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Research article

The index of strong rotundity

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Abstract: The index of strong rotundity is introduced. This index is used to determine how far an element of the unit sphere of a real Banach space is from being a strongly exposed point of the unit ball. This index is computed for Hilbert spaces. Characterizations of the set of rotund points and the set of smooth points are provided for a better understanding of the construction of the index of strong rotundity. Finally, applications to the stereographic projection are provided.

Keywords: exposed point; rotund point; smooth point; Banach space; Hilbert space
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1. Introduction

The irruption of the moduli of convexity and smoothness [7] in the literature of the geometry of the real Banach spaces was a huge revolution that brought strong implications to longstanding open problems such as the Banach-Mazur conjecture for rotations or the fixed-point property problem. The main purpose of this manuscript is to introduce an index that measures a convexity stronger than strict convexity but weaker than uniform convexity. All Banach spaces considered throughout this manuscript will be over the reals. A point x in the unit sphere S_X of a Banach space X is said to be a strongly exposed point of the unit ball B_X if there exists x^* in the unit sphere S_{X^*} of the dual space X^* verifying the following property: If $(x_n)_{n\in\mathbb{N}} \subseteq \mathsf{B}_X$ is such that $(x^*(x_n))_{n\in\mathbb{N}}$ converges to 1, then $(x_n)_{n\in\mathbb{N}}$ converges to x. The functional x^* is said to strongly expose x on B_X and it is trivial that $(x, x^*) \in \Pi_X$, where $\Pi_X := \{(x, x^*) \in \mathsf{S}_X \times \mathsf{S}_{X^*} : x^*(x) = 1\}$. We will let

$$\Pi_X^{se} := \{(x, x^*) \in \mathsf{S}_X \times \mathsf{S}_{X^*} : x^* \text{ strongly exposes } x \text{ on } \mathsf{B}_X\}.$$

A weaker notion than a strongly exposed point is that of an exposed point. A point $x \in \mathsf{S}_X$ is said to be an exposed point of B_X if there exists $x^* \in \mathsf{S}_{X^*}$ in such a way that $(x^*)^{-1}(\{1\}) \cap \mathsf{B}_X = \{x\}$. This time the functional x^* is called a supporting functional that exposes x on B_X. We will let $\Pi_X^e :=$

$\{(x, x^*) \in \mathsf{S}_X \times \mathsf{S}_{X^*} : x^*$ exposes x on $\mathsf{B}_X\}$. Observe that $\Pi_X^{se} \subseteq \Pi_X^e \subseteq \Pi_X$. Another trivial fact is the following: for every surjective linear isometry T between Banach spaces X, Y and for every $(x, x^*) \in \mathsf{S}_X \times \mathsf{S}_{X^*}$, we have that $(x, x^*) \in \Pi_X$ or Π_X^e or Π_X^{se} if and only if $(T(x), T^*(x^*)) \in \Pi_Y$ or Π_Y^e or Π_Y^{se}, respectively. In other words, the previous notions are invariant under surjective linear isometries. Another geometrical notion employed in this manuscript is that of a rotund point. A point $x \in \mathsf{S}_X$ in the unit sphere of a Banach space X is said to be a rotund point of the unit ball of X if x is contained in no non-trivial segment of the unit sphere, in other words, $\{x\}$ is a maximal proper face of B_X. The set of rotund points of B_X is denoted by $\operatorname{rot}(\mathsf{B}_X)$. In view of the Hahn-Banach Separation Theorem, the set of rotund points can be described as follows:

$$\operatorname{rot}(\mathsf{B}_X) = \{x \in \mathsf{S}_X : \text{if } x^* \in \mathsf{S}_{X^*} \text{ is so that } (x, x^*) \in \Pi_X, \text{ then } (x, x^*) \in \Pi_X^e\}.$$

We refer the reader to [2, 3] for a wider perspective on the above concepts and some other geometrical properties related with renormings. The duality mapping [4, 5] of a Banach space X is the set-valued map defined as

$$\begin{array}{rccl} J: & X & \to & \mathcal{P}(X^*) \\ & x & \mapsto & J(x) := \{x^* \in X^* : \|x^*\| = \|x\| \text{ and } x^*(x) = \|x^*\|\|x\|\}. \end{array}$$

A point x in the unit sphere S_X of X is said to be a smooth point [8] of the unit ball B_X of X provided that $J(x)$ is a singleton. The subset of smooth points of B_X is typically denoted by $\operatorname{smo}(\mathsf{B}_X)$. Rotund points and smooth points are dual notions.

The main goal of this manuscript is to introduce an index that measures accurately how far a point in the unit sphere is from being a strongly exposed point of the unit ball. For this, we first establish characterizations of the set of rotund points and the set of smooth points. Then we introduce the index of strong rotundity and show the most basic properties related to such index. We compute the index of strong rotundity of a Hilbert space. Finally, we construct a new set of pairs contained in Π_X^e for which the stereographic projection [13, 14] in a Banach space is a homeomorphism.

2. Results

We will begin by providing a new characterization of the set of rotund points of the unit ball of a Banach space. As usual, if X is a vector space and $x, y \in X$, then $\operatorname{st}(x, y) := x + \mathbb{R}(y - x)$ is the straight line passing through x, y and $[x, y] := x + [0, 1](y - x)$ is the segment joining x, y.

Theorem 2.1. *For a Banach space X,* $\operatorname{rot}(\mathsf{B}_X) = \{x \in \mathsf{S}_X : \forall y \in \mathsf{S}_X \ \operatorname{st}(x, y) \cap \mathsf{S}_X = \{x, y\}\}$.

Proof. We will prove both inclusions in two simple steps.

- $\subseteq$ Let $x \in \operatorname{rot}(\mathsf{B}_X)$. Fix an arbitrary $y \in \mathsf{S}_X \setminus \{x\}$. If there exists $z \in \operatorname{st}(x, y) \cap \mathsf{S}_X$ different from x, y, then we end up with three points aligned in the unit sphere, so the whole segment containing x, y, z lies entirely in the unit sphere. However, this contradicts the fact that $\{x\}$ is a maximal proper face of B_X.
- $\supseteq$ Conversely, let $x \in \mathsf{S}_X$ satisfying that $\operatorname{st}(x, y) \cap \mathsf{S}_X = \{x, y\}$ for all $y \in \mathsf{S}_X$. If $x \notin \operatorname{rot}(\mathsf{B}_X)$, then we can find $y \in \mathsf{S}_X \setminus \{x\}$ such that $[x, y] \subseteq \mathsf{S}_X$, contradicting that $\operatorname{st}(x, y) \cap \mathsf{S}_X = \{x, y\}$ since $[x, y] \subseteq \mathsf{S}_X$.

The next result is a characterization of the set of smooth points in similar terms as the previous theorem. However, a technical lemma is needed first.

Lemma 2.1. *Let X be a Banach space. Let $x \in \mathsf{S}_X$ and consider a straight line $L \subseteq X$ containing x such that $L \cap (\mathsf{S}_X \setminus \{x\}) = \varnothing$. Then $L \cap \mathsf{U}_X = \varnothing$, where U_X is the open unit ball of X.*

Proof. Suppose on the contrary that there exists $u \in L \cap \mathsf{U}_X$. Consider the continuous function

$$\begin{array}{rcl} (-\infty, 0] & \to & \mathbb{R} \\ t & \mapsto & \|u + t(x-u)\|. \end{array}$$

If $t = 0$, then $\|u\| < 1$. Note that

$$\frac{1+\|u\|}{\|x-u\|} \geq 1 > \|u\|,$$

therefore, if $t < -\frac{1+\|u\|}{\|x-u\|}$, then

$$\|u + t(x-u)\| \geq |\|u\| - \|t(x-u)\|| = |\|u\| - |t|\|x-u\|| = |t|\|x-u\| - \|u\| > 1.$$

Bolzano's Theorem assures the existence of $s \in (-\infty, 0)$ such that $\|u + s(x-u)\| = 1$, that is, $u + s(x-u) \in L \cap \mathsf{S}_X$, meaning that $u + s(x-u) = x$, hence $(1-s)u = (1-s)x$, so $u = x$. This is a contradiction because $\|u\| < 1 = \|x\|$.

Theorem 2.2. *For a Banach space X with* $\dim(X) \geq 2$*,*

$$\operatorname{smo}(\mathsf{B}_X) = \left\{ x \in \mathsf{S}_X : \exists x^* \in \mathsf{S}_{X^*}\ (x, x^*) \in \Pi_X \text{ and } \forall z \in (x^*)^{-1}(\{1\})\ \ \operatorname{st}(-x, z) \cap (\mathsf{S}_X \setminus \{-x\}) \neq \varnothing \right\}.$$

Proof. We will prove both inclusions in two steps.

- $\subseteq$ Let $x \in \operatorname{smo}(\mathsf{B}_X)$. There exists $x^* \in \mathsf{S}_{X^*}$ satisfying that $J(x) = \{x^*\}$. Fix an arbitrary $z \in (x^*)^{-1}(\{1\})$. Suppose on the contrary that $\operatorname{st}(-x, z) \cap (\mathsf{S}_X \setminus \{-x\}) = \varnothing$. Note that, in this case, $\|z\| > 1$ because if $\|z\| = 1$, then $z \in \operatorname{st}(-x, z) \cap (\mathsf{S}_X \setminus \{-x\})$. By bearing in mind Lemma 2.1, $\operatorname{st}(-x, z) \cap \mathsf{U}_X = \varnothing$. The Hahn-Banach Separation Theorem allows the existence of $y^* \in \mathsf{S}_{X^*}$ such that $\operatorname{st}(-x, z) \subseteq (y^*)^{-1}(\{1\})$. Then $y^*(-x) = 1$, meaning that $-y^*(x) = 1$, so $-y^* \in J(x) = \{x^*\}$, reaching the contradiction that $y^*(z) = 1$ and $y^*(z) = -x^*(z) = -1$.
- $\supseteq$ Take $x \in \mathsf{S}_X$ for which there exists $x^* \in J(x)$ satisfying that $\operatorname{st}(-x, z) \cap (\mathsf{S}_X \setminus \{-x\}) \neq \varnothing$ for all $z \in (x^*)^{-1}(\{1\})$. Suppose on the contrary that $x \notin \operatorname{smo}(\mathsf{B}_X)$. There exists a 2-dimensional subspace Y of X containing x for which x is not a smooth point of B_Y. Let $J_Y : Y \to \mathcal{P}(Y^*)$ denote the dual mapping of Y. Notice that $J_Y(x)$ is a nontrivial segment that lies entirely in S_{Y^*}. Thus we can write $J_Y(x) = [a^*, b^*]$ where $a^* \neq b^*$ both are in S_{Y^*}. Notice that $\mathsf{B}_Y \subseteq (x^*|_Y)^{-1}([-1,1]) \cap (a^*)^{-1}([-1,1]) \cap (b^*)^{-1}([-1,1])$. Also, $x^*|_Y \in J_Y(x) = [a^*, b^*]$, therefore, either $a^* \neq x^*|_Y$ or $b^* \neq x^*|_Y$. Let us assume without any loss of generality that $a^* \neq x^*|_Y$. Then $(a^*)^{-1}(\{-1\})$ is not parallel to $(x^*|_Y)^{-1}(\{1\})$, hence the straight line $(a^*)^{-1}(\{-1\})$ intersects $(x^*|_Y)^{-1}(\{1\})$. Choose any $z \in (x^*|_Y)^{-1}(\{1\})$ with $(a^*)(z) < -1$. On the one hand, $\operatorname{st}(-x, z) \subseteq Y$, thus $\operatorname{st}(-x, z) \cap (\mathsf{S}_X \setminus \{-x\}) \subseteq \operatorname{st}(-x, z) \cap (\mathsf{S}_Y \setminus \{-x\})$. On the other hand, we will prove that $\operatorname{st}(-x, z) \cap (\mathsf{S}_Y \setminus \{-x\}) = \varnothing$. Indeed, pick any $t \in \mathbb{R} \setminus \{0\}$ and distinguish the following two cases:
  - $t > 0$. In this case, $a^*(tz + (1-t)(-x)) = t(a^*(z) + 1) - 1 < -1$. Since $\mathsf{B}_Y \subseteq (a^*)^{-1}([-1,1])$, we conclude that $tz + (1-t)(-x) \notin \mathsf{B}_Y$.

- $t < 0$. In this case, $x^*|_Y(tz + (1-t)(-x)) = 2t - 1 < -1$. Since $\mathsf{B}_Y \subseteq (x^*|_Y)^{-1}([-1,1])$, we conclude that $tz + (1-t)(-x) \notin \mathsf{B}_Y$.

As a consequence, we end up having the contradiction that $\operatorname{st}(-x, z) \cap (\mathsf{S}_X \setminus \{-x\}) = \varnothing$.

Let us introduce next the index of strong rotundity of a Banach space. First, we recall that, for a Banach space X, U_X stands for the open unit ball, and $\mathsf{U}_X(x,\varepsilon)$ denotes the open ball of center $x \in X$ and radius $\varepsilon \geq 0$.

Definition 2.1. *(Index of strong rotundity) Let X be a Banach space. The local index of strong rotundity of X at* $(x, x^*) \in \Pi_X$ *is defined as*

$$\begin{array}{rrcl} \eta_X(\cdot,(x,x^*)): & [0,2] & \to & [0,2] \\ & \varepsilon & \mapsto & \eta_X(\varepsilon,(x,x^*)) := d\Big((x^*)^{-1}(\{1\}), \mathsf{B}_X \setminus \mathsf{U}_X(x,\varepsilon)\Big). \end{array} \tag{2.1}$$

The index of strong rotundity of X at is defined as

$$\begin{array}{rrcl} \eta_X: & [0,2] & \to & [0,2] \\ & \varepsilon & \mapsto & \eta_X(\varepsilon) := \inf\{\eta_X(\varepsilon,(x,x^*)) : (x,x^*) \in \Pi_X\}. \end{array} \tag{2.2}$$

The index of strong rotundity will serve, among other things, to characterize the strongly exposed points of the unit ball. Later on, we will relate the index of strong rotundity to the modulus of local uniform rotundity. Now, the following lemma unveils the most basic properties satisfied by the index of rotundity.

Lemma 2.2. *Let X be a Banach space. Let* $(x, x^*) \in \Pi_X$. *Then:*

(1) $\eta_X(0,(x,x^*)) = 0$.
(2) $(x^*)^{-1}(\{-1\}) \cap \mathsf{B}_X \subseteq \mathsf{B}_X \setminus \mathsf{U}_X(x,2)$, *hence* $\eta_X(2,(x,x^*)) \leq 2$.
(3) If $(x^*)^{-1}(\{-1\}) \cap \mathsf{B}_X = \mathsf{B}_X \setminus \mathsf{U}_X(x,2)$, *then* $\eta_X(2,(x,x^*)) = 2$.
(4) If $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq 2$, *then* $\eta_X(\varepsilon_1,(x,x^*)) \leq \eta_X(\varepsilon_2,(x,x^*))$.
(5) If $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq 2$, *then* $\eta_X(\varepsilon_1) \leq \eta_X(\varepsilon_2)$.
(6) $\Pi_X^{se} = \{(x,x^*) \in \Pi_X : \forall \varepsilon \in (0,2]\ \ \eta_X(\varepsilon,(x,x^*)) > 0\}$.

Proof. We will only prove the second, third, and last items.

(2) For every $z \in (x^*)^{-1}(\{-1\}) \cap \mathsf{B}_X$, $2 \geq \|z - x\| \geq |x^*(z-x)| = |x^*(z) - x^*(x)| = 2$, meaning that $z \in \mathsf{B}_X \setminus \mathsf{U}_X(x,2)$.

(3) On the one hand, if $u \in (x^*)^{-1}(\{1\})$ and $v \in (x^*)^{-1}(\{-1\}) \cap \mathsf{B}_X$, then $\|u - v\| \geq |x^*(u-v)| = |x^*(u) - x^*(v)| = 2$, thus $d\Big((x^*)^{-1}(\{1\}), (x^*)^{-1}(\{-1\}) \cap \mathsf{B}_X\Big) = 2$. On the other hand,

$$2 \geq \eta_X(2,(x,x^*)) := d\Big((x^*)^{-1}(\{1\}), \mathsf{B}_X \setminus \mathsf{U}_X(x,\varepsilon)\Big) = d\Big((x^*)^{-1}(\{1\}), (x^*)^{-1}(\{-1\}) \cap \mathsf{B}_X\Big) = 2.$$

(6) Fix an arbitrary $(x,x^*) \in \Pi_X^{se}$. Suppose on the contrary that there exists $\varepsilon \in (0,2]$ for which $\eta_X(\varepsilon,(x,x^*)) = 0$. Then we can find two sequences $(y_n)_{n\in\mathbb{N}} \subseteq (x^*)^{-1}(\{1\})$ and $(x_n)_{n\in\mathbb{N}} \subseteq \mathsf{B}_X \setminus \mathsf{U}_X(x,\varepsilon)$ such that $\|y_n - x_n\| \to 0$ as $n \to \infty$. Notice that $x^*(x_n) \to 1$ as $n \to \infty$ but $(x_n)_{n\in\mathbb{N}}$ does not converge to x because $x_n \notin \mathsf{U}_X(x,\varepsilon)$ for each $n \in \mathbb{N}$, contradicting that $(x,x^*) \in \Pi_X^{se}$. Conversely, take any $(x,x^*) \in \Pi_X$ satisfying that $\eta_X(\varepsilon,(x,x^*)) > 0$ for all $\varepsilon \in (0,2]$. Assume to

the contrary that $(x, x^*) \notin \Pi_X^{se}$. There exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathsf{B}_X$ in such a way that $x^*(x_n) \to 1$ as $n \to \infty$ but $(x_n)_{n\in\mathbb{N}}$ does not converge to x. By passing to an appropriate subsequence $(x_{n_k})_{k\in\mathbb{N}}$, we can find $\varepsilon \in (0, 2]$ such that $\|x_{n_k} - x\| \geq \varepsilon$ for all $k \in \mathbb{N}$. Notice that $\left(x^*(x_{n_k})\right)_{k\in\mathbb{N}}$ still converges to 1, so, by assuming that $x^*(x_{n_k}) \neq 0$ for all $k \in \mathbb{N}$, we have that $\left\|\frac{x_{n_k}}{x^*(x_{n_k})} - x_{n_k}\right\| \to 0$, meaning that $\eta_X(\varepsilon, (x, x^*)) = d\left((x^*)^{-1}(\{1\}), \mathsf{B}_X \setminus \mathsf{U}_X(x, \varepsilon)\right) = 0$, which contradicts our initial assumption.

Notice that, under the settings of Lemma 2.2, it does not always hold that $(x^*)^{-1}(\{-1\}) \cap \mathsf{B}_X = \mathsf{B}_X \setminus \mathsf{U}_X(x, 2)$.

Proposition 2.1. *Let X be a Banach space. Let* $(x, x^*) \in \Pi_X^e$. *If* $x \notin \operatorname{rot}(\mathsf{B}_X)$*, then* $(x^*)^{-1}(\{-1\}) \cap \mathsf{B}_X \subsetneq \mathsf{B}_X \setminus \mathsf{U}_X(x, 2)$.

Proof. In accordance with Lemma 2.2(2), $(x^*)^{-1}(\{-1\}) \cap \mathsf{B}_X \subseteq \mathsf{B}_X \setminus \mathsf{U}_X(x, 2)$. Since $(x, x^*) \in \Pi_X^e$, we have that $(x^*)^{-1}(\{1\}) \cap \mathsf{B}_X = \{x\}$, so $(x^*)^{-1}(\{-1\}) \cap \mathsf{B}_X = \{-x\}$. Since $x \notin \operatorname{rot}(\mathsf{B}_X)$, there exists a non-trivial segment of the unit sphere containing x, that is, there exists $y \in \mathsf{S}_X \setminus \{x\}$ with $\left\|\frac{x+y}{2}\right\| = 1$, in other words, $\|x + y\| = 2$, meaning that $-y \in \mathsf{B}_X \setminus \mathsf{U}_X(x, 2)$. Finally, $-y \notin \{-x\} = (x^*)^{-1}(\{-1\}) \cap \mathsf{B}_X$. As a consequence, $(x^*)^{-1}(\{-1\}) \cap \mathsf{B}_X \subsetneq \mathsf{B}_X \setminus \mathsf{U}_X(x, 2)$.

In ℓ_∞, we will let $\mathbf{1}$ to denote the constant sequence of general term 1. Also, $(e_n)_{n\in\mathbb{N}}$ will denote the sequence of canonical unit vectors, that is, $e_n(m) = \delta_{nm}$ for all $n, m \in \mathbb{N}$. And

$$\begin{array}{rccl} \delta_n : & \ell_\infty & \to & \mathbb{R} \\ & x & \mapsto & \delta_n(x) := x(n) \end{array} \tag{2.3}$$

is the n^{th}-coordinate functional. Observe that $(\mathbf{1}, \delta_1), (e_1, \delta_1) \in \Pi_{\ell_\infty}$.

Proposition 2.2. $\eta_{\ell_\infty}(\varepsilon, (\mathbf{1}, \delta_1)) = 0$ *for all* $\varepsilon \in [0, 2]$ *and* $\eta_{\ell_\infty}(2, (e_1, \delta_1)) = 2$.

Proof. According to Lemma 2.2(4), it only suffices to prove that $\eta_{\ell_\infty}(2, (\mathbf{1}, \delta_1)) = 0$. For this, we will show that $(\delta_1)^{-1}(\{1\}) \cap (\mathsf{B}_{\ell_\infty} \setminus \mathsf{U}_{\ell_\infty}(\mathbf{1}, 2)) \neq \varnothing$. Indeed,

$$(1, -1, -1, -1, \dots) \in (\delta_1)^{-1}(\{1\}) \cap (\mathsf{B}_{\ell_\infty} \setminus \mathsf{U}_{\ell_\infty}(\mathbf{1}, 2)).$$

On the other hand, $\mathsf{B}_{\ell_\infty} \setminus \mathsf{U}_{\ell_\infty}(e_1, 2) = (\delta_1)^{-1}(\{-1\}) \cap \mathsf{B}_{\ell_\infty}$, thus

$$\eta_{\ell_\infty}(2, (e_1, \delta_1)) = d\left((\delta_1)^{-1}(\{1\}), \mathsf{B}_{\ell_\infty} \setminus \mathsf{U}_{\ell_\infty}(e_1, 2)\right) = d\left((\delta_1)^{-1}(\{1\}), (\delta_1)^{-1}(\{-1\}) \cap \mathsf{B}_{\ell_\infty}\right) = 2$$

in view of Lemma 2.2(3).

Recall [1,6] that, given a Banach space X with unit sphere S_X and unit ball B_X, a closed subspace $Y \subseteq X$ is said to be an L^p-summand subspace of X, where $1 \leq p \leq \infty$, if Y is L^p-complemented in X, that is, there exists a closed subspace $Z \subseteq X$ such that $X = Y \oplus_p Z$, in the sense that $\|y + z\|^p = \|y\|^p + \|z\|^p$ for all $y \in Y$ and all $z \in Z$. A point $x \in X$ is said to be an L^p-summand vector of X provided that $\mathbb{R}x$ is an L^p-summand subspace of X. In accordance with [1], every unit L^2-summand vector is a rotund point as well as a smooth point of the unit ball.

Theorem 2.3. *Let X be a Banach space with* $\dim(X) \geq 2$. *For every* $(x, x^*) \in \Pi_X$ *such that x is an* L^2*-summand vector of X and for every* $\varepsilon \in [0, 2]$,

$$\eta_X(\varepsilon, (x, x^*)) = \frac{\varepsilon^2}{2}.$$

Proof. First off, notice that $\ker(x^*)$ is the L^2-complement of $\mathbb{R}x$. For every $z \in (x^*)^{-1}(\{1\})$ and every $y \in \mathsf{B}_X \setminus \mathsf{U}_X(x,\varepsilon)$, $\|z-y\| \geq |x^*(z-y)| = |x^*(z)-x^*(y)| = 1-x^*(y)$. Fix an arbitrary $y \in \mathsf{B}_X \setminus \mathsf{U}_X(x,\varepsilon)$ and write $y = x^*(y)x+m_y$ with $m_y \in \ker(x^*)$. Notice that $x^*(y)^2+\|m_y\|^2 \leq 1$ and $(1-x^*(y))^2+\|m_y\|^2 = \|x-y\|^2 \geq \varepsilon^2$. Then $(1-x^*(y))^2 \geq \varepsilon^2 - \|m_y\|^2 \geq \varepsilon^2 + x^*(y)^2 - 1$. In other words, $(1-x^*(y))^2 + 1 - x^*(y)^2 \geq \varepsilon^2$, that is, $2-2x^*(y) \geq \varepsilon^2$, meaning that $1 - x^*(y) \geq \frac{\varepsilon^2}{2}$. Going back to the beginning, for every $z \in (x^*)^{-1}(\{1\})$ and every $y \in \mathsf{B}_X \setminus \mathsf{U}_X(x,\varepsilon)$, $\|z-y\| \geq 1 - x^*(y) \geq \frac{\varepsilon^2}{2}$, which implies that $\eta_X(\varepsilon,(x,x^*)) \geq \frac{\varepsilon^2}{2}$. Finally, since $\dim(X) \geq 2$, we can take any $y \in \mathsf{S}_X \cap \mathsf{S}_X(x,\varepsilon)$ and $z := x + m_y$, where $m_y := y - x^*(y)x$. Simply notice that $\|z-y\| = \|(x+m_y)-y\| = \|(x+y-x^*(y)x)-y\| = \|x-x^*(y)x\| = 1-x^*(y) = \frac{\varepsilon^2}{2}$.

Corollary 2.1. *If X is a Hilbert space with* $\dim(X) \geq 2$*, then* $\eta_X(\varepsilon) = \frac{\varepsilon^2}{2}$.

Proof. By bearing in mind [6], a Banach space is a Hilbert space if and only every unit vector is an L^2-summand vector. Therefore, by applying Theorem 2.3, $\eta_X(\varepsilon,(x,x^*)) = \frac{\varepsilon^2}{2}$ for every $(x,x^*) \in \Pi_X$, obtaining the desired result.

Another index can be defined which lies in between the local modulus of convexity and the index of strong rotundity. Indeed, if X is a Banach space, then we define at $(x,x^*) \in \Pi_X$ the following function:

$$\begin{array}{rccl} \upsilon_X(\cdot,(x,x^*)): & [0,2] & \to & [0,2] \\ & \varepsilon & \mapsto & \upsilon_X(\varepsilon,(x,x^*)) := \inf\{1-x^*(y) : \|y\| \leq 1,\ \|x-y\| \geq \varepsilon)\}. \end{array} \tag{2.4}$$

We remind the reader that the modulus of local convexity [12] at $x \in \mathsf{S}_X$ is given by

$$\begin{array}{rccl} \delta_X(\cdot,x): & [0,2] & \to & [0,1] \\ & \varepsilon & \mapsto & \delta_X(\varepsilon,x) := \inf\left\{1-\left\|\frac{x+y}{2}\right\| : \|y\| \leq 1,\ \|x-y\| \geq \varepsilon)\right\}. \end{array} \tag{2.5}$$

Theorem 2.4. *Let X be a Banach space. For every $(x,x^*) \in \Pi_X$ and every $\varepsilon \in [0,2]$,*

$$2\delta_X(\varepsilon,x) \leq \upsilon_X(\varepsilon,(x,x^*)) \leq \eta_X(\varepsilon,(x,x^*)).$$

Proof. On the one hand, for every $z \in (x^*)^{-1}(\{1\})$ and every $y \in \mathsf{B}_X \setminus \mathsf{U}_X(x,\varepsilon)$,

$$\|z-y\| \geq x^*(z-y) = 1 - x^*(y) \geq \upsilon_X(\varepsilon,(x,x^*)).$$

As a consequence, $\eta_X(\varepsilon,(x,x^*)) \geq \upsilon_X(\varepsilon,(x,x^*))$. On the other hand, for every $y \in \mathsf{B}_X \setminus \mathsf{U}_X(x,\varepsilon)$,

$$\delta_X(\varepsilon,x) \leq 1 - \left\|\frac{x+y}{2}\right\| \leq 1 - x^*\left(\frac{x+y}{2}\right) = 1 - \frac{1+x^*(y)}{2} = \frac{1-x^*(y)}{2}.$$

Therefore, $\delta_X(\varepsilon,x) \leq \frac{1}{2}\upsilon_X(\varepsilon,(x,x^*))$.

Let us finally tackle some applications to the stereographic projection. According to [9, 10], if X is a Banach space and $(x,x^*) \in \Pi_X^e$, then

$$\begin{array}{rcl} \mathsf{S}_X \setminus \{-x\} & \to & (x^*)^{-1}(\{1\}) \\ y & \mapsto & -x + \frac{2}{x^*(y)+1}(y+x) \end{array} \tag{2.6}$$

is a well-defined and continuous function known as stereographic projection.

Definition 2.2. *(Stereographic projection pair) Let X be a Banach space. Let $(x, x^*) \in \Pi_X$. We will say that (x, x^*) is a stereographic projection pair provided that the following conditions are satisfied:*

- $\mathrm{st}(-x, y) \cap \mathsf{S}_X = \{-x, y\}$ *for all* $y \in \mathsf{S}_X$.
- $\mathrm{st}(-x, z) \cap (\mathsf{S}_X \setminus \{-x\}) \neq \varnothing$ *for every* $z \in (x^*)^{-1}(\{1\})$.
- *If* $(y_j)_{j\in\mathbb{N}} \subseteq \mathsf{S}_X \setminus \{-x\}$ *is a sequence converging to* $-x$*, then* $\left\|\frac{y_j+x}{x^*(y_j)+1}\right\| \to \infty$ *as* $j \to \infty$.

The set of stereographic projection pairs will be denoted by Π_X^{sp}.

According to Theorem 2.1, the first condition in the above definition is equivalent to the fact that $-x \in \mathrm{rot}(\mathsf{B}_X)$. The second condition is equivalent to the fact that $x \in \mathrm{smo}(\mathsf{B}_X)$ in view of Theorem 2.2. As a consequence, if $(x, x^*) \in \Pi_X^{sp}$, then $x \in \mathrm{rot}(\mathsf{B}_X) \cap \mathrm{smo}(\mathsf{B}_X)$, hence $(x, x^*) \in \Pi_X^e$. Nevertheless, observe that the last condition of the previous definition is an unusual geometrical property in the sense that it only works for sequences in the unit sphere, but not for sequences in the unit ball. Indeed, if we take $y_j := t_j x$ for all $j \in \mathbb{N}$, where $(t_j)_{j\in\mathbb{N}} \subseteq (-1, 1)$ converges to -1, then

$$\left\|\frac{y_j + x}{x^*(y_j) + 1}\right\| = \left\|\frac{t_j x + x}{t_j + 1}\right\| = 1$$

for all $j \in \mathbb{N}$.

By bearing in mind [9, Lemma 2.1], if $x \in \mathsf{S}_X$ is an L^2-summand vector of a Banach space X and $x^* \in \mathsf{S}_{X^*}$ satisfies that $\ker(x^*)$ is the L^2-complement of $\mathbb{R}x$, then (x, x^*) is a stereographic projection pair. Since a Banach space is a Hilbert space if and only every unit vector is an L^2-summand vector [6], we obtain the following theorem whose proof we omit.

Theorem 2.5. *If X is a Hilbert space, then* $\Pi_X = \Pi_X^{sp}$.

The following theorem generalizes and improves [9, Lemma 2.1]. First, a technical remark is needed, which is a simple limit from a Calculus course.

Remark 2.1. *For each* $p \in (1, \infty)$, $\lim\limits_{x\to -1^+} \dfrac{1 - (-x)^p}{(1 + x)^p} = +\infty$.

Theorem 2.6. *Let X be a Banach space with* $\dim(X) \geq 2$. *Let* $(x, x^*) \in \Pi_X$. *If x is an* L^p*-summand vector of X, for* $1 < p < \infty$, *and* $\ker(x^*)$ *is the* L^p*-complement of* $\mathbb{R}x$*, then* (x, x^*) *is a stereographic projection pair.*

Proof. In the first place, x is an L^p-summand vector of any 2-dimensional subspace containing it. Therefore, any 2-dimensional subspace containing x is linearly isometric to ℓ_p^2, which is rotund and smooth. Since rotund points and smooth points are 2-dimensional properties, we conclude that $x \in \mathrm{rot}(\mathsf{B}_X) \cap \mathrm{smo}(\mathsf{B}_X)$. By applying Theorem 2.1 and Theorem 2.2, we conclude that the first two conditions of Definition 2.2 are satisfied. Let us prove the third condition. Take any sequence $(y_j)_{j\in\mathbb{N}} \subseteq \mathsf{S}_X \setminus \{-x\}$ converging to $-x$. Let us write $y_j = x^*(y_j)x + m_j$, where $m_j \in \ker(x^*)$. Since $\left(x^*(y_j)\right)_{j\in\mathbb{N}}$ converges to -1, we may assume that $-1 < x^*(y_j) < 0$ for all $j \in \mathbb{N}$. Notice then that $1 = (-x^*(y_j))^p + \|m_j\|^p$ for all $j \in \mathbb{N}$. Then

$$\left\|\frac{y_j + x}{x^*(y_j) + 1}\right\|^p = \frac{\left(1 + x^*(y_j)\right)^p + \|m_j\|^p}{\left(x^*(y_j) + 1\right)^p} = \frac{\left(1 + x^*(y_j)\right)^p + 1 - (-x^*(y_j))^p}{\left(x^*(y_j) + 1\right)^p} = 1 + \frac{1 - (-x^*(y_j))^p}{\left(x^*(y_j) + 1\right)^p} \to \infty$$

as $j \to \infty$ in view of Remark 2.1.

The final result in this manuscript shows that stereographic projection pairs make possible that stereographic projections in Banach spaces be homeomorphisms, improving [9, Theorem 2.2]. However, let us recall first the following well-known topological fact [11].

Remark 2.2. *Let X, Y be topological spaces. Let $f : X \to Y$ be injective. Let $x \in X$. Suppose that for every net $(x_i)_{i\in I} \subseteq X$ such that $(f(x_i))_{i\in I}$ converges to $f(x)$, there exists a subnet $(z_j)_{j\in J}$ of $(x_i)_{i\in I}$ convergent to x. Then $f^{-1} : f(X) \to X$ is continuous at $f(x)$.*

One can easily understand that, under the settings of Remark 2.2, if X, Y are both first countable, then Remark 2.2 remains true if we switch nets with sequences.

Theorem 2.7. *Let X be a Banach space. If $(x, x^*) \in \Pi_X^{sp}$, then the stereographic projection* (2.6) *is an homeomorphism.*

Proof. First off, let us denote by ϕ to the stereographic projection (2.6). We already know that ϕ is well defined, continuous, and $\phi(y) \in \mathrm{st}(-x, y)$ for all $y \in \mathsf{S}_X \setminus \{-x\}$. Let us check now that ϕ is surjective. Fix an arbitrary $z \in (x^*)^{-1}(\{1\})$. If $z = x$, then $\phi(x) = x$. So let us assume that $z \neq x$. Since $(x, x^*) \in \Pi_X^{sp}$, by definition we have that $\mathrm{st}(-x, z) \cap (\mathsf{S}_X \setminus \{-x\}) \neq \varnothing$. Let $u \in \mathbb{R} \setminus \{0\}$ such that $y := -x + u(z + x) \in \mathsf{S}_X \setminus \{-x\}$. We will show that $\phi(y) = z$. Indeed

$$\phi(y) = -x + 2\frac{y + x}{x^*(y) + 1} = -x + 2\frac{-x + u(z + x) + x}{x^*(-x + u(z + x)) + 1} = -x + 2\frac{u(z + x)}{2u} = -x + (z + x) = z.$$

Next step is to prove that ϕ is one-to-one. Indeed, take $y_1, y_2 \in \mathsf{S}_X \setminus \{-x\}$ with $\phi(y_1) = \phi(y_2)$. Then $y_2 = -x + \frac{x^*(y_2)+1}{x^*(y_1)+1}(y_1 + x) \in \mathrm{st}(-x, y_1) \cap \mathsf{S}_X = \{-x, y_1\}$, meaning that $y_1 = y_2$. Let us finally prove that ϕ^{-1} is continuous. We will rely on Remark 2.2 for sequences. Fix an arbitrary $y \in \mathsf{S}_X \setminus \{-x\}$. Take a sequence $(y_i)_{i\in\mathbb{N}} \subseteq \mathsf{S}_X \setminus \{-x\}$ such that $(\phi(y_i))_{i\in\mathbb{N}}$ converges to $\phi(y)$. We will show the existence of a subsequence $\left(y_{i_j}\right)_{j\in\mathbb{N}}$ convergent to y. Indeed, there exists a subsequence $\left(y_{i_j}\right)_{j\in\mathbb{N}}$ such that $\left(x^*(y_{i_j})\right)_{j\in\mathbb{N}}$ is convergent to some $r \in [-1, 1]$. Then $\left(\phi(y_{i_j})\right)_{j\in\mathbb{N}}$ converges to $\phi(y)$. This is equivalent to saying that $\left(\frac{y_{i_j}+x}{x^*(y_{i_j})+1}\right)_{j\in\mathbb{N}}$ converges to $\frac{y+x}{x^*(y)+1}$. Since $\left(x^*(y_{i_j}) + 1\right)_{j\in\mathbb{N}}$ is convergent to $r + 1$, we conclude that $\left((x^*(y_{i_j}) + 1)\frac{y_{i_j}+x}{x^*(y_{i_j})+1}\right)_{j\in\mathbb{N}}$ converges to $(r+1)\frac{y+x}{x^*(y)+1}$, in other words, $\left(y_{i_j} + x\right)_{j\in\mathbb{N}}$ converges to $(r+1)\frac{y+x}{x^*(y)+1}$, which is equivalent to stating that $\left(y_{i_j}\right)_{j\in\mathbb{N}}$ converges to $-x + (r + 1)\frac{y+x}{x^*(y)+1}$. Next, observe that $r \neq -1$ since otherwise we obtain that $\left(y_{i_j}\right)_{j\in\mathbb{N}}$ converges to $-x$, reaching the contradiction that $\left\|\frac{y_{i_j}+x}{x^*(y_{i_j})+1}\right\| \to \infty$ as $j \to \infty$ by bearing in mind that $(x, x^*) \in \Pi_X^{sp}$. As a consequence, $-x + (r + 1)\frac{y+x}{x^*(y)+1} \in \mathrm{st}(-x, y) \cap \mathsf{S}_X = \{-x, y\}$, that is, either $-x + (r + 1)\frac{y+x}{x^*(y)+1} = -x$ or $-x + (r + 1)\frac{y+x}{x^*(y)+1} = y$. If $-x + (r + 1)\frac{y+x}{x^*(y)+1} = -x$, then $y = -x$, which is impossible since $y \in \mathsf{S}_X \setminus \{-x\}$. Thus, $-x + (r + 1)\frac{y+x}{x^*(y)+1} = y$. By relying on Remark 2.2, we conclude that ϕ^{-1} is continuous at b.

3. Conclusions

The use of indices in the literature of Geometry of Banach Spaces has always been very useful to determine the exact shape of the unit ball. The most known indices are the modulus of convexity, the modulus of smoothness, the index of rotundity, and the Bishop-Phelps-Bollabás index, among others.

The index of strong rotundity is a novel concept introduced in this manuscript. This index serves to determine how far a point of the unit sphere is from being an strongly exposed point. As expected, Hilbert spaces have a very particular index of strong rotundity. Finally, applications to the stereographic projection are provided, in particular, a novel set of pairs (the stereographic projection pairs) are introduced in this manuscript which guarantee that the stereographic projection is an homeomorphism.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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