14) into (3.17), we get

$$q^p[n]_q\left|a_{p+n}\right|\leq[p]_q|b|\left|\overline{Q}_1\right|\sum_{i=1}^{n}\left|a_{p+n-i}\right|,\quad p,n\in\mathbb{N}.$$

For $n=1,2,3$, we have

$$\begin{aligned}\left|a_{p+1}\right|&\leq\frac{[p]_q}{q^p}\left|b\overline{Q}_1\right|,\\\left|a_{p+2}\right|&\leq\frac{[p]_q\left|b\overline{Q}_1\right|}{q^p[2]_q}\left(1+\left|a_{p+1}\right|\right)\\&\leq\frac{[p]_q\left|b\overline{Q}_1\right|}{q^p[2]_q}\left(1+\frac{[p]_q}{q^p}\left|b\overline{Q}_1\right|\right)\end{aligned}$$

and

$$\begin{aligned}
\left|a_{p+3}\right| &\leq \frac{[p]_q \left|b\overline{Q}_1\right|}{q^p [3]_q}\left(1+\left|a_{p+1}\right|+\left|a_{p+2}\right|\right)\\
&\leq \frac{[p]_q \left|b\overline{Q}_1\right|}{q^p [3]_q}\left(1+\frac{[p]_q}{q^p}\left|b\overline{Q}_1\right|+\frac{[p]_q \left|b\overline{Q}_1\right|}{q^p [2]_q}\left(1+\frac{[p]_q}{q^p}\left|b\overline{Q}_1\right|\right)\right)\\
&= \frac{[p]_q}{q^p}\left|b\overline{Q}_1\right|\left(\frac{\left(1+\frac{[p]_q}{q^p}\left|b\overline{Q}_1\right|\right)\left([2]_q+\frac{[p]_q}{q^p}\left|b\overline{Q}_1\right|\right)}{[3]_q [2]_q}\right),
\end{aligned}$$

respectively. Applying the equality (3.15) and using the mathematical induction principle, we obtain

$$\begin{aligned}
\left|a_{p+n}\right| &\leq \frac{\prod_{k=2}^{n+1}\left([k-2]_q+\frac{[p]_q}{q^p}\left|b\overline{Q}_1\right|\right)}{[n]_q!}\\
&= \frac{\prod_{k=2}^{n+1}\left([k-2]_q+\frac{[p]_q}{q^p}\left|b\left((L-N)\frac{\tau}{2}\right)\right|\right)}{[n]_q!}.
\end{aligned}$$

This evidently completes the proof of Theorem 8. □

The initial coefficient bounds $\left|a_{p+1}\right|$ and $\left|a_{p+2}\right|$ for the functions $f \in \mathcal{K}_p(L,N,q,b)$ are investigated in Theorem 9 using the Lemma 2.

Theorem 9. *Let $f \in \mathcal{K}_p(L,N,q,b)$ be given by (1.1), $-1 \leq N < L \leq 1$. Then*

$$\begin{aligned}
\left|a_{p+1}\right| &\leq \frac{[p]_q^2 |b| (L-N)\tau}{2[p+1]_q},\\
\left|a_{p+2}\right| &\leq \frac{b^2 [p]_q (L-N)\tau}{8[p+2]_q}\left(5-N+[p]_q b^2(L-N)\right).
\end{aligned}$$

These bounds are sharp.

Proof. Let $f \in \mathcal{K}_p(L,N,q,b)$, and be of the form (1.1). Then

$$1-\frac{1}{b}+\frac{1}{b[p]_q}\left(1+\frac{zD_q^2 f(z)}{D_q f(z)}\right) \prec \overline{p}(L,N;z), \tag{3.18}$$

where

$$\overline{p}(L,N,z)=\frac{2L\tau^2z^2+(L-1)\tau z+2}{2N\tau^2z^2+(N-1)\tau z+2}.$$

By applying the concepts of subordination, there exists a function w with

$$w(0)=0 \text{ and } |w(z)|<1,$$

such that

$$1-\frac{1}{b}+\frac{1}{b\left[p\right]_q}\left(1+\frac{zD_q^2 f(z)}{D_q f(z)}\right)=\overline{p}\left(L,N;w(z)\right). \tag{3.19}$$

Let

$$\begin{aligned}
w\left(z\right) &= \frac{h(z)-1}{h(z)+1} \\
&= \frac{c_1 z+c_2 z^2+c_3 z^3+\cdots}{2+c_1 z+c_2 z^2+\cdots} \\
&= \frac{1}{2}c_1 z+\frac{1}{2}\left(c_2-\frac{1}{2}c_1^2\right)z^2+\frac{1}{2}\left(c_3-c_1 c_2+\frac{1}{4}c_1^3\right)z^3+\cdots. \qquad (3.20)
\end{aligned}$$

Since

$$\overline{p}\left(L,N;z\right)=1+\sum_{n=1}^{\infty}\overline{Q}_n z^n,$$

then

$$\begin{aligned}
\overline{p}\left(L,N;w(z)\right) &= 1+\overline{Q}_1\left\{\frac{1}{2}c_1 z+\frac{1}{2}\left(c_2-\frac{1}{2}c_1^2\right)z^2\cdots\right\}+\overline{Q}_2\left\{\frac{1}{2}c_1 z+\frac{1}{2}\left(c_2-\frac{1}{2}c_1^2\right)z^2\cdots\right\}^2+\cdots \\
&= 1+\frac{\overline{Q}_1 c_1}{2}z+\left(\frac{1}{2}\left(c_2-\frac{1}{2}c_1^2\right)\overline{Q}_1+\frac{\overline{Q}_2 c_1^2}{4}\right)z^2+\cdots. \qquad (3.21)
\end{aligned}$$

Also consider the function

$$\overline{p}\left(L,N;z\right)=\frac{2L\tau^2 z^2+\left(L-1\right)\tau z+2}{2N\tau^2 z^2+\left(N-1\right)\tau z+2}.$$

Let $\tau z=\alpha_0$. Then

$$\begin{aligned}
\overline{p}\left(L,N,z\right) &= \frac{2L\alpha_0^2+\left(L-1\right)\alpha_0+2}{2N\alpha_0^2+\left(N-1\right)\alpha_0+2} \\
&= \frac{L\alpha_0^2+\frac{(L-1)}{2}\alpha_0+1}{N\alpha_0^2+\frac{(N-1)}{2}\alpha_0+1} \\
&= \left(L\alpha_0^2+\frac{(L-1)}{2}\alpha_0+1\right)\left[1+\frac{1}{2}\left(1-N\right)\alpha_0+\left(\frac{N^2-6N+1}{4}\right)\alpha_0^2+\cdots\right] \\
&= 1+\frac{1}{2}\left(L-N\right)\alpha_0+\frac{1}{4}\left(L-N\right)\left(5-N\right)\alpha_0^2+\cdots.
\end{aligned}$$

This implies that

$$\overline{p}\left(L,N;z\right)=1+\frac{1}{2}\left(L-N\right)\tau z+\frac{1}{4}\left(L-N\right)\left(5-N\right)\tau^2 z^2+\cdots. \tag{3.22}$$

It is simple to observe from (3.21) that

$$\overline{p}\left(L,N;w(z)\right)=1+\frac{1}{4}\left(L-N\right)\tau c_1 z+\left(\frac{1}{4}\left(L-N\right)\tau\left(c_2-\frac{1}{2}c_1^2\right)+\frac{\left(L-N\right)\left(5-N\right)\tau^2 c_1^2}{16}\right)z^2+\cdots. \tag{3.23}$$

Since $f \in \mathcal{K}_p(L,N,q,b)$, then

$$1-\frac{1}{b}+\frac{1}{b\,[p]_q}\left(1+\frac{zD_q^2 f(z)}{D_q f(z)}\right)=1+\frac{[p+1]_q}{b\,[p]_q}a_{p+1}z+\frac{1}{b\,[p]_q}\left(2\,[p+2]_q\,a_{p+2}-\frac{[p+1]_q^2}{[p]_q}a_{p+1}^2\right)z^2+\cdots. \tag{3.24}$$

It is simple to show that by utilizing (3.19) and comparing the coefficients from (3.23) and (3.24), we get

$$a_{p+1}=\frac{b\,[p]_q^2\,(L-N)\,\tau c_1}{4\,[p+1]_q}.$$

Applying modulus on both side, we have

$$\left|a_{p+1}\right|\leq\frac{[p]_q^2\,|b|\,(L-N)\,\tau}{2\,[p+1]_q}.$$

Now again comparing the coefficients from (3.7) and (3.8), we have

$$\frac{2\,[p+2]_q}{b\,[p]_q}a_{p+2}=\frac{1}{4}(L-N)\,\tau\left(c_2-\frac{1}{2}c_1^2\right)+\frac{(L-N)\,(5-N)\,\tau^2c_1^2}{16}+\frac{(p+1)^2}{bp^2}a_{p+1}^2,$$

$$\left|a_{p+2}\right|=\frac{b\,[p]_q\,(L-N)\,\tau}{8\,[p+2]_q}\left|c_2-\frac{v}{2}c_1^2\right|,$$

where

$$v=1-\frac{\tau}{2}\left(5-N+[p]_q\,b^2(L-N)\right),$$

it shows that $v>2$ which is satisfied by the relation $L>N$. Hence, by applying Lemma 2, we obtain the required result.

Result is sharp for the function

$$\begin{aligned} f_*(z) &= z^p\exp\left(b\,[p]_q^2\int_0^z\frac{\overline{p}\,(L,N,t)-1}{t}dt\right)\\ &= z^p+\frac{b\,[p]_q^2\,(L-N)\,\tau}{2}z^{p+1}+\frac{b\,[p]_q^2\,(L-N)\,(5-N)\,\tau^2}{8}z^{p+2}+\cdots, \end{aligned} \tag{3.25}$$

where $\overline{p}\,(L,N,)$ defined in (1.4). □

Fekete-Szegö problem $\left|a_{p+2}-\mu a_{p+1}^2\right|$ for the functions $f\in\mathcal{K}_p(L,N,q,b)$ are investigated in Theorem 10.

Theorem 10. *Let $f\in\mathcal{K}_p(L,N,q,b)$ and be of the form (1.1). Then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \frac{[p]_q\,|b|\,(L-N)\,|\tau|}{4([p+1]_q-[p]_q+1)\,[p+2]_q}$$

$$\times \quad \max\left\{2, \left|\tau\left(\begin{array}{c} -(L-N)[p]_q^2 b + N - 5 \\ + \frac{[p]_q^3 b [p+2]_q \left([p+1]_q - [p]_q + 1\right)(L-N)}{[p+1]_q^2}\mu \end{array}\right)\right|\right\}.$$

This result is sharp.

Proof. Since $f \in \mathcal{K}_p(L,N,b)$, we have

$$1 - \frac{1}{b} + \frac{1}{b[p]_q}\left(1 + \frac{zD_q^2 f(z)}{D_q f(z)}\right) = \overline{p}(L,N;w(z)), \quad z \in \mathcal{U},$$

where w is Schwarz function such that $w(0)$ and $|w(z)| < 1$ in $\mathcal{U}$. Therefore

$$\begin{aligned} 1 - \frac{1}{b} + \frac{1}{b[p]_q}\left(1 + \frac{zD_q^2 f(z)}{D_q f(z)}\right) &= h(z), \\ 1 + \frac{zD_q^2 f(z)}{D_q f(z)} &= \left(\frac{1}{b} - 1\right) b[p]_q + b[p]_q\left(1 + h_1 z + h_2 z^2 + \cdots\right), \\ 1 + \frac{zD_q^2 f(z)}{D_q f(z)} &= \left([p]_q + b[p]_q h_1 z + b[p]_q h_2 z^2 + \cdots\right), \\ 1 + \frac{zD_q^2 f(z)}{D_q f(z)} &= [p]_q\left[1 + bh_1 z + bh_2 z^2 + \cdots\right], \end{aligned}$$

and after some simple calculation, we have

$$\begin{aligned} &[p]_q\left(1 + [p-1]_q\right) z^{p-1} + ([p+1]_q\left([p]_q + 1\right) a_{p+1} z^p + [p+2]_q\left([p+1]_q + 1\right) a_{p+2} z^{p+1} + \cdots \\ &= [p]_q\left\{[p]_q z^{p-1} + [p+1]_q a_{p+1} z^p + [p+2]_q a_{p+2} z^{p+1} + \cdots\right\}\left(1 + bh_1 z + bh_2 z^2 + \cdots\right) \\ &= [p]_q^2 z^{p-1} + \left([p]_q ([p+1]_q a_{p+1} + [p]_q^2 bh_1\right) z^p \\ &+ \left\{[p]_q [p+2]_q a_{p+2} + [p]_q [p+1]_q bh_1 a_{p+1} + [p]_q^2 bh_2\right\} z^{p+1}. \end{aligned}$$

Comparing the coefficients of both sides, we get

$$a_{p+1} = \frac{[p]_q^2 bh_1}{[p+1]_q}, \quad a_{p+2} = \frac{[p]_q b}{[p+2]_q\left([p+1]_q - [p]_q + 1\right)}\left([p+1]_q h_1 a_{p+1} + h_2\right).$$

This implies that

$$\begin{aligned} a_{p+2} - \mu a_{p+1}^2 &= \frac{[p]_q |b|}{([p+1]_q - [p]_q + 1)[p+2]_q} \\ &\times \left(h_2 + \left(1 - \frac{[p+2]_q [p]_q\left([p+1]_q - [p]_q + 1\right)}{[p+1]_q^2}\mu\right)[p]_q^2 bh_1^2\right) \\ &= \frac{[p]_q |b|}{([p+1]_q - [p]_q + 1)[p+2]_q}\left(h_2 - vh_1^2\right), \end{aligned}$$

where

$$v=\left(\frac{[p+2]_q\,[p]_q\left([p+1]_q-[p]_q+1\right)}{[p+1]_q^2}\mu-1\right)[p]_q^2\,b.$$

By using (iv) of Lemma 1 for

$$v=\left(\frac{[p+2]_q\,[p]_q\left([p+1]_q-[p]_q+1\right)}{[p+1]_q^2}\mu-1\right)[p]_q^2\,b,$$

we have the required result. The equality

$$\begin{aligned}\left|a_{p+2}-\mu a_{p+1}^2\right| &= \frac{[p]_q\,|b|\,(L-N)\,|\tau|^2}{4([p+1]_q-[p]_q+1)\,[p+2]_q}\\ &\times \left|(L-N)\,[p]_q^2\,b-N+5-\frac{[p]_q^3\,b\left([p+1]_q-[p]_q+1\right)[p+2]_q\,(L-N)}{[p+1]_q^2}\mu\right|\end{aligned}$$

holds for f_* given in (3.25). Consider the function f_0: $\mathcal{U}\to\mathbb{C}$ be defined as:

$$\begin{aligned}f_0(z) &= z^p\exp\left([p]_q^2\,b\int_0^z\frac{\overline{p}\left(L,N;t^2\right)-1}{t}dt\right)\\ &= z^p+\frac{\tau\,[p]_q^2\,b}{2}\,(L-N)\,z^{p+2}+\cdots,\end{aligned}$$

where, $\overline{p}\,(L,N;z)$ is defined in (1.4). Hence and

$$1-\frac{1}{b}+\frac{1}{b\,[p]_q}\left(1+\frac{zD_q^2f(z)}{D_qf(z)}\right)=\overline{p}\left(L,N;z^2\right).$$

□

Theorem 11. *Let $f\in\mathcal{A}_p$, be given by (1.1). If $f\in\mathcal{K}_p\,(L,N,b)$, then*

$$\left|a_{p+n}\right|\leq\frac{[p]_q\prod\limits_{k=2}^{n+1}\left([k-2]_q+[2]_q\,\frac{[p]_q}{q^p}\left|b\left((L-N)\,\frac{\tau}{2}\right)\right|\right)}{[n]_q!\left([p+n]_q\right)},\quad p,n\in\mathbb{N}.$$

Proof. We can obtain Theorem 11, by using the same technique of Theorem 8. □

4. Conclusions

In this article, we have used the ideas of cardioid domain, multivalent analytic functions, and q-calculus operator theory to define the new subfamilies of multivalent q-starlike and q-convex functions. In Section 1, we discussed some basic concepts from geometric functions, analytic functions, multivalent functions, q-calculus operator theory, and the idea of the cardioid domain. We also define two new classes of p-valent starlike, convex functions connected with the cardioid domain using the q-difference operator. The already known lemmas are presented in Section 2. In Section 3, for the class $\mathcal{S}_p^*(L,N,q,b)$, we investigated sharp coefficient bounds, Fekete-Szegö functional, and coefficient inequalities. Same type of results also studied for the class $\mathcal{S}_p^*(L,N,q,b)$. The research also demonstrated how the parameters, including some new discoveries, expand and enhance the results.

For future studies, researchers can use a number of ordinary differential and q-analogous of difference and integral operators and can define a number of new subclasses of multivalent functions. By applying the ideas of this article, many new results can be found. The idea presented in this article can be implemented on papers [33–35], and researchers can discuss the new properties of multivalent functions associated with the cardioid domain.

Use of AI tools declaration

The authors declare that they did not employ any artificial intelligence in the execution of this work.

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Conflict of interest

All the authors claim to have no conflicts of interest.

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