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Research article

Positive radial solutions for the problem with Minkowski-curvature operator on an exterior domain

Zhongzi Zhao* **and Meng Yan**

School of Mathematics and Statistics, Xidian University, Xi'an 710071, China

* **Correspondence:** Email: 15193193403@163.com.

Abstract: We are concerned with the problem with Minkowski-curvature operator on an exterior domain

$$\begin{cases} -\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = \lambda K(|x|)\frac{f(u)}{u^\gamma} & \text{in } B^c, \\ \frac{\partial u}{\partial n}|_{\partial B^c} = 0, \quad \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases} \tag{P}$$

where $0 \leq \gamma < 1$, $B^c = \{x \in \mathbb{R}^N : |x| > R\}$ is a exterior domain in $\mathbb{R}^N$, $N > 2$, $R > 0$, $K \in C([R,\infty),(0,\infty))$ is such that $\int_R^\infty rK(r)dr < \infty$, the function $f : [0,\infty) \to (0,\infty)$ is a continuous function such that $\lim\limits_{s\to\infty} \frac{f(s)}{s^{\gamma+1}} = 0$ and $\lambda > 0$ is a parameter. We show that problem (P) has at least one positive radial solution for all $\lambda > 0$. The proof of our main result is based upon the method of sub and super solutions.

Keywords: mean curvature operator; exterior domain; singular; positive radial solutions; sub and super solutions
Mathematics Subject Classification: 35A01, 35B09, 35J93, 53A10

1. Introduction

In this paper, we are concerned with the problem with Minkowski-curvature operator on an exterior domain

$$\begin{cases} -\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = \lambda K(|x|)\frac{f(u)}{u^\gamma} & \text{in } B^c, \\ \frac{\partial u}{\partial n}|_{\partial B^c} = 0, \quad \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.1}$$

where $0 \leq \gamma < 1$, $B^c = \{x \in \mathbb{R}^N : |x| > R\}$ is a exterior domain in $\mathbb{R}^N$, $N > 2$, $R > 0$, the function $f : [0,\infty) \to (0,\infty)$ is a continuous function such that $\lim\limits_{s\to\infty} \frac{f(s)}{s^{\gamma+1}} = 0$ and $\lambda > 0$ is a parameter. Assume that

(K1) $K \in C([R,\infty),(0,\infty))$ is such that $\int_R^\infty rK(r)dr < \infty$.

The prescribed mean curvature equation

$$-\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = H(x,u) \quad \text{in } \Omega \tag{1.2}$$

with some boundary value condition is related to problems on a flat Minkowski space with a Lorentzian metric in differential geometry and the theory of classical relativity, see R. Bartnik and L. Simon [1], C. Gerhardt [6] and A. E. Treibergs [15].

If Ω is a strictly convex bounded domain in $\mathbb{R}^N$ with C^2 boundary $\partial\Omega$, there are some classic papers on the existence of solutions of Eq (1.2) with Dirichlet, Neuuman or periodic boundary conditions using the fixed point theory or topological methods (mostly bifurcation technique), see Bereanu, Jebelean and Mawhin [2, 3], Bereanu, Jebelean and Torres [4], Ma, Gao and Lu [10], Obersnel, Omari and Rivetti [12] and the references therein.

The presence in the existing literature of very few results about the (1.2) in a exterior domain is in sharp contrast with the wide number of works that are available in the bounded domains setting. The likely reasons are that the singular coefficient and singular weight will occur in the problems considered on the exterior of a ball and the concave-convex properties of solutions are uncertain owing to the influence of the mean curvature operator and coefficient function, see [16].

Yang, Lee and Sim [16] concerned with the existence of nodal radial solutions of the following problem

$$\begin{cases} -\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = \lambda k(|x|)g(u) & \text{in } B^c, \\ u|_{\partial B^c} = 0, \quad \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases} \tag{1.3}$$

where $g : \mathbb{R} \to \mathbb{R}$ a continuous and odd function satisfying $g(s)s > 0$ for $s \neq 0$. As the main assumption on the nonlinearity they required $g(0) = 0$, which guarantees that 0 is the trivial solution of the problem (1.3). Therefore, the bifurcation from the trivial solution can be considered in [16].

Although Yang, Lee and Sim [16] obtained the existence result of the radial solutions of the problem with mean curvature operator on the exterior domain of a ball; it is worth noticing, however, that the application to problem (1.1) of the method described in [16] is not feasible as nonlinearity in (1.1) is singular at 0.

Existence of positive radial solutions of the prescribed mean curvature problem (1.1) with singular nonlinearity on the exterior domain of a ball have not been introduced yet as far as the authors know.

The novelty of this paper is twofold: we develop a method of sub-super solutions for a singular problem with mean curvature operator, and provide a existence result of the positive radial solutions for problem (1.1) which is a prescribed mean curvature problem with singular nonlinearity on the exterior domain of a ball.

In order to study the radial solutions of (1.1), we transform problem (1.1) into the one-dimensional problem via consecutive transformations $r = |x|$ and $t = (\frac{r}{R})^{-(N-2)}$ as follows

$$\begin{cases} -\Big(p(t)\varphi(\frac{1}{p(t)}u'(t))\Big)' = \lambda h(t)\frac{f(u(t))}{(u(t))^\gamma}, \quad t \in (0,1), \\ u(0) = u'(1) = 0, \end{cases} \tag{1.4}$$

where $\varphi(y)=\frac{y}{\sqrt{1-y^2}}$ with $y\in(-1,1)$, p and h can be obtained as

$$p(t)=\frac{R}{N-2}t^{-\frac{N-1}{N-2}},\ \ h(t)=p^2(t)K(Rt^{-\frac{1}{N-2}}). \tag{1.5}$$

Notice that $p(t)>0, p'(t)\le 0,\ t\in[0,1]$ and

$$h\in H=\{q\in C((0,1),(0,\infty)):\int_0^1 sq(s)ds<\infty\}$$

since K satisfies $\int_R^\infty rK(r)dr<\infty$ (see [16]). We assume that

(F1) $f(s)>0$ for all $s\ge 0$;

(F2)$\lim\limits_{s\to\infty}\frac{f(s)}{s^{\gamma+1}}=0$.

Theorem 1.1. Let (F1), (F2) and (K1) hold. Then problem (1.1) has at least one positive radial solution for all $\lambda>0$.

Example 1.1. Consider the problem (1.1) with

$$K(r)=\frac{1}{r^3}$$

and

$$f(s)=s^\gamma+2.$$

Obviously, $\lim\limits_{s\to 0}\frac{f(s)}{s^\gamma}=\infty$, i.e. nonlinearity $\frac{f(s)}{s^\gamma}$ is singular at 0. It is easy to check that K and f satisfies

$$\int_R^\infty rK(r)dr=\frac{1}{R}<\infty,\ \ \lim_{s\to\infty}\frac{f(s)}{s^{\gamma+1}}=\lim_{s\to\infty}\frac{s^\gamma+2}{s^{\gamma+1}}=0$$

and $(F1)$. According to Theorem 1.1, the problem (1.1) has at least one positive radial solution for all $\lambda>0$.

As a by-product of the proof of Theorem 1.1, we also proved the following theorem.

Theorem 1.2. Assume that $h\in H$. Then for any fixed $M>0$, the problem

$$\begin{cases} -\Big(p(t)\varphi(\frac{1}{p(t)}u'(t))\Big)'=\frac{Mh(t)}{(u(t))^\gamma}, & t\in(0,1),\\ u(0)=u'(1)=0 \end{cases} \tag{1.6}$$

have a unique positive solution $w\in C([0,1],[0,\infty))\cap C^1((0,1],\mathbb{R})$.

Remark 1.1. For the existence of solutions of the problems with a elliptic operator on an unbounded domain, see Dai et al. [5], Iaia [7,8], Ko et al. [9] and Ma et al. [11] and the references therein.

2. An auxiliary problem

Motivated by [13], we consider the perturbation problem

$$\begin{cases} -\Big(p(t)\varphi(\frac{1}{p(t)}u'(t))\Big)' = \frac{Mh(t)}{(u(t))^{\gamma}}, & t \in (0,1), \\ u(0) = n, u'(1) = 0, \end{cases} \tag{2.1$_n$}$$

where $n \geq 0$ is any constant, $M > 0$ is a fixed constant, p, h are defined by (1.5).

Lemma 2.1. For each fixed constant $n \geq 0$, problem $(2.1)_n$ has at most one positive solution.

Proof. Assume on the contrary that the functions u_1 and u_2 are all the positive solutions of $(2.1)_n$ and $u_1 \not\equiv u_2$ on $[0,1]$. Then there exists $t_0 \in (0,1]$ such that $u_1(t_0) \neq u_2(t_0)$. Without loss of generality, we assume that $u_1(t_0) > u_2(t_0)$. Then there exists an interval $[a,b] \subset [0,1]$ such that b is the point of positive maximum of $u_1(t) - u_2(t)$ on $[a,b]$ and

$$u_1(t) > u_2(t) \text{ for } t \in (a,b], \ \ u_1'(b) = u_2'(b), \ \ u_1(a) = u_2(a).$$

Integrating both sides of the equation in $(2.1)_n$ over $[t,b] \subset [a,b]$, we can obtain

$$u_i'(t) = p(t)\varphi^{-1}\Big(\frac{p(b)}{p(t)}\varphi(\frac{1}{p(b)}u_i'(b)) + \frac{1}{p(t)}\int_t^b \frac{Mh(r)}{(u_i(r))^{\gamma}}dr\Big), \ \ i = 1,2.$$

Integrating both sides of the above equalities from a to b, we have

$$u_i(b) - u_i(a) = \int_a^b p(t)\varphi^{-1}\Big(\frac{p(b)}{p(t)}\varphi(\frac{1}{p(b)}u_i'(b)) + \frac{1}{p(t)}\int_t^b \frac{Mh(r)}{(u_i(r))^{\gamma}}dr\Big)dt, \ \ i = 1,2.$$

By a simple calculation, we can obtain

$$\begin{aligned} u_1(b) &= u_1(a) + \int_a^b p(t)\varphi^{-1}\Big(\frac{p(b)}{p(t)}\varphi(\frac{1}{p(b)}u_1'(b)) + \frac{1}{p(t)}\int_t^b \frac{Mh(r)}{(u_1(r))^{\gamma}}dr\Big)dt \\ &< u_2(a) + \int_a^b p(t)\varphi^{-1}\Big(\frac{p(b)}{p(t)}\varphi(\frac{1}{p(b)}u_2'(b)) + \frac{1}{p(t)}\int_t^b \frac{Mh(r)}{(u_2(r))^{\gamma}}dr\Big)dt = u_2(b). \end{aligned}$$

This is a contradiction. The proof of Lemma 2.1 is complete. □

Let

$$D_n := \{v \in C[0,1] : v(r) \geq n \text{ on } [0,1]\}.$$

Now, we consider the following problem

$$\begin{cases} -\Big(p(t)\varphi(\frac{1}{p(t)}u'(t))\Big)' = \frac{Mh(t)}{(v(t))^{\gamma}}, & t \in (0,1), \\ u(0) = n, u'(1) = 0, \end{cases} \tag{2.2$_n$}$$

where $v \in D_n$.

Lemma 2.2. For each fixed $n > 0$ and each $v \in D_n$, $(2.2)_n$ has a unique solution $u \in C[0,1] \cap C^1(0,1]$ satisfying $u(t) \geq n$ on $[0,1]$.

Proof. Define $A : D_n \to D_n$ by

$$(Av)(t) = n + \int_0^t p(r)\varphi^{-1}\Big(\frac{1}{p(r)}\int_r^1 \frac{Mh(\tau)}{(v(\tau))^\gamma}d\tau\Big)dr. \tag{2.3$_n$}$$

Obviously, $\varphi^{-1}(y) = \frac{y}{\sqrt{1+y^2}}$ and $\varphi^{-1}(y) \le y$ for any $y \in \mathbb{R}^+$. Then for any $v \in D_n$,

$$\begin{aligned}
0 &\le \int_0^t p(r)\varphi^{-1}\Big(\frac{1}{p(r)}\int_r^1 \frac{Mh(\tau)}{(v(\tau))^\gamma}d\tau\Big)dr \\
&\le \int_0^t p(r)\varphi^{-1}\Big(\frac{1}{p(r)}\int_r^1 \frac{Mh(\tau)}{n^\gamma}d\tau\Big)dr \\
&\le \frac{M}{n^\gamma}\int_0^t p(r)\frac{1}{p(r)}\int_r^1 h(\tau)d\tau dr \\
&\le \frac{M}{n^\gamma}\int_0^1\int_r^1 h(\tau)d\tau dr \\
&\le \frac{M}{n^\gamma}\int_0^1 \tau h(\tau)d\tau < +\infty
\end{aligned}$$

and $(Av)(t)$ is continuous on $[0, 1]$. This suggests that $Av \in D_n$ and A is well defined.

Let $u = Av$, then $u \in C[0, 1] \cap C^1(0, 1]$, $u(0) = n, u'(1) = 0$ and

$$u'(t) = p(t)\varphi^{-1}\Big(\frac{1}{p(t)}\int_t^1 \frac{Mh(r)}{(v(r))^\gamma}dr\Big), \quad t \in (0, 1).$$

It is easy to verify

$$-\Big(p(t)\varphi(\frac{1}{p(t)}u'(t))\Big)' = \frac{Mh(t)}{(v(t))^\gamma}, \quad t \in (0, 1).$$

Therefore, $u(t)$ is a solution of problem $(2.2)_n$. The proof of the uniqueness of solution of $(2.2)_n$ is similar to that of Lemma 2.1. □

From the definition of mapping A, we can easily get the following lemma.

Lemma 2.3. Let $A : D_n \to D_n$ be the mapping defined by $(2.3)_n$. Then, for any $v_1, v_2 \in D_n$ with $v_1(t) \le v_2(t)$ on $[0, 1]$,

$$n \le (Av_2)(t) \le (Av_1)(t) \le (An)(t), \quad t \in [0, 1].$$

We can verify that $A : D_n \to D_n$ is a compact continuous mapping. For any fixed $n > 0$, the problem $(2.1)_n$ has at least one solution $u(t, n) \ge n, t \in [0, 1]$, by Schauder fixed point theorem, see [13]. Furthermore, the uniqueness of the solution $u(t, n)$ of $(2.2)_n$ is guaranteed by Lemma 2.1. Applying the method of similarity to the proof of Lemma 2.1, we can obtain the following result.

Lemma 2.4. Let $u(t, n)$ be the unique solution of $(2.1)_n$. Then for $n_1 > n_2 > 0$, we have

$$0 \le u(t, n_1) - u(t, n_2) \le n_1 - n_2, \quad t \in [0, 1].$$

The proof of Theorem 1.2. Let $\{n_j\}_{j=1}^\infty$ be a decreasing sequence which converges to 0. We known that $(2.1)_{n_j}$ has a unique solution $u(t, n_j)$. From Lemma 2.4, for each $j < k$,

$$0 \le u(t, j) - u(t, k) \le n_j - n_k, \quad t \in [0, 1]. \tag{2.4}$$

Then there exists $u \in C[0,1]$ such that

$$\lim_{j\to\infty} u(t,j) = u(t) \geq 0 \ \text{ uniformly on } \ [0,1].$$

Claim. For any $t \in (0,1]$, $u(t) > 0$.

Let $\delta = \max\limits_{0\leq t\leq 1} u(t,1)$. From (2.4), we have

$$u(t,j) \leq \delta, \ \ t \in [0,1], \ \ j = 1,2,\cdots.$$

This suggests that for any $j = 1,2,\cdots$

$$\begin{aligned} u(t,j) &= n_j + \int_0^t p(r)\varphi^{-1}\Big(\frac{1}{p(r)}\int_r^1 \frac{Mh(\tau)}{(u(\tau,j))^\gamma}d\tau\Big)dr \\ &> \int_0^t p(r)\varphi^{-1}\Big(\frac{1}{p(r)}\int_r^1 \frac{Mh(\tau)}{\delta^\gamma}d\tau\Big)dr \\ &=: \rho, \ \ t \in (0,1]. \end{aligned}$$

Passing to the limit, we have

$$u(t) \geq \rho > 0, \ \ t \in (0,1].$$

From the Monotone convergence theorem (see [14]),

$$u(t) = \int_0^t p(r)\varphi^{-1}\Big(\frac{1}{p(r)}\int_r^1 \frac{Mh(\tau)}{(u(\tau))^\gamma}d\tau\Big)dr.$$

It is easy to verify that u is a solution of $(2.1)_0$, i.e., u is a solution of (1.6). The uniqueness of the solution u is guaranteed by Lemma 2.1. □

3. The method of sub-super solutions

In this section, we will develop a method of sub-super solutions for (1.4) which is a singular problem with mean curvature operator. By a *subsolution* of (1.4) we mean a function $\alpha \in C[0,1] \cap C^1(0,1]$ such that

$$\begin{cases} -\Big(p(t)\varphi(\frac{1}{p(t)}\alpha'(t))\Big)' \leq \lambda h(t)\frac{f(u(t))}{(\alpha(t))^\gamma}, & t \in (0,1), \\ \alpha(0) = \alpha'(1) = 0, \end{cases} \tag{3.1}$$

and by a *supersolution* of (1.4) we mean a function $\beta \in C[0,1] \cap C^1(0,1]$ such that

$$\begin{cases} -\Big(p(t)\varphi(\frac{1}{p(t)}\beta'(t))\Big)' \geq \lambda h(t)\frac{f(u(t))}{(\beta(t))^\gamma}, & t \in (0,1), \\ \beta(0) = \beta'(1) = 0. \end{cases} \tag{3.2}$$

Theorem 3.1. Suppose there exist a subsolution α and a supersolution β of (1.4) such that $0 < \alpha \leq \beta$ on $(0,1)$, then (1.4) has at least one positive solution u satisfying $\alpha \leq u \leq \beta$ on $[0,1]$.

Proof. Step 1. Construction of a modified problem. Take a sequence of subintervals of $(0,1)$, say $\{I_j\}_{j=1}^{\infty}$, such that

$$I_1 \subset\subset I_2 \subset\subset \cdots \subset\subset I_j \subset\subset I_{j+1} \subset\subset \cdots$$

and $\bigcup_{j=1}^{\infty} I_j = (0,1)$. For any $j = 1, 2, \cdots$, we are consider the problem

$$\begin{cases} -\Big(p(t)\varphi(\frac{1}{p(t)}u')\Big)' = \lambda h(t)\frac{f(u)}{u^{\gamma}}, & t \in I_j, \\ u(t) = \alpha(t), & t \in \partial I_j. \end{cases} \tag{3.3$_j$}$$

Let $\alpha_j = \min_{\bar{I}_j} \alpha$ and $\beta_j = \max_{\bar{I}_j} \beta$. Define $\bar{g}_j : I_j \times (0,\infty) \to \mathbb{R}^+$ by

$$\bar{g}_j(t,s) = \begin{cases} h(t)f(\alpha_j)/\alpha_j^{\gamma}, & s \le \alpha_j, \\ h(t)f(s)/s^{\gamma}, & \alpha_j < s < \beta_j, \\ h(t)f(\beta_j)/\beta_j^{\gamma}, & s \ge \beta_j. \end{cases}$$

Then we consider the modified problem

$$\begin{cases} -\Big(p(t)\varphi(\frac{1}{p(t)}u')\Big)' = \lambda \bar{g}_j(t,u), & t \in I_j, \\ u(t) = \alpha(t), & t \in \partial I_j. \end{cases} \tag{3.4$_j$}$$

Obviously the restrictions of the functions α and β on I_j are the subsolution and supersolution of $(3.4)_j$, respectively. In other words, for any $t \in I_j$, α_j and β_j satisfy

$$-\Big(p(t)\varphi(\frac{1}{p(t)}\alpha_j')\Big)' \le \lambda h(t)\frac{f(\alpha_j)}{\alpha_j^{\gamma}} = \lambda \bar{g}_j(t,\alpha_j),$$

and

$$-\Big(p(t)\varphi(\frac{1}{p(t)}\beta_j')\Big)' \ge \lambda h(t)\frac{f(\beta_j)}{\beta_j^{\gamma}} = \lambda \bar{g}_j(t,\beta_j).$$

Step 2. Every solution u_j of $(3.4)_j$ satisfies $\alpha \le u_j \le \beta$ on I_j. Let $w_j = (u_j - \alpha_j) \in C^1(\bar{I}_j)$ and $I_j^- = \{t \in I_j : u_j(t) \le \alpha_j(t)\}$. By a calculation, we obtain

$$\begin{aligned} \int_{I_j^-} p(t)\varphi\Big(\frac{1}{p(t)}u_j'(t)\Big)(u_j-\alpha_j)'dt &= \int_{I_j} \Big(p(t)\varphi\Big(\frac{1}{p(t)}u_j'(t)\Big)\Big)'(u_j-\alpha_j)^- dt \\ &= -\lambda \int_{I_j} \bar{g}_j(t,u_j)(u_j-\alpha_j)^- dt \\ &= \lambda \int_{I_j^-} \bar{g}_j(t,u_j)(u_j-\alpha_j)dt, \end{aligned}$$

and

$$-\int_{I_j^-} p(t)\varphi\Big(\frac{1}{p(t)}\alpha_j'(t)\Big)(u_j-\alpha_j)'dt = -\int_{I_j} \Big(p(t)\varphi\Big(\frac{1}{p(t)}\alpha_j'(t)\Big)\Big)'(u_j-\alpha_j)^- dt$$

$$\leq \lambda \int_{I_j} \bar{g}_j(t,\alpha_j)(u_j-\alpha_j)^- dt$$
$$= -\lambda \int_{I_j^-} \bar{g}_j(t,\alpha_j)(u_j-\alpha_j)dt.$$

Summing up we obtain

$$\int_{I_j^-} p(t)\Big[\varphi\Big(\frac{1}{p(t)}u_j'(t)\Big)-\varphi\Big(\frac{1}{p(t)}\alpha_j'(t)\Big)\Big](u_j-\alpha_j)'dt$$
$$\leq \lambda \int_{I_j^-}\Big(\bar{g}_j(t,u_j)-\bar{g}_j(t,\alpha_j)\Big)(u_j-\alpha_j)dt = 0.$$

The strict monotonicity of the function $y \mapsto \frac{y/p}{\sqrt{1-|y/p|^2}}$ yields $(u_j-\alpha_j)' = 0$ a.e. in I_j^-. Hence, $(u_j-\alpha_j)^- = 0$ and $u_j \geq \alpha_j$ on I_j. In a completely similar way we can obtain that $u_j \leq \beta_j$ on I_j.

Step 3. Problem $(3.3)_j$ *has at least one solution* u_j*, with* $\alpha \leq u_j \leq \beta$ *on* I_j*.* Define an operator $\mathcal{T} : C^1(\bar{I}_j) \to C^1(\bar{I}_j)$ which sends any function $v \in C^1(\bar{I}_j)$ onto the unique solution $u \in C^1(\bar{I}_j)$, of the problem

$$\begin{cases} -\Big(p(t)\varphi(\frac{1}{p(t)}u')\Big)' = \lambda \bar{g}_j(t,v), & t \in I_j, \\ u(t) = \alpha(t), & t \in \partial I_j. \end{cases} \qquad (3.5)_j$$

Similar to Section 2, the operator $\mathcal{T}$ is completely continuous. Clearly, u_j is a solution of $(3.5)_j$ if and only if u_j is a fixed point of $\mathcal{T}$. Since for any $t \in I_j \subset\subset (0,1)$, there exist the constant $\varepsilon_2 > \varepsilon_1 > 0$ and $c = c(I_j, \varepsilon_1, \varepsilon_2) > 0$ such that $\varepsilon_1 < \min_{I_j} \alpha < \max_{I_j} \beta < \varepsilon_2$ and

$$|\bar{g}(t,u_j)| \leq c \quad \text{for all } \ (t,u) \in \bar{I}_j \times [\varepsilon_1, \varepsilon_2],$$

and then

$$\deg(\mathcal{I} - \mathcal{T}, [\varepsilon_1, \varepsilon_2], 0) = 1,$$

where $\mathcal{I}$ is the identity operator. By Step 2 we know that u_j satisfies $\alpha \leq u_j \leq \beta$ on I_j and hence it is a solution of $(3.4)_j$ as well.

Step 4. Complete the proof of result. We claim that, for fixed k, there exists $C_k > 0$ such that $\|u_j\|_{C^2(\bar{I}_k)} \leq C_k$ for all $j \geq k+1$. In fact, take J_k such that $I_k \subset\subset J_k \subset\subset I_{k+1}$. Define $g_j(t) = h(t)\frac{f(u_j(t))}{(u_j(t))^\gamma}$. Then

$$-(p\varphi(u_j'/p))' = \lambda g_j \quad \text{on } \ J_k.$$

Since $\{u_j\}_{j\geq k+1}$ are uniformly bounded on $\bar{I}_{k+1}$, we know that there exists $c_k > 0$ such that

$$\|g_j\|_{C(\bar{J}_k)} < c_k \quad \text{for all } \ j \geq k+1.$$

This suggests that there exists a constant $C_k > 0$ such that $\|u_j\|_{C^2(\bar{I}_k)} \leq C_k$ for all $j \geq k+1$.

Since the embedding $C^2(\bar{I}_k) \hookrightarrow C^1(\bar{I}_k)$ is compact, for each k, sequence $\{u_j\}_{j=1}^\infty$ has a subsequence, renamed $\{u_j\}_{j=1}^\infty$, which converges to u in $C^1(\bar{I}_k)$. This implies that $u \in C^1(\bar{I}_k)$ for every k. Consequently, $u \in C^1(0,1)$. For any k, we have

$$\int_{I_k} p(t)\Big(\varphi\Big(\frac{1}{p(t)}u_j'(t)\Big)\Big)\phi'(t)dt = \lambda \int_{I_k} h(t)\frac{f(u_j(t))}{(u_j(t))^\gamma}\phi(t)dt$$

for all $\phi \in C_0^\infty(I_k)$ and $j \geq k+1$. By taking the limit of the sequence converging in $C^1(\bar{I}_k)$,

$$\int_{I_k} p(t)\Big(\varphi\big(\frac{1}{p(t)}u'(t)\big)\Big)\phi'(t)dt = \lambda \int_{I_k} h(t)\frac{f(u(t))}{(u(t))^\gamma}\phi(t)dt$$

for all $\phi \in C_0^\infty(I_k)$. Also, since $\alpha(t) \leq u_j(t) \leq \beta(t)$ for all j, we have $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in (0,1)$. Thus, from $\alpha(0) = \beta(0) = 0$ and $\alpha'(1) = \beta'(1) = 0$, we know that $u \in C[0,1] \cap C^1(0,1]$. This completes the proof of Theorem 3.1. □

4. The proof of Theorem 1.1

Firstly, we construct a positive supersolution β of (1.4). Let $f^*(s) = \max_{0\leq r\leq s} f(r)$. Obviously, $f^*(s)$ is nondecreasing and

$$\frac{f^*(s)}{s^{\gamma+1}} \to 0 \quad \text{as} \quad s \to \infty$$

since $\frac{f(s)}{s^{\gamma+1}} \to 0$ as $s \to \infty$. Then there exists a constant $M_\lambda \gg 1$ such that

$$\frac{f^*(M_\lambda \|w\|_\infty)}{(M_\lambda \|w\|_\infty)^{\gamma+1}} \leq \frac{1}{\lambda \|w\|_\infty^{\gamma+1}},$$

where w is the unique positive solution of (1.6) with M_λ. Then

$$\begin{aligned}-\Big(p(t)\varphi(\frac{1}{p(t)}w'(t))\Big)' &= \frac{M_\lambda h(t)}{w^\gamma} \geq \lambda h(t)\frac{f^*(M_\lambda\|w\|_\infty)}{w^\gamma} \geq \lambda h(t)\frac{f^*(\|w\|_\infty)}{w^\gamma}\\ &\geq \lambda h(t)\frac{f^*(w)}{w^\gamma} \geq \lambda h(t)\frac{f(w)}{w^\gamma}.\end{aligned}$$

This suggests that w is a positive supersolution of (1.4).

Next we construct a positive subsolution α of (1.4). For any fixed constant $m > 0$, the problem

$$\begin{cases} -\Big(p(t)\varphi(\frac{1}{p(t)}u')\Big)' = mh(t), \qquad t \in (0,1),\\ u(0) = 0, u'(1) = 0 \end{cases} \tag{4.1$_m$}$$

has a solution v_m. We can verify that $v_m \to 0$ as $m \to 0$. Since $\frac{f(s)}{s^\gamma} \to \infty$ as $s \to 0$, for any fixed $\lambda > 0$, there exists a sufficiently small $m_\lambda \ll 1$ such that

$$m_\lambda \leq \lambda \frac{f(v)}{v^\gamma},$$

where v is the solution of $(4.1)_{m_\lambda}$. Then

$$-\Big(p(t)\varphi(\frac{1}{p(t)}v'(t))\Big)' = m_\lambda h(t) \leq \lambda h(t)\frac{f(v)}{v^\gamma}.$$

This suggests that v is a positive subsolution of (1.4) such that $v \leq w$ for sufficiently small m_λ.

From Theorem 3.1, (1.4) has a positive solution for all $\lambda > 0$. This completes the proof of Theorem 1.1.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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