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Research article

Robustness analysis of exponential synchronization in complex dynamic networks with random perturbations

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Abstract: This article discusses the robustness of exponential synchronization (ESy) of complex dynamic networks (CDNs) with random perturbations. Using the Gronwall-Bellman lemma and partial inequality techniques, by solving the transcendental equation, the maximum perturbation intensity of the CDN is estimated. This implies that the disturbed system achieves ESy if the disturbance intensity is within the range of our estimation. We illustrate the theoretical results with two numerical examples.

Keywords: complex dynamic network; exponential synchronization; random perturbations; robustness
Mathematics Subject Classification: 93B35, 93C73

1. Introduction

With the development of modern information technology, more and more attention is paid to the research of complex dynamic networks (CDNs). CDNs are composed of the same or different nodes and edges. In recent years, CDNs have been extensively studied and used in various domains, such as physics [1], communication networks [2], biological engineering [3], etc., and typical applications of CDNs include the internet [4, 5], neural networks [6], cell networks [7] and so on.

Synchronization is a common nonlinear phenomenons, which widely exists in human society and nature. The history of synchronization can be traced back to Huygens's pendulum experiments in 1665 [8]. Synchronization has wide applications in many fields, such as artificial intelligence, secret communication, etc., [9–11]. At present, the synchronization types are roughly divided into full synchronization [12], cluster synchronization [13, 14], Mittag-Leffler synchronization [15, 16], projection synchronization [17], etc. Generally speaking, the convergence rate of exponential synchronization (ESy) is faster than that of asymptotic synchronization.

It should be noted that due to the diverse characteristics of CDNs, it is difficult for most CDNs to achieve synchronization by themselves, and the realization of synchronization still needs to rely on some external controls. Therefore, many researchers have designed some effective and feasible controllers to make CDNs achieve synchronization. According to the known literature, the controllers can be divided into nonlinear controllers [18, 19] and linear controllers [20–22]. In [23], Wu et al. designed a synchronization controller that provides a new judgment basis for ESy of CDNs. As we know, event triggering control strategy has been widely used in many fields, such as cyber-attacks [24–26]. Moreover, the event triggering strategy is used to study the prescribed-time synchronization of a piecewise smooth system in [27, 28].

However, in the process of CDN synchronization, time delays and random disturbances are inevitable in the process of network information transmission. The existence of time delays makes the communication between nodes of a CDN have information exchange delays, which will reduce or destroy the synchronization of the CDN [29, 30]. As for random disturbance, the perturbation process is quite complex. The existence of noise will lead to the deviation of information transmission in CDNs. For the disturbed CDNs, the synchronization phenomenon has attracted much attention in recent years [22, 31–33]. Dong et al. studied non-fragile synchronization of CDNs with mixed delays and random perturbations in [22]. Zhou et al. explored CDN synchronization involving random perturbations and other uncertainties in [31]. Zhang et al. studied a new global exponential synchronization criterion for CDNs with neutral terms and random perturbations in [32]. Wang et al. discussed ESy of CDNs with Markov jumps at uncertain transition rates and random interference in [33].

As we all know, CDNs may lose synchronization when they are interfered with by external noise, if the noise interference intensity exceeds a certain limit [34]. We know that CDNs may still have ESy if the intensity of noise interference is small when ESy has been achieved in CDNs [35]. In the literature [11–13, 17–23, 29, 31–38], there have been some interesting results on synchronization of CDNs disturbed by external noise. The Lyapunov stability theory and linear matrix inequality (LMI) technique have become the most powerful tools to study such complex systems. In addition, the robust stability of CDN synchronization was discussed in [39, 40]. It should be emphasized that the above literature focuses on the stability of synchronization rather than the robustness of synchronization. Hence, an interesting question is how much noise intensity can a CDN with ESy withstand without losing synchronization under the controller?

Robustness is the property that controls the system to maintain certain performance under the perturbation of certain parameters (structure, size). It is of great significance to the designs and applications of complex systems. What's more, in recent years, many interesting results have emerged in the research on the robustness of stability. For example, in [6], Shen et al. studied the robustness of the stability of recurrent neural networks for the first time. In [7, 41], Fang et al. discussed the robustness of fuzzy cellular neural networks with deviating argument. In [42–44], Si et al. explored the robustness of dynamical systems with piecewise constant parameters. However, although ESy of CDNs has been extensively studied, few works discuss the robustness of ESy. The robustness of ESy in CDNs with deviation arguments was first explored in [45].

Guided by the above works, this article discusses the robustness of ESy in CDNs interfered with by external noise. The main works and contents are as follows.

• Compared with [29, 32, 33], this paper considers for the first time the robustness of ESy of CDNs with random perturbations. This paper provides a judgment basis for ESy of CDNs with external random interference.

• In [45], the robustness of ESy in CDNs with deviation arguments was first considered. As far as we know, CDN is inevitably affected by deviation arguments and random disturbances in the process of information transmission. Therefore, by using the Gronwall-Bellman lemma and inequality technique, we obtain the upper bound of noise intensity that preserves ESy. By solving the transcendental equation, the allowable range of ESy in CDNs with external noise is estimated.

• In [6,41,42], the robustness of the stability of neural networks with random disturbances has been studied in depth. What is different is that we consider the robustness of ESy for CDNs with random perturbations.

The framework of this article is roughly as follows: Section 2 explains the preliminary knowledge and the models. Section 3 presents the main results. In Section 4, two simulations are given. Section 5 provides a brief summary of this paper.

2. Preliminaries and model

2.1. Notations

Unless otherwise specified, this article uses the following symbols. $\Re^N$ and $\Re^{N\times N}$ represent sets of N-dimensional Euclidean spaces and $N \times N$-dimensional matrices composed of spaces or sets, respectively. l_N is an identity matrix of order N, and 0_N represents an $N \times N$ zero matrix. Define $\theta \otimes \vartheta$ as the Kronecker product of matrices θ and ϑ. Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geqslant 0})$ be a complete probability space with filtering $\{\mathfrak{F}_t\}_{t\geqslant 0}$, and the filter contains all P-null sets and is right-continuous. $\omega(t)$ is a scalar Brownian motion defined in a complete probability space. $\| * \|$ means the Euclidean norm of a real vector. $\mathbb{E}\{\cdot\}$ stands for a mathematical expectation operator with respect to a given probability measure P.

Usually, a graph $G = (\mathcal{V}, \mathcal{E}, \mathfrak{A})$ has three basic elements. $\mathcal{V} = \{1, \dots, N\}$ shows the node set, and the edge set $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$. The coupled moment matrix is $\mathfrak{A} = (a_{ij})_{N\times N}$, where a_{ij} represents the coupling weight from the i-th node to the j-th node in the CDN. If the information is from the first j-th node of the CDN to the i-th node ($i \neq j$), then the $a_{ij} \neq 0$; otherwise, $a_{ij} = 0$.

2.2. Problem formulation

Consider the CDN model with random disturbance composed of N coupled nodes

$$\begin{cases} du_i(t) =[f(u_i(t)) + c\sum_{j=1}^{N} a_{ij}u_j(t) + v_i(t)]dt + \sigma\sum_{j=1}^{N} b_{ij}u_j(t)d\omega(t), \\ u_i(t_0) =u_0 \in \Re^N, \end{cases} \tag{2.1}$$

where $u_i(t) = (u_{i1}, \dots, u_{iN})^T \in \Re^N$ is the state vector of the i-th node. The nonlinear function $f : \Re^N \to \Re^N$ is a continuous vector-valued function, c is called the coupling strength, and $v_i(t) \in \Re^N$ is called the control input vector of the i-th node (the latter needed to design the controller). a_{ij}, b_{ij} respectively represent the coupling matrix $\mathfrak{A}, \mathfrak{B}$, satisfying $a_{ii} = -\sum_{j=1,j\neq i}^{N} a_{ij}$ and $b_{ii} = -\sum_{j=1,j\neq i}^{N} b_{ij}$. σ indicates random interference intensity.

The dynamic equation of a single node of CDN (2.1) is

$$\dot{s}(t) = f(s(t)), \tag{2.2}$$

where $s(t)$ is allowed to be defined by any expected state, i.e., equilibrium point, periodic orbit, chaotic attractor, etc.

When the system CDN (2.1) is undisturbed, it will further degrade to

$$\begin{cases} \dot{r}_i(t) = f(r_i(t)) + c\sum_{j=1}^{N} a_{ij}r_j(t) + v_i(t), \\ r_i(t_0) = r_0 \in \Re^N. \end{cases} \tag{2.3}$$

Similarly, (2.2) also represents the single node dynamics equation of the system (2.3).

The synchronization error of systems (2.1) and (2.2) is defined as

$$p_i(t) = u_i(t) - s(t).$$

Subtracting (2.2) from system (2.1) gives the error system

$$\begin{cases} dp_i(t) =[g(p_i(t)) + c\sum_{j=1}^{N} a_{ij}p_j(t) + v_i(t)]dt + \sigma\sum_{j=1}^{N} b_{ij}p_j(t)d\omega(t), \\ p_i(t_0) =p_0 \in \Re^N, \end{cases} \tag{2.4}$$

where $g(p_i(t)) = f(u_i(t)) - f(s(t))$. The function g satisfies the following assumptions.

($\mathfrak{M}$**1**) The nonlinear function $g(*)$ will satisfy the following condition:

$$\|g(\mu) - g(\nu)\| \leq k\|\mu - \nu\|,\ g(0) = 0, \tag{2.5}$$

where k is a constant.

To synchronize (2.1) and (2.2), the linear controller is

$$v_i(t) = \mathbb{W}p_i(t),\ i = 1, \ldots, N, \tag{2.6}$$

where $\mathbb{W} \in \Re^{N\times N}$ is the feedback gain matrix.

Substituting (2.6) into (2.4), we have

$$\begin{cases} dp_i(t) =[g(p_i(t)) + c\sum_{j=1}^{N} a_{ij}p_j(t) + \mathbb{W}p_i(t)]dt + \sigma\sum_{j=1}^{N} b_{ij}p_j(t)d\omega(t), \\ p_i(t_0) =p_0 \in \Re^N. \end{cases} \tag{2.7}$$

For the convenience of writing, the above component form is expressed as $p(t) = (p_1(t), p_2(t), \ldots, p_N(t)) \in \Re^{N\times N}$, $\mathfrak{A} = (a_{ij})_{N\times N}$, $\mathfrak{B} = (b_{ij})_{N\times N}$, $g(p(t)) = (g^T(p_1(t)), g^T(p_2(t)), \ldots, g^T(p_N(t)))^T$, $A = l_N \otimes \mathfrak{A}$, $B = l_N \otimes \mathfrak{B}$, $W = l_N \otimes \mathbb{W}$.

Then, the error system (2.7) is rewritten in matrix form as

$$\begin{cases} dp(t) =[g(p(t)) + cAp(t) + Wp(t)]dt + \sigma Bp(t)d\omega(t), \\ p(t_0) =p_0 \in \Re^{N\times N}. \end{cases} \tag{2.8}$$

Similarly, we define the synchronization error of systems (2.3) and (2.2) as

$$h_i(t) = r_i(t) - s(t)$$

and subtract (2.2) from (2.3) to get the error system

$$\begin{cases} \dot{h}(t) = g(h(t)) + cAh(t) + Wh(t), \\ h(t_0) = h_0 \in \Re^{N\times N}, \end{cases} \tag{2.9}$$

where $g(h(t)) = f(r(t)) - f(s(t))$. Similarly, the component form is $h(t) = (h_1(t), h_2(t), \ldots, h_N(t))^T$, $g(h(t)) = (g^T(h_1(t)), g^T(h_2(t)), \ldots, g^T(h_N(t)))^T$, and the remaining symbols are the same as above.

For convenience, we express exponential stability is ESt. Mean square exponential stability is MSESt. Exponential synchronization is ESy. Mean square exponential synchronization is MSESy. Almost sure exponential stability is ASESt. Almost sure exponential synchronization is ASESy.

Next, the definition of ESt of the error system (2.9) is given.

Definition 1. If the error system (2.9) has ESt, then the CDN (2.3) with (2.2) is said to have ESy. Namely, for any initial value $h_0 \in \Re^N$, there exist two scalars $\alpha > 0$, $\beta > 0$ such that for any $t \in \Re^+$, t satisfies

$$\|h(t)\| \le \alpha \|h_0\| \exp(-\beta(t - t_0)).$$

For the error system (2.8) subject to random interference, we have the following two definitions.

Definition 2. [46,47] CDN (2.8) is said to be ASESt if $\forall t_0 \in \Re_+$, $p_0 \in \Re^n$, the Lyapunov exponent

$$\limsup_{t\to\infty} \frac{1}{t}(\ln \|p_i(t, t_0, p_0)\|) < 0, \ i = 1, 2, \ldots, N.$$

Then, the CDN (2.7) with (2.2) is said to be ASESy if $\exists M > 0$ and scalar $\alpha > 0, \beta > 0$ such that

$$\|p_i(t)\| \le M \exp(-\alpha\beta(t - t_0)), \ i = 1, 2, \ldots, N,$$

where $M > 0$ is the constant related to the initial value state.

Definition 3. [33,46] CDN (2.8) is said to be MSESt, if $\forall t_0 \in \Re_+$, $p_0 \in \Re^n$, the Lyapunov exponent

$$\limsup_{t\to\infty} (\ln \frac{1}{t}(\mathbb{E}\|p_i(t, t_0, p_0)\|^2)) < 0, \ i = 1, 2, \ldots, N.$$

Then, the CDN (2.7) with (2.2) is said to be MSESy if $\exists G > 0$ and scalar $\alpha > 0, \beta > 0$ such that

$$\mathbb{E}\|h_i(t)\|^2 \le G \exp(-\alpha\beta(t - t_0)), \ i = 1, 2, \ldots, N,$$

where $G > 0$ is the constant related to the initial value state.

From Definitions 2 and 3, we can see that MSESt can be derived from ASESt, but vice versa is not true. However, if $\mathfrak{M}1$ is assumed to be true, the MSESt of the error system (2.8) implies the ASESt of the error system CDN (2.8) [46].

3. Main results

Under assumption $\mathfrak{M}1$, the error system (2.8) has an independent state $p_i(t, t_0, p_0)$ on $t > t_0$ for any initial state (t_0, p_0). Obviously, $p = 0$ is the equilibrium point of the error system (2.8). For now, we are interested in the question of, once a CDN has achieved ESy under a controller, how much noise intensity the CDN can withstand without losing synchronization.

The controller is a set of rules or algorithms that send instructions to the system to bring the system to the desired state, and the control rules are established under the premise of ensuring the stability of the system.

As we know, controllers can be divided into linear controllers and nonlinear controllers. A linear controller is relatively simple in structure. So, we focus on nonlinear controllers based on linear controllers. In the following, we discuss the robustness of CDNs with external noise interference to maintain ESy a under linear and a nonlinear controller, respectively.

3.1. The linear feedback controller

Now, we will explore the effect of random disturbance on the ESt of the error system (2.8).

Theorem 1. If assumption $\mathfrak{M}1$ holds, and the error system (2.9) is ESt, then the error system (2.8) is ESt. That is to say, systems (2.1) and (2.2) can realize the ESy under the linear controller (2.6), if $|\sigma| < \bar{\sigma}$, where $\bar{\sigma}$ is is a unique positive solution of the following transcendental equation.

$$\frac{2\alpha^2\sigma^2\|B\|^2}{\beta(1-\varepsilon)}\exp\left\{4\Delta\left[\frac{3\Delta}{\varepsilon}(k^2+c^2\|A\|^2+\|W\|^2)+\frac{\sigma^2\|B\|^2}{1-\varepsilon}\right]\right\}+2\alpha^2\exp(-2\beta\Delta)=1, \tag{3.1}$$

where $\varepsilon \in (0, 1)$ is an adjustable parameter, and $\Delta > \ln(2\alpha^2)/(2\beta) > 0$.

Proof. For convenience, $h(t, t_0, h_0),\ p(t, t_0, p_0)$ are denoted by $h(t),\ p(t)$, respectively. Based on (2.8) and (2.9) and the initial value $h_0 = p_0$, one has

$$\begin{aligned} p(t)-h(t) = &\int_{t_0}^{t}[g(p(s))-g(h(s))+cA(p(s)-h(s))+W(p(s)-h(s))]ds \\ &-\int_{t_0}^{t}\sigma Bp(s)d\omega(s). \end{aligned} \tag{3.2}$$

Therefore, when $t \leq t_0 + 2\Delta$, according to the *Hölder* inequality, assumption $\mathfrak{M}1$ and ESt of the error system (2.9),

$$\begin{aligned} \mathbb{E}\|p(t)-h(t)\|^2 \leq &\frac{1}{\varepsilon}\mathbb{E}\left\|\int_{t_0}^{t}[g(p(s))-g(h(s))+cA(p(s)-h(s))+W(p(s)-h(s))]ds\right\|^2 \\ &+\frac{1}{1-\varepsilon}\mathbb{E}\left\|\int_{t_0}^{t}\sigma Bp(s)d\omega(s)\right\|^2 \\ \leq &\frac{1}{\varepsilon}\mathbb{E}\int_{t_0}^{t}1^2\,ds\int_{t_0}^{t}\left\|[g(p(s))-g(h(s))+cA(p(s)-h(s))+W(p(s)-h(s))]\right\|^2ds \\ &+\frac{\sigma^2\|B\|^2}{1-\varepsilon}\int_{t_0}^{t}\mathbb{E}\left\|p(s)-h(s)+h(s)\right\|^2ds \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2\Delta}{\varepsilon} \int_{t_0}^{t} \mathbb{E}\Big\|\big[g(p(s)) - h(p(s)) + cA(p(s) - h(s)) + W(p(s) - h(s))\big]\Big\|^2 ds \\
&\quad + \frac{\sigma^2\|B\|^2}{1-\varepsilon} \int_{t_0}^{t} \mathbb{E}\Big\|p(s) - h(s) + h(s)\Big\|^2 ds \\
&\leq \Big[\frac{6\Delta}{\varepsilon}\Big(k^2 + c^2\|A\|^2 + \|W\|^2\Big) + \frac{2\sigma^2\|B\|^2}{1-\varepsilon}\Big] \int_{t_0}^{t} \mathbb{E}\|p(s) - h(s)\|^2 ds \\
&\quad + \frac{2\sigma^2\|B\|^2}{1-\varepsilon} \int_{t_0}^{t} \mathbb{E}\|h(s)\|^2 ds \\
&\leq \Big[\frac{6\Delta}{\varepsilon}\Big(k^2 + c^2\|A\|^2 + \|W\|^2\Big) + \frac{2\sigma^2\|B\|^2}{1-\varepsilon}\Big] \int_{t_0}^{t} \mathbb{E}\|p(s) - h(s)\|^2 ds \\
&\quad + \frac{\alpha^2\sigma^2\|B\|^2\|h_0\|^2}{(1-\varepsilon)\beta}.
\end{aligned} \tag{3.3}
$$

When $t_0 + \Delta \leq t \leq t_0 + 2\Delta$, using the Gronwall-Bellman inequality [46],

$$
\begin{aligned}
\mathbb{E}\|p(t) - h(t)\|^2 &\leq \frac{\alpha^2\sigma^2\|B\|^2\|h_0\|^2}{(1-\varepsilon)\beta} \exp\Big\{\Big[\frac{6\Delta}{\varepsilon}\Big(k^2 + c^2\|A\|^2 + \|W\|^2\Big) + \frac{2\sigma^2\|B\|^2}{1-\varepsilon}\Big](t - t_0)\Big\} \\
&\leq \frac{\alpha^2\sigma^2\|B\|^2}{(1-\varepsilon)\beta} \exp\Big\{4\Delta\Big[\frac{3\Delta}{\varepsilon}\Big(k^2 + c^2\|A\|^2 + \|W\|^2\Big) + \frac{\sigma^2\|B\|^2}{1-\varepsilon}\Big]\Big\} \\
&\quad \times \Big(\sup_{t_0 \leq t \leq t_0+\Delta} \mathbb{E}\|h(t)\|^2\Big).
\end{aligned} \tag{3.4}
$$

For $t_0 + \Delta \leq t \leq t_0 + 2\Delta$, with ESt of (3.4) and (2.9), one has

$$
\begin{aligned}
\mathbb{E}\|p(t)\|^2 &= \mathbb{E}\|p(t) - h(t) + h(t)\|^2 \\
&\leq 2\mathbb{E}\|p(t) - h(t)\|^2 + 2\mathbb{E}\|h(t)\|^2 \\
&\leq \frac{2\alpha^2\sigma^2\|B\|^2}{(1-\varepsilon)\beta} \exp\Big\{4\Delta\Big[\frac{3\Delta}{\varepsilon}\Big(k^2 + c^2\|A\|^2 + \|W\|^2\Big) + \frac{\sigma^2\|B\|^2}{1-\varepsilon}\Big]\Big\}\Big(\sup_{t_0 \leq t \leq t_0+\Delta} \mathbb{E}\|h(t)\|^2\Big) \\
&\quad + 2\alpha^2\|h_0\|^2 \exp(-2\beta(t - t_0))) \\
&\leq \Big\{\frac{2\alpha^2\sigma^2\|B\|^2}{(1-\varepsilon)\beta} \exp\Big\{4\Delta\Big[\frac{3\Delta}{\varepsilon}\Big(k^2 + c^2\|A\|^2 + \|W\|^2\Big) + \frac{\sigma^2\|B\|^2}{1-\varepsilon}\Big]\Big\} + 2\alpha^2 \exp(-2\beta\Delta)\Big\} \\
&\quad \times \Big(\sup_{t_0 \leq t \leq t_0+\Delta} \mathbb{E}\|h(t)\|^2\Big).
\end{aligned} \tag{3.5}
$$

By (3.1), when $|\sigma| < \bar{\sigma}$,

$$
T < 1,
$$

where

$$
T = \frac{2\alpha^2\sigma^2\|B\|^2}{(1-\varepsilon)\beta} \exp\Big\{4\Delta\Big[\frac{3\Delta}{\varepsilon}\Big(k^2 + c^2\|A\|^2 + \|W\|^2\Big) + \frac{\sigma^2\|B\|^2}{1-\varepsilon}\Big]\Big\} + 2\alpha^2 \exp(-2\beta\Delta).
$$

When $t_0 + \Delta \leq t \leq t_0 + 2\Delta$, let

$$
\frac{\partial T}{\partial \varepsilon} = 0,
$$

that is,

$$\begin{aligned}\frac{\partial T}{\partial \varepsilon} =&\frac{2\alpha^2\sigma^2\|B\|^2}{\beta}\Big[\frac{1}{(1-\varepsilon)^2}-\frac{12\Delta^2}{\varepsilon^2(1-\varepsilon)}\Big(k^2+c^2\|A\|^2+\|W\|^2\Big)+\frac{4\Delta\sigma^2\|B\|^2}{(1-\varepsilon)^3}\Big]\\ &\times\exp\Big\{4\Delta\Big[\frac{3\Delta}{\varepsilon}\Big(k^2+c^2\|A\|^2+\|W\|^2\Big)+\frac{\sigma^2\|B\|^2}{1-\varepsilon}\Big]\Big\}.\end{aligned} \tag{3.6}$$

From (3.6),

$$\begin{aligned}&\varepsilon^3-\Big[1-12\Delta^2(k^2+c^2\|A\|^2+\|W\|^2)+4\Delta\sigma^2\|B\|^2\Big]\varepsilon^2-24\Delta^2(k^2+c^2\|A\|^2+\|W\|^2)\varepsilon\\ &\quad+12\Delta^2(k^2+c^2\|A\|^2+\|W\|^2)=0.\end{aligned} \tag{3.7}$$

Then, (3.7) has a real root ε_0, and substituting $\varepsilon=\varepsilon_0$ into T, we has that T is a strictly monotonically increasing function of σ. So, when $\varepsilon=\varepsilon_0\in(0,1)$, Eq (3.1) have unique solution $|\bar{\sigma}|=\sigma_{\max}$.
Let

$$\Gamma=-\ln T/\Delta.$$

So, $\Gamma>0$, and from (3.6), one has

$$\sup_{t_0+\Delta\le t\le t_0+2\Delta}\mathbb{E}\|p(t)\|^2\le\exp(-\Gamma\Delta)\Big(\sup_{t_0\le t\le t_0+\Delta}\mathbb{E}\|p(t)\|^2\Big). \tag{3.8}$$

According to the above analysis, the curves of solutions starting from different initial values are the same. Therefore, for any positive integer $\eta=1,2,\ldots$, when $t\ge t_0+(\eta-1)\Delta$, one has

$$p(t,t_0,h_0)=p(t,t_0+(\eta-1)\Delta,p(t_0+(\eta-1)\Delta,t_0,h_0)). \tag{3.9}$$

From (3.8) and (3.9),

$$\begin{aligned}\sup_{t_0+\eta\Delta\le t\le t_0+(\eta+1)\Delta}\mathbb{E}\|p(t,t_0,p_0)\|^2 =&\Big(\sup_{t_0+(\eta-1)\Delta+\Delta\le t\le t_0+(\eta-1)\Delta+2\Delta}\mathbb{E}\|p(t,t_0+(\eta-1)\Delta,p(t_0+(\eta-1)\Delta,t_0,p_0))\|^2\Big)\\ \le&\exp(-\Gamma\Delta)\Big(\sup_{t_0+(\eta-1)\Delta\le t\le t_0+\eta\Delta}\mathbb{E}\|p(t,t_0,p_0)\|^2\Big)\\ &\cdots\\ \le&\exp(-\Gamma\eta\Delta)\Big(\sup_{t_0\le t\le t_0+\Delta}\mathbb{E}\|p(t,t_0,p_0)\|^2\Big)\\ =:&\,G\exp(-\Gamma\eta\Delta),\end{aligned}$$

where $G=\sup_{t_0\le t\le t_0+\Delta}\mathbb{E}\|p(t,t_0,p_0)\|^2$, so for any $t>t_0+\Delta$, there is a positive integer η where $t_0+\eta\Delta\le t\le t_0+(\eta+1)\Delta$. We have

$$\begin{aligned}\mathbb{E}\|p(t,t_0,p_0)\|^2\le&\,G\exp(-\Gamma t+\Gamma t_0+\Gamma\Delta)\\ =&\,(G\exp(\Gamma\Delta))\exp(-\Gamma(t-t_0)).\end{aligned} \tag{3.10}$$

This is also true when $t_0\le t\le t_0+\Delta$. Thus, the error system (2.8) is ASESt.

Corollary 1. When $\varepsilon=1/2$, the error system (2.8) is MSESt and also ASESt, if $|\sigma|<\bar{\sigma}$, where $\bar{\sigma}$ is the only positive root of the following transcendental equation.

$$\frac{4\alpha^2\sigma^2\|B\|^2}{\beta}\exp\Big\{4\Delta\Big[6\Delta(k^2+c^2\|A\|^2+\|W\|^2)+2\sigma^2\|B\|^2\Big]\Big\}+2\alpha^2\exp(-2\beta\Delta)=1,$$

where $\Delta>\ln(2\alpha^2)/(2\beta)>0$.

Remark 1. Theorem 1 states that the error system (2.9) is ESt, and when the perturbation intensity σ is in the range we derive, the systems (2.1) and (2.2) under a linear controller (2.6) can maintain ESy.

The proof of Theorem 1 is finished.

3.2. The nonlinear feedback controller

Similar (2.6), the nonlinear controller is

$$v_i(t) = -f(u_i(t)) + f(s(t)) + \bar{R}p_i(t),\ i = 1, \ldots, N, \tag{3.11}$$

where $\bar{R} \in \Re^{N\times N}$ represents the feedback gain matrix.

Substitute (3.11) into (2.4), and one has

$$\begin{cases} dp_i(t) =[c \sum_{j=1}^{N} a_{ij}p_j(t) + \bar{R}p_i(t)]dt + \sigma \sum_{j=1}^{N} b_{ij}p_j(t)d\omega(t), \\ p_i(t_0) =p_0 \in \Re^N, \end{cases} \tag{3.12}$$

where $R = I_N \otimes \bar{R}$, and the remaining symbols are the same as in (2.7). Then, the system (3.12) is rewritten in matrix form as

$$\begin{cases} dp(t) = [cAp(t) + Rp(t)]dt + \sigma Bp(t)d\omega(t), \\ p(t_0) = p_0 \in \Re^{N\times N}. \end{cases} \tag{3.13}$$

The error system (3.13) without disturbance will become

$$\begin{cases} \dot{h}(t) = cAh(t) + Rh(t), \\ h(t_0) = h_0 \in \Re^{N\times N}. \end{cases} \tag{3.14}$$

Based on assumption $\mathfrak{M}1$, we explore the influence on the ESt of the error system (3.13) with random perturbation.

Theorem 2. If assumption $\mathfrak{M}1$ holds, and the error system (3.14) is ESt, then the error system (3.13) is ESt. That is, the systems (2.1) and (2.2) can realize the ESy under the nonlinear controller (3.11), if $|\sigma| < \bar{\sigma}$, where $\bar{\sigma}$ is the only positive solution of the following transcendental equation.

$$\frac{\alpha^2\sigma^2\|B\|^2}{\beta(1-\lambda)} \exp\left\{4\Delta\left[\frac{2\Delta}{\lambda}(c^2\|A\|^2 + \|R\|^2) + \frac{\sigma^2\|B\|^2}{1-\lambda}\right]\right\} + 2\alpha^2 \exp(-2\beta\Delta) = 1, \tag{3.15}$$

where λ is an adjustable parameter and $\lambda \in (0, 1), \Delta > \ln(2\alpha^2)/(2\beta) > 0$.

Proof. Define $h(t) = p(t, t_0, p_0),\ p(t) = p(t, t_0, p_0)$, and the differential equations of systems (3.13) and (3.14) are converted into integral equations . The initial value $h_0 = p_0$, and we have

$$p(t) - h(t) = \int_{t_0}^{t} [cA(p(s) - h(s)) + R(p(s) - h(s))]ds - \int_{t_0}^{t} \sigma Bp(s)d\omega(s). \tag{3.16}$$

Therefore, when $t \le t_0 + 2\Delta$, according to the *Hölder* inequality, assumption $\mathfrak{M}1$ and the ESt of system (3.14),

$$\begin{aligned}
\mathbb{E}\|p(t)-h(t)\|^2 =&\mathbb{E}\Big\| \int_{t_0}^{t} \Big[cA(p(s)-h(s)) + R(p(s)-h(s))\Big]ds - \int_{t_0}^{t} \sigma Bp(s)d\omega(s)\Big\|^2 \\
\le&\frac{1}{\lambda}\mathbb{E}\Big\| \int_{t_0}^{t} \Big[cA(p(s)-h(s)) + R(p(s)-h(s))\Big]ds\Big\|^2 \\
&+ \frac{1}{1-\lambda}\mathbb{E}\Big\| \int_{t_0}^{t} \sigma Bp(s)d\omega(s)\Big\|^2 \\
\le&\frac{1}{\lambda}\mathbb{E}\int_{t_0}^{t} 1^2\, ds \int_{t_0}^{t} \Big\|cA(p(s)-h(s)) + R(p(s)-h(s))\Big\|^2 ds \\
&+ \frac{\sigma^2\|B\|^2}{1-\lambda}\int_{t_0}^{t}\mathbb{E}\Big\|p(s)-h(s)+h(s)\Big\|^2 ds \\
\le&\frac{2\Delta}{\lambda}\int_{t_0}^{t}\mathbb{E}\Big\|cA(p(s)-h(s)) + R(p(s)-h(s))\Big\|^2 ds \\
&+ \frac{2\sigma^2\|B\|^2}{1-\lambda}\int_{t_0}^{t}\mathbb{E}\|p(s)-h(s)\|^2 + \mathbb{E}\|h(s)\|^2 ds \\
\le&\Big[\frac{4\Delta}{\lambda}(c^2\|A\|^2 + \|R\|^2) + \frac{2\sigma^2\|B\|^2}{1-\lambda}\Big]\int_{t_0}^{t}\mathbb{E}\|p(s)-h(s)\|^2 ds \\
&+ \frac{2\sigma^2\|B\|^2}{1-\lambda}\int_{t_0}^{t}\mathbb{E}\|h(s)\|^2 ds \\
\le&\Big[\frac{4\Delta}{\lambda}(c^2\|A\|^2 + \|R\|^2) + \frac{2\sigma^2\|B\|^2}{1-\lambda}\Big]\int_{t_0}^{t}\mathbb{E}\|p(s)-h(s)\|^2 ds \\
&+ \frac{\alpha^2\sigma^2\|B\|^2\|h(t_0)\|^2}{\beta(1-\lambda)}.
\end{aligned} \tag{3.17}$$

For $t_0 + \Delta \le t \le t_0 + 2\Delta$, using the Gronwall-Bellman inequality [46],

$$\begin{aligned}
\mathbb{E}\|p(t)-h(t)\|^2 \le&\frac{\alpha^2\sigma^2\|B\|^2|h(t_0)|^2}{\beta(1-\lambda)}\exp\Big\{\Big[\frac{4\Delta}{\lambda}(c^2\|A\|^2 + \|R\|^2) + \frac{2\sigma^2\|B\|^2}{1-\lambda}\Big](t-t_0)\Big\} \\
\le&\frac{\alpha^2\sigma^2\|B\|^2}{\beta(1-\lambda)}\exp\Big\{4\Delta\Big[\frac{2\Delta}{\lambda}(c^2\|A\|^2 + \|R\|^2) + \frac{\sigma^2\|B\|^2}{1-\lambda}\Big]\Big\} \\
&\times\Big(\sup_{t_0\le t\le t_0+\Delta}\mathbb{E}\|h(t)\|^2\Big).
\end{aligned} \tag{3.18}$$

So, when $t_0 + \Delta \le t \le t_0 + 2\Delta$, from (3.18), with the ESt of system (3.14), we have

$$\begin{aligned}
\mathbb{E}\|p(t)\|^2 =&\mathbb{E}\|p(t) - h(t) + h(t)\|^2 \\
\le&2\mathbb{E}\|p(t) - h(t)\|^2 + 2\mathbb{E}\|h(t)\|^2 \\
\le&\frac{\alpha^2\sigma^2\|B\|^2}{\beta(1-\lambda)} \exp\left\{4\Delta\left[\frac{2\Delta}{\lambda}(c^2\|A\|^2 + \|R\|^2) + \frac{\sigma^2\|B\|^2}{1-\lambda}\right]\right\}\left(\sup_{t_0\le t\le t_0+\Delta} \mathbb{E}\|h(t)\|^2\right) \\
&+ 2\alpha^2\|h(t_0)\|^2 \exp(-2\beta(t - t_0)) \\
\le&\left\{\frac{\alpha^2\sigma^2\|B\|^2}{\beta(1-\lambda)} \exp\left\{4\Delta\left[\frac{2\Delta}{\lambda}(c^2\|A\|^2 + \|R\|^2) + \frac{\sigma^2\|B\|^2}{1-\lambda}\right]\right\}\right. \\
&\left.+ 2\alpha^2 \exp(-2\beta\Delta)\right\}\left(\sup_{t_0\le t\le t_0+\Delta} \mathbb{E}\|h(t)\|^2\right).
\end{aligned} \tag{3.19}$$

By (3.15), when $|\sigma| < \bar{\sigma}$,

$$\mathfrak{L} < 1,$$

where

$$\mathfrak{L} =\frac{\alpha^2\sigma^2\|B\|^2}{\beta(1-\lambda)} \exp\left\{4\Delta\left[\frac{2\Delta}{\lambda}(c^2\|A\|^2 + \|R\|^2) + \frac{\sigma^2\|B\|^2}{1-\lambda}\right]\right\} + 2\alpha^2 \exp(-2\beta\Delta).$$

When $t_0 + \Delta \le t \le t_0 + 2\Delta$, let

$$\frac{\partial\mathfrak{L}}{\partial\lambda} = 0,$$

that is,

$$\begin{aligned}
\frac{\partial\mathfrak{L}}{\partial\lambda} =& \frac{\alpha^2\sigma^2\|B\|^2}{\beta}\left[\frac{1}{(1-\lambda)^2} - \frac{8\Delta^2}{\lambda^2(1-\lambda)}\left(c^2\|A\|^2 + \|R\|^2\right) + \frac{4\Delta\sigma^2\|B\|^2}{(1-\lambda)^3}\right] \\
&\times \exp\left\{4\Delta\left[\frac{2\Delta}{\lambda}\left(c^2\|A\|^2 + \|R\|^2\right) + \frac{\sigma^2\|B\|^2}{1-\lambda}\right]\right\}.
\end{aligned} \tag{3.20}$$

From (3.20),

$$\begin{aligned}
&\lambda^3 - \left[1 - 8\Delta^2(c^2\|A\|^2 + \|R\|^2) + 4\Delta\sigma^2\|B\|^2\right]\lambda^2 - 16\Delta^2(c^2\|A\|^2 + \|R\|^2)\lambda \\
&+ 8\Delta^2(c^2\|A\|^2 + \|R\|^2) = 0.
\end{aligned} \tag{3.21}$$

For (3.21) there is a real root λ_0, and substituting $\lambda = \lambda_0$ into $\mathfrak{L}$, we get that $\mathfrak{L}$ is a strictly monotonically increasing function of σ. So, when $\lambda = \lambda_0 \in (0, 1)$, Eq (3.15) has unique root $|\bar{\sigma}| = \sigma_{\max}$.

Let

$$\Lambda = -\ln \mathfrak{L}/\Delta.$$

Hence, $\Lambda > 0$, and from (3.19),

$$\sup_{t_0+\Delta\le t\le t_0+2\Delta} \mathbb{E}\|p(t)\|^2 \le \exp(-\Lambda\Delta)\left(\sup_{t_0\le t\le t_0+\Delta} E\|p(t)\|^2\right). \tag{3.22}$$

Furthermore, the curves of solutions starting from different initial values are the same, so for any positive integer $\varrho = 1, 2, \ldots$, when $t \ge t_0 + (\varrho - 1)\Delta$, we have

$$p(t, t_0, p_0) = p(t, t_0 + (\varrho - 1)\Delta, p(t_0 + (\varrho - 1)\Delta, t_0, p_0)). \tag{3.23}$$

From (3.22) and (3.23), we have

$$\begin{aligned}\sup_{t_0+\varrho\Delta\leq t\leq t_0+(\varrho+1)\Delta}\mathbb{E}\|p(t,t_0,p_0)\|^2 =&\Big(\sup_{t_0+(\varrho-1)\Delta+\Delta\leq t\leq t_0+(\varrho-1)\Delta+2\Delta}\mathbb{E}\|p\big(t,t_0+(\varrho-1)\Delta,h(t_0+(\varrho-1)\Delta,t_0,p_0)\big)\|^2\Big)\\ &\leq \exp(-\Lambda\Delta)\Big(\sup_{t_0+(\varrho-1)\Delta\leq t\leq t_0+\varrho\Delta}\mathbb{E}\|p(t,t_0,p_0)\|^2\Big)\\ &\quad\cdots\\ &\leq \exp(-\Lambda\varrho\Delta)\Big(\sup_{t_0\leq t\leq t_0+\Delta}\mathbb{E}\|p(t,t_0,p_0)\|^2\Big)\\ &=:\mathcal{G}\exp(-\Lambda\varrho\Delta),\end{aligned}$$

where $\mathcal{G}=\sup_{t_0\leq t\leq t_0+\Delta}\mathbb{E}\|p(t,t_0,p_0)\|^2$, so for any $t>t_0+\Delta$, we have a positive integer ϱ such that $t_0+\varrho\Delta\leq t\leq t_0+(\varrho+1)\Delta$. We have

$$\begin{aligned}\mathbb{E}\|p(t,t_0,p_0)\|^2 &\leq \mathcal{G}\exp(-\Lambda t+\Lambda t_0+\Lambda\Delta)\\ &=(\mathcal{G}\exp(\Lambda\Delta))\exp(-\Lambda(t-t_0)).\end{aligned}\tag{3.24}$$

Obviously, (3.24) also holds for $t_0\leq t\leq t_0+\Delta$. Similar to Corollary 1, when $\lambda=1/2$, if $|\sigma|<\bar{\sigma}$,

$$\frac{2\alpha^2\sigma^2\|B\|^2}{\beta}\exp\left\{8\Delta\Big[2\Delta(c^2\|A\|^2+\|R\|^2)+\sigma^2\|B\|^2\Big]\right\}+2\alpha^2\exp(-2\beta\Delta)=1,$$

where $\bar{\sigma}$ is the solution to the transcendental equation, and $\Delta>\ln(2\alpha^2)/(2\beta)>0$.

The proof of Theorem 2 is finished.

Remark 2. Theorem 2 states that when the error system (3.13) is ESt, the disturbed system (3.14) can remain ESt as long as the perturbation intensity σ is less than the upper bound of our estimate. Then, the systems (2.1) and (2.2) subject to random perturbation still maintain ESy under the nonlinear controller (3.11).

Remark 3. According to Theorems 1 and 2, the maximum upper bound of the disturbed intensity is estimated by solving the transcendental equation with MATLAB. If the intensity of the disturbance is less than the bounds estimated in this paper, the CDN is ESt. Then, systems (2.1) and (2.2) are ESy under the controller.

In Table 1, we represent Random perturbations as RP, Deviating argument as DA and Asymptotic synchronization as ASSy, respectively.

Table 1. Other articles are compared with this article.

	Controller	**RP**	**DA**	**ESy**	**ASSy**	**Robustness**
Wang (2018) [33]	✓	✓	-	✓	-	-
Zhang (2018) [29]	✓	✓	-	✓	-	-
Shen (2016) [20]	impulsive control	-	✓	-	✓	-
Lia (2022) [45]	✓	-	✓	✓	-	✓
This paper	✓	✓	-	✓	-	✓

4. Examples

In this section, we will provide some examples to illustrate the validity of the results.

Example 1. We will consider a CDN composed of four nodes under linear cotroller

$$\begin{cases} \dot{r}_i(t) = g(r_i(t)) + c\sum_{j=1}^{4} a_{ij} r_j(t) + v_i(t), \\ r_i(t_0) = r_0 \in \Re^N, \\ \dot{s}(t) = g(s(t)), \\ v_i(t) = \mathbb{W} p_i(t). \end{cases} \tag{4.1}$$

When the error system (4.1) is subjected to random perturbation, it will become the following form:

$$\begin{cases} dr_i(t) =[g(r_i(t)) + c\sum_{j=1}^{4} a_{ij} r_j(t) + v_i(t)]dt + \sigma\sum_{j=1}^{4} b_{ij} r_j(t) d\omega(t), \\ r_i(t_0) =r_0 \in \Re^N, \\ \dot{s}(t) =g(s(t)), \\ v_i(t) =\mathbb{W} p_i(t). \end{cases} \tag{4.2}$$

Let $p_i(t) = r_i(t) - s(t)$, where $i \in 1, 2, 3, 4$, and $r_i(t) = (r_{i1}(t), r_{i2}(t), r_{i3}(t), r_{i4}(t))^T \in \Re^4$ is the state vector of the i-th node of the CDN. σ is the system disturbance intensity, and $\omega(t)$ is a scalar Brownian motion defined in a complete probability space.

In addition, the error systems (4.1) and (4.2) are respectively written in matrix form:

$$\begin{cases} \dot{p}(t) = g(p(t)) + cAp(t) + Wp(t), \\ p(t_0) = p_0 \in \Re^{N\times N}, \end{cases} \tag{4.3}$$

and

$$\begin{cases} dp(t) =[g(p(t)) + cAp(t) + Wp(t)]dt + \sigma Bp(t)d\omega(t), \\ p(t_0) =p_0 \in \Re^{N\times N}. \end{cases} \tag{4.4}$$

Consider the coupling matrices

$$A = \begin{pmatrix} -1 & 0.1 & 1 & -0.1 \\ -0.1 & -1 & 0.1 & 1 \\ 1 & -0.1 & -1 & 0.1 \\ 0.1 & 1 & -0.1 & -1 \end{pmatrix}, B = \begin{pmatrix} -0.2 & -0.1 & -0.2 & -0.1 \\ -0.1 & 0.2 & 0.1 & -0.2 \\ 0.05 & -0.01 & 0.05 & -0.01 \\ -0.1 & -0.2 & -0.1 & 0.2 \end{pmatrix},$$

$$W = \begin{pmatrix} -0.05 & 0 & 0 & -1 \\ 0 & 0.05 & 1 & 0 \\ 0 & -0.2 & -1 & 0 \\ 0.1 & 0 & 0 & -0.1 \end{pmatrix}.$$

Activation function $f(*) = \sin(*)$. Coupling strength is $c = 1.8$. The intensity of the disturbance coefficient is $\sigma = 0.052$. The initial values of the four-nodes and isolated node are $h_1 = (-0.1, 0.2, -0.1, 0.2)^T, h_2 = (-0.12, 0.11, 0.11, -0.12)^T, h_3 = (0.1, -0.2, 0.1, -0.2)^T, h_4 = (0.12, -0.11, -0.11, 0.12)^T$ and $s(t) = (0.1, -0.2, -0.1, 0.2)^T$ respectively.

According to Theorem 1, the error system (4.4) is ESt when $\alpha = 0.9$ and $\beta = 1.6$.

Choose $\Delta = 0.4 > \ln(1.62)/3.2 = 0.3859$, and by solving Eq (3.1),

$$\frac{0.162\sigma^2}{1.6(1-\varepsilon)} \exp\left(1.6\left[\frac{1.2}{\varepsilon}(k^2 + 4.264544) + \frac{0.0001^2\sigma^2}{1-\varepsilon}\right]\right) + 2 \times 0.81 \times \exp(-1.28) = 1, \tag{4.5}$$

where k is a given constant, we can get the curve of $(\sigma, \varepsilon(\lambda))$ as shown in Figure 4.

Remark 4. Figure 1 shows the state of CDN (2.1) undisturbed by external random noise. That is, CDNS (2.1) and (2.2) are ESy under a linear controller. When the interference intensity of external random noise is $\sigma = 0.09$, CDN (2.1) and (2.2) are ESy as shown in Figure 2. Figure 3 shows that when the random disturbance intensity is $\sigma = 0.5$, its solution is not in the curve shown in Figure 4, which does not meet the condition of Theorem 1. That is, CDN (2.1) and (2.2) are not ESy.

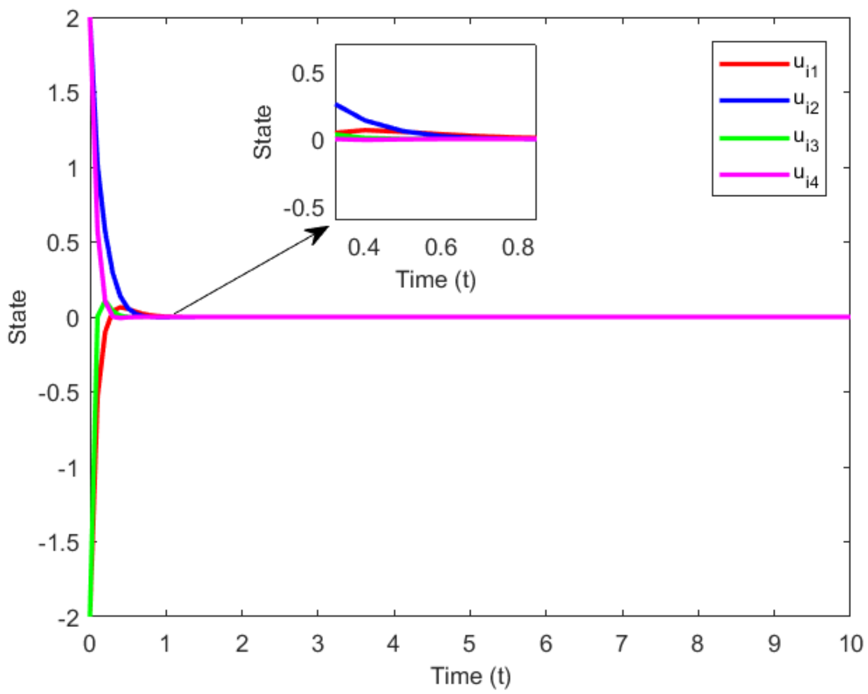

Figure 1. Undisturbed state of CDN (4.3) under linear control.

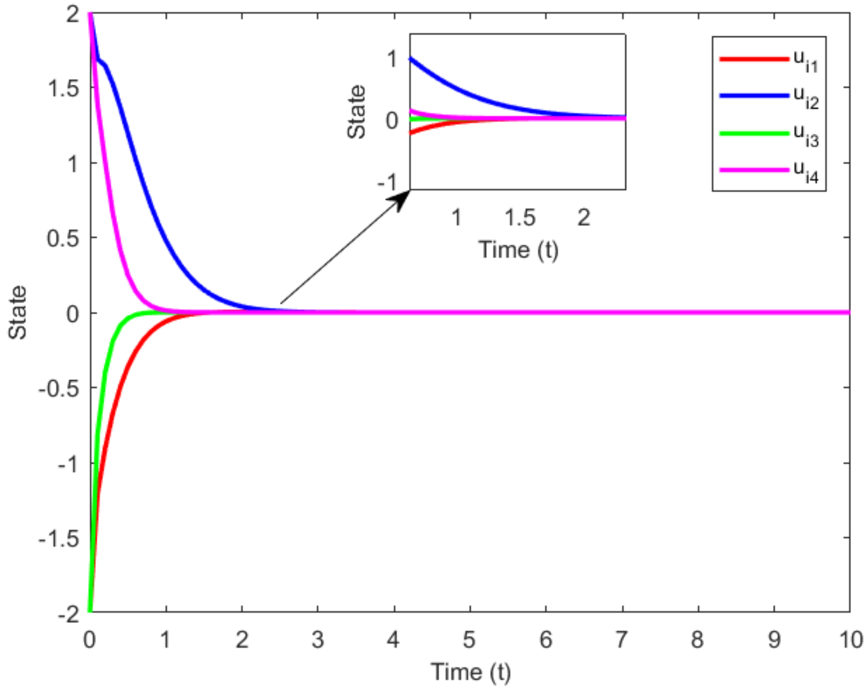

Figure 2. CDN (4.4) state with disturbance coefficient σ = 0.09 under linear controller.

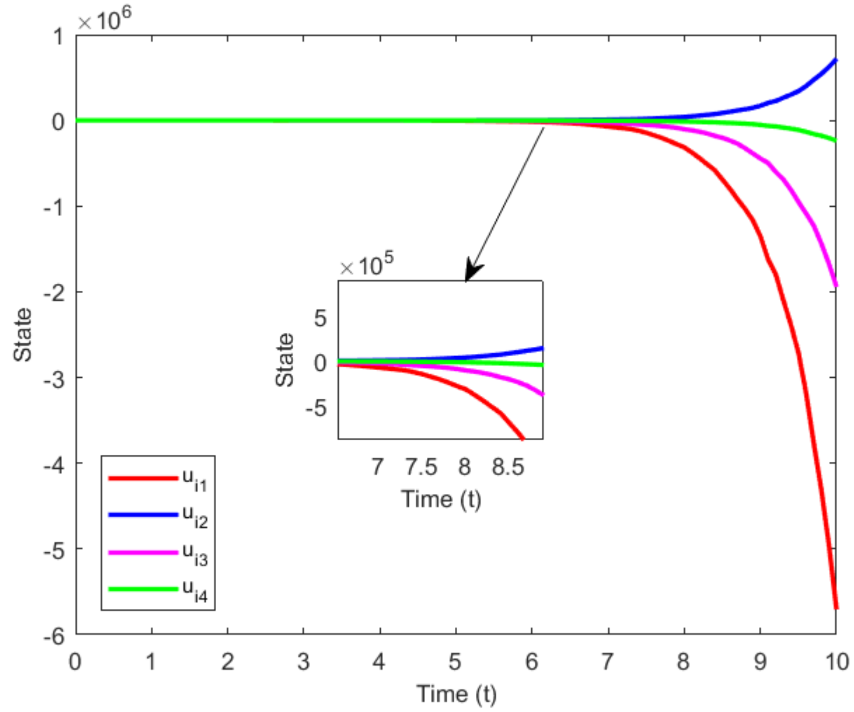

Figure 3. CDN (4.4) state with disturbance coefficient σ = 0.5 under linear controller.

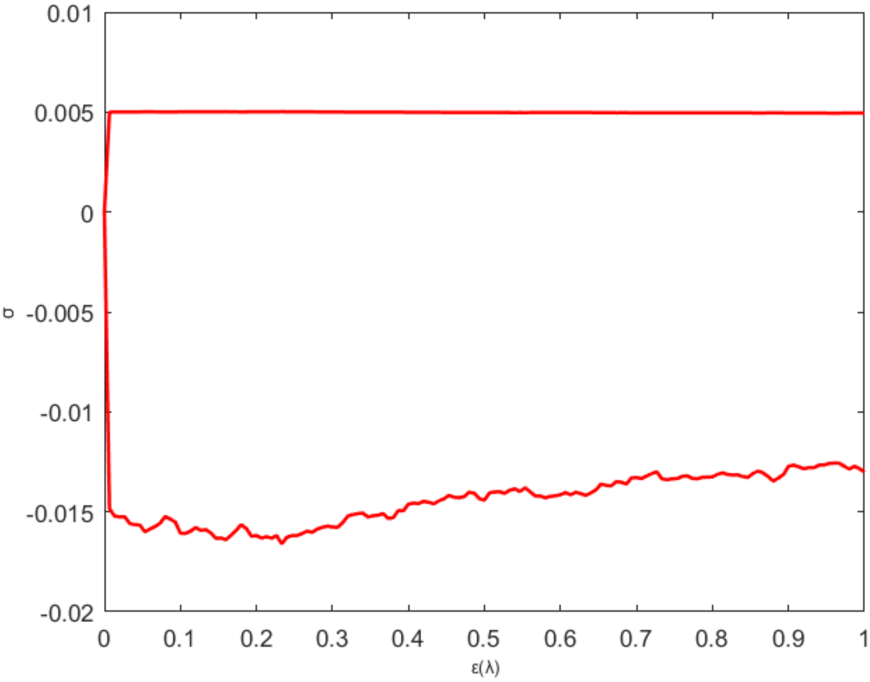

Figure 4. The stable region with $(\sigma,\varepsilon(\lambda))$ in CDN.

Example 2. We will consider a CDN composed of four nodes under nonlinear cotroller

$$\begin{cases} \dot{r}_i(t) = g(r_i(t)) + c\sum_{j=1}^{4} a_{ij} r_j(t) + v_i(t), \\ r_i(t_0) = r_0 \in \Re^N, \\ \dot{s}(t) = g(s(t)), \\ v_i(t) = -g(r_i(t)) + g(s(t)) + \bar{R}p_i(t). \end{cases} \tag{4.6}$$

The disturbed error system (4.6) will become

$$\begin{cases} \dot{r}_i(t) = [g(r_i(t)) + c\sum_{j=1}^{4} a_{ij} r_j(t) + v_i(t)]dt + \sigma \sum_{j=1}^{4} b_{ij} r_j(t) d\omega(t), \\ r_i(t_0) = r_0 \in \Re^N, \\ \dot{s}(t) = g(s(t)), \\ v_i(t) = -g(r_i(t)) + g(s(t)) + \bar{R}p_i(t). \end{cases} \tag{4.7}$$

Let $p_i(t) = r_i(t) - s(t)$, where $i \in 1,2,3,4$, and $r_i(t) = (r_{i1}(t), r_{i2}(t), r_{i3}(t), r_{i4}(t))^T \in \Re^4$ is the state vector of the i-th node of the CDN. σ is the system disturbance intensity, and $\omega(t)$ is a scalar Brownian motion defined in a complete probability space.

Moreover, the matrix forms of the error system (4.6) and the error system (4.7) are, respectively,

$$\begin{cases} \dot{p}(t) = cAp(t) + Rp(t), \\ p(t_0) = p_0 \in \Re^{N\times N}, \end{cases} \tag{4.8}$$

and

$$\begin{cases} dp(t) = [cAp(t) + Rp(t)]dt + \sigma Bp(t)d\omega(t), \\ p(t_0) = p_0 \in \Re^{N\times N}. \end{cases} \tag{4.9}$$

Consider the coupling matrices

$$A = \begin{pmatrix} 0.2 & 0.31 & 0 & -0.2 \\ -0.1 & 0.12 & 0 & 0 \\ 0 & 0 & -0.2 & 0.2 \\ 0 & -0.3 & 0.01 & 0.1 \end{pmatrix}, B = \begin{pmatrix} 0.01 & 0.2 & 0 & -0.1 \\ 1.2 & 0 & -1.3 & 0 \\ 0 & -0.1 & 0.2 & 0.2 \\ -0.1 & 0 & 1.2 & 1.3 \end{pmatrix},$$

$$R = \begin{pmatrix} -0.02 & 0 & 0.1 & -0.1 \\ -0.02 & 0 & 0 & -0.3 \\ 0.1 & -0.2 & -0.2 & 1 \\ 0.2 & -0.5 & 0 & -1 \end{pmatrix}.$$

The coupling strength is $c = 1.2$. Activation function $f(*) = \sin(*)$. The intensity of the disturbance coefficient is $\sigma = 0.013$. The initial values of the four-nodes and isolated node are $h_1 = (-1, 0.1, -0.1, -1)^T, h_2 = (0.2, -0.1, -0.1, 0.2)^T, h_3 = (1, -0.1, 0.1, -1)^T, h_4 = (-0.2, 0.1, 0.1, -0.2)^T$ and $s(t) = (0.2, 0.1, 0.2, -0.1)^T$ respectively.

According to Theorem 1, the error system (4.9) is ESt when $\alpha = 1.1$ and $\beta = 1.2$.

Choose $\Delta = 0.6 > \ln(2.42)/2.4 = 0.5303$, and by solving Eq (3.15),

$$\frac{1.21 \times 0.0035^2\sigma^2}{1.2(1-\varepsilon)} \exp\left(2.4\left[\frac{1.2}{\varepsilon}(1.44 \times 0.024^2 + 0.000009) + \frac{0.0035^2\sigma^2}{1-\varepsilon}\right]\right) + 2.42 \times \exp(-1.44) = 1, \tag{4.10}$$

where k is a given constant, we can get the curve of $(\sigma, \varepsilon(\lambda))$ as shown in Figure 4.

Remark 5. Figure 5 shows the state of CDN (2.6) undisturbed by random noise. That is, CDN (2.1) and (2.2) are ESy under a linear controller. When the interference intensity of external random noise is $\sigma = 0.013$, CDN (2.1) and (2.2) are ESy, as shown in Figure 6. Figure 7 shows that when the random disturbance intensity is $\sigma = 0.8$, its solution is not in the curve shown in Figure 3, which does not meet the condition of Theorem 2. That is, CDN (2.1) and (2.2) are not ESy.

Remark 6. (1) Step size Δ will directly affect the accuracy and stability of the numerical solution. In general, the smaller the step size is, the more accurate the numerical solution, but the amount of computation will also increase. Therefore, we choose the appropriate step size during the example process to increase the utility of our simulation examples.

(2) The initial value h_0, p_0 needs to be as close to the real solution as possible; otherwise, the numerical solution may diverge or converge to the wrong solution.

(3) The eigenvalues of the coefficient matrices A and B in Examples 1–4 and the control gain matrices W and R in the controllers should be negative real numbers, thus making the CDN system stable. Otherwise, the system may become unstable and fail to synchronize.

(4) The parameter size of coupling coefficient c needs to be adjusted according to the intensity of the interfered with CDN.

(5) The increment $d\omega$ of Brownian motion should be an independent identically distributed random variable subject to the standard normal distribution.

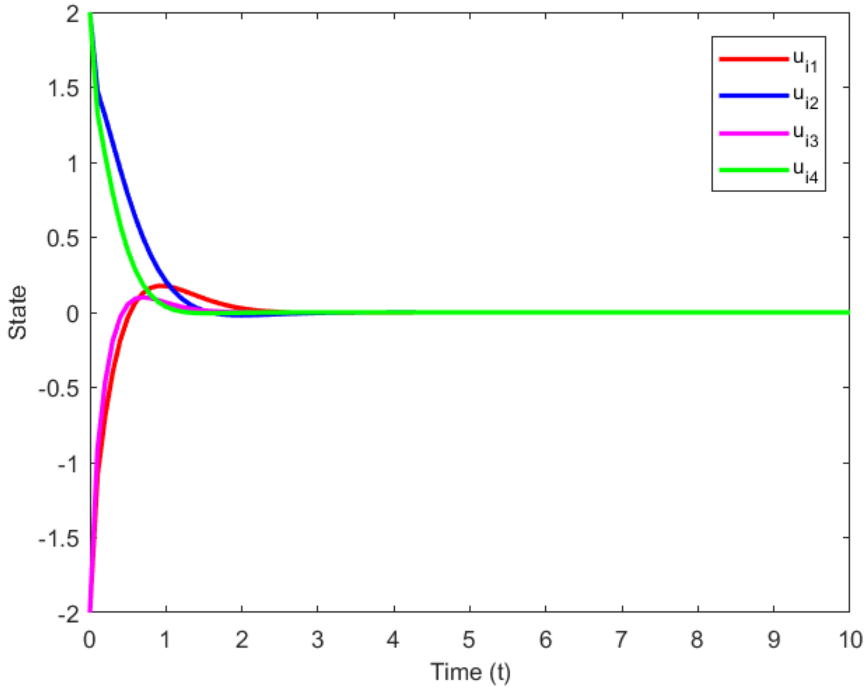

Figure 5. Undisturbed state of CDN (4.8) under linear control.

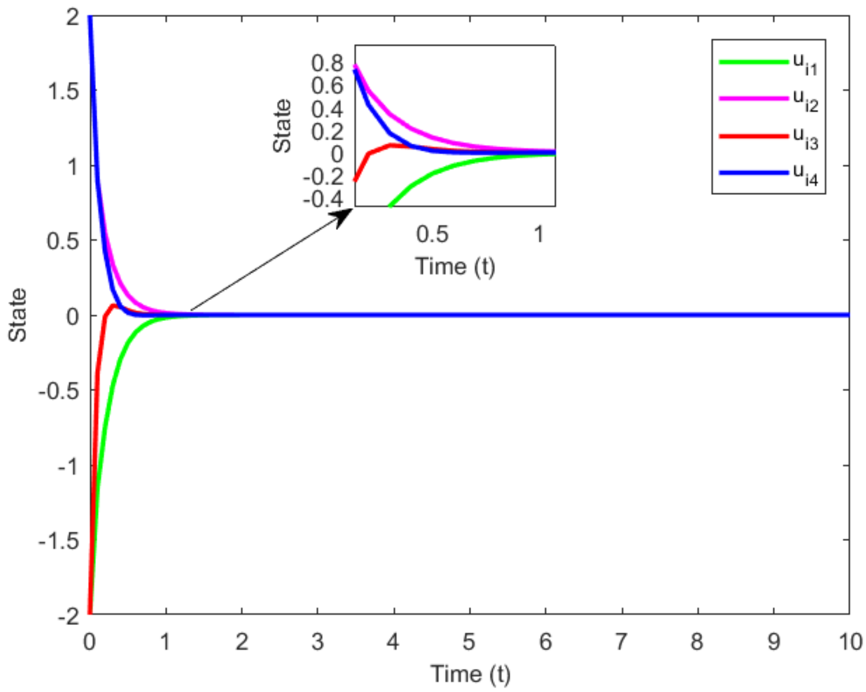

Figure 6. CDN (4.9) state with disturbance coefficient $\sigma = 0.013$ under linear controller.

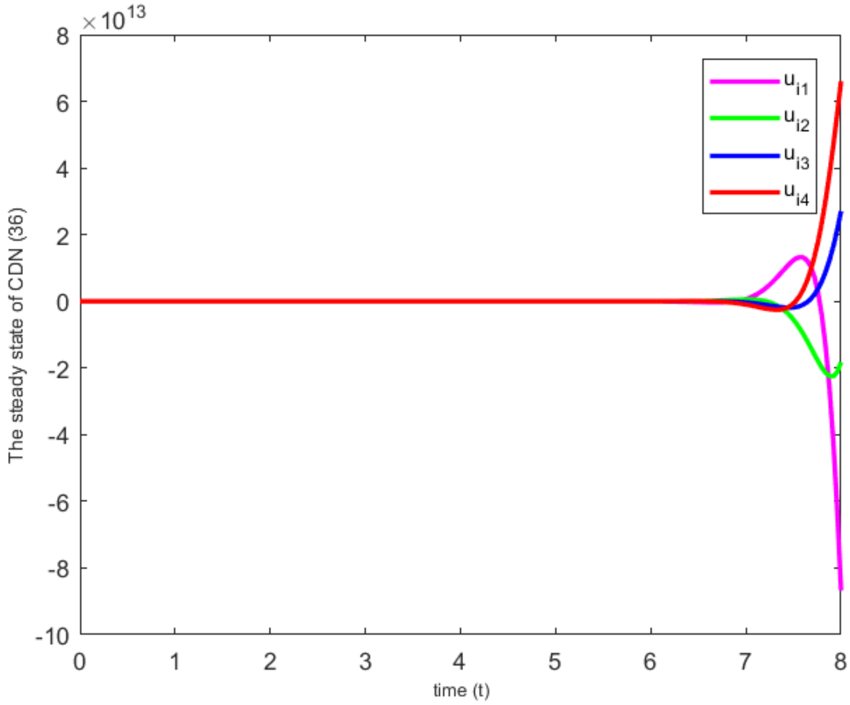

Figure 7. CDN (4.9) state with disturbance coefficient $\sigma = 0.8$ under linear controller.

Remark 7. In [6], the robustness of the stability of recurrent neural networks was studied for the first time. In [45], Lia et al. studied for the first time the robustness of ESy in CDNs with deviation parameters. According to [20, 31, 32, 34], we know that information of nodes in CDNs will inevitably be interfered with by external random noise during transmission. Therefore, it is a necessary research topic that CDNs can withstand the interference of external noise without losing synchronization. Compared with [45], this paper is a supplement to that work. The research results of this paper and [45] provide a theoretical basis for the analysis and design of CDNs.

5. Conclusions

This paper studies the robustness of ESy in CDNs with random perturbations. For CDNs that have achieved ESy under the controllers (linear and nonlinear controllers), how much the CDNs can withstand the interference of external noise without losing synchronization was the topic of this paper. Using the Gronwall-Bellman lemma and inequality techniques, the maximum intensity of CDN disturbance by external noise was estimated by solving transcendental equations, and the sufficient conditions for ESy of CDNs with random perturbations were obtained. The results show that systems (2.8) and (3.13) are ESt when the strength of the random disturbance is less than the upper bound we derived, and thus systems (2.1) and (2.2) are ESy. The results of this paper provide a theoretical reference for the design and application of ESy controllers in CDNs. Since the Lipschitz condition used is conservative, the derivation in this paper is a conservative bound. In the future, we will try to use the less conservative Lipschitz condition to expand the upper limit of the disturbance strength, such that the CDN is more stable and thus better able to achieve synchronization. In view of the methods and techniques analyzed in this paper, we will further consider the simultaneous effects of time delay and random perturbation on the robustness of ESy of CDNs in the next stage. The less conservative Lipschitz condition is used to analyze the upper bound of time delays and random disturbances, which makes the obtained results less conservative, and makes it easier to realize synchronization of CDNs or further consider other more complex models: For example, synchronization of fractional-order CDNs or impulsive fractional-order CDNs.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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