AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 20612–20653.
DOI: 10.3934/math.20231051
Received: 16 February 2023
Revised: 13 May 2023
Accepted: 16 May 2023
Published: 26 June 2023

Research article

An optimization strategy with SV-neutrosophic quaternion information and probabilistic hesitant fuzzy rough Einstein aggregation operator

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Abstract: The single valued neutrosophic probabilistic hesitant fuzzy rough Einstein aggregation operator (SV-NPHFRE-AO) is an extension of the neutrosophic probabilistic hesitant fuzzy rough set theory. It is a powerful decision-making tool that combines the concepts of neutrosophic logic, probability theory, hesitant fuzzy sets, rough sets, and Einstein aggregation operators. SV-NPHFRE-AO can be applied in many fields, including livestock decision making. Making judgments about a wide range of issues, including feed formulation, breeding program design, disease diagnostics, and market analysis, is part of the process of managing livestock. By combining data from many sources, SV-NPHFRE-AO can assist decision-makers in livestock management in integrating and evaluating diverse criteria, which can result in more informed choices. It also provides a more accurate and comprehensive representation of decision-making problems by considering the multiple criteria involved and the relationships between them. The single valued neutrosophic set (SV-NS) aggregation operators (AOs) based on Einstein properties using hesitant fuzzy sets (HFSs) and probabilistic hesitant fuzzy sets (PHFSs) with rough sets (RSs) are proposed in this study and can handle a large volume of data, making them suitable for complex and large-scale livestock decision-making problems. We first defined SV-neutrosophic probabilistic hesitant fuzzy rough weighted averaging (SV-NPHFRWA), SV-neutrosophic probabilistic hesitant fuzzy rough weighted geometric (SV-NPHFRWG), SV-neutrosophic probabilistic hesitant fuzzy rough ordered weighted averaging (SV-NPHFROWA) and SV-neutrosophic probabilistic hesitant fuzzy rough hybrid weighted averaging (SV-NPHFRHWA) AOs. Then, based on Einstein properties, we extended these operators and developed the single-valued neutrosophic probabilistic hesitant fuzzy rough Einstein weighted averaging (SV-NPHFREWA) operator. Additionally, an illustrative scenario to show the applicability of the suggested decision-making approach is provided, along with a sensitivity analysis and comparison analysis, which demonstrate that its outcomes are realistic and reliable. We also provide another relation between criteria and alternatives of decision-making using neutrosophic information with quaternion context. By using such type of operators, livestock managers can make more informed decisions, leading to better animal health, higher productivity, and increased profitability.

Keywords: neutrosophic information; Einstein aggregation operators; probabilistic hesitant information; rough sets; multi-criteria decision-making
Mathematics Subject Classification: 03B52, 03E72

1. Introduction

Fuzzy set theory (FST) is a potent tool that can be used to handle difficult decision-making problems involving numerous criteria [1]. This method is known as multi-criteria decision-making (MCDM). There are several situations in livestock management where MCDM with FST can be used. It can be used in feed formulation to choose the best mixture of feed ingredients that meets the nutritional needs of the animals while lowering the cost. The technique can be used to assess various feed formulations based on numerous factors such as nutritional value, digestibility, and cost. FST can also be utilized to reflect the uncertainty and ambiguity in the nutritional requirements of the animals. In breeding program design, it can also be used to select the best breeding strategy that maximizes the genetic potential of the animals while minimizing the risk of inbreeding. FST can be used to model the uncertainty and vagueness in the genetic parameters, and the MCDM strategy can be used to evaluate different breeding strategies based on multiple criteria such as genetic diversity, productivity, and genetic risk [2, 3].

In disease diagnosis, MCDM can be used to identify the best treatment strategy that maximizes the chances of recovery while minimizing the risk of side effects. This theory can be used to model the uncertainty and vagueness in the symptoms and diagnostic test results, and the MCDM method can be used to evaluate different treatment strategies based on multiple criteria such as effectiveness, safety, and cost. In market analysis, this type of methodology can be used to identify the best marketing strategy that maximizes the profitability of the livestock business while minimizing the risk of losses. Fuzzy set theory can be used to model the uncertainty and vagueness in the market conditions and customer preferences, and the MCDM technique can be used to evaluate different marketing strategies based on multiple criteria such as profitability, market share, and customer satisfaction. By incorporating the principles of fuzzy set theory and MCDM, livestock managers can make more informed decisions based on multiple criteria and handle the uncertainty and vagueness that are inherent in many livestock management problems [4, 5].

A group of elements (objects) having fuzzy boundaries is represented mathematically as a fuzzy set (FS), which allows for the possibility of a gradual change in an element's belongingness to a group, from fully belonging to not belonging. This concept is put forth in the fuzzy sets theory as a means of mathematically expressing fuzzy concepts that individuals use to explain, among other things, how they view real systems, their preferences, and goals [6, 7]. One can select the most effective food loss and waste drivers to promote sustainability by applying fuzzy decision theory. Systems for reducing food loss and waste face numerous difficulties that make it difficult to select the appropriate course of action. When dealing with unstructured scenarios in decision-making situations, classic or crisp methods may not always be the most effective. In 1965, Zadeh [8] designed fuzzy sets (FS) as a method to deal with this contradiction. In FSs, Zadeh assigns membership grades in the range $[0, 1]$ to a set of components. Since many of the set theoretic components of crisp conditions were given for fuzzy sets, Zadeh's work on this area is noteworthy [9].

An improved version of the fuzzy set (FS) that contains membership and non-membership degrees was the intuitionistic fuzzy set (IFS), which was the subject of Atanassov's [10] research. Over the past few decades, IFSs have been demonstrated to be helpful and frequently used by academics to assess uncertainty and instability in data. A single-valued neutrosophic set (SV-NS), which can deal with inaccurate, ambiguous, and incompatible data issues, was proposed by Wang et al. [11]. SV-NS has received a lot of attention from academics because it is a potent universal systematic approach. Ye discussed the correlation and information energy of SV-NSs in [12]. The use of SV-NSs as a decision-making tool was subsequently investigated by many researchers [13–15]. The idea of a neutrosophic rough set (RS) has been described [16], as a the generalization of fuzzy rough sets (FRSs).

Despite the fact that these methods effectively capture uncertainty, they are unable to simulate circumstances where an expert's refusal to make a conclusion plays a significant role. Consider a scenario where a panel of six experts is asked to choose the best applicant throughout the hiring process, but two of them decline to make a choice. The number of decision makers is regarded to be 4 as opposed to 6, i.e., the refusal-providing professionals are completely disregarded, and the decision is established using the responses given by the 4 determination-providing experts only, when analyzing the informative data using the existing methodologies [17]. This leads to a huge loss of knowledge and might produce subpar outcomes. Zhu and Xu [18] developed probabilistic hesitant fuzzy sets (PHFSs) to handle such refusal-oriented instances. The review makes it obvious that (i) the HFS is a dynamic support structure with a capacity to reduce subjective randomness, (ii) it also makes it easier for experts to gain insight into preferences, and (iii) it ignores the likelihood that each element will occur. Zhou and Xu [19] proposed a generalized structure known as probabilistic hesitant fuzzy information (PHFI) that assigns an occurrence probability to each element in response to the claims stated in the systematic review. By doing this, it is possible to determine the confidence of each element, which serves as potential data for MCDM [20].

When making decisions, especially in MCDM, where several criteria or considerations must be taken into account, aggregation operators (AOs) are essential. The knowledge or preferences of decision-makers or stake holders are combined using aggregate operators to get an overall score or ranking for each alternative under consideration. The ordered weighted averaging (OWA), weighted product, weighted average, and Choquet integral are some common AOs used in MCDM [21–23]. Depending on the preferences of the decision maker and the characteristics of the choice problem, these operators have various qualities and are useful for various decision-making scenarios.

The Einstein aggregation operator (EAO) is a powerful and flexible aggregation operator used in MCDM. It was introduced by C. Carlsson and R. Fuller [24] in 2001 as an extension of the well-known weighted average operator. One of the unique features of the EAO is that it allows the decision maker to specify a set of "aggregation weights" for each criterion, which represent the relative importance of that criterion compared to the others. These weights can be different for each alternative being considered, and can even vary across different levels or dimensions of the criteria. In order to more accurately represent complex and uncertain decision-making problems, neutrosophic information with quaternion context and a hybrid model that combines various uncertainty measures and builds a relationship between alternatives and criteria are also presented. The literature still lacks research on the new operators for decision-making used in this study. The creation of novel AOs is specifically required in order to manage the complexity and inherent uncertainty of decision-making situations employing SV-NPHFRS in a more effective and efficient manner. There is a need for their application to larger and more sophisticated decision-making situations, as the previous literature has mostly concentrated on using these AOs to solve small-scale decision-making challenges. The potential of these AOs in domains other than decision-making, such as forecasting and pattern recognition, also needs to be investigated. The development of new operators for SV-NPHFRS that can better capture the complexity and uncertainty of decision-making problems, as well as assessing their efficacy on larger and more complicated decision-making problems, could thus be the main areas of future research. To fully explore their potential and limitations, research on the use of these operators in contexts other than decision-making is also necessary. Based on SV-neutrosophic rough sets (SV-NRSs), in this article, we propose some new AOs because of the following reasons:

(1) Decision-makers have more discretion as a result of SV-NPHFRE-AO's integration of neutrosophic logic, probability theory, hesitant fuzzy sets, rough sets, and Einstein aggregation operator.

(2) SV-NPHFRE-AO can handle different types of information, such as qualitative and quantitative data, linguistic terms, and probabilistic information, which gives more flexibility in decision-making.

(3) SV-NPHFRE-AO can integrate multiple criteria in decision making, allowing decision-makers to consider multiple factors that are relevant to the decision problem. This is important because many real-world decisions involve multiple factors that must be considered, and SV-NPHFRE-AO can help decision-makers to prioritize and weigh these factors appropriately.

(4) For first evaluation, SV-neutrosophic weighted averaging (SV-NWA) and SV-neutrosophic weighted geometric (SV-NWG) AOs are unable to take into account the familiarity of experts with the investigated items, while SV-NPHFREWA and SV-NPHFREOWA AOs may.

(5) This article sought to address more complicated and advanced data due to the clarity of the SV-NPHFREWA and SV-NPHFREOWA AOs and the fact that they cover the decision-making technique.

(6) We also use the neutrosophic data in quaternion context to find the relationship between the decision making objectives.

(7) Two types of information are typically needed from evaluation specialists in real-world decision-making challenges (called Positive and Negative): Performance of the assessment objects and expertise in the evaluation sectors. All currently used methods only consider positive data and lack negative information in the experts' assessments. The proposed work covers the limitation of all existing drawbacks.

Consequently, the following are the research's results:

- To begin, we recognize from the scientific literature the theory of SV-neutrosophic probabilistic hesitant fuzzy rough sets.
- We undertake the development of new AOs like SV-NPHFREWA and SV-NPHFREOWA AOs.
- An illustrative scenario to show the applicability of the suggested decision-making approach is provided, along with a sensitivity analysis and comparison analysis, that demonstrate that its outcomes are realistic and reliable.

2. Preliminaries

Let us through the basics of fuzzy sets, including intuitionistic fuzzy sets, neutrosophic sets, single-valued neutrosophic sets, and probabilistic fuzzy sets, in this part. Once they have been vetted, these ideas will be applied here.

Definition 1. *For a fixed set β, a FS [8] $\Im$ in β is denoted as*

$$\Im = \{\langle \xi_{\flat}, \upsilon_{\Im}(\xi_{\flat})\rangle | \xi_{\flat} \in \beta\},$$

where each $\xi_{\flat} \in \beta$, the membership grade $\upsilon_{\Im} : \beta \to \Delta$ specifies the degree to which the element $\xi_{\flat} \in \Im$, where $\Delta = [0, 1]$ is the interval.

Definition 2. *For a fixed set β, an IFS [10] $\Im$ in β is mathematically described as*

$$\Im = \{\langle \xi_{\flat}, \upsilon_{\Im}(\xi_{\flat}), \tau_{\Im}(\xi_{\flat})\rangle | \xi_{\flat} \in \beta\}.$$

For each $\xi_{\flat} \in \beta$, the membership grade $\upsilon_{\Im} : \beta \to \Delta$ and the non-membership grade $\tau_{\Im} : \beta \to \Delta$ specify the membership and non-membership of $\xi_{\flat}$ to the intuitionistic fuzzy set $\Im$, respectively, where $\Delta = [0, 1]$ is the unit interval. Moreover, it is required that $0 \leq \upsilon_{\Im}(\xi_{\flat}) + \tau_{\Im}(\xi_{\flat}) \leq 1$, for each $\xi_{\flat} \in \beta$.

Definition 3. *Let $\mathbb{N}$ be a fixed set, and $\xi_{\flat} \in \mathbb{N}$. A neutrosophic set [11] ς in $\mathbb{N}$ is denoted as belongingness function $\upsilon_{\Im}(\xi_{\flat})$, an indeterminacy function $\tau_{\Im}(\xi_{\flat})$ and a non-belongingness function $\Upsilon_{\Im}(\xi_{\flat})$. $\upsilon_{\Im}(\xi_{\flat}), \tau_{\Im}(\xi_{\flat})$ and $\Upsilon_{\Im}(\xi_{\flat})$ are real subsets of $]0^{-,}1^{+}[$ and*

$$\upsilon_{\Im}(\xi_{\flat}), \tau_{\Im}(\xi_{\flat}), \Upsilon_{\Im}(\xi_{\flat}) : \mathbb{N} \longrightarrow]0^{-,}1^{+}[$$

The representation of neutrosophic set (NS) ς is mathematically defined as:

$$\varsigma = \{\langle \xi_{\flat}, \upsilon_{\Im}(\xi_{\flat}), \tau_{\Im}(\xi_{\flat}), \Upsilon_{\Im}(\xi_{\flat}))\rangle | \xi_{\flat} \in \mathbb{N}\},$$

where

$$0^{-} < \xi_{\flat}, \upsilon_{\Im}(\xi_{\flat}) + \tau_{\Im}(\xi_{\flat}) + \Upsilon_{\Im}(\xi_{\flat}) \leq 3^{+}.$$

Definition 4. *[11, 25] Let $\mathbb{N}$ be a fixed set and $\xi_{\flat} \in \mathbb{N}$. A single-valued neutrosophic set (SV-NS) B in $\mathbb{N}$ is defined as truth membership grade $\upsilon_{\Im}(\xi_{\flat})$, an indeterminacy membership grade $\tau_{\Im}(\xi_{\flat})$ and a falsity membership grade $\Upsilon_{\Im}(\xi_{\flat})$. $\upsilon_{\Im}(\xi_{\flat}), \tau_{\Im}(\xi_{\flat})$ and $\Upsilon_{\Im}(\xi_{\flat})$ are real subsets of* $[0, 1]$, *and*

$$\upsilon_{\Im}(\xi_{\flat}), \tau_{\Im}(\xi_{\flat}), \Upsilon_{\Im}(\xi_{\flat}) : \mathbb{N} \longrightarrow [0, 1].$$

The representation of single valued neutrosophic set (SV-NS) B is mathematically defined as:

$$B = \{\langle \xi_b, \upsilon_{\Im}(\xi_b), \tau_{\Im}(\xi_b), \Upsilon_{\Im}(\xi_b)\rangle\rangle \, |\xi_b \in \mathbb{N}\},$$

where

$$0 < \upsilon_{\Im}(\xi_b) + \tau_{\Im}(\xi_b) + \Upsilon_{\Im}(\xi_b) \leq 3.$$

Definition 5. *For a fixed set β, a probabilistic fuzzy set (PFS) [26, 27] $\Im$ in β is presented as*

$$\Im = \{\langle \xi_b, (\xi_b)/\Xi_b\rangle \, |\xi_b \in \beta\},$$

where (ξ_b) a subset of $[0, 1]$, and $(\xi_b)/\Xi_b$ represents the membership degree of $\xi_b \in \beta$ in $\Im$. And Ξ_b represent the possibilities of (ξ_b), with constraint that $\sum_b \Xi_b = 1$.

Definition 6. *A quaternion neutrosophic set (QNS) is given by $\acute{N}_Q : \mathbb{N} \to [0, 1]$ such that*

$$\acute{N}_Q(\xi i + \tau j + \Upsilon k) = min\{\overline{X}(\xi), \overline{Y}(\tau), \overline{Z}(\Upsilon)\}$$

for some $(\xi, \tau, \Upsilon) \in \mathbb{N}_{\mathbb{Q}}$ and $\overline{X}, \overline{Y}, \overline{Z} \in \Re_{N_Q}$, where $\Re_{N_Q}$ is a quaternion neutrosophic set. For better understanding $\overline{X}$, is called the membership function, $\overline{Y}$ is called the indeterminacy function, and $\overline{Z}$ is called the non membership function. Here the function satisfied the following condition for a quaternion neutrosophic set (QNS) as

$$0 \leq \xi + \tau + \Upsilon \leq 3$$

and

$$(\xi, \tau, \Upsilon) \to [0, 1].$$

Definition 7. *Let $\Im_1 = \left(\upsilon_1/\Xi_1, \tau_1/\varpi_1, \Upsilon_1/\phi_1\right)$ and $\Im_2 = \left(\upsilon_2/\Xi_2, \tau_2/\varpi_2, \Upsilon_2/\phi_2\right)$ be two SV-NPFNs. The basic operational laws [28, 29] are defined as*

(1) $$\Im_1 \cup \Im_2 = \left\{ \begin{array}{c} \bigcup\limits_{\substack{\xi_1 \in \upsilon_1, \Xi_1 \in \Xi_1 \\ \xi_2 \in \upsilon_2, \Xi_2 \in \Xi_2}} (\max(\xi_1/\Xi_1, \xi_2/\Xi_2)), \bigcup\limits_{\substack{\Lambda_1 \in \tau_1, \varpi_1 \in \varpi_1 \\ \Lambda_2 \in \tau_2, \varpi_2 \in \varpi_2}} (\min(\Lambda_1/\varpi_1, \Lambda_2/\varpi_2)), \\ \bigcup\limits_{\substack{\varrho_1 \in \Upsilon_1, \phi_1 \in \phi_1 \\ \varrho_2 \in \Upsilon_2, \phi_2 \in \phi_2}} (\min(\varrho_1/\phi_1, \varrho_2/\phi_2)) \end{array} \right\},$$

(2) $$\Im_1 \cap \Im_2 = \left\{ \begin{array}{c} \bigcup\limits_{\substack{\xi_1 \in \upsilon_1, \Xi_1 \in \Xi_1 \\ \xi_2 \in \upsilon_2, \Xi_2 \in \Xi_2}} (\min(\xi_1/\Xi_1, \xi_2/\Xi_2)), \bigcup\limits_{\substack{\Lambda_1 \in \tau_1, \varpi_1 \in \varpi_1 \\ \Lambda_2 \in \tau_2, \varpi_2 \in \varpi_2}} (\max(\Lambda_1/\varpi_1, \Lambda_2/\varpi_2)), \\ \bigcup\limits_{\substack{\varrho_1 \in \Upsilon_1, \phi_1 \in \phi_1 \\ \varrho_2 \in \Upsilon_2, \phi_2 \in \phi_2}} (\max(\varrho_1/\phi_1, \varrho_2/\phi_2)) \end{array} \right\},$$

(3) $\Im_1^c = \left\{\Upsilon_\varsigma/\phi, \tau_\varsigma/\varpi, \upsilon_\varsigma/\Xi\right\}$.

Definition 8. *Let $\Im_1 = (\upsilon_1/\Xi_1, \tau_1/\varpi_1, \Upsilon_1/\phi_1)$ and $\Im_2 = (\upsilon_2/\Xi_2, \tau_2/\varpi_2, \Upsilon_2/\phi_2)$ be two SV-neutrosophic probabilistic hesitant fuzzy numbers (SV-NPHFNs) and $\eta > 0$ $(\in \mathbb{R})$. Then, their operations [30] are presented as:*

(1) $\mathfrak{I}_1 \oplus \mathfrak{I}_2 = \left\{ \begin{array}{c} \bigcup_{\substack{\xi_1 \in \upsilon_1, \xi_2 \in \upsilon_2 \\ \Xi_1 \in \Xi_1, \Xi_2 \in \Xi_2}} (\xi_1 + \xi_2 - \xi_1\xi_2/\Xi_1\Xi_2), \\ \bigcup_{\substack{\Lambda_1 \in \tau_1, \Lambda_2 \in \tau_2 \\ \varpi_1 \in \varpi_1, \varpi_2 \in \varpi_2}} (\Lambda_1\Lambda_2/\varpi_1\varpi_2), \\ \bigcup_{\substack{\varrho_1 \in \Upsilon_1, \varrho_2 \in \Upsilon_2 \\ \phi_1 \in \phi_1, \phi_2 \in \phi_2}} (\varrho_1\varrho_2/\phi_1\phi_2) \end{array} \right\},$

(2) $\mathfrak{I}_1 \otimes \mathfrak{I}_2 = \left\{ \begin{array}{c} \bigcup_{\substack{\xi_1 \in \upsilon_1, \xi_2 \in \upsilon_2 \\ \Xi_1 \in \Xi_1, \Xi_2 \in \Xi_2}} (\xi_1\xi_2/\Xi_1\Xi_2), \\ \bigcup_{\substack{\Lambda_1 \in \tau_1, \Lambda_2 \in \tau_2 \\ \varpi_1 \in \varpi_1, \varpi_2 \in \varpi_2}} (\Lambda_1 + \Lambda_2 - \Lambda_1\Lambda_2/\varpi_1\varpi_2) \\ \bigcup_{\substack{\varrho_1 \in \Upsilon_1, \varrho_2 \in \Upsilon_2 \\ \phi_1 \in \phi_1, \phi_2 \in \phi_2}} (\varrho_1 + \varrho_2 - \varrho_1\varrho_2/\phi_1\phi_2) \end{array} \right\},$

(3) $\eta\mathfrak{I}_1 = \left\{ \bigcup_{\xi_1 \in \upsilon_1, \Xi_1 \in \Xi_1} (1 - (1 - \xi_1)^\eta/\Xi_1), \bigcup_{\Lambda_1 \in \tau, \varpi_1 \in \varpi_1} \left(\Lambda_1^\eta/\varpi_1\right), \bigcup_{\varrho_1 \in \Upsilon_1, \phi_1 \in \phi_1} \left(\varrho_1^\eta/\phi_1\right) \right\},$

(4) $\mathfrak{I}_1^\eta = \left\{ \bigcup_{\xi_1 \in \upsilon_1, \Xi_1 \in \Xi_1} \left(\xi_1^\eta/\Xi_1\right), \bigcup_{\Lambda_1 \in \tau_1, \varpi_1 \in \varpi_1} \left(\Lambda_1^\eta/\varpi_1\right), \bigcup_{\varrho_1 \in \Upsilon, \phi_1 \in \phi_1} (1 - (1 - \varrho_1)^\eta/\phi_1) \right\}.$

Definition 9. *[31, 32] For any SV-NPHFN* $\mathfrak{I} = \left(\upsilon_{\ell_\varsigma}/\Xi_\flat, \tau_{\ell_\varsigma}/\varpi_\flat, \Upsilon_{\ell_\varsigma}/\phi_\flat\right)$, *a score function can defined as*

$$\left\{ \begin{array}{c} p(\mathfrak{I}) = \left(\frac{1}{M_\mathfrak{I}} \sum_{\xi_\daleth \in \upsilon_{\ell_\flat}, \Xi_\daleth \in \Xi_{\ell_\flat}} (\xi_\daleth \cdot \Xi_\daleth)\right) - \left(\frac{1}{N_\mathfrak{I}} \sum_{\Lambda_\daleth \in \tau_{\ell_\varsigma}, \varpi_\daleth \in \varpi_{\ell_\flat}} (\Lambda_\daleth \cdot \varpi_\daleth)\right) \\ - \left(\frac{1}{Z_\mathfrak{I}} \sum_{\varrho_\daleth \in \Upsilon_{\ell_\varsigma}, \phi_\daleth \in \phi_{\ell_\flat}} (\varrho_\daleth \cdot \phi_\daleth)\right) \end{array} \right\} \tag{2.1}$$

where $M_\mathfrak{I}$ *represents the number of elements in* $\upsilon_{\ell_\varsigma}$, $N_\mathfrak{I}$ *represents the number of elements in* τ_{ℓ_ς}, *and* $Z_\mathfrak{I}$ *represents the number of elements in* $\Upsilon_{\ell_\varsigma}$.

Definition 10. *[31] For any SV-NPHFN* $\mathfrak{I} = \left(\upsilon_{\ell_\varsigma}/\Xi_\flat, \tau_{\ell_\varsigma}/\varpi_\flat, \Upsilon_{\ell_\varsigma}/\phi_\flat\right)$, *an accuracy function is defined as*

$$\ell(\mathfrak{I}) = \left\{ \begin{array}{c} \left(\frac{1}{M_\mathfrak{I}} \sum_{\xi_\daleth \in \upsilon_{\ell_\flat}, \Xi_\daleth \in \Xi_{\ell_\flat}} (\xi_\daleth \cdot \Xi_\daleth)\right) + \left(\frac{1}{N_\mathfrak{I}} \sum_{\Lambda_\daleth \in \tau_{\ell_\varsigma}, \varpi_\daleth \in \varpi_{\ell_\flat}} (\Lambda_\daleth \cdot \varpi_\daleth)\right) \\ + \left(\frac{1}{Z_\mathfrak{I}} \sum_{\varrho_\daleth \in \Upsilon_{\ell_\varsigma}, \phi_\daleth \in \phi_{\ell_\flat}} (\varrho_\daleth \cdot \phi_\daleth)\right) \end{array} \right\}$$

where $M_\mathfrak{I}$ *represents the number of elements in* $\upsilon_{\ell_\varsigma}$, $N_\mathfrak{I}$ *represents the number of elements in* τ_{ℓ_ς} *and* $Z_\mathfrak{I}$ *represents the number of elements in* $\Upsilon_{\ell_\varsigma}$.

Definition 11. *Let* $\mathfrak{I}_1 = \left(\upsilon_{\ell_{\flat_1}}/\Xi_{\flat_1}, \tau_{\ell_{\flat_1}}/\varpi_{\flat_1}, \Upsilon_{\ell_{\flat_1}}/\phi_{\flat_1}\right)$ *and* $\mathfrak{I}_2 = \left(\upsilon_{\ell_{\flat_2}}/\Xi_{\flat_2}, \tau_{\ell_{\flat_2}}/\varpi_{\flat_2}, \Upsilon_{\ell_{\flat_2}}/\phi_{\flat_2}\right)$ *be two SV-NPHFNs. Then, by using the above definition, comparison of SV-NPHFNs can be described as*

(1) If $p(\mathfrak{I}_1) > p(\mathfrak{I}_2)$, *then* $\mathfrak{I}_1 > \mathfrak{I}_2$.

(2) If $p(\mathfrak{I}_1) = p(\mathfrak{I}_2)$, *and* $\ell(\mathfrak{I}_1) > \ell(\mathfrak{I}_2)$, *then* $\mathfrak{I}_1 > \mathfrak{I}_2$.

Definition 12. *Let* $\mathfrak{I}_\top = \left(\upsilon_{\ell_{\flat_\top}}/\Xi_{\flat_\top}, \tau_{\ell_{\flat_\top}}/\varpi_{\flat_\top}, \Upsilon_{\ell_{\flat_\top}}/\phi_{\flat_\top}\right)$ $(\top = 1, 2, ..., p)$ *be any group of SV-NPHFNs, and* $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p)^T$ *are the weights of* $\mathfrak{I}_\top \in [0, 1]$ *with* $\sum_{\top=1}^{p} \Gamma_\top = 1$. *Then, we have the following.*

(1) The SV-neutrosophic probabilistic hesitant fuzzy weighted averaging (SV-NPHFWA) AO [33] can be described as SV-NPHFWA

$$(\Im_1, \Im_2, ..., \Im_p) = \begin{pmatrix} \bigcup\limits_{\substack{\xi_\top \in \upsilon_{\Im_\top} \\ \Xi_{\Im_\top} \in \Xi_{\Im_\top}}} 1 - \Pi_{\top=1}^{p} \left(1 - \xi_{\Im_\top}\right)^{\Gamma_\top} / \Pi_{\top=1}^{p} \Xi_{\Im_\top}, \\ \bigcup\limits_{\substack{\Lambda_{\Im_\top} \in \tau_{\Im_\top} \\ \varpi_{\Im_\top} \in \varpi_{\Im_\top}}} \Pi_{\top=1}^{p} \left(\Lambda_{\Im_\top}\right)^{\Gamma_\top} / \Pi_{\top=1}^{p} \varpi_{\Im_\top}, \\ \bigcup\limits_{\substack{\varrho_{\Im_\top} \in \Upsilon_{\Im_\top} \\ \phi_{\Im_\top} \in \phi_{\Im_\top}}} \Pi_{\top=1}^{p} \left(\varrho_{\Im_\top}\right)^{\Gamma_\top} / \Pi_{\top=1}^{p} \phi_{\Im_\top} \end{pmatrix}. \tag{2.2}$$

(2) The SV-neutrosophic probabilistic hesitant fuzzy weighted geometric (SV-NPHFWG) AO [33] can be described as SV-NPHFWG

$$(\Im_1, \Im_2, ..., \Im_p) = \begin{pmatrix} \bigcup\limits_{\substack{\xi_\top \in \upsilon_{\Im_\top} \\ \Xi_{\Im_\top} \in \Xi_{\Im_\top}}} \Pi_{\top=1}^{p} \left(\xi_{\Im_\top}\right)^{\Gamma_\top} / \Pi_{\top=1}^{p} \Xi_{\Im_\top}, \\ \bigcup\limits_{\substack{\Lambda_{\Im_\top} \in \tau_{\Im_\top} \\ \varpi_{\Im_\top} \in \varpi_{\Im_\top}}} 1 - \Pi_{\top=1}^{p} \left(1 - \Lambda_{\Im_\top}\right)^{\Gamma_\top} / \Pi_{\top=1}^{p} \varpi_{\Im_\top}, \\ \bigcup\limits_{\substack{\varrho_{\Im_\top} \in \Upsilon_{\Im_\top} \\ \phi_{\Im_\top} \in \phi_{\Im_\top}}} 1 - \Pi_{\top=1}^{p} \left(1 - \varrho_{\Im_\top}\right)^{\Gamma_\top} / \Pi_{\top=1}^{p} \phi_{\Im_\top} \end{pmatrix}. \tag{2.3}$$

Definition 13. *Let* $\Im_\top = \left(\upsilon_{\ell_{\flat_\top}} / \Xi_{\flat_\top}, \tau_{\ell_{\flat_\top}} / \varpi_{\flat_\top}, \Upsilon_{\ell_{\flat_\top}} / \phi_{\flat_\top}\right)$ $(\top = 1, 2, ..., p)$ *be any group of SV-NPHFNs,* $\Im_{\Lambda(\top)}$ *be the jth largest in them and* $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p)^T$ *be the weights of* $\Im_\top \in [0, 1]$ *with* $\sum_{\top=1}^{p} \Gamma_\top = 1$. *Then, the SV-neutrosophic probabilistic hesitant fuzzy ordered weighted averaging (SV-NPHFOWA) AO can be described as*

$$\begin{aligned} & SV - NPHFOWA(\Im_1, \Im_2, ..., \Im_p) \\ = \; & \Gamma_1 \Im_{\Lambda(1)} \oplus \Gamma_2 \Im_{\Lambda(2)} \oplus ..... \oplus \Gamma_p \Im_{\Lambda(p)} \\ = \; & \begin{pmatrix} \bigcup\limits_{\substack{\xi_{\Im_{\Lambda(\top)}} \in \upsilon_{\Im_{\Lambda(\top)}} \\ \Xi_{\Im_{\Lambda(\top)}} \in \Xi_{\Im_{\Lambda(\top)}}}} 1 - \Pi_{\top=1}^{p} \left(1 - \xi_{\Im_{\Lambda(\top)}}\right)^{\Gamma_\top} / \Pi_{\top=1}^{p} \Xi_{\Im_{\Lambda(\top)}}, \\ \bigcup\limits_{\substack{\Lambda_{\Im_{\Lambda(\top)}} \in \tau_{\Im_{\Lambda(\top)}} \\ \varpi_{\Im_{\Lambda(\top)}} \in \varpi_{\Im_{\Lambda(\top)}}}} \Pi_{\top=1}^{p} \left(\Lambda_{\Im_{\Lambda(\top)}}\right)^{\Gamma_\top} / \Pi_{\top=1}^{p} \varpi_{\Im_{\Lambda(\top)}}, \\ \bigcup\limits_{\substack{\Upsilon_{\Im_{\Lambda(\top)}} \in \xi_{\Im_{\Lambda(\top)}} \\ \phi_{\Im_{\Lambda(\top)}} \in \phi_{\Im_{\Lambda(\top)}}}} \Pi_{\top=1}^{p} \left(\Upsilon_{\Im_{\Lambda(\top)}}\right)^{\Gamma_\top} / \Pi_{\top=1}^{p} \phi_{\Im_{\Lambda(\top)}} \end{pmatrix}. \end{aligned} \tag{2.4}$$

Definition 14. *Let* $\Im_\top = \left(\upsilon_{\ell_{\flat_\top}} / \Xi_{\flat_\top}, \tau_{\ell_{\flat_\top}} / \varpi_{\flat_\top}, \Upsilon_{\ell_{\flat_\top}} / \phi_{\flat_\top}\right)$ $(\top = 1, 2, ..., p)$ *be any group of SV-NHFNs,* $\Im_{\Lambda(\top)}$ *is the jth largest of* $\Im_{\Lambda(\top)} = p\Gamma_\top \Im_\top$ $(\top = 1, 2, ..., p)$. *Then, the following aggregation operator with aggregation-associated vector* $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p)^T$ *where* $\Gamma_\beth \in [0, 1]$, $\sum_{\top=1}^{p} \Gamma_\top = 1$, *can be described:*

SV-neutrosophic probabilistic hesitant fuzzy hybrid weighted averaging (SV-NPHFHWA) AO [33]

$$\begin{aligned} & SV-NHFHWA(\Im_1,\Im_2,....,\Im_p) \\ = \ & \Gamma_1\Im_{\Lambda(1)}\oplus\Gamma_2\Im_{\Lambda(2)}\oplus.....\oplus\Gamma_p\Im_{\Lambda(p)} \\ = \ & \begin{pmatrix} \bigcup\limits_{\substack{\xi_{\Im_{\Lambda(\top)}}\in\upsilon_{\Im_{\Lambda(\top)}} \\ \Xi_{\Im_{\Lambda(\top)}}\in\Xi_{\Im_{\Lambda(\top)}}}} 1-\Pi_{\top=1}^{p}\left(1-\xi_{\Im_{\Lambda(\top)}}\right)^{\Gamma_\top}/\Pi_{\top=1}^{p}\Xi_{\Im_{\Lambda(\top)}}, \\ \bigcup\limits_{\substack{\Lambda_{\Im_{\Lambda(\top)}}\in\tau_{\Im_{\Lambda(\top)}} \\ \varpi_{\Im_{\Lambda(\top)}}\in\varpi_{\Im_{\Lambda(\top)}}}} \Pi_{\top=1}^{p}\left(\Lambda_{\Im_{\Lambda(\top)}}\right)^{\Gamma_\top}/\Pi_{\top=1}^{p}\varpi_{\Im_{\Lambda(\top)}}, \\ \bigcup\limits_{\substack{\varrho_{\Im_{\Lambda(\top)}}\in\Upsilon_{\Im_{\Lambda(\top)}} \\ \phi_{\Im_{\Lambda(\top)}}\in\phi_{\Im_{\Lambda(\top)}}}} \Pi_{\top=1}^{p}\left(\varrho_{\Im_{\Lambda(\top)}}\right)^{\Gamma_\top}/\Pi_{\top=1}^{p}\phi_{\Im_{\Lambda(\top)}} \end{pmatrix}. \end{aligned} \tag{2.5}$$

3. Construction of single valued neutrosophic probabilistic hesitant fuzzy rough sets

The single valued neutrosophic probabilistic hesitant fuzzy rough set (SV-NPHFRS) is a hybrid structure of rough set that we present here. We also introduce the SV-NPHFRS's scoring and accuracy features, as well as its basic operational rules.

Definition 15. *Assume universal set $\ddot{U}$ and let $\Im \in SV-NPHFRS(\ddot{U}\times\ddot{U})$ be a $SV-NPHF$ relation. The pair $(\ddot{U},\Im)$ represents a $SV-NPHF$ approximation space. Assume ℓ be any subset of $SV-NPHS(\ddot{U})$ i.e., $\ell \subseteq SV-NPHS(\ddot{U})$. Then, on the bases of $SV-NPHF$ approximation space $(\ddot{U},\Im)$, the lower and upper approximations of ℓ are represented as $\overline{\Im}(\ell)$ and $\underline{\Im}(\ell)$, given as the following:*

$$\overline{\Im}(\ell)=\{\left\langle\sigma,\upsilon_{\overline{\Im}(\ell)}(\sigma)/\Xi_{\ell(\sigma)},\Lambda_{\overline{\Im}(\ell)}(\sigma)/\varpi_{\ell(\sigma)},\Upsilon_{\overline{\Im}(\ell)}(\sigma)/\phi_{\ell(\sigma)}\right\rangle|\sigma\in\ddot{U}\},$$

$$\underline{\Im}(\ell)=\{\langle\sigma,\upsilon_{\underline{\Im}(\ell)}(\sigma)/\Xi_{\ell(\sigma)},\Lambda_{\underline{\Im}(\ell)}(\sigma)/\varpi_{\ell(\sigma)},\Upsilon_{\underline{\Im}(\ell)}(\sigma)/\phi_{\ell(\sigma)}\rangle|\sigma\in\ddot{U}\},$$

where

$$\begin{aligned} \upsilon_{\overline{\Im}(\ell)}(\sigma)/\Xi_{\ell(\sigma)} &= \vee_{t\in\ddot{U}}[\upsilon_{\Im}(\sigma,t)/\Xi_{\ell(\sigma)}\vee\upsilon_{\Im}(t)/\Xi_{\ell(\sigma)}], \\ \Lambda_{\overline{\Im}(\ell)}(\sigma)/\varpi_{\ell(\sigma)} &= \wedge_{t\in\ddot{U}}[\Lambda_{\Im}(\sigma,t)/\varpi_{\ell(\sigma)}\wedge\Lambda_{\Im}(t)/\varpi_{\ell(\sigma)}], \\ \Upsilon_{\overline{\Im}(\ell)}(\sigma)/\phi_{\ell(\sigma)} &= \wedge_{t\in\ddot{U}}[\Upsilon_{\Im}(\sigma,t)/\phi_{\ell(\sigma)}\wedge\Upsilon_{\Im}(t)/\phi_{\ell(\sigma)}], \\ \upsilon_{\underline{\Im}(\ell)}(\sigma)/\Xi_{\ell(\sigma)} &= \wedge_{t\in\ddot{U}}[\upsilon_{\Im}(\sigma,t)/\Xi_{\ell(\sigma)}\wedge\upsilon_{\Im}(t)/\Xi_{\ell(\sigma)}], \\ \Lambda_{\underline{\Im}(\ell)}(\sigma)/\varpi_{\ell(\sigma)} &= \wedge_{t\in\ddot{U}}[\Lambda_{\Im}(\sigma,t)/\varpi_{\ell(\sigma)}\wedge\Lambda_{\Im}(t)/\varpi_{\ell(\sigma)}], \\ \Upsilon_{\underline{\Im}(\ell)}(\sigma)/\phi_{\ell(\sigma)} &= \vee_{t\in\ddot{U}}[\Upsilon_{\Im}(\sigma,t)/\phi_{\ell(\sigma)}\vee\Upsilon_{\Im}(t)/\phi_{\ell(\sigma)}], \end{aligned}$$

such that

$$0<\upsilon_{\overline{\Im}(\ell)}(\sigma)/\Xi_{\ell(\sigma)}+\Lambda_{\overline{\Im}(\ell)}(\sigma)/\varpi_{\ell(\sigma)}+\Upsilon_{\overline{\Im}(\ell)}(\sigma)/\phi_{\ell(\sigma)}\le 3,$$

and

$$0<\upsilon_{\underline{\Im}(\ell)}(\sigma)/\Xi_{\ell(\sigma)}+\Lambda_{\underline{\Im}(\ell)}(\sigma)/\varpi_{\ell(\sigma)}+\Upsilon_{\underline{\Im}(\ell)}(\sigma)/\phi_{\ell(\sigma)}\le 3.$$

As $\underline{\Im}(\ell)$ *and* $\overline{\Im}(\ell)$ *are* $SV-NFSs$*, so* $\overline{\Im}(\ell), \underline{\Im}(\ell) : SV-NFS(\ddot{U}) \longrightarrow SV-NFS(\ddot{U})$ *are upper and lower approximation operators. So, the pair*

$$\Im(\ell) = (\underline{\Im}(\ell), \overline{\Im}(\ell)) = \left\{ \begin{array}{c} \{\sigma, \langle\left((\upsilon_{\underline{\Im}(\ell)}(\sigma)/\Xi_{\ell(\sigma)}, \Lambda_{\underline{\Im}(\ell)}(\sigma)/\varpi_{\ell(\sigma)}, \Upsilon_{\underline{\Im}(\ell)}(\sigma)/\phi_{\ell(\sigma)})\right), \\ \left((\upsilon_{\overline{\Im}(\ell)}(\sigma)/\Xi_{\ell(\sigma)}, \Lambda_{\overline{\Im}(\ell)}(\sigma)/\varpi_{\ell(\sigma)}, \Upsilon_{\overline{\Im}(\ell)}(\sigma)/\phi_{\ell(\sigma)})\right) \end{array} \right\} |\sigma \in \ddot{U}\}$$

is called a $SV-NPHF$ *rough set. For simplicity, it can be denoted as*

$$\Im(\ell) = (\underline{g}(\ell), \overline{\Im}(\ell)) = \left\{ \begin{array}{c} (\underline{\Psi}_{\ell(\sigma)}/\Xi_{\ell(\sigma)}, \underline{\Phi}_{\ell(\sigma)}/\varpi_{\ell(\sigma)}, \underline{\Omega}_{\ell(\sigma)}/\phi_{\ell(\sigma)}), \\ (\overline{\upsilon}_{\ell(\sigma)}/\Xi_{\ell(\sigma)}, \overline{\Lambda}_{\ell(\sigma)}/\varpi_{\ell(\sigma)}, \overline{\Upsilon}_{\ell(\sigma)}/\phi_{\ell(\sigma)}) \end{array} \right\}.$$

Definition 16. *Assume* $\Re_1 = \left(\begin{array}{c} \upsilon_{\ell_{\Re_1}}/\Xi_{\ell(\sigma_1)}, \Lambda_{\ell_{\Re_1}}/\varpi_{\ell(\sigma_1)}, \\ \Upsilon_{\ell_{\Re_1}}/\phi_{\ell(\sigma_1)} \end{array} \right)$ *and* $\Re_2 = \left(\begin{array}{c} \upsilon_{\ell_{\Re_2}}/\Xi_{\ell(\sigma_2)}, \Lambda_{\ell_{\Re_2}}/\varpi_{\ell(\sigma_2)}, \\ \Upsilon_{\ell_{\Re_2}}/\phi_{\ell(\sigma_2)} \end{array} \right)$ *be two SV-NPHFNs. The following are the basic set theoretic operations:*

(1) $$\Re_1 \cup \Re_2 = \left\{ \bigcup_{\substack{\mho_1 \in \upsilon_{\ell_{\Re_1}}/\Xi_{\ell(\sigma_1)} \\ \mho_2 \in \upsilon_{\ell_{\Re_2}}/\Xi_{\ell(\sigma_2)}}} \max(\mho_1, \mho_2), \bigcup_{\substack{\nu_1 \in \Lambda_{\ell_{\Re_1}}/\varpi_{\ell(\sigma_1)} \\ \nu_2 \in \Lambda_{\ell_{\Re_2}}/\varpi_{\ell(\sigma_2)}}} \min(\nu_1, \nu_2), \bigcup_{\substack{\lambda_1 \in \Upsilon_{\ell_{\Re_1}}/\phi_{\ell(\sigma_1)} \\ \lambda_2 \in \Upsilon_{\ell_{\Re_2}}/\phi_{\ell(\sigma_2)}}} \min(\lambda_1, \lambda_2) \right\},$$

(2) $$\Re_1 \cap \Re_2 = \left\{ \bigcup_{\substack{\mho_1 \in \upsilon_{\ell_{\Re_1}}/\Xi_{\ell(\sigma_1)} \\ \mho_2 \in \upsilon_{\ell_{\Re_2}}/\Xi_{\ell(\sigma_2)}}} \min(\mho_1, \mho_2), \bigcup_{\substack{\nu_1 \in \Lambda_{\ell_{\Re_1}}/\varpi_{\ell(\sigma_1)} \\ \nu_2 \in \Lambda_{\ell_{\Re_2}}/\varpi_{\ell(\sigma_2)}}} \max(\nu_1, \nu_2), \bigcup_{\substack{\lambda_1 \in \Upsilon_{\ell_{\Re_1}}/\phi_{\ell(\sigma_1)} \\ \lambda_2 \in \Upsilon_{\ell_{\Re_2}}/\phi_{\ell(\sigma_2)}}} \max(\lambda_1, \lambda_2) \right\},$$

(3) $\Re_1^{\hat{c}} = \left\{ \Upsilon_{\ell_{\Re_1}}/\phi_{\ell(\sigma_1)}, \Lambda_{\ell_{\Re_1}}/\varpi_{\ell(\sigma_1)}, \upsilon_{\ell_{\Re_1}}/\Xi_{\ell(\sigma_1)} \right\}.$

Definition 17. *Assume* $\Re$ *is the universal set and* $\hbar \subseteq \Re \times \Re$ *is a (crisp) relation. Then, we have the following basic crisp sets properties.*
(1) $\hbar$ *is reflexive if* $(\Theta, \Theta) \in \hbar$*, for each* $\Theta \in \Re$;
(2) $\hbar$ *is symmetric if* $\forall \Theta, a \in \Re$, $(\Theta, a) \in \hbar$*, then* $(a, \Theta) \in \hbar$;
(3) $\hbar$ *is transitive if* $\forall\ \Theta, a, b \in \Re, (\Theta, a) \in \Re$ *and* $(a, b) \in \hbar \longrightarrow (\Theta, b) \in \hbar$.
Here, the reflexive, transitive and symmetric properties are same as the crisp sets.

Definition 18. *Assume* F *is the universal set. Then, any subset* $\hbar \in SV-NPHFRS(F \times F)$ *is said to be an SV- neutrosophic probabilistic hesitant fuzzy relation. The pair* $(F, \hbar)$ *is called a SV-NPHFRS approximation space. If for any* $\Lambda \subseteq SV-NPHFRS(F)$*, then the upper and lower approximations of* Λ *with respect to SV-NPHFRS approximation space* $(F, \hbar)$ *are two SV-NPHFRSs, which are denoted by* $\overline{\Im}(\Lambda)$ *and* $\underline{\Im}(\Lambda)$ *and defined as,*

$$\begin{aligned} \overline{\Im}(\Lambda) &= \left\{ \left\langle \Theta, \upsilon_{\ell_{\overline{\Im}(\Lambda)}}(\Theta)/\Xi_{\ell_{\overline{\Im}(\Lambda)}}, \Lambda_{\ell_{\overline{\Im}(\Lambda)}}(\Theta)/\varpi_{\ell_{\overline{\Im}(\Lambda)}}, \Upsilon_{\ell_{\overline{\Im}(\Lambda)}}(\Theta)/\phi_{\ell_{\overline{\Im}(\Lambda)}} \right\rangle |\Theta \in F \right\}, \\ \underline{\Im}(\Lambda) &= \left\{ \left\langle \Theta, \upsilon_{\ell_{\underline{\Im}(\Lambda)}}(\Theta)/\Xi_{\ell_{\underline{\Im}(\Lambda)}}, \Lambda_{\ell_{\underline{\Im}(\Lambda)}}(\Theta)/\varpi_{\ell_{\underline{\Im}(\Lambda)}}, \Upsilon_{\ell_{\underline{\Im}(\Lambda)}}(\Theta)/\varpi_{\ell_{\underline{\Im}(\Lambda)}} \right\rangle |\Theta \in F \right\}, \end{aligned}$$

where

$$\upsilon_{\ell_{\overline{\Im}(\Lambda)}}(\Theta)/\Xi_{\ell_\Theta} = \bigvee_{\chi \in F} \left[\upsilon_{\ell_\hbar}(\Theta, \chi) \bigvee \upsilon_{\ell_\Lambda}(\chi) \right] / \bigvee_{\chi \in F} \left[\Xi_{\ell_\hbar}(\Theta, c) \bigvee \Xi_{\ell_\Lambda}(\chi) \right],$$

$$\begin{aligned}
\Lambda_{\ell_{\overline{\mathfrak{I}}(\Lambda)}}(\Theta)/\varpi_{\ell_\Theta} &= \bigwedge_{\chi\in F}\left[\Lambda_{\ell_\hbar}(\Theta,\chi)\bigwedge\Lambda_{\ell_\Lambda}(\chi)\right]/\bigwedge_{\chi\in F}\left[\varpi_{\ell_\hbar}(\Theta,c)\bigwedge\varpi_{\ell_\Lambda}(\chi)\right],\\
\upsilon_{\ell_{\underline{\mathfrak{I}}(\Lambda)}}(\Theta)/\Xi_{\ell_\Theta} &= \bigwedge_{\chi\in F}\left[\upsilon_{\ell_\hbar}(\Theta,\chi)\bigwedge\upsilon_{\ell_\Lambda}(\chi)\right]/\bigwedge_{\chi\in F}\left[\Xi_{\ell_\hbar}(\Theta,\chi)\bigwedge\Xi_{\ell_\Lambda}(\chi)\right],\\
\Lambda_{\ell_{\underline{\mathfrak{I}}(\Lambda)}}(\Theta)/\varpi_{\ell_\Theta} &= \bigvee_{\chi\in F}\left[\Lambda_{\ell_\hbar}(\Theta,\chi)\bigvee\Lambda_{\ell_\Lambda}(\chi)\right]/\bigvee_{\chi\in F}\left[\varpi_{\ell_\hbar}(\Theta,c)\bigvee\varpi_{\ell_\Lambda}(\chi)\right],\\
\Upsilon_{\ell_{\overline{\mathfrak{I}}(\Lambda)}}(\Theta)/\Xi_{\ell_\Theta} &= \bigvee_{\chi\in F}\left[\Upsilon_{\ell_\hbar}(\Theta,\chi)\bigvee\Upsilon_{\ell_\Lambda}(\chi)\right]/\bigvee_{\chi\in F}\left[\phi_{\ell_\hbar}(\Theta,c)\bigvee\phi_{\ell_\Lambda}(\chi)\right],\\
\Upsilon_{\ell_{\underline{\mathfrak{I}}(\Lambda)}}(\Theta)/\Xi_{\ell_\Theta} &= \bigwedge_{\chi\in F}\left[\Upsilon_{\ell_\hbar}(\Theta,\chi)\bigwedge\Upsilon_{\ell_\Lambda}(\chi)\right]/\bigwedge_{\chi\in F}\left[\phi_{\ell_\hbar}(\Theta,\chi)\bigwedge\phi_{\ell_\Lambda}(\chi)\right],
\end{aligned}$$

such that

$$0<\left(\max(\upsilon_{\ell_{\overline{\mathfrak{I}}(\Lambda)}}(\Theta))\right)+\left(\min(\Lambda_{\ell_{\overline{\mathfrak{I}}(\Lambda)}}(\Theta))\right)+\left(\min(\Upsilon_{\ell_{\overline{\mathfrak{I}}(\Lambda)}}(\Theta))\right)\leq 3,$$

and

$$0<\left(\min(\upsilon_{\ell_{\underline{\mathfrak{I}}(\Lambda)}}(\Theta)\right)+\left(\max(\Lambda_{\ell_{\underline{\mathfrak{I}}(\Lambda)}}(\Theta))\right)+\left(\max(\Lambda_{\ell_{\underline{\mathfrak{I}}(\Lambda)}}(\Theta))\right)\leq 3.$$

As $\left(\overline{\mathfrak{I}}(\Lambda),\underline{\mathfrak{I}}(\Lambda)\right)$ *are* $SV-NPHFRS$, $\overline{\mathfrak{I}}(\Lambda),\underline{\mathfrak{I}}(\Lambda):SV-NPHFRS(F)\rightarrow SV-NPHFRS(F)$ *are upper and lower approximation operators. The pair*

$$\hbar(\Lambda)=\left(\underline{\mathfrak{I}}(\Lambda),\overline{\mathfrak{I}}(\Lambda)\right)=\left\{\left\langle\begin{array}{c}\Theta,\left(\upsilon_{\ell_{\underline{\mathfrak{I}}(\Lambda)}}(\Theta)/\Xi_{\ell_{\underline{\mathfrak{I}}(\Lambda)}},\Lambda_{\ell_{\underline{\mathfrak{I}}(\Lambda)}}(\Theta)/\varpi_{\ell_{\underline{\mathfrak{I}}(\Lambda)}},\Upsilon_{\ell_{\underline{\mathfrak{I}}(\Lambda)}}(\Theta)/\phi_{\ell_{\underline{\mathfrak{I}}(\Lambda)}}\right),\\ \Theta,\left(\upsilon_{\ell_{\overline{\mathfrak{I}}(\Lambda)}}(\Theta)/\Xi_{\ell_{\overline{\mathfrak{I}}(\Lambda)}},\Lambda_{\ell_{\overline{\mathfrak{I}}(\Lambda)}}(\Theta)/\varpi_{\ell_{\hbar(\Lambda)}},\Upsilon_{\ell_{\overline{\mathfrak{I}}(\Lambda)}}(\Theta)/\phi_{\ell_{\overline{\mathfrak{I}}(\Lambda)}}\right)\end{array}\right\rangle|\Theta\in\Lambda\right\}$$

will be called SV*-neutrosophic hesitant fuzzy rough set. For simplicity,*

$$\hbar(\Lambda)=\left\{\begin{array}{c}\langle\Theta,\left(\upsilon_{\ell_{\underline{\mathfrak{I}}(\Lambda)}}(\Theta)//\Xi_{\ell_{\underline{\mathfrak{I}}(\Lambda)}},\Lambda_{\ell_{\underline{\mathfrak{I}}(\Lambda)}}(\Theta)/\varpi_{\ell_{\underline{\mathfrak{I}}(\Lambda)}},\Upsilon_{\ell_{\underline{\mathfrak{I}}(\Lambda)}}(\Theta)/\phi_{\ell_{\underline{\mathfrak{I}}(\Lambda)}}\right),\\ \left(\upsilon_{\ell_{\overline{\mathfrak{I}}(\Lambda)}}(\Theta)//\Xi_{\ell_{\overline{\mathfrak{I}}(\Lambda)}},\Lambda_{\ell_{\overline{\mathfrak{I}}(\Lambda)}}(\Theta)/\varpi_{\ell_{\overline{\mathfrak{I}}(\Lambda)}},\Upsilon_{\ell_{\overline{\mathfrak{I}}(\Lambda)}}(\Theta)/\phi_{\ell_{\overline{\mathfrak{I}}(\Lambda)}}\right)\rangle|\Theta\in\Lambda\end{array}\right\}$$

is represented as

$$\hbar(\Lambda)=\left((\underline{\Psi}/\underline{\eth},\underline{\Phi}/\underline{\partial},\underline{\Omega}/\underline{\Xi}),(\overline{\upsilon}/\overline{\Xi},\overline{\Lambda}/\overline{\varpi},\overline{\Upsilon}/\overline{\phi})\right)$$

and is known as SV-NPHFRSs.

Definition 19. *Here, we define the SV-neutrosophic probabilistic hesitant fuzzy rough weighted averaging (SV-NPHFRWA) AO by using Eq (2.2) as*

$$\left\{\begin{array}{c}(\overline{\mathfrak{I}}_1,\overline{\mathfrak{I}}_2,....,\overline{\mathfrak{I}}_p)\\(\underline{\mathfrak{I}}_1,\underline{\mathfrak{I}}_2,....,\underline{\mathfrak{I}}_p)\end{array}\right\}\tag{3.1}$$

$$
= \left(\begin{array}{c}
\left\{ \begin{array}{c}
\left(\bigcup\limits_{\substack{\xi_{\overline{\mathfrak{I}}^{\top}} \in \upsilon_{\overline{\mathfrak{I}}_{\top}} \\ \Xi_{\overline{\mathfrak{I}}_{\top}} \in \Xi_{\overline{\mathfrak{I}}_{\top}}}} 1 - \Pi_{\top=1}^{p} \left(1 - \xi_{\overline{\mathfrak{I}}_{\top}} \right)^{\Gamma_{\top}} / \Pi_{\top=1}^{p} \Xi_{\overline{\mathfrak{I}}_{\top}} \right) \\
\left(\bigcup\limits_{\substack{\underline{\alpha}_{\top} \in \upsilon_{\underline{\mathfrak{I}}_{\top}} \\ \Xi_{\underline{\mathfrak{I}}_{\top}} \in \Xi_{\underline{\mathfrak{I}}_{\top}}}} 1 - \Pi_{\top=1}^{p} \left(1 - \xi_{\underline{\mathfrak{I}}_{\top}} \right)^{\Gamma_{\top}} / \Pi_{\top=1}^{p} \Xi_{\underline{\mathfrak{I}}_{\top}} \right)
\end{array} \right\}, \\
\left\{ \begin{array}{c}
\left(\bigcup\limits_{\substack{\Lambda_{\overline{\mathfrak{I}}_{\top}} \in \tau_{\overline{\mathfrak{I}}_{\top}} \\ \varpi_{\overline{\mathfrak{I}}_{\top}} \in \varpi_{\overline{\mathfrak{I}}_{\top}}}} \Pi_{\top=1}^{p} \left(\Lambda_{\overline{\mathfrak{I}}_{\top}} \right)^{\Gamma_{\top}} / \Pi_{\top=1}^{p} \varpi_{\overline{\mathfrak{I}}_{\top}} \right) \\
\left(\bigcup\limits_{\substack{\Lambda_{\underline{\mathfrak{I}}_{\top}} \in \tau_{\underline{\mathfrak{I}}_{\top}} \\ \varpi_{\underline{\mathfrak{I}}_{\top}} \in \varpi_{\underline{\mathfrak{I}}_{\top}}}} \Pi_{\top=1}^{p} \left(\Lambda_{\underline{\mathfrak{I}}_{\top}} \right)^{\Gamma_{\top}} / \Pi_{\top=1}^{p} \varpi_{\underline{\mathfrak{I}}_{\top}} \right)
\end{array} \right\}, \\
\left\{ \begin{array}{c}
\left(\bigcup\limits_{\substack{\varrho_{\overline{\mathfrak{I}}_{\top}} \in \Upsilon_{\overline{\mathfrak{I}}_{\top}} \\ \phi_{\overline{\mathfrak{I}}_{\top}} \in \phi_{\overline{\mathfrak{I}}_{\top}}}} \Pi_{\top=1}^{p} \left(\varrho_{\overline{\mathfrak{I}}_{\top}} \right)^{\Gamma_{\top}} / \Pi_{\top=1}^{p} \phi_{\overline{\mathfrak{I}}_{\top}} \right) \\
\left(\bigcup\limits_{\substack{\varrho_{\underline{\mathfrak{I}}_{\top}} \in \Upsilon_{\underline{\mathfrak{I}}_{\top}} \\ \phi_{\underline{\mathfrak{I}}_{\top}} \in \phi_{\underline{\mathfrak{I}}_{\top}}}} \Pi_{\top=1}^{p} (\varrho_{\mathfrak{I}_{\top}})^{\Gamma_{\top}} / \Pi_{\top=1}^{p} \phi_{\underline{\mathfrak{I}}_{\top}} \right)
\end{array} \right\}
\end{array} \right). \tag{3.2}
$$

Definition 20. *The SV-NPHFRWG operator using Eq (2.3) can be described as*

$$
\left\{ \begin{array}{c} (\overline{\mathfrak{I}}_1, \overline{\mathfrak{I}}_2, ..., \overline{\mathfrak{I}}_p) \\ (\underline{\mathfrak{I}}_1, \underline{\mathfrak{I}}_2, ..., \underline{\mathfrak{I}}_p) \end{array} \right\} \tag{3.3}
$$

$$
= \left(\begin{array}{c}
\left\{ \begin{array}{c}
\left(\bigcup\limits_{\substack{\xi_{\overline{\mathfrak{I}}^{\top}} \in \upsilon_{\overline{\mathfrak{I}}_{\top}} \\ \Xi_{\overline{\mathfrak{I}}_{\top}} \in \Xi_{\overline{\mathfrak{I}}_{\top}}}} \Pi_{\top=1}^{p} \left(\xi_{\overline{\mathfrak{I}}_{\top}} \right)^{\Gamma_{\top}} / \Pi_{\top=1}^{p} \Xi_{\overline{\mathfrak{I}}_{\top}} \right) \\
\left(\bigcup\limits_{\substack{\xi_{\mathfrak{I}^{\top}} \in \upsilon_{\underline{\mathfrak{I}}_{\top}} \\ \Xi_{\underline{\mathfrak{I}}_{\top}} \in \Xi_{\underline{\mathfrak{I}}_{\top}}}} \Pi_{\top=1}^{p} \left(\xi_{\underline{\mathfrak{I}}_{\top}} \right)^{\Gamma_{\top}} / \Pi_{\top=1}^{p} \Xi_{\underline{\mathfrak{I}}_{\top}} \right)
\end{array} \right\}, \\
\left\{ \begin{array}{c}
\left(\bigcup\limits_{\substack{\Lambda_{\overline{\mathfrak{I}}_{\top}} \in \tau_{\overline{\mathfrak{I}}_{\top}} \\ \varpi_{\overline{\mathfrak{I}}_{\top}} \in \varpi_{\overline{\mathfrak{I}}_{\top}}}} 1 - \Pi_{\top=1}^{p} \left(1 - \Lambda_{\overline{\mathfrak{I}}_{\top}} \right)^{\Gamma_{\top}} / \Pi_{\top=1}^{p} \varpi_{\overline{\mathfrak{I}}_{\top}} \right) \\
\left(\bigcup\limits_{\substack{\Lambda_{\underline{\mathfrak{I}}_{\top}} \in \tau_{\underline{\mathfrak{I}}_{\top}} \\ \varpi_{\underline{\mathfrak{I}}_{\top}} \in \varpi_{\underline{\mathfrak{I}}_{\top}}}} 1 - \Pi_{\top=1}^{p} \left(1 - \Lambda_{\underline{\mathfrak{I}}_{\top}} \right)^{\Gamma_{\top}} / \Pi_{\top=1}^{p} \varpi_{\underline{\mathfrak{I}}_{\top}} \right)
\end{array} \right\}, \\
\left\{ \begin{array}{c}
\left(\bigcup\limits_{\substack{\varrho_{\overline{\mathfrak{I}}_{\top}} \in \Upsilon_{\overline{\mathfrak{I}}_{\top}} \\ \phi_{\overline{\mathfrak{I}}_{\top}} \in \phi_{\overline{\mathfrak{I}}_{\top}}}} 1 - \Pi_{\top=1}^{p} \left(1 - \varrho_{\overline{\mathfrak{I}}_{\top}} \right)^{\Gamma_{\top}} / \Pi_{\top=1}^{p} \phi_{\overline{\mathfrak{I}}_{\top}} \right) \\
\left(\bigcup\limits_{\substack{\varrho_{\underline{\mathfrak{I}}_{\top}} \in \Upsilon_{\underline{\mathfrak{I}}_{\top}} \\ \phi_{\underline{\mathfrak{I}}_{\top}} \in \phi_{\underline{\mathfrak{I}}_{\top}}}} 1 - \Pi_{\top=1}^{p} \left(1 - \varrho_{\underline{\mathfrak{I}}_{\top}} \right)^{\Gamma_{\top}} / \Pi_{\top=1}^{p} \phi_{\underline{\mathfrak{I}}_{\top}} \right)
\end{array} \right\}
\end{array} \right). \tag{3.4}
$$

Definition 21. *The SV-NPHFROWA operator can be described as, using Eq (2.4),*

$$\left\{\begin{array}{c}(\overline{\mathfrak{I}}_1,\overline{\mathfrak{I}}_2,....,\overline{\mathfrak{I}}_p)\\(\underline{\mathfrak{I}}_1,\underline{\mathfrak{I}}_2,....,\underline{\mathfrak{I}}_p)\end{array}\right\} \tag{3.5}$$

$$= \left\{\begin{array}{c}
\left\{\begin{array}{c}
\left\{\bigcup_{\substack{\xi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\in\upsilon_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\\ \Xi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\in\Xi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}}} 1-\Pi_{\top=1}^{p}\left(1-\xi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\right)^{\Gamma_\top}/\Pi_{\top=1}^{p}\Xi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\right\}\\
\left\{\bigcup_{\substack{\xi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\in\upsilon_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\\ \Xi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\in\Xi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}}} 1-\Pi_{\top=1}^{p}\left(1-\xi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\right)^{\Gamma_\top}/\Pi_{\top=1}^{p}\Xi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\right\}
\end{array}\right\},\\
\left\{\begin{array}{c}
\left\{\bigcup_{\substack{\Lambda_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\in\tau_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\\ \varpi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\in\varpi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}}} \Pi_{\top=1}^{p}\left(\Lambda_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\right)^{\Gamma_\top}/\Pi_{\top=1}^{p}\varpi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\right\}\\
\left\{\bigcup_{\substack{\Lambda_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\in\tau_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\\ \varpi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\in\varpi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}}} \Pi_{\top=1}^{p}\left(\Lambda_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\right)^{\Gamma_\top}/\Pi_{\top=1}^{p}\varpi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\right\}
\end{array}\right\},\\
\left\{\begin{array}{c}
\left\{\bigcup_{\substack{\Upsilon_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\in\xi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\\ \phi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\in\phi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}}} \Pi_{\top=1}^{p}\left(\Upsilon_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\right)^{\Gamma_\top}/\Pi_{\top=1}^{p}\phi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\right\}\\
\left\{\bigcup_{\substack{\Upsilon_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\in\xi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\\ \phi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\in\phi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}}} \Pi_{\top=1}^{p}\left(\Upsilon_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\right)^{\Gamma_\top}/\Pi_{\top=1}^{p}\phi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\right\}
\end{array}\right\}
\end{array}\right\}. \tag{3.6}$$

Definition 22. *The $SV-NPHFRHWA$ operator, using Eq (2.5) is as*

$$\left\{\begin{array}{c}(\overline{\mathfrak{I}}_1,\overline{\mathfrak{I}}_2,....,\overline{\mathfrak{I}}_p)\\(\underline{\mathfrak{I}}_1,\underline{\mathfrak{I}}_2,....,\underline{\mathfrak{I}}_p)\end{array}\right\} \tag{3.7}$$

$$
= \left\{ \begin{array}{c}
\left\{ \begin{array}{c}
\left\{ \bigcup_{\substack{\xi_{\overline{\mathfrak{I}}_{\Lambda(\top)}} \in \upsilon_{\overline{\mathfrak{I}}_{\Lambda(\top)}} \\ \Xi_{\overline{\mathfrak{I}}_{\Lambda(\top)}} \in \Xi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}}} 1 - \Pi_{\top=1}^{p} \left(1 - \xi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\right)^{\Gamma_\top} / \Pi_{\top=1}^{p} \Xi_{\overline{\mathfrak{I}}_{\Lambda(\top)}} \right\} \\
\left\{ \bigcup_{\substack{\xi_{\underline{\mathfrak{I}}_{\Lambda(\top)}} \in \upsilon_{\underline{\mathfrak{I}}_{\Lambda(\top)}} \\ \Xi_{\underline{\mathfrak{I}}_{\Lambda(\top)}} \in \Xi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}}} 1 - \Pi_{\top=1}^{p} \left(1 - \xi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\right)^{\Gamma_\top} / \Pi_{\top=1}^{p} \Xi_{\underline{\mathfrak{I}}_{\Lambda(\top)}} \right\}
\end{array} \right\}, \\
\left\{ \begin{array}{c}
\left\{ \bigcup_{\substack{\Lambda_{\overline{\mathfrak{I}}_{\Lambda(\top)}} \in \tau_{\overline{\mathfrak{I}}_{\Lambda(\top)}} \\ \varpi_{\overline{\mathfrak{I}}_{\Lambda(\top)}} \in \varpi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}}} \Pi_{\top=1}^{p} \left(\Lambda_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\right)^{\Gamma_\top} / \Pi_{\top=1}^{p} \varpi_{\overline{\mathfrak{I}}_{\Lambda(\top)}} \right\} \\
\left\{ \bigcup_{\substack{\Lambda_{\underline{\mathfrak{I}}_{\Lambda(\top)}} \in \tau_{\underline{\mathfrak{I}}_{\Lambda(\top)}} \\ \varpi_{\underline{\mathfrak{I}}_{\Lambda(\top)}} \in \varpi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}}} \Pi_{\top=1}^{p} \left(\Lambda_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\right)^{\Gamma_\top} / \Pi_{\top=1}^{p} \varpi_{\underline{\mathfrak{I}}_{\Lambda(\top)}} \right\}
\end{array} \right\}, \\
\left\{ \begin{array}{c}
\left\{ \bigcup_{\substack{\varrho_{\overline{\mathfrak{I}}_{\Lambda(\top)}} \in \Upsilon_{\overline{\mathfrak{I}}_{\Lambda(\top)}} \\ \phi_{\overline{\mathfrak{I}}_{\Lambda(\top)}} \in \phi_{\overline{\mathfrak{I}}_{\Lambda(\top)}}}} \Pi_{\top=1}^{p} \left(\varrho_{\overline{\mathfrak{I}}_{\Lambda(\top)}}\right)^{\Gamma_\top} / \Pi_{\top=1}^{p} \phi_{\overline{\mathfrak{I}}_{\Lambda(\top)}} \right\} \\
\left\{ \bigcup_{\substack{\varrho_{\underline{\mathfrak{I}}_{\Lambda(\top)}} \in \Upsilon_{\underline{\mathfrak{I}}_{\Lambda(\top)}} \\ \phi_{\underline{\mathfrak{I}}_{\Lambda(\top)}} \in \phi_{\underline{\mathfrak{I}}_{\Lambda(\top)}}}} \Pi_{\top=1}^{p} \left(\varrho_{\underline{\mathfrak{I}}_{\Lambda(\top)}}\right)^{\Gamma_\top} / \Pi_{\top=1}^{p} \phi_{\underline{\mathfrak{I}}_{\Lambda(\top)}} \right\}
\end{array} \right\}
\end{array} \right\}. \tag{3.8}
$$

(3.9)

Einstein operational laws based on SV-NPHFSs

The application of t-norms in FS theory at the intersection of two FSs is widely recognized. T-conorms are used to model disjunction or union. These are a straightforward explanation of the conjunction and disjunction in mathematical fuzzy logic syntax, and they are utilized in MCDM to combine criteria. The Einstein sum ($\oplus_\epsilon$) and Einstein product ($\otimes_\epsilon$) are case studies of t-conorms and t-norms, respectively, and are stated in the single valued neutrosophic probabilistic hesitant fuzzy (SV-NPHF) environment as follows.

$$
\tilde{N} \oplus_\epsilon \check{S} = \frac{\tilde{N} + \check{S}}{1 + \tilde{N}\check{S}}, \tilde{N} \otimes_\epsilon \check{S} = \frac{\tilde{N}\check{S}}{1 + \left(1 - \tilde{N}\right)\left(1 - \check{S}\right)}.
$$

Based on the above Einstein operations, we give the following new operations on SV-NPHF environment.

Definition 23. *Let* $\mathfrak{I}_1 = \left(\upsilon_{\ell_{\mathfrak{b}_1}}/\Xi_{\mathfrak{b}_1}, \tau_{\ell_{\mathfrak{b}_1}}/\varpi_{\mathfrak{b}_1}, \Upsilon_{\ell_{\mathfrak{b}_1}}/\phi_{\mathfrak{b}_1}\right)$ *and* $\mathfrak{I}_2 = \left(\upsilon_{\ell_{\mathfrak{b}_2}}/\Xi_{\mathfrak{b}_2}, \tau_{\ell_{\mathfrak{b}_2}}/\varpi_{\mathfrak{b}_2}, \Upsilon_{\ell_{\mathfrak{b}_2}}/\phi_{\mathfrak{b}_2}\right)$ *be two SV-NPHFEs and* $\eta > 0(\in \mathbb{R})$. *Then,*

$$\textbf{(1)}\ \mathfrak{I}_1 \oplus \mathfrak{I}_2 = \left\{ \begin{array}{c} \bigcup\limits_{\substack{\xi_1 \in \upsilon_{\ell_{\mathfrak{b}_1}}(l_{\mathfrak{b}}), \xi_2 \in \upsilon_{\ell_{\mathfrak{b}_2}}(l_{\mathfrak{b}}) \\ \Xi_1 \in \Xi_{\mathfrak{b}_1}, \Xi_2 \in \Xi_{\mathfrak{b}_2}}} \left(\frac{\xi_1+\xi_2}{1+\xi_1\xi_2}/\Xi_1\Xi_2\right), \\ \bigcup\limits_{\substack{\Lambda_1 \in \tau_{\ell_{\mathfrak{b}_1}}(l_{\mathfrak{b}}), \Lambda_2 \in \tau_{\ell_{\mathfrak{b}_2}}(l_{\mathfrak{b}}) \\ \varpi_1 \in \varpi_{\mathfrak{b}_1}, \varpi_2 \in \varpi_{\mathfrak{b}_2}}} \left(\frac{\Lambda_1\Lambda_2}{1+(1-\Lambda_1)(1-\Lambda_2)}/\varpi_1\varpi_2\right), \\ \bigcup\limits_{\substack{\varrho_1 \in \Upsilon_{\ell_{\mathfrak{b}_1}}(l_{\mathfrak{b}}), \varrho_2 \in \Upsilon_{\ell_{\mathfrak{b}_2}}(l_{\mathfrak{b}}) \\ \phi_1 \in \phi_{\mathfrak{b}_1}, \phi_2 \in \phi_{\mathfrak{b}_2}}} \left(\frac{\varrho_1\varrho_2}{1+(1-\varrho_1)(1-\varrho_2)}/\phi_1\phi_2\right) \end{array} \right\}.$$

$$\textbf{(2)}\ \mathfrak{I}_1 \otimes \mathfrak{I}_2 = \left\{ \begin{array}{c} \bigcup\limits_{\substack{\xi_1 \in \upsilon_{\ell_{\mathfrak{b}_1}}(l_{\mathfrak{b}}), \xi_2 \in \upsilon_{\ell_{\mathfrak{b}_2}}(l_{\mathfrak{b}}) \\ \Xi_1 \in \Xi_{\mathfrak{b}_1}, \Xi_2 \in \Xi_{\mathfrak{b}_2}}} \left(\frac{\xi_1\xi_2}{1+(1-\xi_1)(1-\xi_2)}/\Xi_1\Xi_2\right), \\ \bigcup\limits_{\substack{\Lambda_1 \in \tau_{\ell_{\mathfrak{b}_1}}(l_{\mathfrak{b}}), \Lambda_2 \in \tau_{\ell_{\mathfrak{b}_2}}(l_{\mathfrak{b}}) \\ \varpi_1 \in \varpi_{\mathfrak{b}_1}, \varpi_2 \in \varpi_{\mathfrak{b}_2}}} \left(\frac{\Lambda_1+\Lambda_2}{1+\Lambda_1\Lambda_2}/\varpi_1\varpi_2\right), \\ \bigcup\limits_{\substack{\Upsilon_1 \in \xi_{\ell_{\mathfrak{b}_1}}(l_{\mathfrak{b}}), \Upsilon_2 \in \xi_{\ell_{\mathfrak{b}_2}}(l_{\mathfrak{b}}) \\ \phi_1 \in \phi_{\mathfrak{b}_1}, \phi_2 \in \phi_{\mathfrak{b}_2}}} \left(\frac{\Upsilon_1+\Upsilon_2}{1+\Upsilon_1\Upsilon_2}/\phi_1\phi_2\right) \end{array} \right\}.$$

$$\textbf{(3)}\ \eta\mathfrak{I}_1 = \left\{ \begin{array}{c} \bigcup\limits_{\xi_1 \in \upsilon_{\ell_{\mathfrak{b}_1}}(l_{\mathfrak{b}}), \Xi_1 \in \Xi_{\mathfrak{b}_1}} \left(\frac{(1+\xi_1)^\eta-(1-\xi_1)^\eta}{(1+\xi_1)^\eta+(1-\xi_1)^\eta}/\Xi_1\right), \\ \bigcup\limits_{\Lambda_1 \in \tau_{\ell_{\mathfrak{b}_1}}(l_{\mathfrak{b}}), \varpi_1 \in \varpi_{\mathfrak{b}_1}} \left(\frac{2\Lambda_1^\eta}{(2-\Lambda_1)^\eta+(\Lambda_1)^\eta}/\varpi_1\right), \\ \bigcup\limits_{\varrho_1 \in \Upsilon_{\ell_{\mathfrak{b}_1}}(l_{\mathfrak{b}}), \phi_1 \in \phi_{\mathfrak{b}_1}} \left(\frac{2\varrho_1^\eta}{(2-\varrho_1)^\eta+(\varrho_1)^\eta}/\phi_1\right) \end{array} \right\}.$$

$$\textbf{(4)}\ \mathfrak{I}_1^\eta = \left\{ \begin{array}{c} \bigcup\limits_{\xi_1 \in \upsilon_{\ell_{\mathfrak{b}_1}}(l_{\mathfrak{b}}), \Xi_1 \in \Xi_{\mathfrak{b}_1}} \left(\frac{2\xi_1^\eta}{(2-\xi_1)^\eta+(\xi_1)^\eta}/\Xi_1\right), \\ \bigcup\limits_{\Lambda_1 \in \tau_{\ell_{\mathfrak{b}_1}}(l_{\mathfrak{b}}), \varpi_1 \in \varpi_{\mathfrak{b}_1}} \left(\frac{(1+\Lambda_1)^\eta-(1-\Lambda_1)^\eta}{(1+\Lambda_1)^\eta+(1-\Lambda_1)^\eta} - (1-\Lambda_1)^\eta/\varpi_1\right), \\ \bigcup\limits_{\varrho_1 \in \Upsilon_{\ell_{\mathfrak{b}_1}}(l_{\mathfrak{b}}), \phi_1 \in \phi_{\mathfrak{b}_1}} \left(\frac{(1+\varrho_1)^\eta-(1-\varrho_1)^\eta}{(1+\varrho_1)^\eta+(1-\varrho_1)^\eta} - (1-\varrho_1)^\eta/\phi_1\right) \end{array} \right\}.$$

4. SV-Neutrosophic probabilistic hesitant fuzzy rough Einstein aggregation operators

In this section, we develop several new Einstein operators for SV-NPHFRNs, namely the single valued neutrosophic probabilistic hesitant fuzzy rough Einstein weighted averaging (SV-NPHFREWA) operator, the single valued neutrosophic probabilistic hesitant fuzzy rough Einstein ordered weighted averaging (SV-NPHFREOWA) operator.

Single-valued neutrosophic probabilistic hesitant fuzzy rough Einstein weighted averaging operator

Definition 24. *Let* $\Im_\top = \left\{ \begin{array}{c} \left(\upsilon_{\bar{\flat}_\top}/\Xi_{\bar{\flat}_\top}, \tau_{\bar{\flat}_\top}/\varpi_{\bar{\flat}_\top}, \Upsilon_{\bar{\flat}_\top}/\phi_{\bar{\flat}_\top}\right) \\ \left(\upsilon_{\underline{\flat}_\top}/\Xi_{\underline{\flat}_\top}, \tau_{\underline{\flat}_\top}/\varpi_{\underline{\flat}_\top}, \Upsilon_{\underline{\flat}_\top}/\phi_{\underline{\flat}_\top}\right) \end{array} \right\}$ $(\top = 1, 2, ..., p)$ *be a collection of SV-NPHFRNs.*

$\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p)^T$ are the weights of $\Im_\top \in [0, 1]$ with $\sum_{\top=1}^{p} \Gamma_\top = 1$. Then, SV-NPHFREWA: SV-NPHFRN^p ⟶SV-NPHFRN such that

$$SV-NPHFREWA \left\{ \begin{array}{c} (\overline{\Im}_1, \overline{\Im}_2, ..., \overline{\Im}_p) \\ (\underline{\Im}_1, \underline{\Im}_2, ..., \underline{\Im}_p) \end{array} \right\} = \left\{ \begin{array}{c} \left(\Gamma_1 \cdot_\varepsilon \overline{\Im}_1\right)\left(\Gamma_1 \cdot_\varepsilon \underline{\Im}_1\right) \oplus_\varepsilon \\ \left(\Gamma_2 \cdot_\varepsilon \overline{\Im}_2\right)\left(\Gamma_2 \cdot_\varepsilon \underline{\Im}_2\right) \oplus_\varepsilon ... \oplus_\varepsilon \\ \left(\Gamma_p \cdot_\varepsilon \overline{\Im}_p\right)\left(\Gamma_p \cdot_\varepsilon \underline{\Im}_p\right) \end{array} \right\}$$

is called the single valued neutrosophic probabilistic hesitant fuzzy rough Einstein weighted averaging operator.

Theorem 1. *Let* $\Im_\top = \left\{ \begin{array}{c} \left(\upsilon_{\bar{\flat}_\top}/\Xi_{\bar{\flat}_\top}, \tau_{\bar{\flat}_\top}/\varpi_{\bar{\flat}_\top}, \Upsilon_{\bar{\flat}_\top}/\phi_{\bar{\flat}_\top}\right) \\ \left(\upsilon_{\underline{\flat}_\top}/\Xi_{\underline{\flat}_\top}, \tau_{\underline{\flat}_\top}/\varpi_{\underline{\flat}_\top}, \Upsilon_{\underline{\flat}_\top}/\phi_{\underline{\flat}_\top}\right) \end{array} \right\}$ $(\top = 1, 2, ..., p)$ *be a collection of SV-NPHFRNs.*

Then, we can achieve the following:

$$SV-NPHFREWA \left\{ \begin{array}{c} (\overline{\Im}_1, \overline{\Im}_2, ..., \overline{\Im}_p) \\ (\underline{\Im}_1, \underline{\Im}_2, ..., \underline{\Im}_p) \end{array} \right\} \tag{4.1a}$$

$$= \left(\begin{array}{c} \left\{ \begin{array}{c} \left(\bigcup\limits_{\substack{\xi_{\overline{\Im}_\top} \in \upsilon_{\overline{\Im}_\top} \\ \Xi_{\overline{\Im}_\top} \in \Xi_{\overline{\Im}_\top}}} \frac{\Pi_{\top=1}^{p}\left(1+\xi_{\overline{\Im}_\top}\right)^{\Gamma_\top} - \Pi_{\top=1}^{p}\left(1-\xi_{\overline{\Im}_\top}\right)^{\Gamma_\top}}{\Pi_{\top=1}^{p}\left(1+\xi_{\overline{\Im}_\top}\right)^{\Gamma_\top} + \Pi_{\top=1}^{p}\left(1-\xi_{\overline{\Im}_\top}\right)^{\Gamma_\top}} / \Pi_{\top=1}^{p} \Xi_{\overline{\Im}_\top} \right) \\ \left(\bigcup\limits_{\substack{\xi_{\underline{\Im}_\top} \in \upsilon_{\underline{\Im}_\top} \\ \Xi_{\underline{\Im}_\top} \in \Xi_{\underline{\Im}_\top}}} \frac{\Pi_{\top=1}^{p}\left(1+\xi_{\underline{\Im}_\top}\right)^{\Gamma_\top} - \Pi_{\top=1}^{p}\left(1-\xi_{\underline{\Im}_\top}\right)^{\Gamma_\top}}{\Pi_{\top=1}^{p}\left(1+\xi_{\underline{\Im}_\top}\right)^{\Gamma_\top} + \Pi_{\top=1}^{p}\left(1-\xi_{\underline{\Im}_\top}\right)^{\Gamma_\top}} / \Pi_{\top=1}^{p} \Xi_{\underline{\Im}_\top} \right) \end{array} \right\}, \\ \left\{ \begin{array}{c} \left(\bigcup\limits_{\substack{\Lambda_{\overline{\Im}_\top} \in \tau_{\overline{\Im}_\top} \\ \varpi_{\overline{\Im}_\top} \in \varpi_{\overline{\Im}_\top}}} \frac{2\Pi_{\top=1}^{p}\left(\Lambda_{\overline{\Im}_\top}\right)^{\Gamma_\top}}{\Pi_{\top=1}^{p}\left(2-\Lambda_{\overline{\Im}_\top}\right)^{\Gamma_\top} + \Pi_{\top=1}^{p}\left(\Lambda_{\overline{\Im}_\top}\right)^{\Gamma_\top}} / \Pi_{\top=1}^{p} \varpi_{\overline{\Im}_\top} \right) \\ \bigcup\limits_{\substack{\Lambda_{\underline{\Im}_\top} \in \tau_{\underline{\Im}_\top} \\ \varpi_{\underline{\Im}_\top} \in \varpi_{\underline{\Im}_\top}}} \frac{2\Pi_{\top=1}^{p}\left(\Lambda_{\underline{\Im}_\top}\right)^{\Gamma_\top}}{\Pi_{\top=1}^{p}\left(2-\Lambda_{\underline{\Im}_\top}\right)^{\Gamma_\top} + \Pi_{\top=1}^{p}\left(\Lambda_{\underline{\Im}_\top}\right)^{\Gamma_\top}} / \Pi_{\top=1}^{p} \varpi_{\underline{\Im}_\top} \end{array} \right\}, \\ \left\{ \begin{array}{c} \left(\bigcup\limits_{\substack{\Upsilon_{\overline{\Im}_\top} \in \xi_{\overline{\Im}_\top} \\ \phi_{\overline{\Im}_\top} \in \phi_{\overline{\Im}_\top}}} \frac{2\Pi_{\top=1}^{p}\left(\Upsilon_{\overline{\Im}_\top}\right)^{\Gamma_\top}}{\Pi_{\top=1}^{p}\left(2-\Upsilon_{\overline{\Im}_\top}\right)^{\Gamma_\top} + \Pi_{\top=1}^{p}\left(\Upsilon_{\overline{\Im}_\top}\right)^{\Gamma_\top}} / \Pi_{\top=1}^{p} \phi_{\overline{\Im}_\top} \right) \\ \left(\bigcup\limits_{\substack{\Upsilon_{\underline{\Im}_\top} \in \xi_{\underline{\Im}_\top} \\ \phi_{\underline{\Im}_\top} \in \phi_{\underline{\Im}_\top}}} \frac{2\Pi_{\top=1}^{p}\left(\Upsilon_{\underline{\Im}_\top}\right)^{\Gamma_\top}}{\Pi_{\top=1}^{p}\left(2-\Upsilon_{\underline{\Im}_\top}\right)^{\Gamma_\top} + \Pi_{\top=1}^{p}\left(\Upsilon_{\underline{\Im}_\top}\right)^{\Gamma_\top}} / \Pi_{\top=1}^{p} \phi_{\underline{\Im}_\top} \right) \end{array} \right\} \end{array} \right) \tag{4.1b}$$

where $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p)^T$ are the weights of $\Im_\top$ with $\Gamma_\top \in [0,1]$ with $\sum_{\top=1}^{p} \Gamma_\top = 1$.

We will demonstrate the theorem by mathematical induction. For $p = 2$.

Proof.

$$SV-NPHFREWA\left\{ \begin{array}{c} (\overline{\Im}_1, \overline{\Im}_2) \\ (\underline{\Im}_1, \underline{\Im}_2) \end{array} \right\} = \left\{ \begin{array}{c} \left(\Gamma_1 \cdot_\varepsilon \overline{\Im}_1\right)\left(\Gamma_1 \cdot_\varepsilon \underline{\Im}_1\right) \oplus_\varepsilon \\ \left(\Gamma_2 \cdot_\varepsilon \overline{\Im}_2\right)\left(\Gamma_2 \cdot_\varepsilon \underline{\Im}_2\right) \end{array} \right\}.$$

Since both $\left(\Gamma_1 \cdot_\varepsilon \overline{\Im}_1\right)\left(\Gamma_1 \cdot_\varepsilon \underline{\Im}_1\right)$ and $\left(\Gamma_2 \cdot_\varepsilon \overline{\Im}_2\right)\left(\Gamma_2 \cdot_\varepsilon \underline{\Im}_2\right)$ is also a SV-NPHFRN, $\left\{ \begin{array}{c} \left(\Gamma_1 \cdot_\varepsilon \overline{\Im}_1\right)\left(\Gamma_1 \cdot_\varepsilon \underline{\Im}_1\right) \oplus_\varepsilon \\ \left(\Gamma_2 \cdot_\varepsilon \overline{\Im}_2\right)\left(\Gamma_2 \cdot_\varepsilon \underline{\Im}_2\right) \end{array} \right\}$ is a SV-NPHFRN.

$$\left(\Gamma_1 \cdot_\varepsilon \overline{\Im}_1\right)\left(\Gamma_1 \cdot_\varepsilon \underline{\Im}_1\right)$$

$$= \left(\begin{array}{c} \left\{ \begin{array}{c} \left(\bigcup_{\substack{\xi_{\overline{\Im}_1} \in \upsilon_{\overline{\Im}_1} \\ \Xi_{\overline{\Im}_1} \in \Xi_{\overline{\Im}_1}}} \frac{\left(1+\xi_{\overline{\Im}_1}\right)^{\Gamma_1} - \left(1-\xi_{\overline{\Im}_1}\right)^{\Gamma_1}}{\left(1+\xi_{\overline{\Im}_1}\right)^{\Gamma_1} + \left(1-\xi_{\overline{\Im}_1}\right)^{\Gamma_1}} / \Xi_{\overline{\Im}_1} \right) \\ \left(\bigcup_{\substack{\xi_{\underline{\Im}_1} \in \upsilon_{\underline{\Im}_1} \\ \Xi_{\underline{\Im}_1} \in \Xi_{\underline{\Im}_1}}} \frac{\left(1+\xi_{\underline{\Im}_1}\right)^{\Gamma_1} - \left(1-\xi_{\underline{\Im}_1}\right)^{\Gamma_1}}{\left(1+\xi_{\underline{\Im}_1}\right)^{\Gamma_1} + \left(1-\xi_{\underline{\Im}_1}\right)^{\Gamma_1}} / \Xi_{\underline{\Im}_1} \right) \end{array} \right\}, \\ \left\{ \begin{array}{c} \left(\bigcup_{\substack{\Lambda_{\overline{\Im}_1} \in \tau_{\overline{\Im}_1} \\ \varpi_{\overline{\Im}_1} \in \varpi_{\overline{\Im}_1}}} \frac{2\left(\Lambda_{\overline{\Im}_1}\right)^{\Gamma_1}}{\left(2-\Lambda_{\overline{\Im}_1}\right)^{\Gamma_1} + \left(\Lambda_{\overline{\Im}_1}\right)^{\Gamma_1}} / \varpi_{\overline{\Im}_1} \right) \\ \left(\bigcup_{\substack{\Lambda_{\underline{\Im}_1} \in \tau_{\underline{\Im}_1} \\ \varpi_{\underline{\Im}_1} \in \varpi_{\underline{\Im}_1}}} \frac{2\left(\Lambda_{\underline{\Im}_1}\right)^{\Gamma_1}}{\left(2-\Lambda_{\underline{\Im}_1}\right)^{\Gamma_1} + \left(\Lambda_{\underline{\Im}_1}\right)^{\Gamma_1}} / \varpi_{\underline{\Im}_1} \right) \end{array} \right\}, \\ \left\{ \begin{array}{c} \left(\bigcup_{\substack{\Upsilon_{\overline{\Im}_1} \in \xi_{\overline{\Im}_1} \\ \phi_{\overline{\Im}_1} \in \phi_{\overline{\Im}_1}}} \frac{2\left(\Upsilon_{\overline{\Im}_1}\right)^{\Gamma_1}}{\left(2-\Upsilon_{\overline{\Im}_1}\right)^{\Gamma_1} + \left(\Upsilon_{\overline{\Im}_1}\right)^{\Gamma_1}} / \phi_{\overline{\Im}_1} \right) \\ \left(\bigcup_{\substack{\Upsilon_{\underline{\Im}_1} \in \xi_{\underline{\Im}_1} \\ \phi_{\underline{\Im}_1} \in \phi_{\underline{\Im}_1}}} \frac{2\left(\Upsilon_{\underline{\Im}_1}\right)^{\Gamma_1}}{\left(2-\Upsilon_{\underline{\Im}_1}\right)^{\Gamma_1} + \left(\Upsilon_{\underline{\Im}_1}\right)^{\Gamma_1}} / \phi_{\underline{\Im}_1} \right) \end{array} \right\} \end{array} \right).$$

$$= \begin{pmatrix} \left(\Gamma_{2}\cdot_{\varepsilon}\overline{\mathfrak{I}}_{2}\right)\left(\Gamma_{2}\cdot_{\varepsilon}\underline{\mathfrak{I}}_{2}\right) \\ \left\{ \begin{pmatrix} \bigcup_{\substack{\xi_2\in\upsilon_{\overline{\mathfrak{I}}_2}\\ \Xi_{\overline{\mathfrak{I}}_2}\in\Xi_{\overline{\mathfrak{I}}_2}}} \frac{\left(1+\xi_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}-\left(1-\xi_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}}{\left(1+\xi_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}+\left(1-\xi_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}}/\Xi_{\overline{\mathfrak{I}}_2} \end{pmatrix} \begin{pmatrix} \bigcup_{\substack{\xi_2\in\upsilon_{\underline{\mathfrak{I}}_2}\\ \Xi_{\underline{\mathfrak{I}}_2}\in\Xi_{\underline{\mathfrak{I}}_2}}} \frac{\left(1+\xi_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}-\left(1-\xi_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}}{\left(1+\xi_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}+\left(1-\xi_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}}/\Xi_{\underline{\mathfrak{I}}_2} \end{pmatrix} \right\}, \\ \left\{ \begin{pmatrix} \bigcup_{\substack{\Lambda_{\overline{\mathfrak{I}}_2}\in\tau_{\overline{\mathfrak{I}}_2}\\ \varpi_{\overline{\mathfrak{I}}_2}\in\varpi_{\overline{\mathfrak{I}}_2}}} \frac{2\left(\Lambda_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}}{\left(2-\Lambda_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}+\left(\Lambda_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}}/\varpi_{\overline{\mathfrak{I}}_2} \end{pmatrix} \begin{pmatrix} \bigcup_{\substack{\Lambda_{\underline{\mathfrak{I}}_2}\in\tau_{\underline{\mathfrak{I}}_2}\\ \varpi_{\underline{\mathfrak{I}}_2}\in\varpi_{\underline{\mathfrak{I}}_2}}} \frac{2\left(\Lambda_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}}{\left(2-\Lambda_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}+\left(\Lambda_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}}/\varpi_{\underline{\mathfrak{I}}_2} \end{pmatrix} \right\}, \\ \left\{ \begin{pmatrix} \bigcup_{\substack{\Upsilon_{\overline{\mathfrak{I}}_2}\in\xi_{\overline{\mathfrak{I}}_2}\\ \phi_{\overline{\mathfrak{I}}_2}\in\phi_{\overline{\mathfrak{I}}_2}}} \frac{2\left(\Upsilon_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}}{\left(2-\Upsilon_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}+\left(\Upsilon_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}}/\phi_{\overline{\mathfrak{I}}_2} \end{pmatrix} \begin{pmatrix} \bigcup_{\substack{\Upsilon_{\underline{\mathfrak{I}}_2}\in\xi_{\underline{\mathfrak{I}}_2}\\ \phi_{\underline{\mathfrak{I}}_2}\in\phi_{\underline{\mathfrak{I}}_2}}} \frac{2\left(\Upsilon_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}}{\left(2-\Upsilon_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}+\left(\Upsilon_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}}/\phi_{\underline{\mathfrak{I}}_2} \end{pmatrix} \right\} \end{pmatrix}.$$

Then

$$SV-NPHFREWA\left\{\begin{matrix}(\overline{\mathfrak{I}}_1,\overline{\mathfrak{I}}_2)\\(\underline{\mathfrak{I}}_1,\underline{\mathfrak{I}}_2)\end{matrix}\right\}=\left\{\begin{matrix}\left(\Gamma_{1}\cdot_{\varepsilon}\overline{\mathfrak{I}}_{1}\right)\left(\Gamma_{1}\cdot_{\varepsilon}\underline{\mathfrak{I}}_{1}\right)\oplus_{\varepsilon}\\ \left(\Gamma_{2}\cdot_{\varepsilon}\overline{\mathfrak{I}}_{2}\right)\left(\Gamma_{2}\cdot_{\varepsilon}\underline{\mathfrak{I}}_{2}\right)\end{matrix}\right\}$$

$$= \begin{pmatrix} \left\{ \begin{pmatrix} \bigcup_{\substack{\xi_1\in\upsilon_{\overline{\mathfrak{I}}_1}\\ \Xi_{\overline{\mathfrak{I}}_1}\in\Xi_{\overline{\mathfrak{I}}_1}}} \frac{\left(1+\xi_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1}-\left(1-\xi_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1}}{\left(1+\xi_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1}+\left(1-\xi_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1}}/\Xi_{\overline{\mathfrak{I}}_1} \end{pmatrix} \begin{pmatrix} \bigcup_{\substack{\xi_1\in\upsilon_{\underline{\mathfrak{I}}_1}\\ \Xi_{\underline{\mathfrak{I}}_1}\in\Xi_{\underline{\mathfrak{I}}_1}}} \frac{\left(1+\xi_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1}-\left(1-\xi_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1}}{\left(1+\xi_{\mathfrak{I}_1}\right)^{\Gamma_1}+\left(1-\xi_{\mathfrak{I}_1}\right)^{\Gamma_1}}/\Xi_{\underline{\mathfrak{I}}_1} \end{pmatrix} \right\}, \\ \left\{ \begin{pmatrix} \bigcup_{\substack{\Lambda_{\overline{\mathfrak{I}}_1}\in\tau_{\overline{\mathfrak{I}}_1}\\ \varpi_{\overline{\mathfrak{I}}_1}\in\varpi_{\overline{\mathfrak{I}}_1}}} \frac{2\left(\Lambda_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1}}{\left(2-\Lambda_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1}+\left(\Lambda_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1}}/\varpi_{\overline{\mathfrak{I}}_1} \end{pmatrix} \begin{pmatrix} \bigcup_{\substack{\Lambda_{\underline{\mathfrak{I}}_1}\in\tau_{\mathfrak{I}_1}\\ \varpi_{\underline{\mathfrak{I}}_1}\in\varpi_{\underline{\mathfrak{I}}_1}}} \frac{2\left(\Lambda_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1}}{\left(2-\Lambda_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1}+\left(\Lambda_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1}}/\varpi_{\underline{\mathfrak{I}}_1} \end{pmatrix} \right\}, \\ \left\{ \begin{pmatrix} \bigcup_{\substack{\Upsilon_{\overline{\mathfrak{I}}_1}\in\xi_{\overline{\mathfrak{I}}_1}\\ \phi_{\overline{\mathfrak{I}}_1}\in\phi_{\overline{\mathfrak{I}}_1}}} \frac{2\left(\Upsilon_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1}}{\left(2-\Upsilon_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1}+\left(\Upsilon_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1}}/\phi_{\overline{\mathfrak{I}}_1} \end{pmatrix} \begin{pmatrix} \bigcup_{\substack{\Upsilon_{\underline{\mathfrak{I}}_1}\in\xi_{\underline{\mathfrak{I}}_1}\\ \phi_{\underline{\mathfrak{I}}_1}\in\phi_{\underline{\mathfrak{I}}_1}}} \frac{2\left(\Upsilon_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1}}{\left(2-\Upsilon_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1}+\left(\Upsilon_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1}}/\phi_{\underline{\mathfrak{I}}_1} \end{pmatrix} \right\} \end{pmatrix} \oplus_{\varepsilon}$$

$$\left(\begin{array}{c} \left\{ \left(\bigcup_{\substack{\xi_2 \in \upsilon_{\overline{\mathfrak{J}}_2} \\ \Xi_{\overline{\mathfrak{J}}_2} \in \Xi_{\overline{\mathfrak{J}}_2}}} \frac{\left(1+\xi_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2} - \left(1-\xi_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2}}{\left(1+\xi_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(1-\xi_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2}} / \Xi_{\overline{\mathfrak{J}}_2} \right) \left(\bigcup_{\substack{\xi_2 \in \upsilon_{\underline{\mathfrak{J}}_2} \\ \Xi_{\underline{\mathfrak{J}}_2} \in \Xi_{\underline{\mathfrak{J}}_2}}} \frac{\left(1+\xi_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2} - \left(1-\xi_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}}{\left(1+\xi_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(1-\xi_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}} / \Xi_{\underline{\mathfrak{J}}_2} \right) \right\}, \\ \left\{ \left(\bigcup_{\substack{\Lambda_{\overline{\mathfrak{J}}_2} \in \tau_{\overline{\mathfrak{J}}_2} \\ \varpi_{\overline{\mathfrak{J}}_2} \in \varpi_{\overline{\mathfrak{J}}_2}}} \frac{2\left(\Lambda_{\mathfrak{J}_2}\right)^{\Gamma_2}}{\left(2-\Lambda_{\mathfrak{J}_2}\right)^{\Gamma_2} + \left(\Lambda_{\mathfrak{J}_2}\right)^{\Gamma_2}} / \varpi_{\mathfrak{J}_2} \right) \left(\bigcup_{\substack{\Lambda_{\underline{\mathfrak{J}}_2} \in \tau_{\underline{\mathfrak{J}}_2} \\ \varpi_{\underline{\mathfrak{J}}_2} \in \varpi_{\underline{\mathfrak{J}}_2}}} \frac{2\left(\Lambda_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}}{\left(2-\Lambda_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(\Lambda_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}} / \varpi_{\underline{\mathfrak{J}}_2} \right) \right\}, \\ \left\{ \left(\bigcup_{\substack{\Upsilon_{\overline{\mathfrak{J}}_2} \in \xi_{\overline{\mathfrak{J}}_2} \\ \phi_{\overline{\mathfrak{J}}_2} \in \phi_{\overline{\mathfrak{J}}_2}}} \frac{2\left(\Upsilon_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2}}{\left(2-\Upsilon_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(\Upsilon_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2}} / \phi_{\overline{\mathfrak{J}}_2} \right) \left(\bigcup_{\substack{\Upsilon_{\underline{\mathfrak{J}}_2} \in \xi_{\underline{\mathfrak{J}}_2} \\ \phi_{\underline{\mathfrak{J}}_2} \in \phi_{\underline{\mathfrak{J}}_2}}} \frac{2\left(\Upsilon_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}}{\left(2-\Upsilon_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(\Upsilon_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}} / \phi_{\underline{\mathfrak{J}}_2} \right) \right\} \end{array} \right)$$

$$= \left(\begin{array}{c} \left\{ \begin{array}{c} \left(\bigcup_{\substack{\xi_1 \in \upsilon_{\overline{\mathfrak{J}}_1}, \Xi_{\overline{\mathfrak{J}}_1} \in \Xi_{\overline{\mathfrak{J}}_1} \\ \xi_2 \in \upsilon_{\overline{\mathfrak{J}}_2}, \Xi_{\overline{\mathfrak{J}}_2} \in \Xi_{\overline{\mathfrak{J}}_2}}} \frac{\frac{\left(1+\xi_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1} - \left(1-\xi_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1}}{\left(1+\xi_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1} + \left(1-\xi_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1}} + \frac{\left(1+\xi_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2} - \left(1-\xi_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2}}{\left(1+\xi_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(1-\xi_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2}}}{1 + \left(\frac{\left(1+\xi_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1} - \left(1-\xi_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1}}{\left(1+\xi_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1} + \left(1-\xi_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1}} \right) \cdot_{\varepsilon} \left(\frac{\left(1+\xi_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2} - \left(1-\xi_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2}}{\left(1+\xi_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(1-\xi_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2}} \right)} / \Xi_{\overline{\mathfrak{J}}_1} \Xi_{\overline{\mathfrak{J}}_2} \right) \\ \left(\bigcup_{\substack{\xi_1 \in \upsilon_{\underline{\mathfrak{J}}_1}, \Xi_{\underline{\mathfrak{J}}_1} \in \Xi_{\underline{\mathfrak{J}}_1} \\ \xi_2 \in \upsilon_{\underline{\mathfrak{J}}_2}, \Xi_{\underline{\mathfrak{J}}_2} \in \Xi_{\underline{\mathfrak{J}}_2}}} \frac{\frac{\left(1+\xi_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1} - \left(1-\xi_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1}}{\left(1+\xi_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1} + \left(1-\xi_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1}} + \frac{\left(1+\xi_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2} - \left(1-\xi_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}}{\left(1+\xi_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(1-\xi_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}}}{1 + \left(\frac{\left(1+\xi_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1} - \left(1-\xi_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1}}{\left(1+\xi_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1} + \left(1-\xi_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1}} \right) \cdot_{\varepsilon} \left(\frac{\left(1+\xi_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2} - \left(1-\xi_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}}{\left(1+\xi_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(1-\xi_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}} \right)} / \Xi_{\underline{\mathfrak{J}}_1} \Xi_{\underline{\mathfrak{J}}_2} \right) \end{array} \right\}, \\ \left\{ \begin{array}{c} \left(\bigcup_{\substack{\Lambda_{\overline{\mathfrak{J}}_1} \in \tau_{\overline{\mathfrak{J}}_1}, \varpi_{\overline{\mathfrak{J}}_1} \in \varpi_{\overline{\mathfrak{J}}_1} \\ \Lambda_{\overline{\mathfrak{J}}_2} \in \tau_{\overline{\mathfrak{J}}_2}, \varpi_{\overline{\mathfrak{J}}_2} \in \varpi_{\overline{\mathfrak{J}}_2}}} \frac{\left(\frac{2\left(\Lambda_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1}}{\left(2-\Lambda_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1} + \left(\Lambda_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1}} \right) \cdot_{\varepsilon} \left(\frac{2\left(\Lambda_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2}}{\left(2-\Lambda_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(\Lambda_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2}} \right)}{1 + \left(1 - \frac{2\left(\Lambda_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1}}{\left(2-\Lambda_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1} + \left(\Lambda_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1}} \right) \cdot_{\varepsilon} \left(1 - \frac{2\left(\Lambda_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2}}{\left(2-\Lambda_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(\Lambda_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2}} \right)} / \varpi_{\overline{\mathfrak{J}}_1} \varpi_{\overline{\mathfrak{J}}_2} \right) \\ \left(\bigcup_{\substack{\Lambda_{\underline{\mathfrak{J}}_1} \in \tau_{\underline{\mathfrak{J}}_1}, \varpi_{\underline{\mathfrak{J}}_1} \in \varpi_{\underline{\mathfrak{J}}_1} \\ \Lambda_{\underline{\mathfrak{J}}_2} \in \tau_{\underline{\mathfrak{J}}_2}, \varpi_{\underline{\mathfrak{J}}_2} \in \varpi_{\underline{\mathfrak{J}}_2}}} \frac{\left(\frac{2\left(\Lambda_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1}}{\left(2-\Lambda_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1} + \left(\Lambda_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1}} \right) \cdot_{\varepsilon} \left(\frac{2\left(\Lambda_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}}{\left(2-\Lambda_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(\Lambda_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}} \right)}{1 + \left(1 - \frac{2\left(\Lambda_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1}}{\left(2-\Lambda_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1} + \left(\Lambda_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1}} \right) \cdot_{\varepsilon} \left(1 - \frac{2\left(\Lambda_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}}{\left(2-\Lambda_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(\Lambda_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}} \right)} / \varpi_{\underline{\mathfrak{J}}_1} \varpi_{\underline{\mathfrak{J}}_2} \right) \end{array} \right\}, \\ \left\{ \begin{array}{c} \left(\bigcup_{\substack{\Upsilon_{\overline{\mathfrak{J}}_1} \in \xi_{\overline{\mathfrak{J}}_1}, \phi_{\overline{\mathfrak{J}}_1} \in \phi_{\overline{\mathfrak{J}}_1} \\ \Upsilon_{\overline{\mathfrak{J}}_2} \in \xi_{\overline{\mathfrak{J}}_2}, \phi_{\overline{\mathfrak{J}}_2} \in \phi_{\overline{\mathfrak{J}}_2}}} \frac{\left(\frac{2\left(\Upsilon_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1}}{\left(2-\Upsilon_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1} + \left(\Upsilon_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1}} \right) \cdot_{\varepsilon} \left(\frac{2\left(\Upsilon_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2}}{\left(2-\Upsilon_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(\Upsilon_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2}} \right)}{1 + \left(1 - \frac{2\left(\Upsilon_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1}}{\left(2-\Upsilon_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1} + \left(\Upsilon_{\overline{\mathfrak{J}}_1}\right)^{\Gamma_1}} \right) \cdot_{\varepsilon} \left(1 - \frac{2\left(\left(\Upsilon_{\overline{\mathfrak{J}}_2}\right)^2\right)^{\Gamma_2}}{\left(2-\Upsilon_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(\Upsilon_{\overline{\mathfrak{J}}_2}\right)^{\Gamma_2}} \right)} / \phi_{\overline{\mathfrak{J}}_1} \phi_{\overline{\mathfrak{J}}_2} \right) \\ \left(\bigcup_{\substack{\Upsilon_{\underline{\mathfrak{J}}_1} \in \xi_{\underline{\mathfrak{J}}_1}, \phi_{\underline{\mathfrak{J}}_1} \in \phi_{\underline{\mathfrak{J}}_1} \\ \Upsilon_{\underline{\mathfrak{J}}_2} \in \xi_{\underline{\mathfrak{J}}_2}, \phi_{\underline{\mathfrak{J}}_2} \in \phi_{\underline{\mathfrak{J}}_2}}} \frac{\left(\frac{2\left(\Upsilon_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1}}{\left(2-\Upsilon_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1} + \left(\Upsilon_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1}} \right) \cdot_{\varepsilon} \left(\frac{2\left(\Upsilon_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}}{\left(2-\Upsilon_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(\Upsilon_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}} \right)}{1 + \left(1 - \frac{2\left(\Upsilon_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1}}{\left(2-\Upsilon_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1} + \left(\Upsilon_{\underline{\mathfrak{J}}_1}\right)^{\Gamma_1}} \right) \cdot_{\varepsilon} \left(1 - \frac{2\left(\left(\Upsilon_{\underline{\mathfrak{J}}_2}\right)^2\right)^{\Gamma_2}}{\left(2-\Upsilon_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2} + \left(\Upsilon_{\underline{\mathfrak{J}}_2}\right)^{\Gamma_2}} \right)} / \phi_{\underline{\mathfrak{J}}_1} \phi_{\underline{\mathfrak{J}}_2} \right) \end{array} \right\} \end{array} \right)$$

$$= \left(\begin{array}{l} \left\{ \begin{array}{l} \left(\bigcup_{\substack{\xi_1 \in \upsilon_{\overline{\mathfrak{I}}_1}, \Xi_{\overline{\mathfrak{I}}_1} \in \Xi_{\overline{\mathfrak{I}}_1} \\ \xi_2 \in \upsilon_{\overline{\mathfrak{I}}_2}, \Xi_{\overline{\mathfrak{I}}_2} \in \Xi_{\overline{\mathfrak{I}}_2}}} \frac{\left(1+\xi_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1} -._{\varepsilon} \left(1+\xi_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2} - \left(1-\xi_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1} ._{\varepsilon} \left(1-\xi_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}}{\left(1+\xi_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1} -._{\varepsilon} \left(1+\xi_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2} + \left(1-\xi_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1} ._{\varepsilon} \left(1-\xi_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}} / \Xi_{\overline{\mathfrak{I}}_1} \Xi_{\overline{\mathfrak{I}}_2} \right) \\ \left(\bigcup_{\substack{\xi_1 \in \upsilon_{\underline{\mathfrak{I}}_1}, \Xi_{\underline{\mathfrak{I}}_1} \in \Xi_{\underline{\mathfrak{I}}_1} \\ \xi_2 \in \upsilon_{\underline{\mathfrak{I}}_2}, \Xi_{\underline{\mathfrak{I}}_2} \in \Xi_{\underline{\mathfrak{I}}_2}}} \frac{\left(1+\xi_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1} -._{\varepsilon} \left(1+\xi_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2} - \left(1-\xi_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1} ._{\varepsilon} \left(1-\xi_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}}{\left(1+\xi_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1} -._{\varepsilon} \left(1+\xi_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2} + \left(1-\xi_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1} ._{\varepsilon} \left(1-\xi_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}} / \Xi_{\underline{\mathfrak{I}}_1} \Xi_{\underline{\mathfrak{I}}_2} \right) \end{array} \right\}, \\ \left\{ \begin{array}{l} \left(\bigcup_{\substack{\Lambda_{\overline{\mathfrak{I}}_1} \in \tau_{\overline{\mathfrak{I}}_1}, \varpi_{\overline{\mathfrak{I}}_1} \in \varpi_{\overline{\mathfrak{I}}_1} \\ \Lambda_{\overline{\mathfrak{I}}_2} \in \tau_{\overline{\mathfrak{I}}_2}, \varpi_{\overline{\mathfrak{I}}_2} \in \varpi_{\overline{\mathfrak{I}}_2}}} \frac{2\left(\Lambda_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1} \left(\Lambda_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}}{\left(2-\Lambda_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1} ._{\varepsilon} \left(2-\Lambda_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2} + \left(\Lambda_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1} ._{\varepsilon} \left(\Lambda_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}} / \varpi_{\overline{\mathfrak{I}}_1} \varpi_{\overline{\mathfrak{I}}_2} \right) \\ \left(\bigcup_{\substack{\Lambda_{\underline{\mathfrak{I}}_1} \in \tau_{\underline{\mathfrak{I}}_1}, \varpi_{\underline{\mathfrak{I}}_1} \in \varpi_{\underline{\mathfrak{I}}_1} \\ \Lambda_{\underline{\mathfrak{I}}_2} \in \tau_{\underline{\mathfrak{I}}_2}, \varpi_{\underline{\mathfrak{I}}_2} \in \varpi_{\underline{\mathfrak{I}}_2}}} \frac{2\left(\Lambda_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1} \left(\Lambda_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}}{\left(2-\Lambda_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1} ._{\varepsilon} \left(2-\Lambda_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2} + \left(\Lambda_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1} ._{\varepsilon} \left(\Lambda_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}} / \varpi_{\underline{\mathfrak{I}}_1} \varpi_{\underline{\mathfrak{I}}_2} \right) \end{array} \right\}, \\ \left\{ \begin{array}{l} \left(\bigcup_{\substack{\Upsilon_{\overline{\mathfrak{I}}_1} \in \xi_{\overline{\mathfrak{I}}_1}, \phi_{\overline{\mathfrak{I}}_1} \in \phi_{\overline{\mathfrak{I}}_1} \\ \Upsilon_{\overline{\mathfrak{I}}_2} \in \xi_{\overline{\mathfrak{I}}_2}, \phi_{\overline{\mathfrak{I}}_2} \in \phi_{\overline{\mathfrak{I}}_2}}} \frac{2\left(\Upsilon_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1} \left(\Upsilon_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}}{\left(2-\Upsilon_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1} ._{\varepsilon} \left(2-\Upsilon_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2} + \left(\Upsilon_{\overline{\mathfrak{I}}_1}\right)^{\Gamma_1} ._{\varepsilon} \left(\Upsilon_{\overline{\mathfrak{I}}_2}\right)^{\Gamma_2}} / \phi_{\overline{\mathfrak{I}}_1} \phi_{\overline{\mathfrak{I}}_2} \right) \\ \left(\bigcup_{\substack{\Upsilon_{\underline{\mathfrak{I}}_1} \in \xi_{\underline{\mathfrak{I}}_1}, \phi_{\underline{\mathfrak{I}}_1} \in \phi_{\underline{\mathfrak{I}}_1} \\ \Upsilon_{\underline{\mathfrak{I}}_2} \in \xi_{\underline{\mathfrak{I}}_2}, \phi_{\underline{\mathfrak{I}}_2} \in \phi_{\underline{\mathfrak{I}}_2}}} \frac{2\left(\Upsilon_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1} \left(\Upsilon_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}}{\left(2-\Upsilon_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1} ._{\varepsilon} \left(2-\Upsilon_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2} + \left(\Upsilon_{\underline{\mathfrak{I}}_1}\right)^{\Gamma_1} ._{\varepsilon} \left(\Upsilon_{\underline{\mathfrak{I}}_2}\right)^{\Gamma_2}} / \phi_{\underline{\mathfrak{I}}_1} \phi_{\underline{\mathfrak{I}}_2} \right) \end{array} \right\} \end{array} \right).$$

Thus, the result holds for $p = 2$.

Assume that the results holds for $p = \chi$

$$SV-NPHFREWA\left\{\begin{array}{c}(\overline{\Im}_1,\overline{\Im}_2,....,\overline{\Im}_\chi)\\(\underline{\Im}_1,\underline{\Im}_2,....,\underline{\Im}_\chi)\end{array}\right\}$$

$$=\left(\begin{array}{c}\left\{\begin{array}{c}\left(\bigcup_{\substack{\xi_\top\in\upsilon_{\overline{\Im}_\top}\\ \Xi_{\overline{\Im}_\top}\in\Xi_{\overline{\Im}_\top}}}\frac{\Pi_{\top=1}^{\chi}\left(1+\xi_{\overline{\Im}_\top}\right)^{\Gamma_\top}-\Pi_{\top=1}^{\chi}\left(1-\xi_{\overline{\Im}_\top}\right)^{\Gamma_\top}}{\Pi_{\top=1}^{\chi}\left(1+\xi_{\overline{\Im}_\top}\right)^{\Gamma_\top}+\Pi_{\top=1}^{\chi}\left(1-\xi_{\overline{\Im}_\top}\right)^{\Gamma_\top}}/\Pi_{\top=1}^{\chi}\Xi_{\overline{\Im}_\top}\right)\\ \left(\bigcup_{\substack{\xi_\top\in\upsilon_{\underline{\Im}_\top}\\ \Xi_{\underline{\Im}_\top}\in\Xi_{\underline{\Im}_\top}}}\frac{\Pi_{\top=1}^{\chi}\left(1+\xi_{\underline{\Im}_\top}\right)^{\Gamma_\top}-\Pi_{\top=1}^{\chi}\left(1-\xi_{\underline{\Im}_\top}\right)^{\Gamma_\top}}{\Pi_{\top=1}^{\chi}\left(1+\xi_{\Im_\top}\right)^{\Gamma_\top}+\Pi_{\top=1}^{\chi}\left(1-\xi_{\Im_\top}\right)^{\Gamma_\top}}/\Pi_{\top=1}^{\chi}\Xi_{\underline{\Im}_\top}\right)\end{array}\right\},\\ \left\{\begin{array}{c}\left(\bigcup_{\substack{\Lambda_{\overline{\Im}_\top}\in\tau_{\overline{\Im}_\top}\\ \varpi_{\overline{\Im}_\top}\in\varpi_{\overline{\Im}_\top}}}\frac{2\Pi_{\top=1}^{\chi}\left(\Lambda_{\overline{\Im}_\top}\right)^{\Gamma_\top}}{\Pi_{\top=1}^{\chi}\left(2-\Lambda_{\overline{\Im}_\top}\right)^{\Gamma_\top}+\Pi_{\top=1}^{\chi}\left(\Lambda_{\overline{\Im}_\top}\right)^{\Gamma_\top}}/\Pi_{\top=1}^{\chi}\varpi_{\overline{\Im}_\top}\right)\\ \left(\bigcup_{\substack{\Lambda_{\underline{\Im}_\top}\in\tau_{\underline{\Im}_\top}\\ \varpi_{\underline{\Im}_\top}\in\varpi_{\underline{\Im}_\top}}}\frac{2\Pi_{\top=1}^{\chi}\left(\Lambda_{\underline{\Im}_\top}\right)^{\Gamma_\top}}{\Pi_{\top=1}^{\chi}\left(2-\Lambda_{\underline{\Im}_\top}\right)^{\Gamma_\top}+\Pi_{\top=1}^{\chi}\left(\Lambda_{\underline{\Im}_\top}\right)^{\Gamma_\top}}/\Pi_{\top=1}^{\chi}\varpi_{\underline{\Im}_\top}\right)\end{array}\right\},\\ \left\{\begin{array}{c}\left(\bigcup_{\substack{\Upsilon_{\overline{\Im}_\top}\in\xi_{\overline{\Im}_\top}\\ \phi_{\overline{\Im}_\top}\in\phi_{\overline{\Im}_\top}}}\frac{2\Pi_{\top=1}^{\chi}\left(\Upsilon_{\overline{\Im}_\top}\right)^{\Gamma_\top}}{\Pi_{\top=1}^{\chi}\left(2-\Upsilon_{\overline{\Im}_\top}\right)^{\Gamma_\top}+\Pi_{\top=1}^{\chi}\left(\Upsilon_{\overline{\Im}_\top}\right)^{\Gamma_\top}}/\Pi_{\top=1}^{\chi}\phi_{\overline{\Im}_\top}\right)\\ \left(\bigcup_{\substack{\Upsilon_{\underline{\Im}_\top}\in\xi_{\underline{\Im}_\top}\\ \phi_{\underline{\Im}_\top}\in\phi_{\underline{\Im}_\top}}}\frac{2\Pi_{\top=1}^{\chi}\left(\Upsilon_{\underline{\Im}_\top}\right)^{\Gamma_\top}}{\Pi_{\top=1}^{\chi}\left(2-\Upsilon_{\underline{\Im}_\top}\right)^{\Gamma_\top}+\Pi_{\top=1}^{\chi}\left(\Upsilon_{\underline{\Im}_\top}\right)^{\Gamma_\top}}/\Pi_{\top=1}^{\chi}\phi_{\underline{\Im}_\top}\right)\end{array}\right\}\end{array}\right).$$

Now we will prove for $p=\chi+1$

$$\begin{aligned}&SV-NPHFREWA\left\{(\overline{\Im}_1,\overline{\Im}_2,....,\overline{\Im}_{\chi+1})(\underline{\Im}_1,\underline{\Im}_2,....,\underline{\Im}_{\chi+1})\right\}\\ =\;&SV-NPHFREWA\left\{\begin{array}{c}(\overline{\Im}_1,\overline{\Im}_2,....,\overline{\Im}_\chi)\\(\underline{\Im}_1,\underline{\Im}_2,....,\underline{\Im}_\chi)\end{array}\right\}\oplus_\varepsilon\left\{\left(\Gamma_{\chi+1}\cdot_\varepsilon\overline{\Im}_{\chi+1}\right)\left(\Gamma_{\chi+1}\cdot_\varepsilon\underline{\Im}_{\chi+1}\right)\right\}\end{aligned}$$

$$
= \left(\begin{array}{l}
\left\{ \begin{array}{l}
\left(\bigcup_{\substack{\xi_{\top} \in \upsilon_{\overline{\Im}_{\top}} \\ \Xi_{\overline{\Im}_{\top}} \in \Xi_{\overline{\Im}_{\top}}}} \frac{\Pi_{\top=1}^{\chi}\left(1+\xi_{\overline{\Im}_{\top}}\right)^{\Gamma_{\top}} - \Pi_{\top=1}^{\chi}\left(1-\xi_{\overline{\Im}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{\chi}\left(1+\xi_{\overline{\Im}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{\chi}\left(1-\xi_{\overline{\Im}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{\chi} \Xi_{\overline{\Im}_{\top}} \right) \\
\left(\bigcup_{\substack{\xi_{\top} \in \upsilon_{\underline{\Im}_{\top}} \\ \Xi_{\underline{\Im}_{\top}} \in \Xi_{\underline{\Im}_{\top}}}} \frac{\Pi_{\top=1}^{\chi}\left(1+\xi_{\underline{\Im}_{\top}}\right)^{\Gamma_{\top}} - \Pi_{\top=1}^{\chi}\left(1-\xi_{\underline{\Im}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{\chi}\left(1+\xi_{\underline{\Im}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{\chi}\left(1-\xi_{\underline{\Im}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{\chi} \Xi_{\underline{\Im}_{\top}} \right)
\end{array} \right\}, \\
\left\{ \begin{array}{l}
\left(\bigcup_{\substack{\Lambda_{\overline{\Im}_{\top}} \in \tau_{\overline{\Im}_{\top}} \\ \varpi_{\overline{\Im}_{\top}} \in \varpi_{\overline{\Im}_{\top}}}} \frac{2\Pi_{\top=1}^{\chi}\left(\Lambda_{\overline{\Im}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{\chi}\left(2-\Lambda_{\overline{\Im}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{\chi}\left(\Lambda_{\overline{\Im}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{\chi} \varpi_{\overline{\Im}_{\top}} \right) \\
\left(\bigcup_{\substack{\Lambda_{\underline{\Im}_{\top}} \in \tau_{\underline{\Im}_{\top}} \\ \varpi_{\underline{\Im}_{\top}} \in \varpi_{\underline{\Im}_{\top}}}} \frac{2\Pi_{\top=1}^{\chi}\left(\Lambda_{\underline{\Im}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{\chi}\left(2-\Lambda_{\underline{\Im}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{\chi}\left(\Lambda_{\underline{\Im}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{\chi} \varpi_{\underline{\Im}_{\top}} \right)
\end{array} \right\}, \\
\left\{ \begin{array}{l}
\left(\bigcup_{\substack{\Upsilon_{\overline{\Im}_{\top}} \in \xi_{\overline{\Im}_{\top}} \\ \phi_{\overline{\Im}_{\top}} \in \xi_{\overline{\Im}_{\top}}}} \frac{2\Pi_{\top=1}^{\chi}\left(\Upsilon_{\overline{\Im}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{\chi}\left(2-\Upsilon_{\overline{\Im}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{\chi}\left(\Upsilon_{\overline{\Im}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{\chi} \phi_{\overline{\Im}_{\top}} \right) \\
\left(\bigcup_{\substack{\Upsilon_{\underline{\Im}_{\top}} \in \xi_{\underline{\Im}_{\top}} \\ \phi_{\underline{\Im}_{\top}} \in \xi_{\underline{\Im}_{\top}}}} \frac{2\Pi_{\top=1}^{\chi}\left(\Upsilon_{\underline{\Im}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{\chi}\left(2-\Upsilon_{\underline{\Im}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{\chi}\left(\Upsilon_{\underline{\Im}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{\chi} \phi_{\underline{\Im}_{\top}} \right)
\end{array} \right\}
\end{array} \right) \oplus
$$

$$
\varepsilon \left(\begin{array}{l}
\left\{ \begin{array}{l}
\left(\bigcup_{\substack{\xi_{\chi+1} \in \upsilon_{\Im_{\chi+1}} \\ \Xi_{\Im_{\chi+1}} \in \Xi_{\Im_{\chi+1}}}} \frac{\left(1+\xi_{\Im_{\chi+1}}\right)^{\Gamma_{\chi+1}} - \left(1-\xi_{\Im_{\chi+1}}\right)^{\Gamma_{\chi+1}}}{\left(1+\xi_{\Im_{\chi+1}}\right)^{\Gamma_{\chi+1}} + \left(1-\xi_{\Im_{\chi+1}}\right)^{\Gamma_{\chi+1}}} / \Xi_{\Im_{\chi+1}} \right) \\
\left(\bigcup_{\substack{\xi_{\chi+1} \in \upsilon_{\underline{\Im}_{\chi+1}} \\ \Xi_{\underline{\Im}_{\chi+1}} \in \Xi_{\underline{\Im}_{\chi+1}}}} \frac{\left(1+\xi_{\underline{\Im}_{\chi+1}}\right)^{\Gamma_{\chi+1}} - \left(1-\xi_{\underline{\Im}_{\chi+1}}\right)^{\Gamma_{\chi+1}}}{\left(1+\xi_{\Im_{\chi+1}}\right)^{\Gamma_{\chi+1}} + \left(1-\xi_{\underline{\Im}_{\chi+1}}\right)^{\Gamma_{\chi+1}}} / \Xi_{\underline{\Im}_{\chi+1}} \right)
\end{array} \right\}, \\
\left\{ \begin{array}{l}
\left(\bigcup_{\substack{\Lambda_{\Im_{\chi+1}} \in \tau_{\Im_{\chi+1}} \\ \varpi_{\Im_{\chi+1}} \in \varpi_{\Im_{\chi+1}}}} \frac{2\left(\Lambda_{\Im_{\chi+1}}\right)^{\Gamma_{\chi+1}}}{\left(2-\Lambda_{\Im_{\chi+1}}\right)^{\Gamma_{\chi+1}} + \left(\Lambda_{\Im_{\chi+1}}\right)^{\Gamma_{\chi+1}}} / \varpi_{\Im_{\chi+1}} \right) \\
\left(\bigcup_{\substack{\Lambda_{\underline{\Im}_{\chi+1}} \in \tau_{\underline{\Im}_{\chi+1}} \\ \varpi_{\underline{\Im}_{\chi+1}} \in \varpi_{\underline{\Im}_{\chi+1}}}} \frac{2\left(\Lambda_{\Im_{\chi+1}}\right)^{\Gamma_{\chi+1}}}{\left(2-\Lambda_{\underline{\Im}_{\chi+1}}\right)^{\Gamma_{\chi+1}} + \left(\Lambda_{\underline{\Im}_{\chi+1}}\right)^{\Gamma_{\chi+1}}} / \varpi_{\underline{\Im}_{\chi+1}} \right)
\end{array} \right\}, \\
\left\{ \begin{array}{l}
\left(\bigcup_{\substack{\Upsilon_{\Im_{\chi+1}} \in \xi_{\Im_{\chi+1}} \\ \phi_{\Im_{\chi+1}} \in \phi_{\Im_{\chi+1}}}} \frac{2\left(\Upsilon_{\Im_{\chi+1}}\right)^{\Gamma_{\chi+1}}}{\left(2-\Upsilon_{\Im_{\chi+1}}\right)^{\Gamma_{\chi+1}} + \left(\Upsilon_{\Im_{\chi+1}}\right)^{\Gamma_{\chi+1}}} / \phi_{\Im_{\chi+1}} \right) \\
\left(\bigcup_{\substack{\Upsilon_{\underline{\Im}_{\chi+1}} \in \xi_{\underline{\Im}_{\chi+1}} \\ \phi_{\underline{\Im}_{\chi+1}} \in \phi_{\underline{\Im}_{\chi+1}}}} \frac{2\left(\Upsilon_{\underline{\Im}_{\chi+1}}\right)^{\Gamma_{\chi+1}}}{\left(2-\Upsilon_{\underline{\Im}_{\chi+1}}\right)^{\Gamma_{\chi+1}} + \left(\Upsilon_{\underline{\Im}_{\chi+1}}\right)^{\Gamma_{\chi+1}}} / \phi_{\underline{\Im}_{\chi+1}} \right)
\end{array} \right\}
\end{array} \right)
$$

$$= \left(\begin{array}{c} \left\{ \begin{array}{c} \left(\bigcup\limits_{\substack{\xi_{\top} \in \upsilon_{\overline{\mathfrak{I}}_{\top}} \\ \Xi_{\overline{\mathfrak{I}}_{\top}} \in \Xi_{\overline{\mathfrak{I}}_{\top}}}} \frac{\Pi_{\top=1}^{\chi+1}\left(1+\xi_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} - \Pi_{\top=1}^{\chi+1}\left(1-\xi_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{\chi+1}\left(1+\xi_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{\chi+1}\left(1-\xi_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{\chi+1} \Xi_{\overline{\mathfrak{I}}_{\top}} \right) \\ \left(\bigcup\limits_{\substack{\xi_{\top} \in \upsilon_{\underline{\mathfrak{I}}_{\top}} \\ \Xi_{\underline{\mathfrak{I}}_{\top}} \in \Xi_{\underline{\mathfrak{I}}_{\top}}}} \frac{\Pi_{\top=1}^{\chi+1}\left(1+\xi_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} - \Pi_{\top=1}^{\chi+1}\left(1-\xi_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{\chi+1}\left(1+\xi_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{\chi+1}\left(1-\xi_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{\chi+1} \Xi_{\underline{\mathfrak{I}}_{\top}} \right) \end{array} \right\}, \\ \left\{ \begin{array}{c} \left(\bigcup\limits_{\substack{\Lambda_{\overline{\mathfrak{I}}_{\top}} \in \tau_{\overline{\mathfrak{I}}_{\top}} \\ \varpi_{\overline{\mathfrak{I}}_{\top}} \in \varpi_{\overline{\mathfrak{I}}_{\top}}}} \frac{2\Pi_{\top=1}^{\chi+1}\left(\Lambda_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{\chi+1}\left(2-\Lambda_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{\chi+1}\left(\Lambda_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{\chi+1} \varpi_{\overline{\mathfrak{I}}_{\top}} \right) \\ \left(\bigcup\limits_{\substack{\Lambda_{\underline{\mathfrak{I}}_{\top}} \in \tau_{\underline{\mathfrak{I}}_{\top}} \\ \varpi_{\underline{\mathfrak{I}}_{\top}} \in \varpi_{\underline{\mathfrak{I}}_{\top}}}} \frac{2\Pi_{\top=1}^{\chi+1}\left(\Lambda_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{\chi+1}\left(2-\Lambda_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{\chi+1}\left(\Lambda_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{\chi+1} \varpi_{\underline{\mathfrak{I}}_{\top}} \right) \end{array} \right\}, \\ \left\{ \begin{array}{c} \left(\bigcup\limits_{\substack{\Upsilon_{\overline{\mathfrak{I}}_{\chi+1}} \in \xi_{\overline{\mathfrak{I}}_{\chi+1}} \\ \phi_{\overline{\mathfrak{I}}_{\chi+1}} \in \phi_{\overline{\mathfrak{I}}_{\chi+1}}}} \frac{2\Pi_{\top=1}^{\chi+1}\left(\Upsilon_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{\chi+1}\left(2-\Upsilon_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{\chi+1}\left(\Upsilon_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{\chi+1} \phi_{\overline{\mathfrak{I}}_{\top}} \right) \\ \left(\bigcup\limits_{\substack{\Upsilon_{\underline{\mathfrak{I}}_{\chi+1}} \in \xi_{\underline{\mathfrak{I}}_{\chi+1}} \\ \phi_{\underline{\mathfrak{I}}_{\chi+1}} \in \phi_{\underline{\mathfrak{I}}_{\chi+1}}}} \frac{2\Pi_{\top=1}^{\chi+1}\left(\Upsilon_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{\chi+1}\left(2-\Upsilon_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{\chi+1}\left(\Upsilon_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{\chi+1} \phi_{\underline{\mathfrak{I}}_{\top}} \right) \end{array} \right\} \end{array} \right).$$

Thus

$$SV-NPHFREWA\left\{(\overline{\mathfrak{I}}_1, \overline{\mathfrak{I}}_2, ..., \overline{\mathfrak{I}}_p)(\underline{\mathfrak{I}}_1, \underline{\mathfrak{I}}_2, ..., \underline{\mathfrak{I}}_p)\right\}$$

$$= \left(\begin{array}{c} \left\{ \begin{array}{c} \left(\bigcup\limits_{\substack{\xi_{\top} \in \upsilon_{\overline{\mathfrak{I}}_{\top}} \\ \Xi_{\overline{\mathfrak{I}}_{\top}} \in \Xi_{\overline{\mathfrak{I}}_{\top}}}} \frac{\Pi_{\top=1}^{p}\left(1+\xi_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} - \Pi_{\top=1}^{p}\left(1-\xi_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{p}\left(1+\xi_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{p}\left(1-\xi_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{p} \Xi_{\overline{\mathfrak{I}}_{\top}} \right) \\ \left(\bigcup\limits_{\substack{\xi_{\top} \in \upsilon_{\underline{\mathfrak{I}}_{\top}} \\ \Xi_{\underline{\mathfrak{I}}_{\top}} \in \Xi_{\underline{\mathfrak{I}}_{\top}}}} \frac{\Pi_{\top=1}^{p}\left(1+\xi_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} - \Pi_{\top=1}^{p}\left(1-\xi_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{p}\left(1+\xi_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{p}\left(1-\xi_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{p} \Xi_{\underline{\mathfrak{I}}_{\top}} \right) \end{array} \right\}, \\ \left\{ \begin{array}{c} \left(\bigcup\limits_{\substack{\Lambda_{\overline{\mathfrak{I}}_{\top}} \in \tau_{\overline{\mathfrak{I}}_{\top}} \\ \varpi_{\overline{\mathfrak{I}}_{\top}} \in \varpi_{\overline{\mathfrak{I}}_{\top}}}} \frac{2\Pi_{\top=1}^{p}\left(\Lambda_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{p}\left(2-\Lambda_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{p}\left(\Lambda_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{p} \varpi_{\overline{\mathfrak{I}}_{\top}} \right) \\ \left(\bigcup\limits_{\substack{\Lambda_{\underline{\mathfrak{I}}_{\top}} \in \tau_{\underline{\mathfrak{I}}_{\top}} \\ \varpi_{\underline{\mathfrak{I}}_{\top}} \in \varpi_{\underline{\mathfrak{I}}_{\top}}}} \frac{2\Pi_{\top=1}^{p}\left(\Lambda_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{p}\left(2-\Lambda_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{p}\left(\Lambda_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{p} \varpi_{\underline{\mathfrak{I}}_{\top}} \right) \end{array} \right\}, \\ \left\{ \begin{array}{c} \left(\bigcup\limits_{\substack{\Upsilon_{\overline{\mathfrak{I}}_{\top}} \in \xi_{\overline{\mathfrak{I}}_{\top}} \\ \phi_{\overline{\mathfrak{I}}_{\top}} \in \phi_{\overline{\mathfrak{I}}_{\top}}}} \frac{2\Pi_{\top=1}^{p}\left(\Upsilon_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{p}\left(2-\Upsilon_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{p}\left(\Upsilon_{\overline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{p} \phi_{\overline{\mathfrak{I}}_{\top}} \right) \\ \left(\bigcup\limits_{\substack{\Upsilon_{\underline{\mathfrak{I}}_{\top}} \in \xi_{\underline{\mathfrak{I}}_{\top}} \\ \phi_{\underline{\mathfrak{I}}_{\top}} \in \phi_{\underline{\mathfrak{I}}_{\top}}}} \frac{2\Pi_{\top=1}^{p}\left(\Upsilon_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}}{\Pi_{\top=1}^{p}\left(2-\Upsilon_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}} + \Pi_{\top=1}^{p}\left(\Upsilon_{\underline{\mathfrak{I}}_{\top}}\right)^{\Gamma_{\top}}} / \Pi_{\top=1}^{p} \phi_{\underline{\mathfrak{I}}_{\top}} \right) \end{array} \right\} \end{array} \right).$$

□

There are some properties which are fulfilled by the SV-NPHFREWA as follows:

Theorem 2. *Let* $\Im_\top = \left\{\begin{array}{c}\left(\upsilon_{\bar{\flat}_\top}/\Xi_{\bar{\flat}_\top}, \tau_{\bar{\flat}_\top}/\varpi_{\bar{\flat}_\top}, \Upsilon_{\bar{\flat}_\top}/\phi_{\bar{\flat}_\top}\right)\\ \left(\upsilon_{\underline{\flat}_\top}/\Xi_{\underline{\flat}_\top}, \tau_{\underline{\flat}_\top}/\varpi_{\underline{\flat}_\top}, \Upsilon_{\underline{\flat}_\top}/\phi_{\underline{\flat}_\top}\right)\end{array}\right\} (\top = 1, 2, ..., p)$

be a collection of SV-NPHFRNs, $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p)^T$ *are the weights of* $\Im_\top$ *with* $\Gamma_\top \in [0, 1]$ *with* $\sum_{\top=1}^{p} \Gamma_\top = 1$; *then we have the following:*

(1) Boundary:

$$\left\{\left(\overline{\Im^-}\right)\left(\underline{\Im}\right)\right\} \leq SV - NPHFREWA\left\{\begin{array}{c}(\overline{\Im}_1, \overline{\Im}_2, ..., \overline{\Im}_p)\\ (\underline{\Im}_1, \underline{\Im}_2, ..., \underline{\Im}_p)\end{array}\right\} \leq \left\{\left(\overline{\Im^+}\right)\left(\underline{\Im}\right)\right\}$$

where

$$\left\{\left(\overline{\Im^-}\right)\left(\underline{\Im}\right)\right\} = \left(\left(\begin{array}{c}\left\{\begin{array}{c}\left(\min_{1\leq\top\leq p}\min_{\overline{\xi}_{\Im_\top}\in\overline{\upsilon}_{\ell_{\flat_\top}}} \overline{\xi}_{\Im_\top}/\overline{\Xi}_{\flat_1}\overline{\Xi}_{\flat_2}.....\overline{\Xi}_{\flat_p}\right)\\ \left(\max_{1\leq\top\leq p}\max_{\underline{\alpha}_{\Im_\top}\in\underline{\upsilon}_{\ell_{\flat_\top}}} \underline{\alpha}_{\Im_\top}/\underline{\Xi}_{\flat_1}\underline{\Xi}_{\flat_2}.....\underline{\Xi}_{\flat_p}\right)\end{array}\right\},\\ \left\{\begin{array}{c}\left(\max_{1\leq\top\leq p}\max_{\overline{\Lambda}_{\Im_\top}\in\overline{\tau}_{\ell_{\flat_\top}}} \overline{\Lambda}_{\Im_\top}/\overline{\varpi}_{\flat_1}\overline{\varpi}_{\flat_2}.....\overline{\varpi}_{\flat_p}\right)\\ \left(\min_{1\leq\top\leq p}\min_{\underline{\Lambda}_{\Im_\top}\in\underline{\tau}_{\ell_{\flat_\top}}} \underline{\Lambda}_{\Im_\top}/\underline{\varpi}_{\flat_1}\underline{\varpi}_{\flat_2}.....\underline{\varpi}_{\flat_p}\right)\end{array}\right\},\\ \left\{\begin{array}{c}\left(\max_{1\leq\top\leq p}\max_{\overline{\Upsilon}_{\Im_\top}\in\overline{\xi}_{\ell_{\flat_\top}}} \overline{\Upsilon}_{\Im_\top}/\overline{\phi}_{\flat_1}\overline{\phi}_{\flat_2}.....\overline{\phi}_{\flat_p}\right)\\ \left(\min_{1\leq\top\leq p}\min_{\underline{\Upsilon}_{\Im_\top}\in\underline{\alpha}_{\ell_{\flat_\top}}} \underline{\Upsilon}_{\Im_\top}/\underline{\flat}_{\flat_1}\underline{\flat}_{\flat_2}.....\underline{\flat}_{\flat_p}\right)\end{array}\right\}\end{array}\right)\right)$$

$$\left\{\left(\overline{\Im^+}\right)\left(\underline{\Im}\right)\right\} = \left(\left(\begin{array}{c}\left\{\begin{array}{c}\left(\max_{1\leq\top\leq p}\max_{\overline{\xi}_{\Im_\top}\in\overline{\upsilon}_{\ell_{\flat_\top}}} \overline{\xi}_{\Im_\top}/\overline{\Xi}_{\flat_1}\overline{\Xi}_{\flat_2}.....\overline{\Xi}_{\flat_p}\right)\\ \left(\min_{1\leq\top\leq p}\min_{\underline{\alpha}_{\Im_\top}\in\underline{\upsilon}_{\ell_{\flat_\top}}} \underline{\alpha}_{\Im_\top}/\underline{\Xi}_{\flat_1}\underline{\Xi}_{\flat_2}.....\underline{\Xi}_{\flat_p}\right)\end{array}\right\},\\ \left\{\begin{array}{c}\left(\min_{1\leq\top\leq p}\min_{\overline{\Lambda}_{\Im_\top}\in\overline{\tau}_{\ell_{\flat_\top}}} \overline{\Lambda}_{\Im_\top}/\overline{\varpi}_{\flat_1}\overline{\varpi}_{\flat_2}.....\overline{\varpi}_{\flat_p}\right)\\ \left(\max_{1\leq\top\leq p}\max_{\underline{\Lambda}_{\Im_\top}\in\underline{\tau}_{\ell_{\flat_\top}}} \underline{\Lambda}_{\Im_\top}/\underline{\varpi}_{\flat_1}\underline{\varpi}_{\flat_2}.....\underline{\varpi}_{\flat_p}\right)\end{array}\right\},\\ \left\{\begin{array}{c}\left(\min_{1\leq\top\leq p}\min_{\overline{\Upsilon}_{\Im_\top}\in\overline{\xi}_{\ell_{\flat_\top}}} \overline{\Upsilon}_{\Im_\top}/\overline{\phi}_{\flat_1}\overline{\phi}_{\flat_2}.....\overline{\phi}_{\flat_p}\right)\\ \left(\max_{1\leq\top\leq p}\max_{\underline{\Upsilon}_{\Im_\top}\in\underline{\alpha}_{\ell_{\flat_\top}}} \underline{\Upsilon}_{\Im_\top}/\underline{\flat}_{\flat_1}\underline{\flat}_{\flat_2}.....\underline{\flat}_{\flat_p}\right)\end{array}\right\}\end{array}\right)\right)$$

(2) Monotonicity: Let $\left(\overline{\Im}^*_\top, \underline{\Im}^*_\top\right) = \left\{\begin{array}{c}\left(\upsilon_{\bar{\flat}_\top}/\Xi_{\bar{\flat}_\top}, \tau_{\bar{\flat}_\top}/\varpi_{\bar{\flat}_\top}, \Upsilon_{\bar{\flat}_\top}/\phi_{\bar{\flat}_\top}\right)\\ \left(\upsilon_{\underline{\flat}_\top}/\Xi_{\underline{\flat}_\top}, \tau_{\underline{\flat}_\top}/\varpi_{\underline{\flat}_\top}, \Upsilon_{\underline{\flat}_\top}/\phi_{\underline{\flat}_\top}\right)\end{array}\right\} (\top = 1, 2, ..., p)$

be a collection of SV-NPHFRNs. If $\left(\overline{\Im}_\top, \underline{\Im}_\top\right) \leq \left(\overline{\Im}^*_\top, \underline{\Im}^*_\top\right)$, while the probabilities are same, then

$$SV - NPHFEWA(\Im_1, \Im_2, ..., \Im_p) \leq SV - NPHFEWA(\Im^*_1, \Im^*_2, ..., \Im^*_p).$$

Proof. Let $f(\varsigma) = \frac{1-\varsigma}{1+\varsigma}, \varsigma \in [0, 1]$. Then, $f'(\varsigma) = \frac{-2}{(1+\varsigma)} < 0$, so $f(\varsigma)$ is a decreasing function.

Let $\left\{\begin{array}{c}\max\left(\overline{\xi}_{\Im_\top}\right)\\ \max\left(\underline{\alpha}_{\Im_\top}\right)\end{array}\right\} = \left\{\begin{array}{c}\left(\max_{1\leq\top\leq p}\max_{\overline{\xi}_{\Im_\top}\in\overline{\upsilon}_{\ell_{\flat_\top}}}\left(\overline{\xi}_{\Im_\top}\right)\right)\\ \left(\min_{1\leq\top\leq p}\min_{\underline{\alpha}_{\Im_\top}\in\underline{\upsilon}_{\ell_{\flat_\top}}}\left(\underline{\alpha}_{\Im_\top}\right)\right)\end{array}\right\}$ and

$$\left\{ \begin{array}{c} \min\left(\overline{\xi}_{\mathfrak{I}_\top}\right) \\ \min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right) \end{array} \right\} = \left\{ \begin{array}{c} \left(\min_{1\le\top\le p} \min_{\overline{\xi}_{\mathfrak{I}_\top}\in\overline{\upsilon}_{\ell_{\flat_\top}}} \left(\overline{\xi}_{\mathfrak{I}_\top}\right)\right) \\ \left(\max_{1\le\top\le p} \max_{\underline{\alpha}_{\mathfrak{I}_\top}\in\underline{\upsilon}_{\ell_{\flat_\top}}} \left(\underline{\alpha}_{\mathfrak{I}_\top}\right)\right) \end{array} \right\}$$

$$\left\{ \begin{array}{c} \left(\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right) \le \left(\overline{\xi}_{\mathfrak{I}_\top}\right) \le \max\left(\overline{\xi}_{\mathfrak{I}_\top}\right) \ forall \top = 1,2,....., p\right) \\ \left(\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right) \le \left(\underline{\alpha}_{\mathfrak{I}_\top}\right) \le \min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right) \ forall \top = 1,2,....., p\right) \end{array} \right\}$$

$$\left\{ \begin{array}{c} f(\max\left(\overline{\xi}_{\mathfrak{I}_\top}\right) \le \left(\overline{\xi}_{\mathfrak{I}_\top}\right) \le f(\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right)) \ forall \top = 1,2,....., p \\ f(\min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right) \le \left(\underline{\alpha}_{\mathfrak{I}_\top}\right) \le f(\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)) \ forall \top = 1,2,....., p \end{array} \right\}$$

$$\left\{ \begin{array}{c} \left(\frac{1-\max\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}{1+\max\left(\overline{\xi}_{\mathfrak{I}_\top}\right)} \le \frac{1-\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}{1+\left(\overline{\xi}_{\mathfrak{I}_\top}\right)} \le \frac{1-\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}{1+\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}\right) \\ \left(\frac{1-\min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}{1+\min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)} \le \frac{1-\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}{1+\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)} \le \frac{1-\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}{1+\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}\right) \end{array} \right\}.$$

Since $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_p)^T$ are the weights of $\left(\overline{\mathfrak{I}}_\top, \underline{\mathfrak{I}}_\top\right)$ with $\Gamma_\top \in [0,1]$ with $\sum_{\top=1}^{p} \Gamma_\top = 1$, we have

$$\left\{ \begin{array}{c} \left\{\left(\frac{1-\max\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}{1+\max\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}\right)^{\Gamma_\top} \le \left(\frac{1-\overline{\xi}_{\mathfrak{I}_\top}}{1+\overline{\xi}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top} \le \left(\frac{1-\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}{1+\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}\right)^{\Gamma_\top}\right\} \\ \left(\frac{1-\min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}{1+\min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}\right)^{\Gamma_\top} \le \left(\frac{1-\underline{\alpha}_{\mathfrak{I}_\top}}{1+\underline{\alpha}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top} \le \left(\frac{1-\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}{1+\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}\right)^{\Gamma_\top} \end{array} \right\},$$

$$\left\{ \begin{array}{c} \left\{\prod_{\top=1}^{p}\left(\frac{1-\max\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}{1+\max\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}\right)^{\Gamma_\top} \le \prod_{\top=1}^{p}\left(\frac{1-\overline{\xi}_{\mathfrak{I}_\top}}{1+\overline{\xi}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top} \le \prod_{\top=1}^{p}\left(\frac{1-\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}{1+\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}\right)^{\Gamma_\top}\right\} \\ \left\{\prod_{\top=1}^{p}\left(\frac{1-\min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}{1+\min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}\right)^{\Gamma_\top} \le \prod_{\top=1}^{p}\left(\frac{1-\underline{\alpha}_{\mathfrak{I}_\top}}{1+\underline{\alpha}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top} \le \prod_{\top=1}^{p}\left(\frac{1-\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}{1+\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}\right)^{\Gamma_\top}\right\} \end{array} \right\},$$

$$\left\{ \begin{array}{c} \left\{\left(\frac{1-\max\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}{1+\max\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}\right)^{\sum_{\top=1}^{p}\Gamma_\top} \le \prod_{\top=1}^{p}\left(\frac{1-\overline{\xi}_{\mathfrak{I}_\top}}{1+\overline{\xi}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top} \le \left(\frac{1-\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}{1+\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}\right)^{\sum_{\top=1}^{p}\Gamma_\top}\right\} \\ \left\{\left(\frac{1-\min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}{1+\min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}\right)^{\sum_{\top=1}^{p}\Gamma_\top} \le \prod_{\top=1}^{p}\left(\frac{1-\underline{\alpha}_{\mathfrak{I}_\top}}{1+\underline{\alpha}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top} \le \left(\frac{1-\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}{1+\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}\right)^{\sum_{\top=1}^{p}\Gamma_\top}\right\} \end{array} \right\}$$

$$\Longleftrightarrow \left\{ \begin{array}{c} \left(\left(\frac{1-\max\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}{1+\max\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}\right) \le \prod_{\top=1}^{p}\left(\frac{1-\overline{\xi}_{\mathfrak{I}_\top}}{1+\overline{\xi}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top} \le \left(\frac{1-\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}{1+\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}\right)\right) \\ \left(\left(\frac{1-\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}{1+\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}\right) \le \prod_{\top=1}^{p}\left(\frac{1-\underline{\alpha}_{\mathfrak{I}_\top}}{1+\underline{\alpha}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top} \le \left(\frac{1-\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}{1+\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}\right)\right) \end{array} \right\}$$

$$\Longleftrightarrow \left\{ \begin{array}{c} \left(\left(\frac{2}{1+\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}\right) \le 1 + \prod_{\top=1}^{p}\left(\frac{1-\overline{\xi}_{\mathfrak{I}_\top}}{1+\overline{\xi}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top} \le \left(\frac{2}{1+\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}\right)\right) \\ \left(\left(\frac{2}{1+\min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}\right) \le 1 + \prod_{\top=1}^{p}\left(\frac{1-\underline{\alpha}_{\mathfrak{I}_\top}}{1+\underline{\alpha}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top} \le \left(\frac{2}{1+\min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}\right)\right) \end{array} \right\}$$

$$\Longleftrightarrow \left\{ \begin{array}{c} \left(\left(\frac{1+\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}{2}\right) \le \frac{1}{1+\prod_{\top=1}^{p}\left(\frac{1-\overline{\xi}_{\mathfrak{I}_\top}}{1+\overline{\xi}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top}} \le \left(\frac{1+\max\left(\overline{\xi}_{\mathfrak{I}_\top}\right)}{2}\right)\right) \\ \left(\left(\frac{1+\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}{2}\right) \le \frac{1}{1+\prod_{\top=1}^{p}\left(\frac{1-\underline{\alpha}_{\mathfrak{I}_\top}}{1+\underline{\alpha}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top}} \le \left(\frac{1+\min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)}{2}\right)\right) \end{array} \right\}$$

$$\Longleftrightarrow \left\{ \begin{array}{l} \left(\left(1 + \min(\overline{\xi}_{\mathfrak{I}_\top})\right) \leq \frac{2}{1+\prod_{\top=1}^{p}\left(\frac{1-\overline{\xi}_{\mathfrak{I}_\top}}{1+\overline{\xi}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top}} \leq \left(1 + \max\left(\overline{\xi}_{\mathfrak{I}_\top}\right)\right) \right) \\ \left(\left(1 + \max(\underline{\alpha}_{\mathfrak{I}_\top})\right) \leq \frac{2}{1+\prod_{\top=1}^{p}\left(\frac{1-\underline{\alpha}_{\mathfrak{I}_\top}}{1+\underline{\alpha}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top}} \leq \left(1 + \min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right)\right) \right) \end{array} \right\}$$

$$\Longleftrightarrow \left\{ \begin{array}{l} \left(\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right) \leq \frac{2}{1+\prod_{\top=1}^{p}\left(\frac{1-\overline{\xi}_{\mathfrak{I}_\top}}{1+\overline{\xi}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top}} - 1 \leq \max\left(\overline{\xi}_{\mathfrak{I}_\top}\right) \right) \\ \left(\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right) \leq \frac{2}{1+\prod_{\top=1}^{p}\left(\frac{1-\underline{\alpha}_{\mathfrak{I}_\top}}{1+\underline{\alpha}_{\mathfrak{I}_\top}}\right)^{\Gamma_\top}} - 1 \leq \min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right) \right) \end{array} \right\}$$

$$\Longleftrightarrow \left\{ \begin{array}{l} \left(\min\left(\overline{\xi}_{\mathfrak{I}_\top}\right) \leq \frac{\prod_{\top=1}^{p}\left(1+\overline{\xi}_{\mathfrak{I}_\top}\right)^{\Gamma_\top} - \prod_{\top=1}^{p}\left(1-\overline{\xi}_{\mathfrak{I}_\top}\right)^{\Gamma_\top}}{\prod_{\top=1}^{p}\left(1+\overline{\xi}_{\mathfrak{I}_\top}\right)^{\Gamma_\top} + \prod_{\top=1}^{p}\left(1-\overline{\xi}_{\mathfrak{I}_\top}\right)^{\Gamma_\top}} \leq \max\left(\overline{\xi}_{\mathfrak{I}_\top}\right) \right) \\ \left(\max\left(\underline{\alpha}_{\mathfrak{I}_\top}\right) \leq \frac{\prod_{\top=1}^{p}\left(1+\underline{\alpha}_{\mathfrak{I}_\top}\right)^{\Gamma_\top} - \prod_{\top=1}^{p}\left(1-\underline{\alpha}_{\mathfrak{I}_\top}\right)^{\Gamma_\top}}{\prod_{\top=1}^{p}\left(1+\underline{\alpha}_{\mathfrak{I}_\top}\right)^{\Gamma_\top} + \prod_{\top=1}^{p}\left(1-\underline{\alpha}_{\mathfrak{I}_\top}\right)^{\Gamma_\top}} \leq \min\left(\underline{\alpha}_{\mathfrak{I}_\top}\right) \right) \end{array} \right\}.$$

Thus,

$$\left\{ \begin{array}{l} \left(\min_{\mathfrak{I}_\top} \overline{\xi} \leq \frac{\prod_{\top=1}^{p}\left(1+\overline{\xi}_{\mathfrak{I}_\top}\right)^{\Gamma_\top} - \prod_{\top=1}^{p}\left(1-\overline{\xi}_{\mathfrak{I}_\top}\right)^{\Gamma_\top}}{\prod_{\top=1}^{p}\left(1+\overline{\xi}_{\mathfrak{I}_\top}\right)^{\Gamma_\top} + \prod_{\top=1}^{p}\left(1-\overline{\xi}_{\mathfrak{I}_\top}\right)^{\Gamma_\top}} \leq \max \overline{\xi}_{\mathfrak{I}_\top} \right) \\ \left(\max \underline{\alpha}_{\mathfrak{I}_\top} \leq \frac{\prod_{\top=1}^{p}\left(1+\underline{\alpha}_{\mathfrak{I}_\top}\right)^{\Gamma_\top} - \prod_{\top=1}^{p}\left(1-\underline{\alpha}_{\mathfrak{I}_\top}\right)^{\Gamma_\top}}{\prod_{\top=1}^{p}\left(1+\underline{\alpha}_{\mathfrak{I}_\top}\right)^{\Gamma_\top} + \prod_{\top=1}^{p}\left(1-\underline{\alpha}_{\mathfrak{I}_\top}\right)^{\Gamma_\top}} \leq \min \underline{\alpha}_{\mathfrak{I}_\top} \right) \end{array} \right\}.$$

Consider $g(y) = \frac{2-y}{y}, y \in (0,1]$. Then, $g'(y) = -\frac{2}{y}$, *i.e.*, $g(y)$ is a decreasing function on $(0,1]$. Let

$$\begin{aligned} \max(\overline{\Lambda}_{\mathfrak{I}_\top}, \underline{\Delta}_{\mathfrak{I}_\top}) &= \left\{ \begin{array}{l} \left(\max_{1\leq\top\leq p} \max_{\overline{\Lambda}_{\mathfrak{I}_\top}\in\tau_{\ell_{b_\top}}} \overline{\Lambda}_{\mathfrak{I}_\top}\right) \\ \left(\min_{1\leq\top\leq p} \min_{\underline{\Delta}_{\mathfrak{I}_\top}\in\tau_{\ell_{b_\top}}} \underline{\Delta}_{\mathfrak{I}_\top}\right) \end{array} \right\}, \\ \min\left(\overline{\Lambda}_{\mathfrak{I}_\top}, \underline{\Delta}_{\mathfrak{I}_\top}\right) &= \left\{ \begin{array}{l} \left(\min_{1\leq\top\leq p} \min_{\overline{\Lambda}_{\mathfrak{I}_\top}\in\tau_{\ell_{b_\top}}} \overline{\Lambda}_{\mathfrak{I}_\top}\right) \\ \left(\max_{1\leq\top\leq p} \max_{\underline{\Delta}_{\mathfrak{I}_\top}\in\tau_{\ell_{b_\top}}} \underline{\Delta}_{\mathfrak{I}_\top}\right) \end{array} \right\}, \end{aligned}$$

$$\left\{ \begin{array}{l} \left(\min\left(\overline{\Lambda}_{\mathfrak{I}_\top}\right) \leq \left(\overline{\Lambda}_{\mathfrak{I}_\top}\right) \leq \max\left(\overline{\Lambda}_{\mathfrak{I}_\top}\right) \text{ for all } \top = 1, 2, \ldots.., p\right) \\ \left(\max\left(\underline{\Delta}_{\mathfrak{I}_\top}\right) \leq \left(\underline{\Delta}_{\mathfrak{I}_\top}\right) \leq \min\left(\underline{\Delta}_{\mathfrak{I}_\top}\right) \text{ for all } \top = 1, 2, \ldots.., p\right) \end{array} \right\}$$

$$\left\{ \begin{array}{l} \left(g\left(\max\left(\overline{\Lambda}_{\mathfrak{I}_\top}\right)\right) \leq g\left(\overline{\Lambda}_{\mathfrak{I}_\top}\right) \leq g\left(\min\left(\overline{\Lambda}_{\mathfrak{I}_\top}\right)\right) \text{ for all } \top = 1, 2, \ldots.., p\right) \\ \left(g\left(\min\left(\underline{\Delta}_{\mathfrak{I}_\top}\right)\right) \leq g\left(\underline{\Delta}_{\mathfrak{I}_\top}\right) \leq g\left(\max\left(\underline{\Delta}_{\mathfrak{I}_\top}\right)\right) \text{ for all } \top = 1, 2, \ldots.., p\right) \end{array} \right\}$$

$$\left\{ \begin{array}{l} \left(\frac{2-\max\left(\overline{\Lambda}_{\mathfrak{I}_\top}\right)}{\max\left(\overline{\Lambda}_{\mathfrak{I}_\top}\right)} \leq \frac{2-\left(\overline{\Lambda}_{\mathfrak{I}_\top}\right)}{\left(\overline{\Lambda}_{\mathfrak{I}_\top}\right)} \leq \frac{2-\min\left(\overline{\Lambda}_{\mathfrak{I}_\top}\right)}{\min\left(\overline{\Lambda}_{\mathfrak{I}_\top}\right)}\right) \\ \left(\frac{2-\min\left(\underline{\Delta}_{\mathfrak{I}_\top}\right)}{\min\left(\underline{\Delta}_{\mathfrak{I}_\top}\right)} \leq \frac{2-\left(\underline{\Delta}_{\mathfrak{I}_\top}\right)}{\left(\underline{\Delta}_{\mathfrak{I}_\top}\right)} \leq \frac{2-\max\left(\underline{\Delta}_{\mathfrak{I}_\top}\right)}{\max\left(\underline{\Delta}_{\mathfrak{I}_\top}\right)}\right) \end{array} \right\}$$

$$\left\{\begin{array}{l}\left(\left(\frac{2-\max\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}{\max\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\overline{\Lambda}_{\mathfrak{J}_{\top}}}{\overline{\Lambda}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\min\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}{\min\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}}\right) \\ \left(\left(\frac{2-\min\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}{\min\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\underline{\Delta}_{\mathfrak{J}_{\top}}}{\underline{\Delta}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\max\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}{\max\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}}\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\prod_{\top=1}^{p}\left(\frac{2-\max\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}{\max\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}} \leq \prod_{\top=1}^{p}\left(\frac{2-\overline{\Lambda}_{\mathfrak{J}_{\top}}}{\overline{\Lambda}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \prod_{\top=1}^{p}\left(\frac{2-\min\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}{\min\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}}\right) \\ \left(\prod_{\top=1}^{p}\left(\frac{2-\min\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}{\min\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}} \leq \prod_{\top=1}^{p}\left(\frac{2-\underline{\Delta}_{\mathfrak{J}_{\top}}}{\underline{\Delta}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \prod_{\top=1}^{p}\left(\frac{2-\max\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}{\max\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}}\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\left(\frac{2-\max\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}{\max\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}\right)^{\sum_{\top=1}^{p}\Gamma_{\top}} \leq \prod_{\top=1}^{p}\left(\frac{2-\overline{\Lambda}_{\mathfrak{J}_{\top}}}{\overline{\Lambda}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\min\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}{\min\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}\right)^{\sum_{\top=1}^{p}\Gamma_{\top}}\right) \\ \left(\left(\frac{2-\min\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}{\min\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}\right)^{\sum_{\top=1}^{p}\Gamma_{\top}} \leq \prod_{\top=1}^{p}\left(\frac{2-\underline{\Delta}_{\mathfrak{J}_{\top}}}{\underline{\Delta}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\max\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}{\max\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}\right)^{\sum_{\top=1}^{p}\Gamma_{\top}}\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\left(\frac{2-\max\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}{\max\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}\right) \leq \prod_{\top=1}^{p}\left(\frac{2-\overline{\Lambda}_{\mathfrak{J}_{\top}}}{\overline{\Lambda}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\min\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}{\min\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}\right)\right) \\ \left(\left(\frac{2-\min\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}{\min\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}\right) \leq \prod_{\top=1}^{p}\left(\frac{2-\underline{\Delta}_{\mathfrak{J}_{\top}}}{\underline{\Delta}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\max\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}{\max\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}\right)\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\frac{2}{\max\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)} \leq 1+\prod_{\top=1}^{p}\left(\frac{2-\overline{\Lambda}_{\mathfrak{J}_{\top}}}{\overline{\Lambda}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \frac{2}{\min\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}\right) \\ \left(\frac{2}{\min\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)} \leq 1+\prod_{\top=1}^{p}\left(\frac{2-\underline{\Delta}_{\mathfrak{J}_{\top}}}{\underline{\Delta}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \frac{2}{\max\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\frac{\max\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}{2} \leq \frac{1}{1+\prod_{\top=1}^{p}\left(\frac{2-\overline{\Lambda}_{\mathfrak{J}_{\top}}}{\overline{\Lambda}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}}} \leq \frac{\min\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}{2}\right) \\ \left(\frac{\min\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}{2} \leq \frac{1}{1+\prod_{\top=1}^{p}\left(\frac{2-\underline{\Delta}_{\mathfrak{J}_{\top}}}{\underline{\Delta}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}}} \leq \frac{\max\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)}{2}\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\max\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right) \leq \frac{2}{1+\prod_{\top=1}^{p}\left(\frac{2-\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}{\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}}} \leq \min\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)\right) \\ \left(\min\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right) \leq \frac{2}{1+\prod_{\top=1}^{p}\left(\frac{2-\underline{\Delta}_{\mathfrak{J}_{\top}}}{\underline{\Delta}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}}} \leq \max\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\max\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right) \leq \frac{2\prod_{\top=1}^{p}\left(\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)\right)^{\Gamma_{\top}}}{\prod_{\top=1}^{p}\left(2-\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)^{\Gamma_{\top}}+\prod_{\top=1}^{p}\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)^{\Gamma_{\top}}} \leq \min\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)\right) \\ \left(\min\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right) \leq \frac{2\prod_{\top=1}^{p}\left(\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)\right)^{\Gamma_{\top}}}{\prod_{\top=1}^{p}\left(2-\underline{\Delta}_{\mathfrak{J}_{\top}}\right)^{\Gamma_{\top}}+\prod_{\top=1}^{p}\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)^{\Gamma_{\top}}} \leq \max\left(\underline{\Delta}_{\mathfrak{J}_{\top}}\right)\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\max_{\mathfrak{J}_{\top}} \overline{\Lambda} \leq \frac{2\prod_{\top=1}^{p}\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)^{\Gamma_{\top}}}{\prod_{\top=1}^{p}\left(2-\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)^{\Gamma_{\top}}+\prod_{\top=1}^{p}\left(\overline{\Lambda}_{\mathfrak{J}_{\top}}\right)^{\Gamma_{\top}}} \leq \min \overline{\Lambda}_{\mathfrak{J}_{\top}}\right) \\ \left(\min \underline{\Lambda}_{\mathfrak{J}_{\top}} \leq \frac{2\prod_{\top=1}^{p}\left(\underline{\Lambda}_{\mathfrak{J}_{\top}}\right)^{\Gamma_{\top}}}{\prod_{\top=1}^{p}\left(2-\underline{\Lambda}_{\mathfrak{J}_{\top}}\right)^{\Gamma_{\top}}+\prod_{\top=1}^{p}\left(\underline{\Lambda}_{\mathfrak{J}_{\top}}\right)^{\Gamma_{\top}}} \leq \max \underline{\Lambda}_{\mathfrak{J}_{\top}}\right)\end{array}\right\}.$$

Consider $g(y) = \frac{2-y}{y}, y \in (0,1]$. Then, $g'(y) = -\frac{2}{y}$, *i.e.*, $g(y)$ is a decreasing function on $(0,1]$. Let $\max\Upsilon_{\mathfrak{J}_{\top}} = \max_{1\leq\top\leq p}\max_{\Upsilon_{\mathfrak{J}_{\top}}\in\xi_{\ell_{\mathfrak{b}_{\top}}}}\Upsilon_{\mathfrak{J}_{\top}}$ and $\min\Upsilon_{\mathfrak{J}_{\top}}=\min_{1\leq\top\leq p}\min_{\Upsilon_{\mathfrak{J}_{\top}}\in\xi_{\ell_{\mathfrak{b}_{\top}}}}$.

$$\begin{aligned}\max(\overline{\Upsilon}_{\mathfrak{J}_{\top}},\underline{\Upsilon}_{\mathfrak{J}_{\top}}) &= \left\{\begin{array}{l}\left(\max_{1\leq\top\leq p}\max_{\overline{\Upsilon}_{\mathfrak{J}_{\top}}\in\xi_{\ell_{\mathfrak{b}_{\top}}}}\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right) \\ \left(\min_{1\leq\top\leq p}\min_{\underline{\Upsilon}_{\mathfrak{J}_{\top}}\in\xi_{\ell_{\mathfrak{b}_{\top}}}}\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)\end{array}\right\}, \\ \min\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}},\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right) &= \left\{\begin{array}{l}\left(\min_{1\leq\top\leq p}\min_{\overline{\Upsilon}_{\mathfrak{J}_{\top}}\in\xi_{\ell_{\mathfrak{b}_{\top}}}}\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right) \\ \left(\max_{1\leq\top\leq p}\max_{\underline{\Upsilon}_{\mathfrak{J}_{\top}}\in\xi_{\ell_{\mathfrak{b}_{\top}}}}\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)\end{array}\right\},\end{aligned}$$

$$\left\{\begin{array}{l}\left(\min\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right) \leq \left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right) \leq \max\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right) \text{ for all } \top = 1,2,.....,p\right) \\ \left(\max\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right) \leq \left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right) \leq \min\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right) \text{ for all } \top = 1,2,.....,p\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(g\left(\max\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)\right) \leq g\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right) \leq g\left(\min\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)\right) \text{ for all } \top = 1,2,.....,p\right) \\ \left(g\left(\min\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)\right) \leq g\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right) \leq g\left(\max\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)\right) \text{ for all } \top = 1,2,.....,p\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\frac{2-\max\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\max\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)} \leq \frac{2-\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)} \leq \frac{2-\min\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\min\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right) \\ \left(\frac{2-\min\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\min\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)} \leq \frac{2-\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)} \leq \frac{2-\max\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\max\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\left(\frac{2-\max\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\max\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\overline{\Upsilon}_{\mathfrak{J}_{\top}}}{\overline{\Upsilon}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\min\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\min\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}}\right) \\ \left(\left(\frac{2-\min\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\min\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\underline{\Upsilon}_{\mathfrak{J}_{\top}}}{\underline{\Upsilon}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\max\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\max\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}}\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\prod_{\top=1}^{p}\left(\frac{2-\max\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\max\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}} \leq \prod_{\top=1}^{p}\left(\frac{2-\overline{\Upsilon}_{\mathfrak{J}_{\top}}}{\overline{\Upsilon}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \prod_{\top=1}^{p}\left(\frac{2-\min\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\min\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}}\right) \\ \left(\prod_{\top=1}^{p}\left(\frac{2-\min\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\min\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}} \leq \prod_{\top=1}^{p}\left(\frac{2-\underline{\Upsilon}_{\mathfrak{J}_{\top}}}{\underline{\Upsilon}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \prod_{\top=1}^{p}\left(\frac{2-\max\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\max\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right)^{\Gamma_{\top}}\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\left(\frac{2-\max\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\max\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right)^{\sum_{\top=1}^{p}\Gamma_{\top}} \leq \prod_{\top=1}^{p}\left(\frac{2-\overline{\Upsilon}_{\mathfrak{J}_{\top}}}{\overline{\Upsilon}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\min\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\min\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right)^{\sum_{\top=1}^{p}\Gamma_{\top}}\right) \\ \left(\left(\frac{2-\min\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\min\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right)^{\sum_{\top=1}^{p}\Gamma_{\top}} \leq \prod_{\top=1}^{p}\left(\frac{2-\underline{\Upsilon}_{\mathfrak{J}_{\top}}}{\underline{\Upsilon}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\max\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\max\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right)^{\sum_{\top=1}^{p}\Gamma_{\top}}\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\left(\frac{2-\max\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\max\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right) \leq \prod_{\top=1}^{p}\left(\frac{2-\overline{\Upsilon}_{\mathfrak{J}_{\top}}}{\overline{\Upsilon}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\min\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\min\left(\overline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right)\right) \\ \left(\left(\frac{2-\min\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\min\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right) \leq \prod_{\top=1}^{p}\left(\frac{2-\underline{\Upsilon}_{\mathfrak{J}_{\top}}}{\underline{\Upsilon}_{\mathfrak{J}_{\top}}}\right)^{\Gamma_{\top}} \leq \left(\frac{2-\max\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}{\max\left(\underline{\Upsilon}_{\mathfrak{J}_{\top}}\right)}\right)\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\frac{2}{\max\left(\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right)} \leq 1+\prod_{\top=1}^{p}\left(\frac{2-\overline{\Upsilon}_{\mathfrak{I}_{\top}}}{\overline{\Upsilon}_{\mathfrak{I}_{\top}}}\right)^{\Gamma_{\top}} \leq \frac{2}{\min\left(\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right)}\right) \\ \left(\frac{2}{\min\left(\underline{\Upsilon}_{\mathfrak{I}_{\top}}\right)} \leq 1+\prod_{\top=1}^{p}\left(\frac{2-\underline{\Upsilon}_{\mathfrak{I}_{\top}}}{\underline{\Upsilon}_{\mathfrak{I}_{\top}}}\right)^{\Gamma_{\top}} \leq \frac{2}{\max\left(\underline{\Upsilon}_{\mathfrak{I}_{\top}}\right)}\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\frac{\max\left(\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right)}{2} \leq \frac{1}{1+\prod_{\top=1}^{p}\left(\frac{2-\overline{\Upsilon}_{\mathfrak{I}_{\top}}}{\overline{\Upsilon}_{\mathfrak{I}_{\top}}}\right)^{\Gamma_{\top}}} \leq \frac{\min\left(\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right)}{2}\right) \\ \left(\frac{\min\left(\underline{\Upsilon}_{\mathfrak{I}_{\top}}\right)}{2} \leq \frac{1}{1+\prod_{\top=1}^{p}\left(\frac{2-\underline{\Upsilon}_{\mathfrak{I}_{\top}}}{\underline{\Upsilon}_{\mathfrak{I}_{\top}}}\right)^{\Gamma_{\top}}} \leq \frac{\max\left(\underline{\Upsilon}_{\mathfrak{I}_{\top}}\right)}{2}\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\max\left(\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right) \leq \frac{2}{1+\prod_{\top=1}^{p}\left(\frac{2-\left(\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right)}{\left(\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right)}\right)^{\Gamma_{\top}}} \leq \min\left(\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right)\right) \\ \left(\min\left(\underline{\Upsilon}_{\mathfrak{I}_{\top}}\right) \leq \frac{2}{1+\prod_{\top=1}^{p}\left(\frac{2-\underline{\Upsilon}_{\mathfrak{I}_{\top}}}{\underline{\Upsilon}_{\mathfrak{I}_{\top}}}\right)^{\Gamma_{\top}}} \leq \max\left(\underline{\Upsilon}_{\mathfrak{I}_{\top}}\right)\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\max\left(\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right) \leq \frac{2\prod_{\top=1}^{p}\left(\left(\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right)\right)^{\Gamma_{\top}}}{\prod_{\top=1}^{p}\left(2-\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right)^{\Gamma_{\top}}+\prod_{\top=1}^{p}\left(\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right)^{\Gamma_{\top}}} \leq \min\left(\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right)\right) \\ \left(\min\left(\underline{\Upsilon}_{\mathfrak{I}_{\top}}\right) \leq \frac{2\prod_{\top=1}^{p}\left(\left(\underline{\Upsilon}_{\mathfrak{I}_{\top}}\right)\right)^{\Gamma_{\top}}}{\prod_{\top=1}^{p}\left(2-\underline{\Upsilon}_{\mathfrak{I}_{\top}}\right)^{\Gamma_{\top}}+\prod_{\top=1}^{p}\left(\underline{\Upsilon}_{\mathfrak{I}_{\top}}\right)^{\Gamma_{\top}}} \leq \max\left(\underline{\Upsilon}_{\mathfrak{I}_{\top}}\right)\right)\end{array}\right\}$$

$$\left\{\begin{array}{l}\left(\max_{\mathfrak{I}_{\top}} \overline{\Upsilon} \leq \frac{2\prod_{\top=1}^{p}\left(\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right)^{\Gamma_{\top}}}{\prod_{\top=1}^{p}\left(2-\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right)^{\Gamma_{\top}}+\prod_{\top=1}^{p}\left(\overline{\Upsilon}_{\mathfrak{I}_{\top}}\right)^{\Gamma_{\top}}} \leq \min \overline{\Upsilon}_{\mathfrak{I}_{\top}}\right) \\ \left(\min \underline{\Upsilon}_{\mathfrak{I}_{\top}} \leq \frac{2\prod_{\top=1}^{p}\left(\underline{\Upsilon}_{\mathfrak{I}_{\top}}\right)^{\Gamma_{\top}}}{\prod_{\top=1}^{p}\left(2-\underline{\Upsilon}_{\mathfrak{I}_{\top}}\right)^{\Gamma_{\top}}+\prod_{\top=1}^{p}\left(\underline{\Upsilon}_{\mathfrak{I}_{\top}}\right)^{\Gamma_{\top}}} \leq \max \underline{\Upsilon}_{\mathfrak{I}_{\top}}\right)\end{array}\right\}.$$

□

5. Multi-attribute decision-making methodology

We present an approach for dealing with unpredictability in multi-attribute group decision-making (MAGDM) with single valued neutrosophic probabilistic hesitant fuzzy rough (SV-NPHFR) information. Consider a decision-making (DM) issue with a collection of variables $\{\Theta_1, \Theta_2, ..., \Theta_n\}$ of n substitutes and a set of n attributes $\left\{\hat{C}_1, \hat{C}_2, ..., \hat{C}_n\right\}$ with $(w_1, w_2, ..., w_n)^T$ the weights, that is, $w_\jmath \in [0,1], \oplus_{\jmath=1}^{n} w_\jmath = 1$. Also, consider a probabilistic terms $\Xi_{\ell_\jmath}$,$\varpi_{\ell_\jmath}$ and $\phi_{\ell_\jmath}$ such that $\oplus_{\jmath=1}^{n}\Xi_{\ell_\jmath} = 1, \oplus_{\jmath=1}^{n}\varpi_{\ell_\jmath} = 1$ and $\oplus_{\jmath=1}^{n}\phi_{\ell_\jmath} = 1$ with the property that $0 \leq \Xi_{\ell_\jmath}, \phi_{\ell_\jmath}, \varpi_{\ell_\jmath} \leq 1$. To test the validity of k$^{\text{th}}$ substitute $\Theta_\jmath$ under the the attribute $\hat{C}_\jmath$, let $\{D_1, D_2, ..., D_\top\}$ be a set of decision makers (DMs), and $(\sigma_1, \sigma_2, ..., \sigma_n)^T$ be DMs weights such that $\sigma_\jmath \in [0,1]$, $\oplus_{\jmath=1}^{n}\sigma_\jmath = 1$. The decision professional information of a certain decision support problem is given as follows:

$$M = \left[\overline{\mathfrak{I}}(\Lambda_{\jmath j}^{\top})\right]_{m\times n}$$

$$= \begin{bmatrix} \left(\overline{\Im}(\Lambda_{11}), \underline{\Im}(\Lambda_{11})\right) & \left(\overline{\Im}(\Lambda_{12}), \underline{\Im}(\Lambda_{12})\right) & \cdots & \left(\overline{\Im}(\Lambda_{1j}), \underline{\Im}(\Lambda_{1j})\right) \\ \left(\overline{\Im}(\Lambda_{21}), \underline{\Im}(\Lambda_{21})\right) & \left(\overline{\Im}(\Lambda_{22}), \underline{\Im}(\Lambda_{22})\right) & \cdots & \left(\overline{\Im}(\Lambda_{2j}), \underline{\Im}(\Lambda_{2j})\right) \\ \left(\overline{\Im}(\Lambda_{31}), \underline{\Im}(\Lambda_{31})\right) & \left(\overline{\Im}(\Lambda_{32}), \underline{\Im}(\Lambda_{32})\right) & \cdots & \left(\overline{\Im}(\Lambda_{3j}), \underline{\Im}(\Lambda_{3j})\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\overline{\Im}(\Lambda_{\jmath 1}), \underline{\Im}(\Lambda_{\jmath 1})\right) & \left(\overline{\Im}(\Lambda_{\jmath 2}), \underline{\Im}(\Lambda_{\jmath 2})\right) & \cdots & \left(\overline{\Im}(\Lambda_{\jmath j}), \underline{\Im}(\Lambda_{\jmath j})\right) \end{bmatrix},$$

where

$$\overline{\Im}(\Lambda_{\jmath j}) = \left\{\left\langle \Theta, \upsilon_{\ell_{\overline{\Im}(\Lambda)}}(\Theta)/\Xi_{\ell_{\overline{\Im}(\Lambda)}}, \Lambda_{\ell_{\overline{\Im}(\Lambda)}}(\Theta)/\varpi_{\ell_{\overline{\Im}(\Lambda)}}, \Upsilon_{\ell_{\overline{\Im}(\Lambda)}}(\Theta)/\Xi_{\ell_{\overline{\Im}(\Lambda)}} \right\rangle | \Theta \in \mathbb{Q}\right\}$$

and

$$\underline{\delta}(\Lambda_{\jmath j}) = \left\{\left\langle \Theta, \upsilon_{\ell_{\underline{\delta}(\Lambda)}}(\Theta)/\Xi_{\ell_{\underline{\delta}(\Lambda)}}, \Lambda_{\ell_{\underline{\delta}(\Lambda)}}(\Theta)/\varpi_{\ell_{\underline{\delta}(\Lambda)}}, \Upsilon_{\ell_{\underline{\delta}(\Lambda)}}(\Theta)/\Xi_{\ell_{\underline{\delta}(\Lambda)}} \right\rangle | \Theta \in \mathbb{Q}\right\}$$

such that $0 < \left(\max(\upsilon_{\ell_{\overline{\Im}(\Lambda)}}(\Theta))\right) + \left(\min(\Lambda_{\ell_{\overline{\Im}(\Lambda)}}(\Theta))\right) + \left(\min(\Upsilon_{\ell_{\overline{\Im}(\Lambda)}}(\Theta))\right) \leq 3$ and $0 < \left(\min(\upsilon_{\ell_{\underline{\delta}(\Lambda)}}(\Theta)\right) + \left(\max(\Lambda_{\ell_{\underline{\delta}(\Lambda)}}(\Theta))\right) + \left(\max(\Upsilon_{\ell_{\underline{\delta}(\Lambda)}}(\Theta))\right) \leq 3$ are the SV-NPHF rough values. The algorithm steps are described as follows for decision making:

Step-1. Construct the assessment matrices for the experts as follows:

$$(E)^{\top} = \begin{bmatrix} \left(\overline{\Im}(\Lambda^{\top}_{11}), \underline{\delta}(\Lambda^{\top}_{11})\right) & \left(\overline{\Im}(\Lambda^{\top}_{12}), \underline{\delta}(\Lambda^{\top}_{12})\right) & \cdots & \left(\overline{\Im}(\Lambda^{\top}_{1j}), \underline{\delta}(\Lambda^{\top}_{1j})\right) \\ \left(\overline{\Im}(\Lambda^{\top}_{21}), \underline{\delta}(\Lambda^{\top}_{21})\right) & \left(\overline{\Im}(\Lambda^{\top}_{22}), \underline{\delta}(\Lambda^{\top}_{22})\right) & \cdots & \left(\overline{\Im}(\Lambda^{\top}_{2j}), \underline{\delta}(\Lambda^{\top}_{2j})\right) \\ \left(\overline{\Im}(\Lambda^{\top}_{31}), \underline{\delta}(\Lambda^{\top}_{31})\right) & \left(\overline{\Im}(\Lambda^{\top}_{32}), \underline{\delta}(\Lambda^{\top}_{32})\right) & \cdots & \left(\overline{\Im}(\Lambda^{\top}_{3j}), \underline{\delta}(\Lambda^{\top}_{3j})\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\overline{\Im}(\Lambda^{\top}_{\jmath 1}), \underline{\delta}(\Lambda^{\top}_{\jmath 1})\right) & \left(\overline{\Im}(\Lambda^{\top}_{\jmath 2}), \underline{\delta}(\Lambda^{\top}_{\jmath 2})\right) & \cdots & \left(\overline{\Im}(\Lambda^{\top}_{\jmath j}), \underline{\delta}(\Lambda^{\top}_{\jmath j})\right) \end{bmatrix}$$

where $\top$ shows the expert numbers.

Step-2. Conduct an analysis on the normalized version of the experts' matrices $(\mathbb{N})^{\top}$, as follows:

$$(\mathbb{N})^{\top} = \begin{cases} \Im(\Lambda_{\jmath j}) = \left(\underline{\delta}\left(\Lambda_{\jmath j}\right), \overline{\Im}\left(\Lambda_{\jmath j}\right)\right) & \text{if For benefit} \\ \left(\Im(\Lambda_{\jmath j})\right)^{c} = \left(\left(\underline{\delta}\left(\Lambda_{\jmath j}\right)\right)^{c}, \left(\overline{\Im}\left(\Lambda_{\jmath j}\right)\right)^{c}\right) & \text{if For cost} \end{cases}$$

Step-3. Use novel list of algebraic AOs to determined the collected decision making expert information.

Step-4. For each option under consideration, compute the aggregated SV-NPHFRNs by evaluating them to the given set of criteria and characteristics.

Step-5. Calculate the relative importances of various options using a scoring formula.

Step-6. In decreasing order, place all of the alternate ratings. When it comes to alternatives, a bigger value indicates superiority.

The algorithm steps for multi-attribute decision making are given in Figure 1:

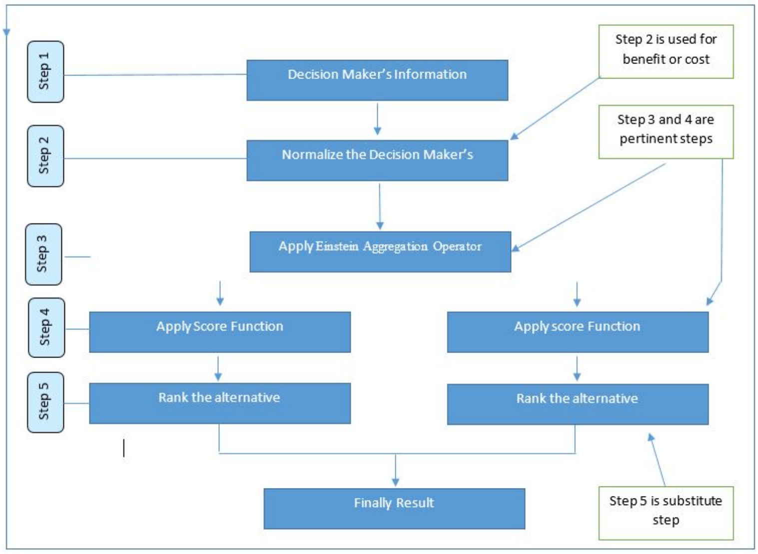

Figure 1. Algorithm flow chart.

6. Case study

Sensor and data technologies: The social, economic, and political spheres are becoming rapidly digitalized. Agriculture, particularly livestock production, is one of the sectors of development that calls for the use of emerging technologies as the world enters various streams of technological progress. However, knowledge and skills are generally lacking, especially in underdeveloped nations. Therefore, it is essential to close this knowledge gap by gathering data and disseminating it to a broad audience in order to spur additional research in the area. It is vital to think about the potential use of these technologies in the period following the introduction of sensors for broader applications, such as the assessment of rangeland condition and animal production. The application of precision agricultural procedures for livestock, including individual animal behavior, grazing conditions, health conditions, and feed intakes, benefits more from technological advancements. It is crucial to evaluate the state of both rangeland resource conditions and livestock grazing behavior for grazing cattle. All around the world, rangeland resources are crucial to the production of cattle. Due to changes in the environment and management practices, forages and pastures are dynamic in terms of quantity, species composition, and chemical composition. For improved management and exploitation, the rangeland resources need to be evaluated and tracked. Traditional evaluations that involve mechanical or manual counting, identification, and chemical composition are time-consuming and labor-intensive. Under field conditions, sensors can be used to monitor the quality of grazing grounds, but they can also be used to better manage grazing stock by gaining an understanding of how animals graze. It is essential to comprehend current technology, such as sensors or bio sensors, in order to supplement or replace old procedures. The goal of this research article is to raise awareness of the technologies that are currently available and their applicability to rangeland resources, particularly in tropical rangelands. In most situations, GPS devices are used in the tropics to evaluate solely the status of the rangeland without taking the grazing cattle into account. The assessment further clarifies the significance of sensor technology for quickly and easily detecting changes in cattle health and movement at the field level.

Sensors (bio sensors), like any other technology, have limitations, such as measurement accuracy and repetitive data complaints. However, the assessment clarifies the usage of sensor technology, which saves time and energy in animal production that would otherwise require a lot of it.

Numerical example

The applications of the suggested methodologies in this research are illustrated and their viability and efficacy are shown using an example concerning investment possibilities for a MADM problem that has been adapted from experts. In Figure 2, some views of cattle farms are shown.

Assume that a standard MADM problem Cattle farm evaluation of developing technology commercialization—needs to be resolved. Θ_1, Θ_2, Θ_3 and Θ_4 on a panel represent four potential developing technology such as, Sensor and data technologies, robots, temperature and moisture sensors, aerial images, and GPS technology respectively. Four factors must be taken into account while making a decision: $(\hat{C}_1)$ Technological development; $(\hat{C}_2)$ Risk premium and prospective markets; $(\hat{C}_3)$ Environment for production, human resources and financial circumstances; and $(\hat{C}_4)$ Opportunities for employment.

Figure 2. Views of cattle farming using technology.

Step-1. The information of professionals are given in Table 1(a)–(d) in the form of SV-NPHFRS.

Table 1(a). Expert details.

	$\hat{C}_1$	$\hat{C}_2$
Θ_1	$\left(\begin{pmatrix}(0.5/0.5, 0.2/0.5), \\ (0.6/0.6, 0.3/0.4), \\ (0.3/1.0)\end{pmatrix}, \begin{pmatrix}(0.9/1.0), \\ (0.6/0.8, 0.5/0.2) \\ (0.2/0.3, 0.5/0.7)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.3/1.0), \\ (0.9/1.0), \\ (0.1/0.3, 0.7/0.7)\end{pmatrix}, \begin{pmatrix}(0.4/1.0), \\ (0.2/1), \\ (0.7/1.0)\end{pmatrix}\right)$
Θ_2	$\left(\begin{pmatrix}(0.4/0.1, 0.6/0.9), \\ (0.7/1.0), \\ (0.8/0.4, 0.9/0.6)\end{pmatrix}, \begin{pmatrix}(0.3/0.2, 0.5/0.8), \\ (0.1/0.5, 0.6/0.5) \\ (0.8/1.0)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.8/1.0), \\ (0.2/0.5, 0.5/0.5), \\ (0.9/1.0)\end{pmatrix}, \begin{pmatrix}(0.6/0.3, 0.7/0.7), \\ (0.2/1.0), \\ (0.5/1.0)\end{pmatrix}\right)$

Table 1(b). Expert details.

	$\hat{C}_3$	$\hat{C}_4$
Θ_1	$\left(\left(\begin{pmatrix}(0.4/0.8, 0.5/0.2), \\ (0.6/1.0), \\ (0.4/1.0)\end{pmatrix}\right), \left(\begin{pmatrix}(0.9/1.0), \\ (0.6/1.0), \\ (0.1/1.0)\end{pmatrix}\right)\right)$	$\left(\left(\begin{pmatrix}(0.2/1.0), \\ (0.3/1.0), \\ (0.2/0.4, 0.3/0.6)\end{pmatrix}\right), \left(\begin{pmatrix}(0.9/1.0), \\ (0.6/0.4, 0.3/0.6), \\ (0.3/1.0)\end{pmatrix}\right)\right)$
Θ_2	$\left(\left(\begin{pmatrix}(0.2/1.0), \\ (0.6/0.7, 0.3/0.3), \\ (0.4/1.0)\end{pmatrix}\right), \left(\begin{pmatrix}(0.2/1.0), \\ (0.8/1.0), \\ (0.2/0.2, 0.5/0.8)\end{pmatrix}\right)\right)$	$\left(\left(\begin{pmatrix}(0.9/1.0), \\ (0.4/1.0), \\ (0.4/1.0)\end{pmatrix}\right), \left(\begin{pmatrix}(0.5/1.0), \\ (0.3/0.6, 0.4/0.4), \\ (0.4/1.0)\end{pmatrix}\right)\right)$

Table 1(c). Expert details.

	$\hat{C}_1$	$\hat{C}_2$
Θ_3	$\left(\begin{array}{c}\left(\left(\begin{array}{c}(0.5/1.0),\\ (0.6/0.6, 0.3/0.4),\\ (0.3/0.7, 0.4/0.3\end{array}\right)\right),\\ \left(\left(\begin{array}{c}(0.5/0.5, 0.2/0.5),\\ (0.3/1.0),\\ (0.8/0.2, 0.9/0.8)\end{array}\right)\right)\end{array}\right)$	$\left(\begin{array}{c}\left(\left(\begin{array}{c}(0.5/0.7, 0.2/0.2, 0.9/0.1),\\ (0.3/1.0),\\ (0.3/0.2, 0.4/0.8)\end{array}\right)\right),\\ \left(\left(\begin{array}{c}(0.7/0.3, 0.3/0.7),\\ (0.3/1.0),\\ (0.3/0.6, 0.4/0.4)\end{array}\right)\right)\end{array}\right)$
Θ_4	$\left(\begin{array}{c}\left(\left(\begin{array}{c}(0.2/1.0),\\ (0.8/1.0),\\ (0.4/1.0)\end{array}\right)\right),\\ \left(\left(\begin{array}{c}(0.2/0.4, 0.5/0.6),\\ (0.3/0.6, 0.3/0.4),\\ (0.1/1.0)\end{array}\right)\right)\end{array}\right)$	$\left(\begin{array}{c}\left(\left(\begin{array}{c}(0.2/1.0),\\ (0.6/0.4, 0.4/0.4, 0.3/0.2),\\ (0.4/1.0)\end{array}\right)\right),\\ \left(\left(\begin{array}{c}(0.9/1.0),\\ (0.3/1.0),\\ (0.7/0.2, 0.1/0.8)\end{array}\right)\right)\end{array}\right)$

Table 1(d). Expert details.

	$\hat{C}_3$	$\hat{C}_4$
Θ_3	$\left(\begin{array}{c}\left(\left(\begin{array}{c}(0.9/1.0),\\ (0.4/1.0),\\ (0.1/1.0)\end{array}\right)\right),\\ \left(\left(\begin{array}{c}(0.2/1.0),\\ (0.9/0.7, 0.3/0.3),\\ (0.3/1.0)\end{array}\right)\right)\end{array}\right)$	$\left(\begin{array}{c}\left(\left(\begin{array}{c}(0.5/1.0),\\ (0.3/1.0),\\ (0.4/1.0)\end{array}\right)\right),\\ \left(\left(\begin{array}{c}(0.5/1.0),\\ (0.2/0.6, 0.4/0.4),\\ (0.4/1.0)\end{array}\right)\right)\end{array}\right)$
Θ_4	$\left(\begin{array}{c}\left(\left(\begin{array}{c}(0.2/1.0),\\ (0.1/1.0),\\ (0.5/0.3, 0.3/0.2, 0.4/0.5)\end{array}\right)\right),\\ \left(\left(\begin{array}{c}(0.8/1.0),\\ (0.7/0.6, 0.4/0.4),\\ (0.4/1.0)\end{array}\right)\right)\end{array}\right)$	$\left(\begin{array}{c}\left(\left(\begin{array}{c}(0.2/0.8, 0.5/0.1, 0.2/0.1),\\ (0.6/1.0),\\ (0.1/1.0)\end{array}\right)\right),\\ \left(\left(\begin{array}{c}(0.5/0.6, 0.2/0.4),\\ (0.6/1.0),\\ (0.3/0.2, 0.4/0.8)\end{array}\right)\right)\end{array}\right)$

Step-2. Expert opinion is of the beneficiary type. As a result, we don't need to normalize the SV-NPHFRNs in this context.

Step-3. In this problem, just one professional is evaluated for the collection of questionable data. As a result, we are not obligated to detect the information gained.

Step-4. The SV-NPHFRWA operator is used to evaluate the alternatives' aggregation details under the given list of attributes:

$\Gamma = (0.34,\ 0.55,\ 0.11)^T$, we obtain the overall preference values Θ_χ of the alternative $L_\chi(\chi = 1,2,3,4)$ by utilizing the suggested SV-NPHFRWA operator.

For illustration,

$$SV-NPHFREWA\left\{\begin{array}{c}(\overline{\Im}_1,\overline{\Im}_2,....,\overline{\Im}_p)\\(\underline{\delta}_1,\underline{\delta}_2,....,\underline{\delta}_p)\end{array}\right\}$$

$$=\left(\begin{array}{c}\left\{\begin{array}{c}\left(\bigcup_{\substack{\xi_{\overline{\Im}_\intercal}\in\upsilon_{\overline{\Im}_\intercal}\\ \Xi_{\overline{\Im}_\intercal}\in\Xi_{\overline{\Im}_\intercal}}}\frac{\Pi_{\intercal=1}^p\left(1+\xi_{\overline{\Im}_\intercal}\right)^{\Gamma_\intercal}-\Pi_{\intercal=1}^p\left(1-\xi_{\overline{\Im}_\intercal}\right)^{\Gamma_\intercal}}{\Pi_{\intercal=1}^p\left(1+\xi_{\overline{\Im}_\intercal}\right)^{\Gamma_\intercal}+\Pi_{\intercal=1}^p\left(1-\xi_{\overline{\Im}_\intercal}\right)^{\Gamma_\intercal}}/\Pi_{\intercal=1}^p\Xi_{\overline{\Im}_\intercal}\right)\\ \left(\bigcup_{\substack{\xi_{\underline{\delta}_\intercal}\in\upsilon_{\underline{\delta}_\intercal}\\ \Xi_{\underline{\delta}_\intercal}\in\Xi_{\underline{\delta}_\intercal}}}\frac{\Pi_{\intercal=1}^p\left(1+\xi_{\underline{\delta}_\intercal}\right)^{\Gamma_\intercal}-\Pi_{\intercal=1}^p\left(1-\xi_{\underline{\delta}_\intercal}\right)^{\Gamma_\intercal}}{\Pi_{\intercal=1}^p\left(1+\xi_{\underline{\delta}_\intercal}\right)^{\Gamma_\intercal}+\Pi_{\intercal=1}^p\left(1-\xi_{\underline{\delta}_\intercal}\right)^{\Gamma_\intercal}}/\Pi_{\intercal=1}^p\Xi_{\underline{\delta}_\intercal}\right)\end{array}\right\},\\ \left\{\begin{array}{c}\left(\bigcup_{\substack{\Lambda_{\overline{\Im}_\intercal}\in\tau_{\overline{\Im}_\intercal}\\ \varpi_{\overline{\Im}_\intercal}\in\varpi_{\overline{\Im}_\intercal}}}\frac{2\Pi_{\intercal=1}^p\left(\Lambda_{\overline{\Im}_\intercal}\right)^{\Gamma_\intercal}}{\Pi_{\intercal=1}^p\left(2-\Lambda_{\overline{\Im}_\intercal}\right)^{\Gamma_\intercal}+\Pi_{\intercal=1}^p\left(\Lambda_{\overline{\Im}_\intercal}\right)^{\Gamma_\intercal}}/\Pi_{\intercal=1}^p\varpi_{\overline{\Im}_\intercal}\right)\\ \left(\bigcup_{\substack{\Lambda_{\underline{\delta}_\intercal}\in\tau_{\underline{\delta}_\intercal}\\ \varpi_{\underline{\delta}_\intercal}\in\varpi_{\underline{\delta}_\intercal}}}\frac{2\Pi_{\intercal=1}^p\left(\Lambda_{\underline{\delta}_\intercal}\right)^{\Gamma_\intercal}}{\Pi_{\intercal=1}^p\left(2-\Lambda_{\underline{\delta}_\intercal}\right)^{\Gamma_\intercal}+\Pi_{\intercal=1}^p\left(\Lambda_{\underline{\delta}_\intercal}\right)^{\Gamma_\intercal}}/\Pi_{\intercal=1}^p\varpi_{\underline{\delta}_\intercal}\right)\end{array}\right\},\\ \left\{\begin{array}{c}\left(\bigcup_{\substack{\Upsilon_{\overline{\Im}_\intercal}\in\xi_{\overline{\Im}_\intercal}\\ \phi_{\overline{\Im}_\intercal}\in\phi_{\overline{\Im}_\intercal}}}\frac{2\Pi_{\intercal=1}^p\left(\Upsilon_{\overline{\Im}_\intercal}\right)^{\Gamma_\intercal}}{\Pi_{\intercal=1}^p\left(2-\Upsilon_{\overline{\Im}_\intercal}\right)^{\Gamma_\intercal}+\Pi_{\intercal=1}^p\left(\Upsilon_{\overline{\Im}_\intercal}\right)^{\Gamma_\intercal}}/\Pi_{\intercal=1}^p\phi_{\overline{\Im}_\intercal}\right)\\ \left(\bigcup_{\substack{\Upsilon_{\underline{\delta}_\intercal}\in\xi_{\underline{\delta}_\intercal}\\ \phi_{\underline{\delta}_\intercal}\in\phi_{\underline{\delta}_\intercal}}}\frac{2\Pi_{\intercal=1}^p\left(\Upsilon_{\underline{\delta}_\intercal}\right)^{\Gamma_\intercal}}{\Pi_{\intercal=1}^p\left(2-\Upsilon_{\Im_\intercal}\right)^{\Gamma_\intercal}+\Pi_{\intercal=1}^p\left(\Upsilon_{\underline{\delta}_\intercal}\right)^{\Gamma_\intercal}}/\Pi_{\intercal=1}^p\phi_{\underline{\delta}_\intercal}\right)\end{array}\right\}\end{array}\right).$$

After obtain the overall preference values Θ_χ of the alternative $L_\chi(\chi = 1,2,3,4)$

$$s(\Theta_\chi)=\frac{1}{2}\left\{\begin{array}{c}\left(\begin{array}{c}\left(\frac{1}{M_\Theta}\sum_{\hbar_\daleth\in\tau_{\ell_{\dot{g}}},\tilde{p}_\daleth\in\tilde{p}_{\ell_{\dot{g}}}}(\hbar_\daleth\cdot\tilde{p}_\daleth)\right)+\\ \left(\frac{1}{M_\Theta}\sum_{\hbar_\daleth\in\tau_{\ell_{\dot{g}}},\tilde{p}_\daleth\in\tilde{p}_{\ell_{\dot{g}}}}(\hbar_\daleth\cdot\tilde{p}_\daleth)\right)\end{array}\right)-\\ \left(\begin{array}{c}\left(\frac{1}{N_\Theta}\sum_{\varrho_\daleth\in\Xi_{\ell_\varsigma},\varpi_\daleth\in\varpi_{\ell_{\dot{g}}}}(\varrho_\daleth\cdot\varpi_\daleth)\right)+\\ \left(\frac{1}{N_\Theta}\sum_{\varrho_\daleth\in\Xi_{\ell_\varsigma},\varpi_\daleth\in\varpi_{\ell_{\dot{g}}}}(\varrho_\daleth\cdot\varpi_\daleth)\right)\end{array}\right)-\\ \left(\begin{array}{c}\left(\frac{1}{Z_\Theta}\sum_{\Re_\daleth\in\Xi_{\ell_\varsigma},\phi_\daleth\in\phi_{\ell_{\dot{g}}}}(\varrho_\daleth\cdot\phi_\daleth)\right)+\\ \left(\frac{1}{Z_\Theta}\sum_{\Re_\daleth\in\Xi_{\ell_\varsigma},\phi_\daleth\in\phi_{\ell_{\dot{g}}}}(\varrho_\daleth\cdot\phi_\daleth)\right)\end{array}\right)\end{array}\right\}.$$

Step-5. Score values of all alternatives under developed aggregation operator are presented using Table 1. The Score values are presented in Table 2.

Table 2. Score values.

Operator	$s(\Theta_1)$	$s(\Theta_2)$	$s(\Theta_3)$	$s(\Theta_4)$
SV-NPHFREWA	0.1517	0.1248	0.1402	0.1419

Step-6. Categorize the alternatives. $\Theta_\chi(\chi = 1, 2, ..., 4)$ are listed in Table 3.

Table 3. Ranking of the alternatives.

Operator	Score	Most Effective Alternative
SV-NPHFREWA	$s(\Theta_1) > s(\Theta_4) > s(\Theta_3) > s(\Theta_2)$	Θ_1

We determined from the preceding mathematical formulation that choice Θ_1 is the best option among the rest, and it is therefore strongly recommended.

6.1. Discussion and comparative studies

The viability of the proposed process, the adaptability of its aggregation to handle certain inputs and outputs, the influence of score functions, sensitivity analysis, superiority, and, finally, the comparison of the proposed methodology with existing methodologies are all covered in this part. Also present the relationship strategy between alternatives and crossposting criteria using quaternion relationship function are presented. The suggested approach was exact and appropriate for all types of input data. The framework that was developed worked well for handling uncertainties. Using Einstein AOs, it contained the PHFS, RS, and NS spaces. We may effectively employ our model in a number of scenarios by extending the gap between the pleasure and displeasure classes by altering the practical significance of particular parameters. We faced a variety of elements and input parameters under the appropriate conditions in various MCDM problems. The suggested approaches are straightforward, easy to comprehend, and easily adaptable to a wide range of options and attributes. It should be highlighted that none of these theories provide a comprehensive explanation of how to handle a scenario in which the attributes are intertwined. The effectiveness of the aforementioned MADM tactics was further confirmed through this comparative analysis and relationship function with other approaches.

Example: We take into account four records of the cattle-forums that are cited in Table 4 data.

Table 4. Quaternion neutrosophic fuzzy relation S (cattle-farm→Alternatives).

S	Θ_1	Θ_2	Θ_3	Θ_4
Forum-1	(0.8,0.3,0.6)	(0.4,0.6,0.9)	(0.9,0.8,0.6)	(0.5,0.5,0.8)
Forum-2	(0.3,0.4,0.8)	(0.5,0.3,0.6)	(0.7,0.9,0.9)	(0.8,0.9,0.3)
Forum-3	(0.3,0.2,0.2)	(0.1,0.6,0.8)	(0.9,0.3,0.4)	(0.3,0.6,0.9)
Forum-4	(0.6,0.4,0.8)	(0.3,0.4,0.1)	(0.2,0.6,0.1)	(0.8,0.3,0.9)

We take into account four entries that are cited in Table 5 data.

Table 5. Quaternion neutrosophic fuzzy relation T (Alternatives→Criteria).

T	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	$\hat{C}_4$
Θ_1	(0.3,0.5,0.9)	(0.7,0.8,0.9)	(0.6,0.7,0.8)	(0.9,0.3,0.5)
Θ_2	(0.6,0.7,0.7)	(0.8,0.3,0.4)	(0.7,0.3,0.4)	(0.5,0.6,0.7)
Θ_3	(0.9,0.6,0.3)	(0.7,0.8,0.3)	(0.0.8,0.3,0.1)	(0.8,0.6,0.2)
Θ_4	(0.8,0.3,0.7)	(0.9,0.3,0.4)	(0.7,0.3,0.2)	(0.6,0.4,0.5)

The Table 6 shows the relationship between Table 4 and Table 5.

Table 6. Quaternion neutrosophic fuzzy relation R (cattle-farm→Criteria).

R	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	$\hat{C}_4$
Forum-1	(0.9,0.5,0.6)	(0.7,0.5,0.6)	(0.8,0.4,0.6)	(0.8,0.3,0.6)
Forum-2	(0.8,0.5,0.5)	(0.7,0.3,0.4)	(0.7,0.3,0.5)	(0.7,0.4,0.5)
Forum-3	(0.9,0.5,0.4)	(0.7,0.6,0.4)	(0.8,0.3,0.4)	(0.8,0.3,0.3)
Forum-4	(0.8,0.6,0.3)	(0.8,0.6,0.3)	(0.7,0.3,0.3)	(0.6,0.3,0.4)

In the Table 7, We find the results of relationship function. The results are found using the following formulas on quaternion neutrosophic numbers:

$$\xi_T(P_i, F_k) = \bigvee_{s \in S} [\xi_T(P_i, T) \bigwedge \xi_R(T, F_k)]$$

$$\tau_T(P_i, F_k) = \bigwedge_{s \in S} [\tau_T(P_i, T) \bigvee \tau_R(T, F_k)]$$

$$\Upsilon_T(P_i, F_k) = \bigwedge_{s \in S} [\Upsilon_T(P_i, T) \bigvee \Upsilon_R(T, F_k)].$$

After getting the information relationship of (cattle-farm→Criteria), we apply the following formula on the neutrosophic quaternion numbers to obtained the Table 7. $S_R = \xi_R - \Im_R.\Theta_R$ by following the rule of the quaternion numbers, $ij = k, jk = i, ki = j$.

Table 7. Results of relationship.

	$\hat{C}_1$	$\hat{C}_2$	$\hat{C}_3$	$\hat{C}_4$
Forum-1	0.6	0.4	0.56	0.62
Forum-2	0.55	0.58	0.55	0.5
Forum-3	0.7	0.46	0.68	0.71
Forum-4	0.58	0.62	0.61	0.48

In this Table 7 we can find that, form cattle-farm-2 and cattle-farm-4, the best criteria is $\hat{C}_2$. For cattle-farm-1, $\hat{C}_4$ is best, and for cattle-farm-3 the best option is $\hat{C}_4$.

In particular, the comparison study was based on the same example situation, where the weight of the traits was the same as above. Table 8 below shows the results utilizing multiple approaches and complete weight data. We apply the single-valued neutrosophic probabilistic hesitant fuzzy Dombi weighted averaging and geometric (SV-NPHFDWA and SV-NPHFDWG) AOs [34], single-valued neutrosophic probabilistic hesitant fuzzy weighted averaging and geometric (SV-NPHFWA and SV-NPHFWG) AOs [35], and single-valued neutrosophic probabilistic hesitant fuzzy rough weighted averaging and geometric (SV-NPHFRWA and SV-NPHFRWG) AOs [36].

Table 8. Comparison analysis.

Operators	Score	Most Effective Alternative
SV-NPHFDWA	Undefined	No Outcome
SV-NPHFDWG	Undefined	No Outcome
SV-NPHFWA	Undefined	No Outcome
SV-NPHFWG	Undefined	No Outcome
SV-NHFROWA (Defined)	Undefined	No Outcome
SV-NHFRHWA (Defined)	Undefined	No Outcome
SV-NHFRHWG (Defined)	Undefined	No Outcome
SV-NPHFRWA	$s(\Theta_1) > s(\Theta_3) > s(\Theta_2) > s(\Theta_4)$	$s(\Theta_1)$
SV-NPHFRWG	$s(\Theta_4) > s(\Theta_1) > s(\Theta_3) > s(\Theta_2)$	$s(\Theta_4)$
SV-NPHFREWA(*Proposed*)	$s(\Theta_1) > s(\Theta_4) > s(\Theta_3) > s(\Theta_2)$	$s(\Theta_1)$
SV-NPHFRWG (Defined)	$s(\Theta_1) > s(\Theta_3) > s(\Theta_4) > s(\Theta_2)$	$s(\Theta_1)$

The desirable alternative is therefore $s(\Theta_1)$ when the SV-NPHFREWA operator is employed. If another types of procedures is used, the best option is $s(\Theta_1)$. For all of the compared methods, the worst alternative is always $s(\Theta_2)$. The other operators failed to give output because our information had rough data that cannot be handled by other operators. The results show the effectiveness and viability of the SV-NPHFREWA and SV-NPHFREWG operators utilizing the compared methods and the suggested method in this paper for the identical single-valued neutrosophic hesitant fuzzy rough information.

6.2. Limitations of the present work

The SV-NPHFRE-AO) has a number of benefits for decision-making, but there are also certain drawbacks and concerns to take into account:

- SV-NPHFRE-AO is a complex method that requires a deep understanding of the underlying concepts and mathematical operations. This can make it challenging to apply in practice, particularly for decision-makers who are not familiar with advanced mathematical methods.
- SV-NPHFRE-AO requires a significant amount of data to be effective. This can be a limitation in decision-making situations where data is scarce or difficult to obtain.
- There is currently no standard approach to using SV-NPHFRE-AO in decision making, which can make it difficult to compare results across different studies or applications.
- SV-NPHFRE-AO is a subjective method that relies on human judgment to some extent. This can introduce bias and inconsistencies into the decision-making process, particularly if there are differences in opinion among decision-makers.
- SV-NPHFRE-AO can be difficult to interpret, particularly for non-experts. This can make it challenging to communicate the results of the decision-making process to stakeholders or other decision-makers.

- SV-NPHFRE-AO may not be scalable to large decision-making problems due to the complexity of the method and the amount of data required.

The limitations and issues of SV-NPHFREA in decision making should be carefully considered before applying the method. Decision-makers should ensure that they have the necessary expertise and data to use SV-NPHFREA effectively, and that they are aware of the limitations and potential biases inherent in the method.

7. Conclusions

We deal with significant and technical data every day. To work more efficiently and compute full information, we developed methodologies and tools for this type of data. Aggregation entails inherent costs to reduce the volume of data to a single value. For situations where each item has a range of probable values specified by MD, indeterminacy, and non-MD, the SV-NPHFRE-AO was created as a potent fusion of an SV-N, PHFS, and RS. More information about these techniques' advantages is provided below.

(1) First, we've established some SV-NPHFRS operational standards based on Einstein operations.

(2) Due to the significance of operating procedures, the weighted averaging and geometric Einstein aggregation operators have also been designed to deal with SV-NHFR data, including SV-NHFRA and SV-NHFRG aggregation data in decision-making issues.

(3) Then, we designed the the weighted averaging Einstein aggregation operators to deal with SV-NPHFR data, including sv-neutrosophic probabilistic hesitant fuzzy rough Einstein weighted averaging (SV-NPHFREWA) aggregation operator.

(4) A number of helpful characteristics have been demonstrated for them.

(5) The MCDM techniques proposed in this paper are also capable of recognizing more correlation between attributes and alternatives, demonstrating that they have a higher accuracy and a larger setpoint than the existing methodologies, which are unable to take into account the inter-relationships of attributes in practical uses. This shows that by using the MCDM methodologies outlined in this paper, even additional links between features may be discovered.

(6) Proposed AOs could be used in studies of two-sided matching decisions with multi-granular and incomplete criterion weight information, as well as studies of consensus reaching with non-cooperative behavior management. This analysis of the constraints imposed by proposed AOs does not take into account the degrees of participation, abstention, or non-membership. Over here, the intended AOs are being implemented with a novel hybrid structure of prioritized, interactive AOs.

(7) Future work will use state-of-the-art decision-making methods like TOPSIS, VIKOR, TODAM, GRA, and EDAS to examine the theoretical basis of SV-NPHFSs for Einstein operations. We will also talk about the ways in which these techniques are used in other areas, including soft computing, robotics, horticulture, intelligent systems, the social sciences, economics, and human resource management.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Contributions from each author are distributed equally.

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through Small Groups Project under grant number (R.G.P.1/109/44).

Conflicts of interest

There are no conflicting interests declared by the authors.

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