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Research article

Existence and stability results for impulsive (k,ψ)-Hilfer fractional double integro-differential equation with mixed nonlocal conditions

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Abstract: This paper investigates a class of nonlinear impulsive fractional integro-differential equations with mixed nonlocal boundary conditions (multi-point and multi-term) that involves (ρ_k,ψ_k)-Hilfer fractional derivative. The main objective is to prove the existence and uniqueness of the solution for the considered problem by means of fixed point theory of Banach's and O'Regan's types, respectively. In this contribution, the transformation of the considered problem into an equivalent integral equation is necessary for our main results. Furthermore, the nonlinear functional analysis technique is used to investigate various types of Ulam's stability results. The applications of main results are guaranteed with three numerical examples.

Keywords: (k,ψ)-Hilfer fractional derivative; impulsive conditions; nonlocal conditions; fixed-point theorems; Ulam stability
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1. Introduction

Fractional calculus (FC) has gained considerable importance in many fields of applied sciences and engineering for solving various differential equations and investigating behaviors of mathematical models simulating real-world problems. Its amazing presence is evident in the modeling of several natural phenomena such as Hamiltonian chaos and fractional dynamics [1], bio-engineering [2], viscoelasticity [3], vibrations and diffusion [4], physics [5, 6], financial economics [7] and references cited therein. For more detailed theoretical aspect of FC, see [8–12] and references therein. One of the significant factors that make FC advantageous to ordinary calculus is that fractional-order (non-integer order) derivatives and integral operators (FDO/FIOs) are more effective for describing real-life problems than integer-order ones. Many researchers have attempted to propose various fractional operators that deal with derivatives and integrals of non-integer orders with successful applications to solve many problems. Different definitions of FDO and FIOs have been employed in research papers, mainly focusing on Riemann-Liouville (RL) [13], Caputo [13], Hadamard [13], Katugampola [13], Erdélyi-Kober [13], Hilfer [6], ψ-RL [13], k-RL [14], ψ-Hilfer [15], (k,ψ)-RL [14], (k,ψ)-Hilfer [16], and so on.

The study of nonlocal boundary value problems (BVP) is expanding quickly. In addition to the theoretical interest, this type of problems can be used to represent several phenomena in engineering, physics, and biological sciences. The nonlocal conditions have been used to describe some properties that occur at various points inside the domain instead of handling initial or boundary conditions. For historical backgrounds, see, e.g., [17–19]. At present, research on fractional differential equations (FDE) under various FDO/FIOs has developed rapidly in numerous directions. Fractional integro-differential equations (FIDEs) are the popular subjects that attract many researchers, some of which studied the existence of the solution for FIDEs. We recommend a series of recent works as in 2017, Jalilian and Ghasmi [20] established the existence and uniqueness of solutions for FIDEs with pantograph type by applying lipchitz condition. In 2020, the authors [21] studied the existence and stability of a class of nonlocal BVPs for integro-differential Langevin equation under the generalized Caputo proportional FDO by means of Banach's, Krasnoselskii's, Schaefer's fixed point theorems, and Ulam's stability approach. In 2021, Sudsutad et al. [22] used fixed point theories of Banach's, Krasnoselskii's, and Leray-Schauder nonlinear alternative types to discuss the existence, uniqueness, and stability of BVP for ψ-Hilfer FIDEs with mixed nonlocal boundary conditions, which include multi-point, fractional derivative multi-order, and fractional integral multi-order conditions:

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi}x(t) = f(t, x(t), I_{0^+}^{\phi;\psi}x(t), \ t \in (0, T], \\ x(0) = 0, \quad \sum_{i=1}^{m} \delta_i x(\eta_i) + \sum_{j=1}^{n} \omega_j \, I_{0^+}^{\beta_j;\psi} x(\theta_j) + \sum_{k=1}^{r} \lambda_k \, {}^{H}\mathfrak{D}_{0^+}^{\mu_k,\rho;\psi} x(\xi_k) = K, \end{cases} \tag{1.1}$$

where the description of parameters can be found in [22]. Later, in 2022, Thaiprayoon et al. [23] studied a class of ψ-Hilfer implicit fractional integro-differential equations with mixed nonlocal conditions:

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi}x(t) = f(t, x(t), {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi}x(t), \ I_{0^+}^{\alpha;\psi}x(t), \ t \in (0, T], \\ \sum_{i=1}^{m} \omega_i x(\eta_i) + \sum_{j=1}^{n} k_j \, {}^{H}\mathfrak{D}_{0^+}^{\beta_j,\rho;\psi} x(\xi_k) + \sum_{r=1}^{k} \sigma_r \, I_{0^+}^{\delta_r;\psi} x(\theta_r) = A, \end{cases} \tag{1.2}$$

where the description of parameters can be found in [23]. The existence and uniqueness of a solution for their problem were verified employing Banach's, Schaefer's, and Krasnosellskii's fixed point theorems, and the analysis of the problem was established via various kinds of Ulam stability results. In 2023, Sitho et al. [24] utilized the Banach contraction principle to show the uniqueness of the solution and the Leray-Schauder nonlinear alternative to prove the existence of solutions for a new class of BVP consisting of a mixed-type ψ_1-Hilfer and ψ_2-Caputo fractional order differential equation with integro-differential nonlocal boundary conditions as follows

$$\begin{cases} {}^{H}\mathfrak{D}^{\alpha,\beta;\psi_1}({}^{C}\mathfrak{D}^{\gamma;\psi_2}\pi)(t) = \Pi(t,\pi(t)), \ 0<\alpha,\beta,\gamma<1, \quad t\in[0,x_1], \\ {}^{C}\mathfrak{D}^{\gamma;\psi_2}\pi(0)=0, \quad \pi(T)=\sum_{i=1}^{m}\lambda_i{}^{C}\mathfrak{D}^{\gamma;\psi_2}\pi(\eta_i)+\sum_{j=1}^{n}\delta_j I^{\mu_j;\psi_2}\pi(\xi_j), \end{cases} \tag{1.3}$$

where the description of parameters can be found in [24]. For more interesting works on existence, uniqueness, and Ulam's stability results of these topics, we refer to [25–30] and reference cited therein. In parallel with FIDEs, impulsive differential equations play an important role in dynamical systems of evolutionary procedures by exhibiting instantaneous state changes at specific moments. It is thus regarded as an effective instrument for comprehending numerous real-world situations in applied sciences and engineering; see [31–33]. Many researchers have attempted FDEs and FIDEs with impulse conditions to develop an excellent qualitative theory. Over the years, they have produced crucial and fascinating insights that greatly aided the mathematical understanding of FDEs with impulse effects. We refer to [34–39] for further fascinating works on the subject.

The inspiration for this paper is based on the previous works mentioned above. The novelty and differences from the others are considered in our work. In this paper, we produce qualitative results of the solutions for the following nonlinear impulsive (ρ_k,ψ_k)-Hilfer FIDEs supplemented with mixed nonlocal boundary conditions:

$$\begin{cases} {}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+}u(t) = f(t,u(t),{}_{\rho_k}I^{\sigma_k;\psi_k}_{t_k}u(t),{}_{\rho_k}I^{\nu_k;\psi_k}_{t_k}u(t)), \ t\in J_k, \ t\neq t_k, \ k=0,1,\dots,m, \\ {}_{\rho_k}I^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+}u(t_k^+) - {}_{\rho_{k-1}}I^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+}u(t_k^-) = \phi_k(u(t_k)), \quad k=1,2,\dots,m, \\ {}^{RL}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+}u(t_k^+) - {}^{RL}_{\rho_{k-1}}\mathfrak{D}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+}u(t_k^-) = \phi_k^*(u(t_k)), \quad k=1,2,\dots,m, \\ u(0)=0, \quad \sum_{i=0}^{m}\mu_i u(\eta_i)+\sum_{l=0}^{n}\lambda_{l\,\rho_l}I^{\theta_l;\psi_l}_{t_l}u(\xi_l)=A, \quad \eta_i,\in(t_i,t_{i+1}], \quad \xi_l\in(t_l,t_{l+1}], \end{cases} \tag{1.4}$$

where ${}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_K;\psi_k}_{t_k^+}$ denotes the (ρ_k,ψ_k)-Hilfer-FDO of order $\alpha_k\in(1,2]$ and type $\beta_k\in[0,1]$, $\rho_k\in R^+$, $J_k:=(t_k,t_{k+1}]\subset(a,b]$ for $k=0,1,2,\dots,m$, with $J_0:=[a,t_1]$, $J:=[a,b]$, $0\le a=t_0<t_1<\cdots<t_m<t_{m+1}=b\le T$, ${}_{\rho_k}I^{q;\psi_k}_{t_k^+}$ is the (ρ_k,ψ_k)-RL-FIO with order $q\in\{\rho_k(2-\gamma_k),\rho_{k-1}(2-\gamma_{k-1}),\nu_k,\sigma_k,\theta_l\}$, $q>0$, $k=1,2,\dots,m$, $l=0,1,\dots,n$, ${}^{RL}_{\rho_k}\mathfrak{D}^{p;\psi_k}_{t_k^+}$, is the (ρ_k,ψ_k)-RL fractional derivative with order $p\in\{\rho_k(\gamma_k-1),\rho_{k-1}(\gamma_{k-1}-1)\}$ with $p\in(1,2)$, $k=1,2,\dots,m$, ${}_{\rho_k}I^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+}u(t_k^+)=\lim_{t\to0^+}{}_{\rho_k}I^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+}u(t_k+h)$, ${}^{RL}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+}u(t_k^+)=\lim_{h\to0^+}{}^{RL}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+}u(t_k+h)$, ϕ_k, $\phi_k^*\in C(R,R)$, $k=1,2,\dots,m$, $f\in C(J\times R^3,R)$, A, μ_i, $\lambda_l\in R$, $\eta_i,\in(t_i,t_{i+1}]$, $\xi_l\in(t_l,t_{l+1}]$, $i=1,2,\dots,m$, and $l=0,1,\dots,n$.

The remaining sections of this work are structured as follows: Section 2 presents the prerequisite and relevant facts for the concepts of the (ρ,ψ)-Hilfer fractional operators, as well as some necessary lemmas that examine the solution of the linear variant of the proposed problem in terms of an integral equation. Section 3 proves the existence of the solution using O'Regan's fixed point theorem, while the uniqueness of the solution is investigated by utilizing Banach's fixed point theorem. Later, various Ulam's stability results, such as Ulam-Hyers (UH), generalized Ulam-Hyers (GUH), Ulam-Hyers-Rassias (UHR), and generalized Ulam-Hyers-Rassias (GUHR), are established to ensure the existence results in Section 4. Finally, some illustrative examples are provided to support the main theoretical results in the last section.

2. Preliminaries

Definition 2.1. *[40] Let $f \in L^1(J,b)$ and an increasing function $\psi(t): J \to R$ with $\psi'(t) \neq 0$ for $t \in [a,b]$. The (ρ,ψ)-RL-FIO of a function f of order $\alpha > 0$ is defined by*

$$ {}_\rho I_{a^+}^{\alpha;\psi} f(t) = \frac{1}{\rho\Gamma_\rho(\alpha)} \int_a^t (\psi(t)-\psi(s))^{\frac{\alpha}{\rho}-1} \psi'(s) f(s) ds, \quad \rho,\alpha \in R^+ := (0,\infty), $$

where $\Gamma_\rho(\cdot)$ is the ρ-Gamma function introduced by Diaz and Pariguan [41],

$$ \Gamma_\rho(z) = \int_0^\infty t^{z-1} e^{-\frac{t^\rho}{\rho}} dt, \quad z \in C, \ Re(z) > 0, \ \rho > 0. \tag{2.1} $$

Some other useful properties of (2.1) *are well known:* $\Gamma_\rho(z+\rho) = z\Gamma_\rho(z)$, $\Gamma_\rho(\rho) = 1$, $\Gamma_\rho(z) = (\rho)^{\frac{z}{\rho}-1}\Gamma(z/\rho)$, $\Gamma(z) = \lim_{\rho\to 1}\Gamma_\rho(z)$.

Definition 2.2. *[16] Let $f \in C^n(J,R)$ and a function $\psi(t) \in C^n(J,R)$ with $\psi'(t) \neq 0$ for $t \in J$. Then, the (ρ,ψ)-RL-FDO of a function f of order α, $\rho \in R^+$, is defined by*

$$ {}_\rho^{RL}\mathfrak{D}_{a^+}^{\alpha;\psi} f(t) = \left(\frac{\rho}{\psi'(t)} \cdot \frac{d}{dt}\right)^n {}_\rho I_{a^+}^{\rho n-\alpha;\psi} f(t) = \delta_\psi^n {}_\rho I_{a^+}^{\rho n-\alpha;\psi} f(t), \quad \delta_\psi^n = \left(\frac{\rho}{\psi'(t)} \cdot \frac{d}{dt}\right)^n, \quad n = \lceil \alpha/\rho \rceil. $$

Definition 2.3. *[16] Let $f \in C^n(J,R)$, $\psi \in C^n(J,R)$, $\psi'(t) \neq 0$, for $t \in J$, α, $\rho \in R^+$, and $\beta \in [0,1]$. The (ρ,ψ)-Hilfer FDO of a function f of order α and type β is given by*

$$ {}_\rho^H\mathfrak{D}_{a^+}^{\alpha,\beta;\psi} f(t) = {}_\rho I_{a^+}^{\beta(\rho n-\alpha);\psi} \delta_\psi^n {}_\rho I_{a^+}^{(1-\beta)(\rho n-\alpha);\psi} f(t) = {}_\rho I_{a^+}^{\beta(\rho n-\alpha);\psi} \left({}_\rho^{RL}\mathfrak{D}_{a^+}^{\gamma_\rho;\psi} f\right)(t), \tag{2.2} $$

where $(1-\beta)(n\rho-\alpha) = n\rho - \gamma_\rho$, $\delta_\psi^n = \left(\frac{\rho}{\psi'(t)}\frac{d}{dt}\right)^n$ and $n = \lceil \alpha/\rho \rceil$.

Lemma 2.1. *[16] Let α, $\rho \in R^+$ and $\beta \in R$, such that $\beta/\rho > -1$. Then we have*

(i) ${}_\rho I_{a^+}^{\alpha;\psi}\left[(\psi(t)-\psi(a))^{\frac{\beta}{\rho}} \right] = \frac{\Gamma_\rho(\beta+\rho)}{\Gamma_\rho(\beta+\rho+\alpha)} (\psi(t)-\psi(a))^{\frac{\beta+\alpha}{\rho}}$.
(ii) ${}_\rho^{RL}\mathfrak{D}_{a^+}^{\alpha;\psi}\left[(\psi(t)-\psi(a))^{\frac{\beta}{\rho}} \right] = \frac{\Gamma_\rho(\beta+\rho)}{\Gamma_\rho(\beta+\rho-\alpha)} (\psi(t)-\psi(a))^{\frac{\beta-\alpha}{\rho}}$.
(iii) ${}_\rho I_{a^+}^{\alpha;\phi} {}_\rho I_{a^+}^{\beta;\psi} f(t) = {}_\rho I_{a^+}^{\alpha+\beta;\psi} f(t) = {}_\rho I_{a^+}^{\beta;\phi} {}_\rho I_{a^+}^{\alpha;\psi} f(t)$.

Lemma 2.2. *[35] If $f \in C^n(J,R)$, ρ, $\alpha \in R^+$, $\beta \in [0,1]$ and $n \in N$, then*

$$ \left({}_\rho I_{a^+}^{\alpha;\psi} {}_\rho^H\mathfrak{D}_{a^+}^{\alpha,\beta;\psi} f\right)(t) = f(t) - \sum_{i=1}^n \frac{(\psi(t)-\psi(a))^{\gamma-i}}{\rho^{i-n}\Gamma_\rho(\rho(\gamma-i+1))} \left[\delta_\psi^{n-i}\left({}_\rho I_{a^+}^{\rho(n-\gamma);\psi} f(a)\right)\right], $$

where $\gamma = \frac{1}{\rho}(\beta(\rho n-\alpha)+\alpha)$ and $n = \lceil \alpha/\rho \rceil$.

For convenience's sake, we set the notation as follows:

$$ \Psi_\psi^u(t,s) = (\psi(t)-\psi(s))^u. $$

Next, we establish the following auxiliary result:

Lemma 2.3. *Let $\nu \in (m-1, m)$, ρ, $\alpha \in R^{+}$, $m \in N$. If $h \in C^{n}([a,b], R)$, then*

$$ {}^{RL}_{\rho}\mathfrak{D}^{\alpha;\psi}_{a^+}\Big[{}_{\rho}I^{\nu;\psi}_{a^+}h(t)\Big] = {}_{\rho}I^{\nu-\alpha;\psi}_{a^+}h(t). \tag{2.3} $$

Proof. By applying Definition 2.2 and (iii) of Lemma 2.1, we have

$$ {}^{RL}_{\rho}\mathfrak{D}^{\alpha;\psi}_{a^+}\Big[{}_{\rho}I^{\nu;\psi}_{a^+}h(t)\Big] = \left(\frac{1}{\psi'(t)}\cdot\frac{d}{dt}\right)^{n}\rho^{n}\Big[{}_{\rho}I^{\rho n-\alpha+\nu;\psi}_{a^+}h(t)\Big]. \tag{2.4} $$

By using Definition 2.1, for $n = 1$, we obtain

$$ \begin{aligned} \left(\frac{1}{\psi'(t)}\cdot\frac{d}{dt}\right)\rho\,{}_{\rho}I^{\rho n-\alpha+\nu;\psi}_{a^+}h(t) &= \frac{\rho}{\psi'(t)}\cdot\frac{d}{dt}\left(\frac{1}{\rho\Gamma_{\rho}(\rho n-\alpha+\nu)}\int_a^t \Psi_{\psi}^{\frac{\rho n-\alpha+\nu}{\rho}-1}(t,s)\psi'(s)h(s)ds\right) \\ &= \frac{1}{\rho\Gamma_{\rho}(\rho n-\alpha+\nu-\rho)}\int_a^t \Psi_{\psi}^{\frac{\rho n-\alpha+\nu-\rho}{\rho}-1}(t,s)\psi'(s)h(s)ds \\ &= {}_{\rho}I^{\rho n-\alpha+\nu-\rho;\psi}_{a^+}h(t). \end{aligned} $$

In the same way, for $n = 2$, we have

$$ \begin{aligned} \left(\frac{1}{\psi'(t)}\cdot\frac{d}{dt}\right)^{2}\rho^{2}\,{}_{\rho}I^{\rho n-\alpha+\nu;\psi}_{a^+}h(t) &= \frac{\rho}{\psi'(t)}\cdot\frac{d}{dt}\left(\frac{1}{\rho\Gamma_{\rho}(\rho n-\alpha+\nu-\rho)}\int_a^t \Psi_{\psi}^{\frac{\rho n-\alpha+\nu-\rho}{\rho}-1}(t,s)\psi'(s)h(s)ds\right) \\ &= \frac{1}{\rho\Gamma_{\rho}(\rho n-\alpha+\nu-2\rho)}\int_a^t \Psi_{\psi}^{\frac{\rho n-\alpha+\nu-2\rho}{\rho}-1}(t,s)\psi'(s)h(s)ds \\ &= {}_{\rho}I^{\rho n-\alpha+\nu-2\rho;\psi}_{a^+}h(t). \end{aligned} $$

Repeating the above method, we obtain

$$ \begin{aligned} \left(\frac{1}{\psi'(t)}\cdot\frac{d}{dt}\right)^{n}\rho^{n}\,{}_{\rho}I^{\rho n-\alpha+\nu;\psi}_{a^+}h(t) &= \frac{\rho}{\psi'(t)}\cdot\frac{d}{dt}\left(\frac{1}{\rho\Gamma_{\rho}(\rho n-\alpha+\nu-(n-1)\rho)}\int_a^t \Psi_{\psi}^{\frac{\rho n-\alpha+\nu-(n-1)\rho}{\rho}-1}(t,s)\psi'(s)h(s)ds\right) \\ &= \frac{1}{\rho\Gamma_{\rho}(\nu-\alpha)}\int_a^t \Psi_{\psi}^{\frac{\nu-\alpha}{\rho}-1}(t,s)\psi'(s)h(s)ds \\ &= {}_{\rho}I^{\nu-\alpha;\psi}_{a^+}h(t). \end{aligned} $$

The proof is completed.

Denote the weighted space

$$ C^{2-\gamma}_{\psi}(J,R) = \Big\{u:(a,b]\to R \,|\, u(a^+) \text{ exists and } \Psi^{2-\gamma}_{\psi}(t,a)u(t)\in C(J,R)\Big\},\ \gamma\in(1,2], $$

where $C^{2-\gamma}_{\psi} = C^{2-\gamma}_{\psi}(J,R)$. The weighted space of piece-wise continuous functions is defined by

$$ \begin{aligned} PC^{2-\gamma_k}_{\psi_k}(J,R) = \Big\{&u:(a,b]\to R \,|\, u\in C^{2-\gamma_k}_{\psi_k},\ k=0,1,2,\ldots,m, \\ &{}_{\rho_k}I^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+}u(t_k^+),\ {}_{\rho_{k-1}}I^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+}u(t_k^-) \text{ exist and} \\ &{}_{\rho_{k-1}}I^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+}u(t_k^-) = {}_{\rho_{k-1}}I^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+}u(t_k),\ k=1,\ldots,m\Big\}. \end{aligned} $$

Observe that $PC = PC^{2-\gamma_k}_{\psi_k}(J,R)$ is a Banach space equipped with

$$ \|u\|_{PC} = \sup_{t\in J}\left|\Psi^{2-\gamma_k}_{\psi_k}(t,t_k)u(t)\right|. $$

Lemma 2.4. *Let* $\alpha_k \in (1,2)$, $\beta_k \in [0,1]$, $\rho_k > 0$, $\mu_k > 0$, $\nu_k > 0$, $\gamma_k = (1/\rho_k)(\beta_k(2\rho_k - \alpha_k) + \alpha_k)$, $\psi_k \in C(J,R)$ *with* $\psi_k' > 0$, $k = 0,1,2,\ldots,m$, $\mathfrak{h} \in C^{2-\gamma_k}_{\psi_k}$. *Then the following linear variant impulsive* (ρ_k,ψ_k)*-Hilfer fractional boundary value problem*

$$\begin{cases} {}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+}u(t) = \mathfrak{h}(t), \quad t \neq t_k,\ k = 0,1,\ldots,m, \\ {}^{RL}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+}u(t_k^+) - {}^{RL}_{\rho_{k-1}}\mathfrak{D}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+}u(t_k^-) = \phi_k(u(t_k)), \quad k = 1,2,\ldots,m, \\ {}_{\rho_k}I^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+}u(t_k^+) - {}_{\rho_{k-1}}I^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+}u(t_k^-) = \phi_k^*(u(t_k)), \quad k = 1,2,\ldots,m, \\ u(0) = 0, \quad \displaystyle\sum_{i=0}^{m+1}\mu_i u(\eta_i) + \sum_{l=0}^{n}\lambda_l {}_{\rho_l}I^{\theta_l;\psi_l}_{t_l}u(\xi_l) = A, \quad \eta_i, \in (t_i,t_{i+1}],\ \xi_l \in (t_l,t_{l+1}], \end{cases} \tag{2.5}$$

satisfies the following integral equation, $u \in PC$, *as*

$$\begin{aligned} u(t) \;=\; & \left\{\frac{\Psi^{\gamma_k-1}_{\psi_k}(t,t_k)}{\Lambda\Gamma_{\rho_k}(\rho_k\gamma_k)} + \frac{\Psi^{\gamma_k-2}_{\psi_k}(t,t_k)}{\Lambda\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right\}\Bigg\{A \\ & -\Bigg(\sum_{i=0}^{m+1}\frac{\mu_i\Psi^{\gamma_i-1}_{\psi_i}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{j=0}^{i-1}\Big({}_{\rho_j}I^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}_{t_j}\mathfrak{h}(t_{j+1}) + \phi_{j+1}(u(t_{j+1}))\Big) \\ & +\sum_{i=0}^{m+1}\frac{\mu_i\Psi^{\gamma_i-2}_{\psi_i}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\Bigg[\sum_{j=0}^{i-1}\Big({}_{\rho_j}I^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}_{t_j}\mathfrak{h}(t_{j+1}) + \phi^*_{j+1}(u(t_{j+1}))\Big) \\ & +\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}_{t_r}\mathfrak{h}(t_{r+1}) + \phi_{r+1}(u(t_{r+1}))\Big)\Bigg] + \sum_{i=0}^{m+1}\mu_i {}_{\rho_i}I^{\alpha_i;\psi_i}_{t_i}\mathfrak{h}(\eta_i) \\ & +\sum_{l=0}^{n}\lambda_l {}_{\rho_l}I^{\alpha_l+\theta_l;\psi_l}_{t_l}\mathfrak{h}(\xi_l) + \sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\Big({}_{\rho_j}I^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}_{t_j}\mathfrak{h}(t_{j+1}) + \phi_{j+1}(u(t_{j+1}))\Big) \\ & +\sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Bigg[\sum_{j=0}^{l-1}\Big({}_{\rho_j}I^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}_{t_j}\mathfrak{h}(t_{j+1}) + \phi^*_{j+1}(u(t_{j+1}))\Big) \\ & +\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}_{t_r}\mathfrak{h}(t_{r+1}) + \phi_{r+1}(u(t_{r+1}))\Big)\Bigg]\Bigg)\Bigg\} \\ & +{}_{\rho_k}I^{\alpha_k;\psi_k}_{t_k}\mathfrak{h}(t) + \frac{\Psi^{\gamma_k-1}_{\psi_k}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\Big({}_{\rho_j}I^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}_{t_j}\mathfrak{h}(t_{j+1}) + \phi_{j+1}(u(t_{j+1}))\Big) \\ & +\frac{\Psi^{\gamma_k-2}_{\psi_k}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\Bigg[\sum_{j=0}^{k-1}\Big({}_{\rho_j}I^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}_{t_j}\mathfrak{h}(t_{j+1}) + \phi^*_{j+1}(u(t_{j+1}))\Big) \\ & +\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}_{t_r}\mathfrak{h}(t_{r+1}) + \phi_{r+1}(u(t_{r+1}))\Big)\Bigg], \end{aligned} \tag{2.6}$$

where

$$\begin{aligned}\Lambda &= \sum_{i=0}^{m+1}\frac{\mu_i\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}+\sum_{i=0}^{m+1}\frac{\mu_i\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=0}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\\ &\quad+\sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}+\sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=0}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}. \end{aligned}\tag{2.7}$$

Proof. Suppose $u \in PC$ is a solution of the impulsive (ρ_k,ψ_k)-Hilfer problem (2.5).

For $t \in [t_0,t_1]$, we have

$$u(t)=\frac{\Psi_{\psi_0}^{\gamma_0-1}(t,t_0)}{\Gamma_{\rho_0}(\rho_0\gamma_0)}c_1+\frac{\Psi_{\psi_0}^{\gamma_0-2}(t,t_0)}{\Gamma_{\rho_0}(\rho_0(\gamma_0-1))}c_2+{}_{\rho_0}I_{t_0}^{\alpha_0;\psi_0}\mathfrak{h}(t),$$

where $c_1={}_{\rho_0}^{RL}\mathfrak{D}_{t_0}^{\rho_0(\gamma_0-1);\psi_0}u(t_0^+)$ and $c_2={}_{\rho_0}I_{t_0}^{\rho_0(2-\gamma_0);\psi_0}u(t_0^+)$. By using Lemma 2.1 and Lemma 2.3, we obtain

$${}_{\rho_0}I_{t_0}^{\rho_0(2-\gamma_0);\psi_0}u(t) = \frac{\Psi_{\psi_0}(t,t_0)}{\rho_0}c_1+c_2+{}_{\rho_0}I_{t_0}^{\alpha_0+\rho_0(2-\gamma_0);\psi_0}\mathfrak{h}(t),\tag{2.8}$$

$${}_{\rho_0}^{RL}\mathfrak{D}_{t_0}^{\rho_0(\gamma_0-1);\psi_0}u(t) = c_1+{}_{\rho_0}I_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}\mathfrak{h}(t).\tag{2.9}$$

Putting $t=t_1$ into (2.8) and (2.9), we have

$${}_{\rho_0}I_{t_0}^{\rho_0(2-\gamma_0);\psi_0}u(t_1) = \frac{\Psi_{\psi_0}(t_1,t_0)}{\rho_0}c_1+c_2+{}_{\rho_0}I_{t_0}^{\alpha_0+\rho_0(2-\gamma_0);\psi_0}\mathfrak{h}(t_1),\tag{2.10}$$

$${}_{\rho_0}^{RL}\mathfrak{D}_{t_0}^{\rho_0(\gamma_0-1);\psi_0}u(t_1) = c_1+{}_{\rho_0}I_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}\mathfrak{h}(t_1).\tag{2.11}$$

For $t\in(t_1,t_2]$, we obtain

$$u(t)=\frac{\Psi_{\psi_1}^{\gamma_1-1}(t,t_1)}{\Gamma_{\rho_1}(\rho_1\gamma_1)}{}_{\rho_1}^{RL}\mathfrak{D}_{t_1}^{\rho_1(\gamma_1-1);\psi_1}u(t_1^+)+\frac{\Psi_{\psi_1}^{\gamma_1-2}(t,t_1)}{\Gamma_{\rho_1}(\rho_1(\gamma_1-1))}{}_{\rho_1}I_{t_1}^{\rho_1(2-\gamma_1);\psi_1}u(t_1^+)+{}_{\rho_1}I_{t_1}^{\alpha_1;\psi_1}\mathfrak{h}(t).$$

From the impulsive conditions, that is ${}_{\rho_1}^{RL}\mathfrak{D}_{t_1^+}^{\rho_1(\gamma_1-1);\psi_1}u(t_1^+)={}_{\rho_0}^{RL}\mathfrak{D}_{t_0^+}^{\rho_0(\gamma_0-1);\psi_0}u(t_1)+\phi_1(u(t_1))$ and ${}_{\rho_1}I_{t_1^+}^{\rho_1(2-\gamma_1);\psi_1}u(t_1^+)={}_{\rho_0}I_{t_0^+}^{\rho_0(2-\gamma_0);\psi_0}u(t_1)+\phi_1^*(u(t_1))$, it implies that

$$\begin{aligned}u(t) &= \left(\frac{\Psi_{\psi_1}^{\gamma_1-1}(t,t_1)}{\Gamma_{\rho_1}(\rho_1\gamma_1)}+\frac{\Psi_{\psi_1}^{\gamma_1-2}(t,t_1)}{\Gamma_{\rho_1}(\rho_1(\gamma_1-1))}\cdot\frac{\Psi_{\psi_0}(t_1,t_0)}{\rho_0}\right)c_1+\frac{\Psi_{\psi_1}^{\gamma_1-2}(t,t_1)}{\Gamma_{\rho_1}(\rho_1(\gamma_1-1))}c_2\\ &\quad+\frac{\Psi_{\psi_1}^{\gamma_1-1}(t,t_1)}{\Gamma_{\rho_1}(\rho_1\gamma_1)}\left({}_{\rho_0}I_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}\mathfrak{h}(t_1)+\phi_1(u(t_1))\right)\\ &\quad+\frac{\Psi_{\psi_1}^{\gamma_1-2}(t,t_1)}{\Gamma_{\rho_1}(\rho_1(\gamma_1-1))}\left({}_{\rho_0}I_{t_0}^{\alpha_0+\rho_0(2-\gamma_0);\psi_0}\mathfrak{h}(t_1)+\phi_1^*(u(t_1))\right)+{}_{\rho_1}I_{t_1}^{\alpha_1;\psi_1}\mathfrak{h}(t).\end{aligned}$$

By applying Lemmas 2.1 and 2.3, we get

$${}_{\rho_1}I_{t_1}^{\rho_1(2-\gamma_1);\psi_1}u(t) = \left(\frac{\Psi_{\psi_1}(t,t_1)}{\rho_1}+\frac{\Psi_{\psi_0}(t_1,t_0)}{\rho_0}\right)c_1+c_2+{}_{\rho_0}I_{t_0}^{\alpha_0+\rho_0(2-\gamma_0);\psi_0}\mathfrak{h}(t_1)+\phi_1^*(u(t_1))$$

$$+\frac{\Psi_{\psi_1}(t,t_1)}{\rho_1}\left({}_{\rho_0}I_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}\mathfrak{h}(t_1)+\phi_1(u(t_1))\right)+{}_{\rho_1}I_{t_1}^{\alpha_1+\rho_1(2-\gamma_1);\psi_1}\mathfrak{h}(t),$$

$${}_{\rho_1}^{RL}\mathfrak{D}_{t_1}^{\rho_1(\gamma_1-1);\psi_1}u(t) = c_1+{}_{\rho_0}I_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}\mathfrak{h}(t_1)+\phi_1(u(t_1))+{}_{\rho_1}I_{t_1}^{\alpha_1-\rho_1(\gamma_1-1);\psi_1}\mathfrak{h}(t).$$

In particular for $t = t_2$, we have

$$\begin{aligned}{}_{\rho_1}I_{t_1}^{\rho_1(2-\gamma_1);\psi_1}u(t_2) &= \left(\frac{\Psi_{\psi_1}(t_2,t_1)}{\rho_1}+\frac{\Psi_{\psi_0}(t_1,t_0)}{\rho_0}\right)c_1+c_2+{}_{\rho_0}I_{t_0}^{\alpha_0+\rho_0(2-\gamma_0);\psi_0}\mathfrak{h}(t_1)+\phi_1^*(u(t_1))\\ &\quad+\frac{\Psi_{\psi_1}(t_2,t_1)}{\rho_1}\left({}_{\rho_0}I_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}\mathfrak{h}(t_1)+\phi_1(u(t_1))\right)+{}_{\rho_1}I_{t_1}^{\alpha_1+\rho_1(2-\gamma_1);\psi_1}\mathfrak{h}(t_2),\\ {}_{\rho_1}^{RL}\mathfrak{D}_{t_1}^{\rho_1(\gamma_1-1);\psi_1}u(t_2) &= c_1+{}_{\rho_0}I_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}\mathfrak{h}(t_1)+\phi_1(u(t_1))+{}_{\rho_1}I_{t_1}^{\alpha_1-\rho_1(\gamma_1-1);\psi_1}\mathfrak{h}(t_2).\end{aligned}$$

Under the impulsive conditions, ${}_{\rho_2}^{RL}\mathfrak{D}_{t_2^+}^{\rho_2(\gamma_2-1);\psi_2}u(t_2^+) = {}_{\rho_1}^{RL}\mathfrak{D}_{t_1^+}^{\rho_1(\gamma_1-1);\psi_1}u(t_2) + \phi_2(u(t_2))$ and ${}_{\rho_2}I_{t_2^+}^{\rho_2(2-\gamma_2);\psi_2}u(t_2^+) = {}_{\rho_1}I_{t_1^+}^{\rho_1(2-\gamma_1);\psi_1}u(t_2)+\phi_2^*(u(t_2))$, for $t\in(t_2,t_3]$, we get

$$\begin{aligned}u(t) &= \left(\frac{\Psi_{\psi_2}^{\gamma_2-1}(t,t_2)}{\Gamma_{\rho_2}(\rho_2\gamma_2)}+\frac{\Psi_{\psi_2}^{\gamma_2-2}(t,t_2)}{\Gamma_{\rho_2}(\rho_2(\gamma_2-1))}\sum_{j=0}^{1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right)c_1+\frac{\Psi_{\psi_2}^{\gamma_2-2}(t,t_2)}{\Gamma_{\rho_2}(\rho_2(\gamma_2-1))}c_2\\ &\quad+\frac{\Psi_{\psi_2}^{\gamma_2-1}(t,t_2)}{\Gamma_{\rho_2}(\rho_2\gamma_2)}\sum_{j=0}^{1}\left({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathfrak{h}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\right)\\ &\quad+\frac{\Psi_{\psi_2}^{\gamma_2-2}(t,t_2)}{\Gamma_{\rho_2}(\rho_2(\gamma_2-1))}\Bigg[\sum_{j=0}^{1}\left({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}H(t_{j+1})+\phi_{j+1}^*(u(t_{j+1}))\right)\\ &\quad+\frac{\Psi_{\psi_1}(t_2,t_1)}{\rho_1}\left({}_{\rho_0}I_{t_0}^{\alpha_0-\rho_0(\gamma_0-1);\psi_0}F_u(t_1)+\phi_1(x(t_1))\right)\Bigg]+{}_{\rho_2}I_{t_2}^{\alpha_2;\psi_2}\mathfrak{h}(t).\end{aligned}$$

Then for $t\in(t_3,t_4]$, we have

$$\begin{aligned}u(t) &= \left(\frac{\Psi_{\psi_3}^{\gamma_3-1}(t,t_3)}{\Gamma_{\rho_3}(\rho_3\gamma_3)}+\frac{\Psi_{\psi_3}^{\gamma_3-2}(t,t_3)}{\Gamma_{\rho_3}(\rho_3(\gamma_3-1))}\sum_{j=0}^{2}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right)c_1+\frac{\Psi_{\psi_3}^{\gamma_3-2}(t,t_3)}{\Gamma_{\rho_3}(\rho_3(\gamma_3-1))}c_2\\ &\quad+\frac{\Psi_{\psi_3}^{\gamma_3-1}(t,t_3)}{\Gamma_{\rho_3}(\rho_3\gamma_3)}\sum_{j=0}^{2}\left({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathfrak{h}(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\right)\\ &\quad+\frac{\Psi_{\psi_3}^{\gamma_3-2}(t,t_3)}{\Gamma_{\rho_3}(\rho_3(\gamma_3-1))}\Bigg[\sum_{j=0}^{2}\left({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathfrak{h}(t_{j+1})+\phi_{j+1}^*(u(t_{j+1}))\right)\\ &\quad+\sum_{j=1}^{2}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathfrak{h}(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\Bigg]+{}_{\rho_3}I_{t_3}^{\alpha_3;\psi_3}\mathfrak{h}(t).\end{aligned}$$

Repeating the above process, for any $t\in(t_k,t_{k+1}]$, $k=0,1,\ldots,m$, one has

$$u(t) = \left(\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right)c_1+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}c_2$$

$$+{}_{\rho_k}I_{t_k}^{\alpha_k;\psi_k}\mathfrak{h}(t) + \frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathfrak{h}(t_{j+1}) + \phi_{j+1}(u(t_{j+1}))\Big)$$
$$+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\Bigg[\sum_{j=0}^{k-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathfrak{h}(t_{j+1}) + \phi_{j+1}^{*}(u(t_{j+1}))\Big)$$
$$+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathfrak{h}(t_{r+1}) + \phi_{r+1}(u(t_{r+1}))\Big)\Bigg]. \tag{2.12}$$

By applying the first boundary condition, $u(0) = 0$, we get $c_2 = 0$. From the second boundary condition, $\sum_{i=0}^{m+1}\mu_i u(\eta_i) + \sum_{l=0}^{n}\lambda_{l\,\rho_l}I_{t_l}^{\theta_l;\psi_l}u(\xi_l) = A$, we have

$$c_1 = \frac{1}{\Lambda}\Bigg\{A - \Bigg(\sum_{i=0}^{m+1}\frac{\mu_i\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathfrak{h}(t_{j+1}) + \phi_{j+1}(u(t_{j+1}))\Big)$$
$$+\sum_{i=0}^{m+1}\frac{\mu_i\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\Bigg[\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathfrak{h}(t_{j+1}) + \phi_{j+1}^{*}(u(t_{j+1}))\Big)$$
$$+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathfrak{h}(t_{r+1}) + \phi_{r+1}(u(t_{r+1}))\Big)\Bigg] + \sum_{i=0}^{m+1}\mu_{i\,\rho_i}I_{t_i}^{\alpha_i;\psi_i}\mathfrak{h}(\eta_i)$$
$$+\sum_{l=0}^{n}\lambda_{l\,\rho_l}I_{t_l}^{\alpha_l+\theta_l;\psi_l}\mathfrak{h}(\xi_l) + \sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}\mathfrak{h}(t_{j+1}) + \phi_{j+1}(u(t_{j+1}))\Big)$$
$$+\sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Bigg[\sum_{j=0}^{l-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}\mathfrak{h}(t_{j+1}) + \phi_{j+1}^{*}(u(t_{j+1}))\Big)$$
$$+\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}\mathfrak{h}(t_{r+1}) + \phi_{r+1}(u(t_{r+1}))\Big)\Bigg]\Bigg)\Bigg\},$$

where Λ is given by (2.7). Taking the values c_1 and c_2 in (2.12), we obtain the solution (2.6).

3. Existence results

By applying Lemma 2.4 and $F_u(t) = f(t, u(t), {}_{\rho_k}I_{t_k}^{\sigma_k;\psi_k}u(t), {}_{\rho_k}I_{t_k}^{\nu_k;\psi_k}u(t))$, we set an operator $Q : PC \to PC$ by

$$(Qu)(t) = \Bigg\{\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Lambda\Gamma_{\rho_k}(\rho_k\gamma_k)} + \frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Lambda\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\Bigg\}\Bigg\{A$$
$$-\Bigg(\sum_{i=0}^{m+1}\frac{\mu_i\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}F_u(t_{j+1}) + \phi_{j+1}(u(t_{j+1}))\Big)$$
$$+\sum_{i=0}^{m+1}\frac{\mu_i\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\Bigg[\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}F_u(t_{j+1}) + \phi_{j+1}^{*}(u(t_{j+1}))\Big)$$

$$+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}F_u(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\Bigg]$$

$$+\sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\left({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}F_u(t_{j+1})+\phi_{j+1}(u(t_{j+1}))\right)$$

$$+\sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Bigg[\sum_{j=0}^{l-1}\left({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}F_u(t_{j+1})+\phi_{j+1}^*(u(t_{j+1}))\right)$$

$$+\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}F_u(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\Bigg]+\sum_{i=0}^{m+1}\mu_{i\,\rho_i}I_{t_i}^{\alpha_i;\psi_i}F_u(\eta_i)$$

$$+\sum_{l=0}^{n}\lambda_{l\,\rho_l}I_{t_l}^{\alpha_l+\theta_l;\psi_l}F_u(\xi_l)\Bigg)\Bigg\}+{}_{\rho_k}I_{t_k}^{\alpha_k;\psi_k}F_u(t)+\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}F_u(t_{j+1})$$

$$+\phi_{j+1}(u(t_{j+1})))+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\Bigg[\sum_{j=0}^{k-1}\left({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}F_u(t_{j+1})+\phi_{j+1}^*(u(t_{j+1}))\right)$$

$$+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}F_u(t_{r+1})+\phi_{r+1}(u(t_{r+1}))\right)\Bigg]. \quad (3.1)$$

Note that, the considered problem (1.4) has a solution if and only if Q has fixed points.

We assign notation for constants that will be used throughout this work

$$\Omega_1 := \frac{\Psi_{\psi_m}(T,t_m)}{|\Lambda|\,\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{|\Lambda|\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}, \quad (3.2)$$

$$\Omega_2 := \sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i+\alpha_i)}+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{j=0}^{i-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}$$

$$+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=0}^{i-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}$$

$$+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\alpha_l+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l+\alpha_l+\theta_l)}+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}$$

$$+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=0}^{l-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}$$

$$+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}$$

$$+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}, \tag{3.3}$$

$$\begin{aligned}\Omega_3 \;:=\; & \frac{\Psi_{\psi_m}^{\frac{\alpha_m}{\rho_m}+2-\gamma_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\alpha_m)}+\frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}\\ & +\frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}\\ & +\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\sum_{r=0}^{j-1}\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\Omega_4 \;:=\; & \sum_{i=0}^{m+1}\frac{i|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=1}^{i-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\\ & +\sum_{l=0}^{n}\frac{l|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=1}^{l-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j},\end{aligned} \tag{3.5}$$

$$\Omega_5 \;:=\; \frac{m\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=1}^{m-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}, \tag{3.6}$$

$$\Omega_6 \;:=\; \sum_{i=0}^{m+1}\frac{i|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}+\sum_{l=0}^{n}\frac{l|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}. \tag{3.7}$$

3.1. Uniqueness result under Banach's fixed point theorem

Lemma 3.1. *(Banach's fixed point theorem [42]) Let D be a non-empty closed subset of a Banach space $\mathfrak{E}$. Then any contraction mapping Q from D into itself has a unique fixed-point.*

Theorem 3.1. *Assume $\psi_k \in C^2(J)$ where $\psi_k'(t) > 0$, $k = 0, 1, 2, \ldots, m$, $t \in J$ and $f \in C(J \times R^3, R)$, ϕ_k, $\phi_k^* \in C(R, R)$, $k = 1, 2, \ldots, m$, corresponding to the following conditions:*

(H_1) *There are real constants $L_i > 0$, $i = 1, 2, 3$, so that, for any $t \in J$ and u_i, $v_i \in R$, $i = 1, 2, 3$,*

$$|f(t,u_1,u_2,u_3)-f(t,v_1,v_2,v_3)| \le \Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\sum_{i=1}^{3}L_i\,|u_i-v_i|\,.$$

(H_2) *There are real constants $I_i > 0$, $i = 1, 2$, so that, for any $t \in J$ and u, $v \in R$, $k = 1, 2, \ldots, m$,*

$$|\phi_k(u)-\phi_k(v)| \le I_1\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,|u-v|\,, \quad \left|\phi_k^*(u)-\phi_k^*(v)\right| \le I_2\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,|u-v|\,.$$

Then, the considered problem (1.4) *has a unique solution on J, if*

$$\Delta_1+\Delta_2<1, \tag{3.8}$$

where

$$\Delta_1 := (\Omega_1\Omega_2 + \Omega_3)(L_1 + \Psi_*^{\sigma_m} L_2 + \Psi_*^{\nu_m} L_3), \tag{3.9}$$

$$\Delta_2 := (\Omega_1\Omega_4 + \Omega_5)I_1 + (\Omega_1\Omega_6 + m\Psi_*^{\gamma_m})I_2, \tag{3.10}$$

$$\Psi_*^{\sigma_m} := \frac{\Psi_{\psi_m}^{\frac{\sigma_m}{\rho_m}}(T, t_m)}{\Gamma_{\rho_m}(\rho_m + \sigma_m)}, \quad \Psi_*^{\nu_m} := \frac{\Psi_{\psi_m}^{\frac{\nu_m}{\rho_m}}(T, t_m)}{\Gamma_{\rho_m}(\rho_m + \nu_m)}, \quad \Psi_*^{\gamma_m} := \frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m - 1))}. \tag{3.11}$$

Proof. Clearly, the considered problem (1.4) is corresponding to fixed-point problem $u = Qu$. Then we will show that Q has a fixed-point by the Banach's fixed-point theorem.

Define constants M_i, $i = 1, 2, 3$, by $M_1 := \sup_{t\in J} |f(t, 0, 0, 0)|$, $M_2 := \max\{\phi_k(0) : k = 1, 2, \ldots, m\}$ and $M_3 := \max\left\{\phi_k^*(0) : k = 1, 2, \ldots, m\right\}$. Let $\mathfrak{B}_{\Re_1} := \{u \in \mathfrak{E} : \|u\| \leq \Re_1\}$ where

$$\Re_1 \geq \frac{(\Omega_1\Omega_2 + \Omega_3)M_1 + (\Omega_1\Omega_4 + \Omega_5)M_2 + (\Omega_1\Omega_6 + m\Psi_*^{\gamma_m})M_3 + \Omega_1 |A|}{1 - (\Delta_1 + \Delta_2)}.$$

The remaining proof is divided into two steps:

Step I. We will prove that $Q\mathfrak{B}_{\Re_1} \subset \mathfrak{B}_{\Re_1}$.

Suppose that $u \in \mathfrak{B}_{\Re_1}$ and $t \in J$, we obtain

$$\begin{aligned}
\left|\Psi_{\psi_k}^{2-\gamma_k}(t, t_k)(Qu)(t)\right| \leq\ & \left\{\frac{\Psi_{\psi_k}(t, t_k)}{|\Lambda|\,\Gamma_{\rho_k}(\rho_k\gamma_k)} + \sum_{j=0}^{k-1} \frac{\Psi_{\psi_j}(t_{j+1}, t_j)}{|\Lambda|\,\rho_j\Gamma_{\rho_k}(\rho_k(\gamma_k - 1))}\right\}\Bigg\{ |A| \\
& + \sum_{i=0}^{m+1} \frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i, t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)} \sum_{j=0}^{i-1} \left({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|F_u(t_{j+1})| + |\phi_{j+1}(u(t_{j+1}))|\right) \\
& + \sum_{i=0}^{m+1} \frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i, t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i - 1))}\Bigg[\sum_{j=0}^{i-1} \left({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|F_u(t_{j+1})| + |\phi_{j+1}^*(u(t_{j+1}))|\right) \\
& + \sum_{j=1}^{i-1} \frac{\Psi_{\psi_j}(t_{j+1}, t_j)}{\rho_j} \sum_{r=0}^{j-1} \left({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|F_u(t_{r+1})| + |\phi_{r+1}(u(t_{r+1}))|\right)\Bigg] \\
& + \sum_{l=0}^{n} \frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l, t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l + \theta_l)} \sum_{j=0}^{l-1} \left({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|F_u(t_{j+1})| + |\phi_{j+1}(u(t_{j+1}))|\right) \\
& + \sum_{l=0}^{n} \frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l, t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l - 1) + \theta_l)}\Bigg[\sum_{j=0}^{l-1} \left({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|F_u(t_{j+1})| + |\phi_{j+1}^*(u(t_{j+1}))|\right) \\
& + \sum_{j=1}^{l-1} \frac{\Psi_{\psi_j}(t_{j+1}, t_j)}{\rho_j} \sum_{r=0}^{j-1} \left({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|F_u(t_{r+1})| + |\phi_{r+1}(u(t_{r+1}))|\right)\Bigg] \\
& + \sum_{i=0}^{m+1} |\mu_i|{}_{\rho_i}I_{t_i}^{\alpha_i;\psi_i}|F_u(\eta_i)| + \sum_{l=0}^{n} |\lambda_l|{}_{\rho_l}I_{t_l}^{\alpha_l+\theta_l;\psi_l}|F_u(\xi_l)|\Bigg\} \\
& + \Psi_{\psi_k}^{2-\gamma_k}(t, t_k){}_{\rho_k}I_{t_k}^{\alpha_k;\psi_k}|F_u(t)| + \frac{\Psi_{\psi_k}(t, t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)} \sum_{j=0}^{k-1} \left({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|F_u(t_{j+1})| + |\phi_{j+1}(u(t_{j+1}))|\right) \\
& + \frac{1}{\Gamma_{\rho_k}(\rho_k(\gamma_k - 1))}\Bigg[\sum_{j=0}^{k-1} \left({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|F_u(t_{j+1})| + |\phi_{j+1}^*(u(t_{j+1}))|\right)
\end{aligned}$$

$$+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|F_u(t_{r+1})|+|\phi_{r+1}(u(t_{r+1}))|\Big)\Bigg]. \quad (3.12)$$

By using the property (i) in Lemma 2.1, we get

$$\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\left|{}_{\rho_k}I_{t_k^+}^{\sigma_k;\psi_k}u(t)\right|\le {}_{\rho_k}I_{t_k^+}^{\sigma_k;\psi_k}(1)(t)\|u\|_{PC}\le\frac{\Psi_{\psi_m}^{\frac{\sigma_m}{\rho_m}}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\sigma_m)}\|u\|_{PC}. \quad (3.13)$$

From the conditions (H_1), (H_2) and (3.13), we can find that

$$\begin{aligned}
|F_u(t)| &\le \left|f(t,u(t),{}_{\rho_k}I_{t_k^+}^{\sigma_k;\psi_k}u(t),{}_{\rho_k}I_{t_k}^{\nu_k;\psi_k}u(t))-f(t,0,0,0)\right|+|f(t,0,0,0)|\\
&\le L_1\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,|u(t)|+L_2\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\left|{}_{\rho_k}I_{t_k^+}^{\sigma_k;\psi_k}u(t)\right|+L_3\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\left|{}_{\rho_k}I_{t_k^+}^{\nu_k;\psi_k}u(t)\right|+M_1\\
&\le \left(L_1+\frac{\Psi_{\psi_m}^{\frac{\sigma_m}{\rho_m}}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\sigma_m)}L_2+\frac{\Psi_{\psi_m}^{\frac{\nu_m}{\rho_m}}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\nu_m)}L_3\right)\|u\|_{PC}+M_1\\
&= (L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u\|_{PC}+M_1, \qquad (3.14)\\
|\phi_k(u(t_k))| &\le |\phi_k(u(t_k))-\phi_k(0)|+|\phi_k(0)|\le I_1\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,|u(t)|+M_2\le I_1\|u\|_{PC}+M_2, \qquad (3.15)\\
\left|\phi_k^*(u(t_k))\right| &\le \left|\phi_k^*(u(t_k))-\phi_k^*(0)\right|+\left|\phi_k^*(0)\right|\le I_2\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\,|u(t)|+M_3\le I_2\|u\|_{PC}+M_3. \qquad (3.16)
\end{aligned}$$

Inserting (3.14)–(3.16) into (3.12), we see that

$$\begin{aligned}
&\left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(Qu)(t)\right|\\
\le\ &\left\{\frac{\Psi_{\psi_k}(t,t_k)}{|\Lambda|\,\Gamma_{\rho_k}(\rho_k\gamma_k)}+\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{|\Lambda|\,\rho_j\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\right\}\Bigg\{|A|+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\\
&\times\sum_{j=0}^{i-1}\left({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}(1)(t_{j+1})\Big[(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u\|_{PC}+M_1\Big]+I_1\|u\|_{PC}+M_2\right)\\
&+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\Bigg[\sum_{j=0}^{i-1}\Bigg({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}(1)(t_{j+1})\Big[(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u\|_{PC}+M_1\Big]\\
&+I_2\|u\|_{PC}+M_3\Bigg)+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Bigg({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}(1)(t_{r+1})\Big[(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u\|_{PC}+M_1\Big]\\
&+I_1\|u\|_{PC}+M_2\Bigg)\Bigg]+\sum_{i=0}^{m+1}|\mu_i|{}_{\rho_i}I_{t_i}^{\alpha_i;\psi_i}(1)(\eta_i)\Big[(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u\|_{PC}+M_1\Big]\\
&+\sum_{l=0}^{n}|\lambda_l|{}_{\rho_l}I_{t_l}^{\alpha_l+\theta_l;\psi_l}(1)(\xi_l)\Big[(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u\|_{PC}+M_1\Big]\\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\Bigg({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}(1)(t_{j+1})\Big[(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u\|_{PC}+M_1\Big]\\
&+I_1\|u\|_{PC}+M_2\Bigg)+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Bigg[\sum_{j=0}^{l-1}\Bigg({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}(1)(t_{j+1})\\
&\times\Big[(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u\|_{PC}+M_1\Big]+I_2\|u\|_{PC}+M_3\Bigg)
\end{aligned}$$

$$+\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}(1)(t_{r+1})\Big[(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u\|_{PC}+M_1\Big]$$

$$+I_1\|u\|_{PC}+M_2\Big)\Big]\Big\}+\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)_{\rho_k}I_{t_k}^{\alpha_k;\psi_k}(1)(t)\Big[(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u\|_{PC}+M_1\Big]$$

$$+\frac{\Psi_{\psi_k}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}(1)(t_{j+1})\Big[(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u\|_{PC}+M_1\Big]+I_1\|u\|_{PC}+M_2\Big)$$

$$+\frac{1}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\Big[\sum_{j=0}^{k-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}(1)(t_{j+1})\Big[(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u\|_{PC}+M_1\Big]+I_2\|u\|_{PC}+M_3\Big)$$

$$+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}(1)(t_{r+1})\Big[(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u\|_{PC}+M_1\Big]+I_1\|u\|_{PC}+M_2\Big)\Big].$$

From the property (i) in Lemma 2.1, it implies that

$$\begin{aligned}
&\Big|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(Qu)(t)\Big| \\
\leq\;& \Big[(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u\|_{PC}+M_1\Big]\Big[\Big(\frac{\Psi_{\psi_m}(T,t_m)}{|\Lambda|\,\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{|\Lambda|\,\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Big) \\
&\times\Big(\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i+\alpha_i)}+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{j=0}^{i-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\alpha_l+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l+\alpha_l+\theta_l)} \\
&+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=0}^{i-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))} \\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=0}^{l-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))} \\
&+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))} \\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}\Big)+\frac{\Psi_{\psi_m}^{\frac{\alpha_m}{\rho_m}+2-\gamma_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\alpha_m)} \\
&+\frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}+\frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))} \\
&+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\sum_{r=0}^{j-1}\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}\Big]+\Big[\Big(\frac{\Psi_{\psi_m}(T,t_m)}{|\Lambda|\,\Gamma_{\rho_m}(\rho_m\gamma_m)} \\
&+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{|\Lambda|\,\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Big)\Big(\sum_{i=0}^{m+1}\frac{i|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}+\sum_{l=0}^{n}\frac{l|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)} \\
&+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=1}^{i-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=1}^{l-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\Big)+\frac{m\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}
\end{aligned}$$

$$
\begin{aligned}
& +\sum_{j=1}^{m-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Big]\Big[I_1\|u\|_{PC}+M_2\Big]+\Big[\Big(\frac{\Psi_{\psi_m}(T,t_m)}{|\Lambda|\,\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{|\Lambda|\,\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Big) \\
& \times\Big(\sum_{i=0}^{m+1}\frac{i|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}+\sum_{l=0}^{n}\frac{l|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Big)+\frac{m}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Big]\Big[I_2\|u\|_{PC}+M_3\Big] \\
& +\Big(\frac{\Psi_{\psi_m}(T,t_m)}{|\Lambda|\,\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{|\Lambda|\,\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Big)|A| \\
= \;& \Omega_1\,|A|+\Big[\Omega_1\Omega_2+\Omega_3\Big]\Big[(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u\|_{PC}+M_1\Big] \\
& +\Big[\Omega_1\Omega_4+\Omega_5\Big]\Big[I_1\|u\|_{PC}+M_2\Big]+\Big[\Omega_1\Omega_6+m\Psi_*^{\gamma_m}\Big]\Big[I_2\|u\|_{PC}+M_3\Big] \\
\le \;& \Big[(\Omega_1\Omega_2+\Omega_3)(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)+(\Omega_1\Omega_4+\Omega_5)I_1+(\Omega_1\Omega_6+m\Psi_*^{\gamma_m})I_2\Big]\Re_1 \\
& +(\Omega_1\Omega_2+\Omega_3)M_1+(\Omega_1\Omega_4+\Omega_5)M_2+(\Omega_1\Omega_6+m\Psi_*^{\gamma_m})M_3+\Omega_1\,|A|\le\Re_1.
\end{aligned}
$$

Hence, $\|Qu\|_{PC}\le\Re_1$, which yields that $Q\mathfrak{B}_{\Re_1}\subset\mathfrak{B}_{\Re_1}$.

Step II. We will prove that Q is a contraction.

Suppose that $u,\,v\in\mathfrak{B}_{\Re_1}$ and $t\in J$, we have

$$
\begin{aligned}
& \Big|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(Qu)(t)-\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(Qv)(t)\Big| \\
\le \;& \Big\{\frac{\Psi_{\psi_k}(t,t_k)}{|\Lambda|\,\Gamma_{\rho_k}(\rho_k\gamma_k)}+\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{|\Lambda|\,\rho_j\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\Big\}\Big\{\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)} \\
& \times\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|F_u(t_{j+1})-F_v(t_{j+1})|+|\phi_{j+1}(u(t_{j+1}))-\phi_{j+1}(v(t_{j+1}))|\Big) \\
& +\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\Big[\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|F_u(t_{j+1})-F_v(t_{j+1})|+|\phi_{j+1}^*(u(t_{j+1}))-\phi_{j+1}^*(v(t_{j+1}))|\Big) \\
& +\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|F_u(t_{r+1})-F_v(t_{r+1})|+|\phi_{r+1}(u(t_{r+1}))-\phi_{r+1}(v(t_{r+1}))|\Big)\Big] \\
& +\sum_{i=0}^{m+1}|\mu_i|{}_{\rho_i}I_{t_i}^{\alpha_i;\psi_i}|F_u(\eta_i)-F_v(\eta_i)|+\sum_{l=0}^{n}|\lambda_l|{}_{\rho_l}I_{t_l}^{\alpha_l+\theta_l;\psi_l}|F_u(\xi_l)-F_v(\xi_l)| \\
& +\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|F_u(t_{j+1})-F_v(t_{j+1})|+|\phi_{j+1}(u(t_{j+1}))-\phi_{j+1}(v(t_{j+1}))|\Big) \\
& +\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Big[\sum_{j=0}^{l-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|F_u(t_{j+1})-F_v(t_{j+1})|+|\phi_{j+1}^*(u(t_{j+1}))-\phi_{j+1}^*(v(t_{j+1}))|\Big) \\
& +\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|F_u(t_{r+1})-F_v(t_{r+1})|+|\phi_{r+1}(u(t_{r+1}))-\phi_{r+1}(v(t_{r+1}))|\Big)\Big]\Big\} \\
& +{}_{\rho_k}I_{t_k}^{\alpha_k;\psi_k}|F_u(t)-F_v(t)|+\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|F_u(t_{j+1})-F_v(t_{j+1})|
\end{aligned}
$$

$$+|\phi_{j+1}(u(t_{j+1}))-\phi_{j+1}(v(t_{j+1}))|\Big)+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\Bigg[\sum_{j=0}^{k-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|F_u(t_{j+1})-F_v(t_{j+1})|$$

$$+|\phi^*_{j+1}(u(t_{j+1}))-\phi^*_{j+1}(v(t_{j+1}))|\Big)+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|F_u(t_{r+1})-F_v(t_{r+1})|$$

$$+|\phi_{r+1}(u(t_{r+1}))-\phi_{r+1}(v(t_{r+1}))|\Big)\Bigg]. \tag{3.17}$$

By applying the property (i) in Lemma 2.1, we have

$$\Psi_{\psi_k}^{2-\gamma_k}(t,t_k){}_{\rho_k}I_{t_k^+}^{\sigma_k;\psi_k}|u(t)-v(t)|\leq{}_{\rho_k}I_{t_k^+}^{\sigma_k;\psi_k}(1)(t)\|u\|_{PC}\leq\frac{\Psi_{\psi_m}^{\frac{\sigma_m}{\rho_m}}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\sigma_m)}\|u-v\|_{PC}. \tag{3.18}$$

Using the conditions (H_1), (H_2) and (3.18), we can find that

$$\begin{aligned}
|F_u(t)-F_v(t)| &\leq \Big|f(t,u(t),{}_{\rho_k}I_{t_k^+}^{\sigma_k;\psi_k}u(t),{}_{\rho_k}I_{t_k}^{\nu_k;\psi_k}u(t))-f(t,v(t),{}_{\rho_k}I_{t_k^+}^{\sigma_k;\psi_k}v(t),{}_{\rho_k}I_{t_k}^{\nu_k;\psi_k}v(t))\Big|\\
&\leq L_1\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)|u(t)-v(t)|+L_2\Psi_{\psi_k}^{2-\gamma_k}(t,t_k){}_{\rho_k}I_{t_k^+}^{\sigma_k;\psi_k}|u(t)-v(t)|\\
&\quad+L_3\Psi_{\psi_k}^{2-\gamma_k}(t,t_k){}_{\rho_k}I_{t_k^+}^{\nu_k;\psi_k}|u(t)-v(t)|\\
&\leq (L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u-v\|_{PC}, &(3.19)\\
|\phi_k(u(t_k))-\phi_k(v(t_k))| &\leq I_1\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)|u(t)-v(t)|\leq I_1\|u-v\|_{PC}, &(3.20)\\
\Big|\phi^*_k(u(t_k))-\phi^*_k(v(t_k))\Big| &\leq I_2\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)|u(t)-v(t)|\leq I_2\|u-v\|_{PC}. &(3.21)
\end{aligned}$$

Inserting (3.19)–(3.21) into (3.17), which yields that

$$\begin{aligned}
&\Big|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)((Qu_n)(t)-(Qu)(t))\Big|\\
\leq\ &\Bigg\{\frac{\Psi_{\psi_m}(T,t_m)}{|\Lambda|\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{|\Lambda|\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg\}\Bigg\{\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\\
&\times\sum_{j=0}^{i-1}\Bigg(\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_J}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u-v\|_{PC}+I_1\|u-v\|_{PC}\Bigg)\\
&+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\Bigg[\sum_{j=0}^{i-1}\Bigg(\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u-v\|_{PC}\\
&+I_2\|u-v\|_{PC}\Bigg)+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Bigg(\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u-v\|_{PC}\\
&+I_1\|u-v\|_{PC}\Bigg)\Bigg]+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i+\alpha_i)}(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u-v\|_{PC}\\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\alpha_l+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l+\alpha_l+\theta_l)}(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u-v\|_{PC}+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}
\end{aligned}$$

$$\times\sum_{j=0}^{l-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_J}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u-v\|_{PC}+I_1\|u-v\|_{PC}\right)$$

$$+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Bigg[\sum_{j=0}^{l-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u-v\|_{PC}\right.$$

$$\left.+I_2\|u-v\|_{PC}\right)+\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left(\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u-v\|_{PC}\right.$$

$$\left.\left.+I_1\|u-v\|_{PC}\right)\Bigg]\Bigg\}+\frac{\Psi_{\psi_m}^{\frac{\alpha_m}{\rho_m}+2-\gamma_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\alpha_m)}(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u-v\|_{PC}$$

$$+\frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_J}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u-v\|_{PC}+I_1\|u-v\|_{PC}\right)$$

$$+\frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg[\sum_{j=0}^{m-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j+\rho_j(2-\gamma_j))}(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u-v\|_{PC}+I_2\|u-v\|_{PC}\right)$$

$$+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left(\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)\|u-v\|_{PC}+I_1\|u-v\|_{PC}\right)\Bigg]$$

$$\leq\ \Big[(\Omega_1\Omega_2+\Omega_3)(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)+(\Omega_1\Omega_4+\Omega_5)I_1+(\Omega_1\Omega_6+m\Psi_*^{\gamma_m})I_2\Big]\|u-v\|_{PC}.$$

It follows that $\|Qu-Qv\|_{PC}\leq[\Delta_1+\Delta_2]\|u-v\|_{PC}$. Condition (3.8) stated that $\Delta_1+\Delta_2<1$. Thus Q is a contraction. By Lemma 3.1, problem (1.4) has a unique solution on J.

3.2. Existence result under O'Regan's fixed point theorem

Lemma 3.2. *(O'Regan's fixed point theorem [43]) Let O be an open subset of a closed, convex set D in a Banach space $\mathfrak{E}$ such that $0\in O$. Moreover, let $Q:\overline{O}\to D$ be such that $Q(\overline{O})$ is bounded and that $Q=Q_1+Q_2$, where $Q_1:\overline{O}\to D$ is continuous and completely continuous and $Q_2:\overline{O}\to D$ is a nonlinear contraction, i.e., there exists a nonnegative nondecreasing function $\Theta:[0,\infty)\to[0,\infty)$, such that $\Theta(w)<w$ for any $w\in R^+$ and $\|Q_2u-Q_2v\|\leq\Theta(\|u-v\|)$ for all u, $v\in\overline{O}$. Then either (a_1). Q has a fixed point $u\in\overline{O}$ or (a_2). there exist a point $u\in\partial K$ and $\theta\in(0,1)$, such that $u=\theta Qu$. Here, $\overline{O}$ and ∂O represent the closure and the boundary of O, respectively.*

Theorem 3.2. *Assume $\psi_k\in C^2(J)$ where $\psi_k'(t)>0$, $k=0,1,2,\ldots,m$, $t\in J$, $f\in C(J\times R^3,R)$, ϕ_k, $\phi_k^*\in C(R,R)$, $k=1,2,\ldots,m$ satisfying the following conditions:*

(H_3) *There exist positive real numbers M_1, M_2 such that*

$$|\phi_k(u)|\leq M_1,\quad |\phi_k^*(u)|\leq M_2,\quad u\in R. \tag{3.22}$$

(H_4) *There exist a continuous nondecreasing function $\Theta:[0,\infty)\to[0,\infty)$ and $g_i\in C(J,R^+)$, $i=1,2,3$,* *such that*

$$|f(t,u,v,w)| \le g_1(t)\Theta\left(\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)|u|\right) + \Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\Big[g_2(t)|v| + g_3(t)|w|\Big], \tag{3.23}$$

for any $u, v, w \in R$, $t \in J$, $k = 1, 2, \dots, m$.

(H_5) *There exist continuous nondecreasing functions $K_i : [0,\infty) \to [0,\infty)$, and Ξ_i, $i = 1, 2$, such that*

$$|\phi_k(u) - \phi_k(v)| \le K_1\left(\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)\,|u-v|\right),\ \left|\phi_k^*(u) - \phi_k^*(v)\right| \le K_2\left(\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)\,|u-v|\right),$$
$$K_1\left(\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)\,|u|\right) \le \Xi_1\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)\,|u|,\ K_2\left(\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)\,|u|\right) \le \Xi_2\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)\,|u|,$$

for any $u, v \in R$, $k = 1, 2, \dots, m$, satisfying $[(\Omega_5 + \Omega_1\Omega_4)\Xi_1 + (m\Psi_*^{\gamma_m} + \Omega_1\Omega_6)\Xi_2] < 1$ where Ω_1, Ω_4, Ω_5, Ω_6 are given by (3.2) *and* (3.5)–(3.7)*, respectively.*

(H_6)

$$\sup_{\Re_2\in(0,\infty)} \frac{\Re_2}{g_1^*\Theta(\Re_2)(\Omega_3 + \Omega_1\Omega_2) + \mathfrak{C}^*} > \frac{1}{1 - [g_2^*\Psi_*^{\sigma_m} + g_3^*\Psi_*^{\nu_m}](\Omega_3 + \Omega_1\Omega_2)}, \tag{3.24}$$

with $[g_2^*\Psi_*^{\sigma_m} + g_3^*\Psi_*^{\nu_m}](\Omega_3 + \Omega_1\Omega_2) < 1$, Ω_i are given by (3.2)–(3.4)*, respectively, and $g_i^* = \sup_{t\in J}|g_i(t)|$, $i = 1, 2, 3$.*

Then the considered problem (1.4) *has at least one solution on J.*

Proof. We will divide the operator $Q : PC \to PC$ defined by (3.1) into two operators, that is Q_1 and Q_2, where $(Qu)(t) = (Q_1u)(t) + (Q_2u)(t)$, for any $t \in J$. The operators Q_1 and Q_2 are defined by

$$\begin{aligned}
(Q_1u)(t) \quad = \quad & \frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}{}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}F_u(t_{j+1}) + \frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=0}^{k-1}{}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}F_u(t_{j+1}) \\
& + {}_{\rho_k}I_{t_k}^{\alpha_k;\psi_k}F_u(t) + \frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}{}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}F_u(t_{r+1}) \\
& - \left\{\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Lambda\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j} + \frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Lambda\Gamma_{\rho_k}(\rho_k\gamma_k)}\right\}\left\{\sum_{i=0}^{m+1}\mu_i{}_{\rho_i}I_{t_i}^{\alpha_i;\psi_i}F_u(\eta_i)\right. \\
& + \sum_{l=0}^{n}\lambda_l{}_{\rho_l}I_{t_l}^{\alpha_l+\theta_l;\psi_l}F_u(\xi_l) + \sum_{i=0}^{m+1}\frac{\mu_i\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{j=0}^{i-1}{}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}F_u(t_{j+1}) \\
& + \sum_{i=0}^{m+1}\frac{\mu_i\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}{}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}F_u(t_{r+1}) \\
& + \sum_{i=0}^{m+1}\frac{\mu_i\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=0}^{i-1}{}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}F_u(t_{j+1}) + \sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)} \\
& \times\sum_{j=0}^{l-1}{}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}F_u(t_{j+1}) + \sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=0}^{l-1}{}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}F_u(t_{j+1}) \\
& \left. + \sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}{}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}F_u(t_{r+1})\right\},
\end{aligned} \tag{3.25}$$

$$
\begin{aligned}
(Q_2u)(t) &= \frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\phi_{j+1}(u(t_{j+1})) + \frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\phi_{r+1}(u(t_{r+1}))\\
&+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=0}^{k-1}\phi^*_{j+1}(u(t_{j+1})) - \left\{\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Lambda\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right.\\
&\left.+\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Lambda\Gamma_{\rho_k}(\rho_k\gamma_k)}\right\}\left\{\sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\phi_{r+1}(u(t_{r+1}))\right.\\
&+\sum_{i=0}^{m+1}\frac{\mu_i\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{j=0}^{i-1}\phi_{j+1}(u(t_{j+1})) + \sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\phi_{j+1}(u(t_{j+1}))\\
&+\sum_{i=0}^{m+1}\frac{\mu_i\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\phi_{r+1}(u(t_{r+1})) + \sum_{i=0}^{m+1}\frac{\mu_i\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=0}^{i-1}\phi^*_{j+1}(u(t_{j+1}))\\
&\left.+\sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=0}^{l-1}\phi^*_{j+1}(u(t_{j+1})) - A\right\}. \qquad (3.26)
\end{aligned}
$$

Next, assume that $\mathfrak{B}_{\Re_2} = \{u \in \mathfrak{E} : ||u||_{PC} \le \Re_2\}$ such that

$$
\frac{\Re_2}{g_1^*\Theta(\Re_2)(\Omega_3+\Omega_1\Omega_2)+\mathfrak{C}^*} > \frac{1}{1-[g_2^*\Psi_*^{\sigma_m}+g_3^*\Psi_*^{\nu_m}](\Omega_3+\Omega_1\Omega_2)}. \qquad (3.27)
$$

Thanks to Theorem 3.1, we see that Q_1 is continuous. For any $t \in J$, we have

$$
\begin{aligned}
&\left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(Q_1u)(t)\right|\\
\le\ & \frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}{}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|F_u(t_{j+1})| + \frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\sum_{j=0}^{m-1}{}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|F_u(t_{j+1})|\\
&+\Psi_{\psi_m}^{2-\gamma_m}(T,t_m)\,{}_{\rho_m}I_{t_m}^{\alpha_m;\psi_m}|F_u(T)| + \sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\sum_{r=0}^{j-1}{}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|F_u(t_{r+1})|\\
&+\left\{\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Lambda|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))} + \frac{\Psi_{\psi_m}(T,t_m)}{|\Lambda|\Gamma_{\rho_m}(\rho_m\gamma_m)}\right\}\left\{\sum_{i=0}^{m+1}|\mu_i|\,{}_{\rho_i}I_{t_i}^{\alpha_i;\psi_i}|F_u(\eta_i)|\right.\\
&+\sum_{l=0}^{n}|\lambda_l|\,{}_{\rho_l}I_{t_l}^{\alpha_l+\theta_l;\psi_l}|F_u(\xi_l)| + \sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{j=0}^{i-1}{}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|F_u(t_{j+1})|\\
&+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}{}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|F_u(t_{r+1})|\\
&+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=0}^{i-1}{}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|F_u(t_{j+1})| + \sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}{}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|F_u(t_{j+1})|
\end{aligned}
$$

$$+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=0}^{l-1}{}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|F_u(t_{j+1})|$$

$$+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}{}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|F_u(t_{r+1})|\Bigg\}. \tag{3.28}$$

From condition (H_4), we obtain

$$|F_u(t)|=\left|f(t,u(t),{}_{\rho_k}I_{t_k^+}^{\sigma_k;\psi_k}u(t),{}_{\rho_k}I_{t_k}^{\nu_k;\psi_k}u(t))\right|\leq g_1^*\Theta(\Re_2)+[g_2^*\Psi_*^{\sigma_m}+g_3^*\Psi_*^{\nu_m}]\Re_2. \tag{3.29}$$

Substituting (3.29) into (3.28) and using the property (i) in Lemma 2.1, we have

$$\begin{aligned}
&\left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(Q_1u)(t)\right|\\
\leq\;& \Big(g_1^*\Theta(\Re_2)+[g_2^*\Psi_*^{\sigma_m}+g_3^*\Psi_*^{\nu_m}]\Re_2\Big)\Bigg(\frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j-\rho_j(\gamma_j-1)+\rho_j)}+\frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\\
&\times\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j+\rho_j(2-\gamma_j)+\rho_j)}+\frac{\Psi_{\psi_m}^{\frac{\alpha_m}{\rho_m}+2-\gamma_m}(T,t_m)}{\Gamma_{\rho_m}(\alpha_m+\rho_m)}+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\\
&\times\sum_{r=0}^{j-1}\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r-\rho_r(\gamma_r-1)+\rho_r)}+\left\{\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Lambda|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}+\frac{\Psi_{\psi_m}(T,t_m)}{|\Lambda|\Gamma_{\rho_m}(\rho_m\gamma_m)}\right\}\Bigg\{\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i,t_i)}{\Gamma_{\rho_i}(\alpha_i+\rho_i)}\\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\alpha_l+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\alpha_l+\theta_l+\rho_l)}+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{j=0}^{i-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j-\rho_j(\gamma_j-1)+\rho_j)}\\
&+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r-\rho_r(\gamma_r-1)+\rho_r)}\\
&+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=0}^{i-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j+\rho_j(2-\gamma_j)+\rho_j)}+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j-\rho_j(\gamma_j-1)+\rho_j)}\\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=0}^{l-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j+\rho_j(2-\gamma_j)+\rho_j)}\\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r-\rho_r(\gamma_r-1)+\rho_r)}\Bigg\}\Bigg)\\
=\;&(g_1^*\Theta(\Re_2)+[g_2^*\Psi_*^{\sigma_m}+g_3^*\Psi_*^{\nu_m}]\Re_2)(\Omega_3+\Omega_1\Omega_2).
\end{aligned}$$

This yields that $Q_1(\mathfrak{B}_{\Re_2})\leq(g_1^*\Theta(\Re_2)+[g_2^*\Psi_*^{\sigma_m}+g_3^*\Psi_*^{\nu_m}]\Re_2)(\Omega_3+\Omega_1\Omega_2)$.

Now, we will prove that Q_1 maps bounded set $\mathfrak{B}_{\Re_2}$ into equicontinuous set of $\mathfrak{E}$. Suppose that τ_1, $\tau_2\in J_k$, $k=0,1,\ldots,m$, under $\tau_1<\tau_2$ and for any $u\in\mathfrak{B}_{\Re_2}$, we obtain that

$$\left|\Psi_{\psi_k}^{2-\gamma_k}(\tau_2,t_k)(Q_1u)(\tau_2)-\Psi_{\psi_k}^{2-\gamma_k}(\tau_1,t_k)(Q_1u)(\tau_1)\right|$$

$$
\begin{aligned}
\leq\ & \Big(g_1^*\Theta(\Re_2)+[g_2^*\Psi_*^{\sigma_m}+g_3^*\Psi_*^{\gamma_m}]\Re_2\Big)\Big(\Omega_3+\Omega_1\Omega_2\Big)\Bigg(\frac{\left|\Psi_{\psi_k}(\tau_2,t_k)-\Psi_{\psi_k}(\tau_1,t_k)\right|}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j-\rho_j(\gamma_j-1)+\rho_j)}\\
&+\frac{\Psi_{\psi_m}^{2-\gamma_m}(\tau_2,t_m)\Psi_{\psi_m}^{\frac{\alpha_m}{\rho_m}}(\tau_2,t_1)}{\alpha_m\Gamma_{\rho_m}(\alpha_m)}+\frac{1}{\alpha_m\Gamma_{\rho_m}(\alpha_m)}\left|\Psi_{\psi_k}^{\frac{\alpha_k}{\rho_k}+2-\gamma_k}(\tau_2,t_k)-\Psi_{\psi_k}^{\frac{\alpha_k}{\rho_k}+2-\gamma_k}(\tau_1,t_k)-\Psi_{\psi_k}^{2-\gamma_k}(\tau_2,t_k)\Psi_{\psi_k}^{\frac{\alpha_k}{\rho_k}}(\tau_2,\tau_1)\right|\\
&+\frac{\left|\Psi_{\psi_k}(\tau_2,t_k)-\Psi_{\psi_k}(\tau_1,t_k)\right|}{|\Lambda|\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg\{\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i,t_i)}{\Gamma_{\rho_i}(\alpha_i+\rho_i)}+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\alpha_l+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\alpha_l+\theta_l+\rho_l)}+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\\
&\times\sum_{j=0}^{i-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j-\rho_j(\gamma_j-1)+\rho_j)}+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r-\rho_r(\gamma_r-1)+\rho_r)}\\
&+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=0}^{i-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j+\rho_j(2-\gamma_j)+\rho_j)}+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j-\rho_j(\gamma_j-1)+\rho_j)}\\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=0}^{l-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j+\rho_j(2-\gamma_j)+\rho_j)}\\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r-\rho_r(\gamma_r-1)+\rho_r)}\Bigg\}\Bigg)\\
=\ & \Big(g_1^*\Theta(\Re_2)+[g_2^*\Psi_*^{\sigma_m}+g_3^*\Psi_*^{\gamma_m}]\Re_2\Big)\Big(\Omega_3+\Omega_1\Omega_2\Big)\Bigg(\frac{\left|\Psi_{\psi_k}(\tau_2,t_k)-\Psi_{\psi_k}(\tau_1,t_k)\right|}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j-\rho_j(\gamma_j-1)+\rho_j)}\\
&+\frac{\Psi_{\psi_m}^{2-\gamma_m}(\tau_2,t_m)\Psi_{\psi_m}^{\frac{\alpha_m}{\rho_m}}(\tau_2,\tau_1)}{\alpha_m\Gamma_{\rho_m}(\alpha_m)}+\frac{1}{\alpha_m\Gamma_{\rho_m}(\alpha_m)}\left|\Psi_{\psi_k}^{\frac{\alpha_k}{\rho_k}+2-\gamma_k}(\tau_2,t_k)-\Psi_{\psi_k}^{\frac{\alpha_k}{\rho_k}+2-\gamma_k}(\tau_1,t_k)-\Psi_{\psi_k}^{2-\gamma_k}(\tau_2,t_k)\Psi_{\psi_k}^{\frac{\alpha_k}{\rho_k}}(\tau_2,\tau_1)\right|\\
&+\frac{\Omega_2\left|\Psi_{\psi_k}(\tau_2,t_k)-\Psi_{\psi_k}(\tau_1,t_k)\right|}{|\Lambda|\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg).
\end{aligned}
$$

Observe that the above result is independent of $u\in\mathfrak{B}_{\Re_2}$. This implies that

$$\left|\Psi_{\psi_k}^{2-\gamma_k}(\tau_2,t_k)(Qu_1)(\tau_2)-\Psi_{\psi_k}^{2-\gamma_k}(\tau_1,t_k)(Qu_1)(\tau_1)\right|\to 0\quad\text{as}\quad\tau_2\to\tau_1.$$

Since Q_1 maps bounded set $\mathfrak{B}_{\Re_2}$ into an equicontinuous set of $\mathfrak{E}$, by the Arzelá-Ascoli theorem, we obtain that Q_1 is completely continuous.

Next, we will prove that Q_2 is a nonlinear contraction. Let $\Theta: R^+\to R^+$ be a continuous nondecreasing function given by $\Theta(\epsilon)=[(\Omega_5+\Omega_1\Omega_4)\Xi_1+(m\Psi_*^{\gamma_m}+\Omega_1\Omega_6)\Xi_2]\epsilon$, for all $\epsilon\geq 0$. It is easy to see that $\Theta(0)=0$. Since $[(\Omega_5+\Omega_1\Omega_4)\Xi_1+(m\Psi_*^{\gamma_m}+\Omega_1\Omega_6)\Xi_2]<1$, we have $\Theta(\epsilon)<\epsilon$ for all $\epsilon>0$. For any $u,v\in\mathfrak{B}_{\Re_2}$, we obtain

$$
\begin{aligned}
&\left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(Q_2u)(t)-\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(Q_2v)(t)\right|\\
\leq\ & \frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}K_1(\|u-v\|_{PC})+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\sum_{r=0}^{j-1}K_1(\|u-v\|_{PC})\\
&+\frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\sum_{j=0}^{m-1}K_2(\|u-v\|_{PC})+\left\{\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Lambda|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}+\frac{\Psi_{\psi_m}(T,t_m)}{|\Lambda|\Gamma_{\rho_m}(\rho_m\gamma_m)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Bigg\{\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}K_1(\|u-v\|_{PC})+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{j=0}^{i-1}K_1(\|u-v\|_{PC})\\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}K_1(\|u-v\|_{PC})+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}K_1(\|u-v\|_{PC})\\
&+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=0}^{i-1}K_2(\|u-v\|_{PC})+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=0}^{l-1}K_2(\|u-v\|_{PC})\Bigg\}\\
\le\;&\Bigg[\Bigg(\frac{m\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=1}^{m-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg)\Xi_1+\frac{m}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Xi_2\\
&+\Bigg\{\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Lambda|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}+\frac{\Psi_{\psi_m}(T,t_m)}{|\Lambda|\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg\}\Bigg\{\Bigg(\sum_{i=0}^{m+1}\frac{i|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}+\sum_{l=0}^{n}\frac{l|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=1}^{l-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=1}^{i-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\Bigg)\Xi_1\\
&+\Bigg(\sum_{i=0}^{m+1}\frac{i|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}+\sum_{l=0}^{n}\frac{l|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Bigg)\Xi_2\Bigg\}\Bigg]\|u-v\|_{PC}\\
=\;&[(\Omega_5+\Omega_1\Omega_4)\Xi_1+(m\Psi_*^{\gamma_m}+\Omega_1\Omega_6)\Xi_2]\|u-v\|_{PC}.
\end{aligned}
$$

By taking $\Theta(\epsilon)=[(\Omega_5+\Omega_1\Omega_4)\Xi_1+(m\Psi_*^{\gamma_m}+\Omega_1\Omega_6)\Xi_2]\epsilon$, we have $\Theta(0)=0$ and $\Theta(\epsilon)<\epsilon$ for all $\epsilon>0$. Then

$$\|Q_2u-Q_2v\|_{PC}\le\Theta(\|u-v\|_{PC}).$$

This yields that Q_2 is a nonlinear contraction.

Next, we will prove that $Q_2(\mathfrak{B}_{\mathfrak{R}_2})$ is bounded. By (H_3), for any $u\in\mathfrak{B}_{\mathfrak{R}_2}$, it follows that

$$
\begin{aligned}
&\left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(Q_2u)(t)\right|\\
\le\;&\Bigg(\frac{m\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=1}^{m-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg)M_1+\frac{m}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}M_2\\
&+\Bigg\{\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Lambda|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}+\frac{\Psi_{\psi_m}(T,t_m)}{|\Lambda|\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg\}\Bigg\{\Bigg(\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=1}^{l-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\\
&+\sum_{i=0}^{m+1}\frac{i|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}+\sum_{l=0}^{n}\frac{l|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=1}^{i-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\Bigg)M_1\\
&+\Bigg(\sum_{i=0}^{m+1}\frac{i|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}+\sum_{l=0}^{n}\frac{l|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Bigg)M_2+|A|\Bigg\}\\
=\;&(\Omega_5+\Omega_1\Omega_4)M_1+(m\Psi_*^{\sigma_m}+\Omega_1\Omega_6)M_2+\Omega_1|A|.
\end{aligned}
$$

Then, $Q_2(\mathfrak{B}_{\mathfrak{R}_2})$ is bounded with the boundedness of the set $Q_1(\mathfrak{B}_{\mathfrak{R}_2})$.

Lastly, we will prove that the assumption (a_2) in Lemma 3.2 is not true. Suppose that (a_2) is true. Then there exists a constant $\theta \in (0,1)$ such that $u = \theta Qu$ for any $u \in \mathfrak{B}_{\Re_2}$. We obtain that $\|u\|_{PC} \le \Re_2$ and

$$\begin{aligned}\left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)u(t)\right| &= \theta\left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(Qu)(t)\right|\\ &\le \left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(Q_1u)(t) + \Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(Q_2u)(t)\right|\\ &\le (g_1^*\Theta(\Re_2) + [g_2^*\Psi_*^{\sigma_m} + g_3^*\Psi_*^{\nu_m}]\Re_2)(\Omega_3+\Omega_1\Omega_2)\\ &\quad +(\Omega_5+\Omega_1\Omega_4)M_1 + (m\Psi_*^{\sigma_m}+\Omega_1\Omega_6)M_2 + \Omega_1|A|,\end{aligned}$$

which implies

$$\begin{aligned}\Re_2 \le\ & (g_1^*\Theta(\Re_2) + [g_2^*\Psi_*^{\sigma_m} + g_3^*\Psi_*^{\nu_m}]\Re_2)(\Omega_3+\Omega_1\Omega_2)\\ &+(\Omega_5+\Omega_1\Omega_4)M_1 + (m\Psi_*^{\sigma_m}+\Omega_1\Omega_6)M_2 + \Omega_1|A|.\end{aligned}$$

Hence,

$$\frac{\Re_2}{g_1^*\Theta(\Re_2)(\Omega_3+\Omega_1\Omega_2)+\mathfrak{C}^*} \le \frac{1}{1-[g_2^*\Psi_*^{\sigma_m}+g_3^*\Psi_*^{\nu_m}](\Omega_3+\Omega_1\Omega_2)},$$

where

$$\mathfrak{C}^* := (\Omega_5+\Omega_1\Omega_4)M_1 + (m\Psi_*^{\sigma_m}+\Omega_1\Omega_6)M_2 + \Omega_1|A|, \tag{3.30}$$

this contradicts the condition (H_6). Therefore, Q_1 and Q_2 satisfy all conditions of Lemma 3.2. Hence, the considered problem (1.4) has a solution on J.

4. Ulam stability results

This section discusses a variety of Ulam-Hyers stability of the considered problem (1.4). Before proving, we will state Ulam-Hyers stability ideas for the considered problem (1.4). Assume that $\chi \in C(J, R^+)$ is a non-decreasing function and $\epsilon > 0$, $\delta \ge 0$, $z \in E$, so that for any $t \in J_k$, $k = 1, 2, \ldots, m$ the following important inequalities are satisfied:

$$\begin{cases}\left|{}^{H}_{\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\beta_k;\psi_k}z(t) - f(t, z(t), {}_{\rho_k}I_{t_k}^{\sigma_k;\psi_k}z(t), {}_{\rho_k}I_{t_k}^{\nu_k;\psi_k}z(t))\right| \le \epsilon,\\ \left|{}_{\rho_k}I_{t_k^+}^{\rho_k(2-\gamma_k);\psi_k}z(t_k^+) - {}_{\rho_{k-1}}I_{t_{k-1}^+}^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}z(t_k^-) - \phi_k(z(t_k))\right| \le \epsilon,\\ \left|{}^{RL}_{\rho_k}\mathfrak{D}_{t_k^+}^{\rho_k(\gamma_k-1);\psi_k}z(t_k^+) - {}^{RL}_{\rho_{k-1}}\mathfrak{D}_{t_{k-1}^+}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}z(t_k^-) - \phi_k^*(z(t_k))\right| \le \epsilon,\end{cases} \tag{4.1}$$

$$\begin{cases}\left|{}^{H}_{\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\beta_k;\psi_k}z(t) - f(t, z(t), {}_{\rho_k}I_{t_k}^{\sigma_k;\psi_k}z(t), {}_{\rho_k}I_{t_k}^{\nu_k;\psi_k}z(t))\right| \le \chi(t),\\ \left|{}_{\rho_k}I_{t_k^+}^{\rho_k(2-\gamma_k);\psi_k}z(t_k^+) - {}_{\rho_{k-1}}I_{t_{k-1}^+}^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}z(t_k^-) - \phi_k(z(t_k))\right| \le \delta,\\ \left|{}^{RL}_{\rho_k}\mathfrak{D}_{t_k^+}^{\rho_k(\gamma_k-1);\psi_k}z(t_k^+) - {}^{RL}_{\rho_{k-1}}\mathfrak{D}_{t_{k-1}^+}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}z(t_k^-) - \phi_k^*(z(t_k))\right| \le \delta,\end{cases} \tag{4.2}$$

$$\begin{cases}\left|{}^{H}_{\rho_k}\mathfrak{D}_{t_k^+}^{\alpha_k,\beta_k;\psi_k}z(t) - f(t, z(t), {}_{\rho_k}I_{t_k}^{\sigma_k;\psi_k}z(t), {}_{\rho_k}I_{t_k}^{\nu_k;\psi_k}z(t))\right| \le \epsilon\chi(t),\\ \left|{}_{\rho_k}I_{t_k^+}^{\rho_k(2-\gamma_k);\psi_k}z(t_k^+) - {}_{\rho_{k-1}}I_{t_{k-1}^+}^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}z(t_k^-) - \phi_k(z(t_k))\right| \le \epsilon\delta,\\ \left|{}^{RL}_{\rho_k}\mathfrak{D}_{t_k^+}^{\rho_k(\gamma_k-1);\psi_k}z(t_k^+) - {}^{RL}_{\rho_{k-1}}\mathfrak{D}_{t_{k-1}^+}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}z(t_k^-) - \phi_k^*(z(t_k))\right| \le \epsilon\delta.\end{cases} \tag{4.3}$$

Definition 4.1. *The considered problem* (1.4) *is said to be Ulam–Hyers* (UH) *stable, if there exists a real constant* $\mathfrak{C}_F > 0$ *so that for every* $\epsilon > 0$ *and for any* $z \in E$ *of* (4.1) *there exists* $u \in E$ *of* (1.4) *that satisfies*

$$|z(t) - u(t)| \leq \mathfrak{C}_F \,\epsilon, \quad t \in J. \tag{4.4}$$

Definition 4.2. *The considered problem* (1.4) *is said to be generalized Ulam-Hyers* (GUH) *stable, if there exists* $\chi \in C(R^+, R^+)$ *via* $\chi(0) = 0$ *so that for every* $\epsilon > 0$ *and for any* $z \in E$ *of* (4.2) *there exists* $u \in E$ *of* (1.4) *that satisfies*

$$|z(t) - u(t)| \leq \chi(\epsilon), \quad t \in J. \tag{4.5}$$

Definition 4.3. *The considered problem* (1.4) *is said to be Ulam-Hyers-Rassias* (UHR) *stable with respect to* (δ, χ)*, if there exists a real constant* $\mathfrak{C}_{F,\chi_F} > 0$ *so that for every* $\epsilon > 0$ *and for any* $z \in E$ *of* (4.3) *there exists* $u \in E$ *of* (1.4) *that satisfies*

$$|z(t) - u(t)| \leq \mathfrak{C}_{F,\chi_F} \,\epsilon(\delta + \chi(t)), \quad t \in J. \tag{4.6}$$

Definition 4.4. *The considered problem* (1.4) *is said to be generalized Ulam-Hyers-Rassias* ($GUHR$) *stable with respect to* (δ, χ)*, if there exists a real constant* $\mathfrak{C}_{F,\chi_F} > 0$ *so that for any* $z \in E$ *of* (4.2) *there exists* $u \in E$ *of* (1.4) *that satisfies*

$$|z(t) - u(t)| \leq \mathfrak{C}_{F,\chi_F}(\delta + \chi(t)), \quad t \in J. \tag{4.7}$$

Remark 4.1. *By Definitions 4.1–4.4, we will find out that:* (R_1) *Definition 4.1* $\Rightarrow$ *Definition 4.2;* (R_2) *Definition 4.3* $\Rightarrow$ *Definition 4.4; and* (R_3) *Definition 4.3* $\Rightarrow$ *Definition 4.1.*

Remark 4.2. *Assume that* $z \in E$ *is the solution of* (4.1). *If there exists* $g \in E$ *with a sequence* g_k *for* $k = 1, 2, \ldots, m$*, depending on a function* z*, such that* (A_1) $|g(t)| \leq \epsilon$, $|g_k| \leq \epsilon$, $t \in J$; (A_2) ${}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+} z(t) = f(t, z(t), {}_{\rho_k}I^{\sigma_k;\psi_k}_{t_k} z(t), {}_{\rho_k}I^{\nu_k;\psi_k}_{t_k} z(t)) + g(t)$, $t \in J$; (A_3) ${}_{\rho_k}I^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}I^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi_k(z(t_k)) + g_k$, $t \in J$*; and* (A_4) ${}^{RL}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+} z(t_k^+) - {}^{RL}_{\rho_{k-1}}\mathfrak{D}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi_k^*(z(t_k)) + g_k$, $t \in J$.

Remark 4.3. *Assume that* $z \in E$ *is the solution of* (4.2). *If there exists* $g \in E$ *and* g_k *for* $k = 1, 2, \ldots, m$*, depending on a function* z*, such that* (B_1) $|g(t)| \leq \chi(t)$, $|g_k| \leq \delta$, $t \in J$; (B_2) ${}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+} z(t) = f(t, z(t), {}_{\rho_k}I^{\sigma_k;\psi_k}_{t_k} z(t), {}_{\rho_k}I^{\nu_k;\psi_k}_{t_k} z(t)) + g(t)$, $t \in J$; (B_3) ${}_{\rho_k}I^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}I^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi_k(z(t_k)) + g_k$*; and* (B_4) ${}^{RL}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+} z(t_k^+) - {}^{RL}_{\rho_{k-1}}\mathfrak{D}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi_k^*(z(t_k)) + g_k$, $t \in J$.

Remark 4.4. *Assume that* $z \in E$ *is the solution of* (4.3). *If there exists* $g \in E$ *and* g_k *for* $k = 1, 2, \ldots, m$*, depending on a function* z*, such that* (C_1) $|g(t)| \leq \epsilon\chi(t)$, $|g_k| \leq \epsilon\delta$, $t \in J$; (C_2) ${}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+} z(t) = f(t, z(t), {}_{\rho_k}I^{\sigma_k;\psi_k}_{t_k} z(t), {}_{\rho_k}I^{\nu_k;\psi_k}_{t_k} z(t)) + g(t)$, $t \in J$; (C_3) ${}_{\rho_k}I^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}I^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi_k(z(t_k)) + g_k$*; and* (C_4) ${}^{RL}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+} z(t_k^+) - {}^{RL}_{\rho_{k-1}}\mathfrak{D}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi_k^*(z(t_k)) + g_k$, $t \in J$.

4.1. Ulam-Hyers stability results

Theorem 4.1. *Assume that $\alpha_k \in (1,2]$, $\beta_k \in [0,1]$, $\rho_k \in R^+$, $\gamma_k = (\beta_k(2\rho_k - \alpha_k) + \alpha_k)/\rho_k$, $\psi_k \in C(J,R)$ where $\psi'_k > 0$, $k = 1,2,\dots,m$ and $f \in C(J \times R^3, R)$. If the assumptions (H_1) and (H_2) and the inequality* (3.8) *hold, then the considered problem* (1.4) *is UH stable on J.*

Proof. Assume that $z \in PC$ is the solution of the problem (4.1). Under the conditions (A_2) and (A_3) of Remark 4.2 and Lemma 2.4, we have

$$\begin{cases} {}^{H}_{\rho_k}\mathfrak{D}^{\alpha_k,\beta_k;\psi_k}_{t_k^+} z(t) = F_z(t) + g(t), \quad t \neq t_k,\ k = 0,1,\dots,m, \\ {}^{RL}_{\rho_k}\mathfrak{D}^{\rho_k(\gamma_k-1);\psi_k}_{t_k^+} z(t_k^+) - {}^{RL}_{\rho_{k-1}}\mathfrak{D}^{\rho_{k-1}(\gamma_{k-1}-1);\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi_k(z(t_k)) + g_k, \quad k = 1,2,\dots,m, \\ {}_{\rho_k}I^{\rho_k(2-\gamma_k);\psi_k}_{t_k^+} z(t_k^+) - {}_{\rho_{k-1}}I^{\rho_{k-1}(2-\gamma_{k-1});\psi_{k-1}}_{t_{k-1}^+} z(t_k^-) = \phi^*_k(z(t_k)) + g_k, \quad k = 1,2,\dots,m, \\ z(0) = 0, \quad \sum_{i=0}^{m+1}\mu_i z(\eta_i) + \sum_{l=0}^{n}\lambda_{l\,\rho_l}I^{\theta_l;\psi_l}_{t_l} z(\xi_l) = A, \quad \eta_i, \in (t_i,t_{i+1}],\ \xi_l \in (t_l,t_{l+1}]. \end{cases} \tag{4.8}$$

Then, the solution of (4.8) is given by

$$\begin{aligned} &z(t) \\ =\ & \left\{\frac{\Psi^{\gamma_k-1}_{\psi_k}(t,t_k)}{\Lambda\Gamma_{\rho_k}(\rho_k\gamma_k)} + \frac{\Psi^{\gamma_k-2}_{\psi_k}(t,t_k)}{\Lambda\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right\}\left\{A - \left(\sum_{i=0}^{m+1}\frac{\mu_i\Psi^{\gamma_i-1}_{\psi_i}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\right.\right. \\ &\times\sum_{j=0}^{i-1}\left({}_{\rho_j}I^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}_{t_j}F_z(t_{j+1}) + \phi_{j+1}(z(t_{j+1}))\right) + \sum_{i=0}^{m+1}\frac{\mu_i\Psi^{\gamma_i-2}_{\psi_i}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\Bigg[\sum_{j=0}^{i-1}\big({}_{\rho_j}I^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}_{t_j}F_z(t_{j+1}) \\ &+\phi^*_{j+1}(z(t_{j+1}))) + \sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}I^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}_{t_r}F_z(t_{r+1}) + \phi_{r+1}(z(t_{r+1}))\right)\Bigg] \\ &+\sum_{l=0}^{n}\frac{\lambda_l\Psi^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}_{\psi_l}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\left({}_{\rho_j}I^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}_{t_j}F_z(t_{j+1}) + \phi_{j+1}(z(t_{j+1}))\right) \\ &+\sum_{l=0}^{n}\frac{\lambda_l\Psi^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}_{\psi_l}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Bigg[\sum_{j=0}^{l-1}\left({}_{\rho_j}I^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}_{t_j}F_z(t_{j+1}) + \phi^*_{j+1}(z(t_{j+1}))\right) \\ &+\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}I^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}_{t_r}F_z(t_{r+1}) + \phi_{r+1}(z(t_{r+1}))\right)\Bigg] + \sum_{i=0}^{m+1}\mu_{i\,\rho_i}I^{\alpha_i;\psi_i}_{t_i}F_z(\eta_i) \\ &\left.\left.+\sum_{l=0}^{n}\lambda_{l\,\rho_l}I^{\alpha_l+\theta_l;\psi_l}_{t_l}F_z(\xi_l)\right)\right\} + {}_{\rho_k}I^{\alpha_k;\psi_k}_{t_k}F_z(t) + \frac{\Psi^{\gamma_k-1}_{\psi_k}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\left({}_{\rho_j}I^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}_{t_j}F_z(t_{j+1}) + \phi_{j+1}(z(t_{j+1}))\right) \\ &+\frac{\Psi^{\gamma_k-2}_{\psi_k}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\Bigg[\sum_{j=0}^{k-1}\left({}_{\rho_j}I^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}_{t_j}F_z(t_{j+1}) + \phi^*_{j+1}(z(t_{j+1}))\right) \\ &+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}I^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}_{t_r}F_z(t_{r+1}) + \phi_{r+1}(z(t_{r+1}))\right)\Bigg] - \left\{\frac{\Psi^{\gamma_k-1}_{\psi_k}(t,t_k)}{\Lambda\Gamma_{\rho_k}(\rho_k\gamma_k)}\right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Lambda\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\Bigg\}\Bigg\{\sum_{i=0}^{m+1}\frac{\mu_i\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}g(t_{j+1})+g_{j+1}\Big)\\
&+\sum_{i=0}^{m+1}\frac{\mu_i\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\Bigg[\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}g(t_{j+1})+g_{j+1}\Big)\\
&+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}g(t_{r+1})+g_{r+1}\Big)\Bigg]+\sum_{i=0}^{m+1}\mu_{i\,\rho_i}I_{t_i}^{\alpha_i;\psi_i}g(\eta_i)\\
&+\sum_{l=0}^{n}\lambda_{l\,\rho_l}I_{t_l}^{\alpha_l+\theta_l;\psi_l}g(\xi_l)+\sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}g(t_{j+1})+g_{j+1}\Big)\\
&+\sum_{l=0}^{n}\frac{\lambda_l\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Bigg[\sum_{j=0}^{l-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}g(t_{j+1})+g_{j+1}\Big)\\
&+\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}g(t_{r+1})+g_{r+1}\Big)\Bigg]\Bigg\}+{}_{\rho_k}I_{t_k}^{\alpha_k;\psi_k}g(t)\\
&+\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}g(t_{j+1})+g_{j+1}\Big)+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\Bigg[\sum_{j=0}^{k-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}g(t_{j+1})+g_{j+1}\Big)\\
&+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}g(t_{r+1})+g_{r+1}\Big)\Bigg]. \qquad (4.9)
\end{aligned}
$$

By applying (A_1) of Remark 4.2 with (H_1) and (H_2), we obtain

$$
\begin{aligned}
&\left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(z(t)-u(t))\right|\\
\le\;&\Bigg\{\frac{\Psi_{\psi_m}(T,t_m)}{|\Lambda|\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Lambda|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg\}\Bigg\{\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\\
&\times\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|F_z(t_{j+1})-F_u(t_{j+1})|+|\phi_{j+1}(z(t_{j+1}))-\phi_{j+1}(u(t_{j+1}))|\Big)\\
&+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\Bigg[\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|F_z(t_{j+1})-F_u(t_{j+1})|+|\phi^*_{j+1}(z(t_{j+1}))-\phi^*_{j+1}(u(t_{j+1}))|\Big)\\
&+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|F_z(t_{r+1})-F_u(t_{r+1})|+|\phi_{r+1}(z(t_{r+1}))-\phi_{r+1}(u(t_{r+1}))|\Big)\Bigg]\\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|F_z(t_{j+1})-F_u(t_{j+1})|+|\phi_{j+1}(z(t_{j+1}))-\phi_{j+1}(u(t_{j+1}))|\Big)\\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Bigg[\sum_{j=0}^{l-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|F_z(t_{j+1})-F_u(t_{j+1})|+|\phi^*_{j+1}(z(t_{j+1}))-\phi^*_{j+1}(u(t_{j+1}))|\Big)\\
&+\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|F_z(t_{r+1})-F_u(t_{r+1})|+|\phi_{r+1}(z(t_{r+1}))-\phi_{r+1}(u(t_{r+1}))|\Big)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=0}^{m+1}|\mu_i|_{\rho_i}I_{t_i}^{\alpha_i;\psi_i}|F_z(\eta_i)-F_u(\eta_i)|+\sum_{l=0}^{n}|\lambda_l|_{\rho_l}I_{t_l}^{\alpha_l+\theta_l;\psi_l}|F_z(\xi_l)-F_u(\xi_l)|\Bigg\}+\Psi_{\psi_m}^{2-\gamma_m}(T,t_m)\\
&\times_{\rho_m}I_{t_m}^{\alpha_m;\psi_m}|F_z(T)-F_u(T)|+\frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|F_z(t_{j+1})-F_u(t_{j+1})|\\
&+|\phi_{j+1}(z(t_{j+1}))-\phi_{j+1}(u(t_{j+1}))|\Big)+\frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg[\sum_{j=0}^{m-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|F_z(t_{j+1})-F_u(t_{j+1})|\\
&+|\phi_{j+1}^*(z(t_{j+1}))-\phi_{j+1}^*(u(t_{j+1}))|\Big)+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|F_z(t_{r+1})-F_u(t_{r+1})|\\
&+|\phi_{r+1}(z(t_{r+1}))-\phi_{r+1}(u(t_{r+1}))|\Big)\Bigg]+\Bigg\{\frac{\Psi_{\psi_m}(T,t_m)}{\Lambda\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Lambda|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg\}\\
&\times\Bigg\{\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|g(t_{j+1})|+|g_{j+1}|\Big)+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\\
&\times\Bigg[\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|g(t_{j+1})|+|g_{j+1}|\Big)+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|g(t_{r+1})|+|g_{r+1}|\Big)\Bigg]\\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|g(t_{j+1})|+|g_{j+1}|\Big)\\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Bigg[\sum_{j=0}^{l-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|g(t_{j+1})|+|g_{j+1}|\Big)\\
&+\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|g(t_{r+1})|+|g_{r+1}|\Big)\Bigg]+\sum_{i=0}^{m+1}|\mu_i|_{\rho_i}I_{t_i}^{\alpha_i;\psi_i}|g(\eta_i)|+\sum_{l=0}^{n}|\lambda_l|_{\rho_l}I_{t_l}^{\alpha_l+\theta_l;\psi_l}|g(\xi_l)|\Bigg\}\\
&+\Psi_{\psi_m}^{2-\gamma_m}(T,t_m)_{\rho_m}I_{t_m}^{\alpha_m;\psi_m}|g(T)|+\frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|g(t_{j+1})|+|g_{j+1}|\Big)\\
&+\frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg[\sum_{j=0}^{m-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|g(t_{j+1})|+|g_{j+1}|\Big)\\
&+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|g(t_{r+1})|+|g_{r+1}|\Big)\Bigg]\\
\leq\;&\Big[(\Omega_1\Omega_2+\Omega_3)(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)+(\Omega_1\Omega_4+\Omega_5)I_1+(\Omega_1\Omega_6+m\Psi_*^{\gamma_m})I_2\Big]\|z-u\|_{PC}\\
&+\epsilon\Bigg\{\frac{\Psi_{\psi_m}(T,t_m)}{\Lambda\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Lambda|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg\}\Bigg\{\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\\
&\times\sum_{j=0}^{i-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}+1\right)+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\Bigg[\sum_{j=0}^{i-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_J}}(t_{j+1},t_J)}{\Gamma_{\rho_j}(\rho_J+\alpha_j+\rho_j(2-\gamma_j))}+1\right)\\
&+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left(\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}+1\right)\Bigg]+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}
\end{aligned}
$$

$$
\begin{aligned}
&\times\sum_{j=0}^{l-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}+1\right)+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Bigg[\sum_{j=0}^{l-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_J}}(t_{j+1},t_J)}{\Gamma_{\rho_j}(\rho_J+\alpha_j+\rho_j(2-\gamma_j))}+1\right)\\
&+\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left(\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}+1\right)\Bigg]+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i+\alpha_i)}+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\alpha_l+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l+\alpha_l+\theta_l)}\Bigg\}\\
&+\Bigg\{\frac{\Psi_{\psi_m}^{\frac{\alpha_m}{\rho_m}+2-\gamma_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m+\alpha_m)}+\frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_j+\alpha_j-\rho_j(\gamma_j-1))}+1\right)\\
&+\frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg[\sum_{j=0}^{m-1}\left(\frac{\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_J}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\rho_J+\alpha_j+\rho_j(2-\gamma_j))}+1\right)\\
&+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left(\frac{\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\rho_r+\alpha_r-\rho_r(\gamma_r-1))}+1\right)\Bigg]\Bigg\}\epsilon\\
=&\ \Big[(\Omega_1\Omega_2+\Omega_3)(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\gamma_m}L_3)+(\Omega_1\Omega_4+\Omega_5)I_1+(\Omega_1\Omega_6+m\Psi_*^{\gamma_m})I_2\Big]\|z-u\|_{PC}\\
&+\epsilon\Big[\Omega_1(\Omega_2+\Omega_4+\Omega_6)+\Omega_3+\Omega_5+m\Psi_*^{\gamma_m}\Big]\\
=&\ [\Delta_1+\Delta_2]\|z-u\|_{PC}+\epsilon[\Omega_1(\Omega_2+\Omega_4+\Omega_6)+\Omega_3+\Omega_5+m\Psi_*^{\gamma_m}].
\end{aligned}
$$

This yields that $\|z-u\|_{PC}\leq\mathfrak{C}_F\epsilon$, where $\mathfrak{C}_F$ is given by

$$\mathfrak{C}_F:=\frac{\Omega_1(\Omega_2+\Omega_4+\Omega_6)+\Omega_3+\Omega_5+m\Psi_*^{\gamma_m}}{1-(\Delta_1+\Delta_2)}. \tag{4.10}$$

Hence, the considered problem (1.4) is UH stable in E.

Corollary 4.1. *By taking* $\chi(\epsilon)=\mathfrak{C}_F\epsilon$ *and* $\chi(0)=0$ *in Theorem 4.1, we obtain the considered problem* (1.4) *is GUH stable.*

4.2. Ulam-Hyers-Rassias stability results

To prove UHR and GUHR stability results, we will require the following assumption:

(U_1) There exist a non-decreasing function $\chi\in C(J,R)$ and a positive real constant $\mathfrak{C}_\chi>0$ such that

$${}_{\rho_k}I_{t_k}^{\alpha_k;\psi_k}\chi(t)\leq\mathfrak{C}_\chi\chi(t).$$

Here we give notation for the constants

$$\Omega_7\ :=\ \sum_{i=0}^{m+1}|\mu_i|+\sum_{l=0}^{n}|\lambda_l|, \tag{4.11}$$

$$\Omega_8\ :=\ \sum_{i=0}^{m+1}\frac{i|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}+\sum_{l=0}^{n}\frac{l|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}, \tag{4.12}$$

$$\Omega_9\ :=\ \Psi_{\psi_m}^{2-\gamma_m}(T,t_m)+\frac{m\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=1}^{m-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}. \tag{4.13}$$

Theorem 4.2. *Assume that* $\alpha_k \in (1,2]$, $\beta_k \in [0,1]$, $\rho_k \in R^+$, $\gamma_k = (\beta_k(2\rho_k - \alpha_k) + \alpha_k)/\rho_k$, $\psi_k \in C(J,R)$ *where* $\psi_k' > 0$, $k = 1,2,\ldots,m$ *and* $f \in C(J \times R^3, R)$. *If the assumptions* (H_1) *and* (H_2) *and the inequality* (3.8) *hold, then the considered problem* (1.4) *is UHR stable with respect to* (δ,χ) *on* J.

Proof. Assume that $z \in E$ is any solution of (4.3) and $u \in E$ is a solution of the considered problem (1.4). By the same argument as in Theorem 4.1, it follows that

$$
\begin{aligned}
&\left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(z(t)-u(t))\right| \\
\leq\ & \left\{\frac{\Psi_{\psi_m}(T,t_m)}{|\Lambda|\Gamma_{\rho_m}(\rho_m\gamma_m)} + \sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Lambda|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\right\}\left\{\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\right. \\
&\times\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|F_z(t_{j+1})-F_u(t_{j+1})| + |\phi_{j+1}(z(t_{j+1}))-\phi_{j+1}(u(t_{j+1}))|\Big) \\
&+\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\Bigg[\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|F_z(t_{j+1})-F_u(t_{j+1})| + |\phi^*_{j+1}(z(t_{j+1}))-\phi^*_{j+1}(u(t_{j+1}))|\Big) \\
&+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|F_z(t_{r+1})-F_u(t_{r+1})| + |\phi_{r+1}(z(t_{r+1}))-\phi_{r+1}(u(t_{r+1}))|\Big)\Bigg] \\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|F_z(t_{j+1})-F_u(t_{j+1})| + |\phi_{j+1}(z(t_{j+1}))-\phi_{j+1}(u(t_{j+1}))|\Big) \\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Bigg[\sum_{j=0}^{l-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|F_z(t_{j+1})-F_u(t_{j+1})| + |\phi^*_{j+1}(z(t_{j+1}))-\phi^*_{j+1}(u(t_{j+1}))|\Big) \\
&+\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|F_z(t_{r+1})-F_u(t_{r+1})| + |\phi_{r+1}(z(t_{r+1}))-\phi_{r+1}(u(t_{r+1}))|\Big)\Bigg] \\
&\left.+\sum_{i=0}^{m+1}|\mu_i|_{\rho_i}I_{t_i}^{\alpha_i;\psi_i}|F_z(\eta_i)-F_u(\eta_i)| + \sum_{l=0}^{n}|\lambda_l|_{\rho_l}I_{t_l}^{\alpha_l+\theta_l;\psi_l}|F_z(\xi_l)-F_u(\xi_l)|\right\} + \Psi_{\psi_m}^{2-\gamma_m}(T,t_m) \\
&\times{}_{\rho_m}I_{t_m}^{\alpha_m;\psi_m}|F_z(T)-F_u(T)| + \frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|F_z(t_{j+1})-F_u(t_{j+1})| \\
&+|\phi_{j+1}(z(t_{j+1}))-\phi_{j+1}(u(t_{j+1}))|\Big) + \frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg[\sum_{j=0}^{m-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|F_z(t_{j+1})-F_u(t_{j+1})| \\
&+|\phi^*_{j+1}(z(t_{j+1}))-\phi^*_{j+1}(u(t_{j+1}))|\Big) + \sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|F_z(t_{r+1})-F_u(t_{r+1})| \\
&+|\phi_{r+1}(z(t_{r+1}))-\phi_{r+1}(u(t_{r+1}))|\Big)\Bigg] + \left\{\frac{\Psi_{\psi_m}(T,t_m)}{\Lambda\Gamma_{\rho_m}(\rho_m\gamma_m)} + \sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Lambda|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\right\} \\
&\times\left\{\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|g(t_{j+1})| + |g_{j+1}|\Big) + \sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\right. \\
&\times\Bigg[\sum_{j=0}^{i-1}\Big({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|g(t_{j+1})| + |g_{j+1}|\Big) + \sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\Big({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|g(t_{r+1})| + |g_{r+1}|\Big)\Bigg]
\end{aligned}
$$

$$+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\left({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|g(t_{j+1})|+|g_{j+1}|\right)$$

$$+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Bigg[\sum_{j=0}^{l-1}\left({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|g(t_{j+1})|+|g_{j+1}|\right)$$

$$+\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|g(t_{r+1})|+|g_{r+1}|\right)\Bigg]+\sum_{i=0}^{m+1}|\mu_i|_{\rho_i}I_{t_i}^{\alpha_i;\psi_i}|g(\eta_i)|+\sum_{l=0}^{n}|\lambda_l|_{\rho_l}I_{t_l}^{\alpha_l+\theta_l;\psi_l}|g(\xi_l)|\Bigg\}$$

$$+\Psi_{\psi_m}^{2-\gamma_m}(T,t_m)_{\rho_m}I_{t_m}^{\alpha_m;\psi_m}|g(T)|+\frac{\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\sum_{j=0}^{m-1}\left({}_{\rho_j}I_{t_j}^{\alpha_j-\rho_j(\gamma_j-1);\psi_j}|g(t_{j+1})|+|g_{j+1}|\right)$$

$$+\frac{1}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg[\sum_{j=0}^{m-1}\left({}_{\rho_j}I_{t_j}^{\alpha_j+\rho_j(2-\gamma_j);\psi_j}|g(t_{j+1})|+|g_{j+1}|\right)$$

$$+\sum_{j=1}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left({}_{\rho_r}I_{t_r}^{\alpha_r-\rho_r(\gamma_r-1);\psi_r}|g(t_{r+1})|+|g_{r+1}|\right)\Bigg].$$

Under (C_1) of Remark 4.4 and (H_1), (H_2) and (U_1), we see that

$$\begin{aligned}
&\left|\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)(z(t)-u(t))\right|\\
\leq\ &\Big[(\Omega_1\Omega_2+\Omega_3)(L_1+\Psi_*^{\sigma_m}L_2+\Psi_*^{\nu_m}L_3)+(\Omega_1\Omega_4+\Omega_5)I_1+(\Omega_1\Omega_6+m\Psi_*^{\gamma_m})I_2\Big]\|z-u\|_{PC}\\
&+\epsilon\left\{\frac{\Psi_{\psi_m}(T,t_m)}{\Lambda\Gamma_{\rho_m}(\rho_m\gamma_m)}+\sum_{j=0}^{m-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j|\Lambda|\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\right\}\Bigg\{\mathfrak{C}_\chi\chi(t)\Bigg(\sum_{i=0}^{m+1}\frac{i|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\\
&+\sum_{l=0}^{n}\frac{l|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}+\sum_{i=0}^{m+1}|\mu_i|+\sum_{l=0}^{n}|\lambda_l|\Bigg)+\delta\Bigg(\sum_{i=0}^{m+1}\frac{i|\mu_i|\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}+\sum_{l=0}^{n}\frac{l|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\Bigg)\\
&+(\mathfrak{C}_\chi\chi(t)+\delta)\Bigg(\sum_{i=0}^{m+1}\frac{|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\sum_{j=1}^{i-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}+\sum_{i=0}^{m+1}\frac{i|\mu_i|\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}+\sum_{l=0}^{n}\frac{l|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\\
&+\sum_{l=0}^{n}\frac{|\lambda_l|\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\sum_{j=1}^{l-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\Bigg)\Bigg\}+\epsilon\Bigg\{\mathfrak{C}_\chi\chi(t)\Bigg(\Psi_{\psi_m}^{2-\gamma_m}(T,t_m)+\frac{m\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}\Bigg)\\
&+\delta\frac{m\Psi_{\psi_m}(T,t_m)}{\Gamma_{\rho_m}(\rho_m\gamma_m)}+(\mathfrak{C}_\chi\chi(t)+\delta)\Bigg(\frac{m}{\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}+\sum_{j=1}^{m-1}\frac{j\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j\Gamma_{\rho_m}(\rho_m(\gamma_m-1))}\Bigg)\Bigg\}\\
\leq\ &[\Delta_1+\Delta_2]\|z-u\|_{PC}+\epsilon\Big\{\Omega_1\Big[(\mathfrak{C}_\chi\chi(t)+\delta)\Omega_4+\mathfrak{C}_\chi\chi(t)(\Omega_7+\Omega_8)+\delta\Omega_8\Big]\\
&+\mathfrak{C}_\chi\chi(t)(m\Psi_*^{\gamma_m}+\Omega_9)+\delta(m\Psi_*^{\gamma_m}+\Omega_5)\Big\}\\
=\ &[\Delta_1+\Delta_2]\|z-u\|_{PC}+\epsilon\Big\{\Omega_1\Big[(\Omega_4+\Omega_7+\Omega_8)\mathfrak{C}_\chi\chi(t)+(\Omega_4+\Omega_8)\delta\Big]\\
&+(m\Psi_*^{\gamma_m}+\Omega_9)\mathfrak{C}_\chi\chi(t)+(m\Psi_*^{\gamma_m}+\Omega_5)\delta\Big\}\\
\leq\ &[\Delta_1+\Delta_2]\|z-u\|_{PC}+\Big\{\Omega_1\Big[(\Omega_4+\Omega_7+\Omega_8)\mathfrak{C}_\chi+\Omega_4+\Omega_8\Big]\\
&+(m\Psi_*^{\gamma_m}+\Omega_9)\mathfrak{C}_\chi+m\Psi_*^{\gamma_m}+\Omega_5\Big\}\epsilon(\delta+\chi(t)).
\end{aligned}$$

It follows that, $\|z-u\|_{PC} \leq \mathfrak{C}_{F,\chi_F}\, \epsilon\, (\delta + \chi(t))$, where

$$\mathfrak{C}_{F,\chi_F} := \frac{\Omega_1[(\Omega_4+\Omega_7+\Omega_8)\mathfrak{C}_\chi + \Omega_4+\Omega_8] + (m\Psi_*^{\gamma_m}+\Omega_9)\mathfrak{C}_\chi + m\Psi_*^{\gamma_m}+\Omega_5}{1-(\Delta_1+\Delta_2)}. \tag{4.14}$$

Therefore, the considered problem (1.4) is UHR stable with respect to (δ,χ) in E.

Corollary 4.2. *By taking $\epsilon = 1$ and $\chi(0) = 0$ in Theorem 4.2, we obtain the considered problem* (1.4) *is GUHR stable.*

5. Applications

Example 5.1. *Consider the following impulsive problem of the form:*

$$\begin{cases} {}^{H}_{\frac{3k+46}{50}}\mathfrak{D}_{t_k^+}^{\frac{2k+8}{7},\frac{3-k}{4};\psi_k} u(t) = f\left(t, u(t), {}_{\frac{3k+46}{50}}I_{t_k}^{\frac{3k+2}{5-k};\psi_k} u(t), {}_{\frac{3k+46}{50}}I_{t_k}^{\frac{2k+3}{8};\psi_k} u(t)\right), \ t \neq t_k, \ k = 0,1,2, \\ {}_{\frac{3k+46}{50}}I_{t_k^+}^{\frac{3k+46}{50}(2-\gamma_k);\psi_k} u(t_k^+) - {}_{\frac{3k+43}{50}}I_{t_{k-1}^+}^{\frac{3k+43}{50}(2-\gamma_{k-1});\psi_{k-1}} u(t_k^-) = \phi_k(u(t_k)), \quad k = 1,2, \\ {}^{RL}_{\frac{3k+46}{50}}\mathfrak{D}_{t_k^+}^{\frac{3k+46}{50}(\gamma_k-1);\psi_k} u(t_k^+) - {}^{RL}_{\frac{3k+43}{50}}\mathfrak{D}_{t_{k-1}^+}^{\frac{3k+43}{50}(\gamma_{k-1}-1);\psi_{k-1}} u(t_k^-) = \phi_k^*(u(t_k)), \quad k = 1,2, \\ u(0) = 0, \quad \displaystyle\sum_{i=0}^{2}\left(\frac{4i+3}{12-2i}\right) u\left(\frac{2i+2}{5}\right) + \sum_{l=0}^{2}\left(\frac{2l+2}{7-2l}\right) {}_{\frac{3l+46}{50}}I_{t_l}^{\frac{2l+3}{4};\psi_l} u\left(\frac{3l+2}{6}\right) = e. \end{cases} \tag{5.1}$$

From the considered problem (5.1), we set $\alpha_k = (2k+8)/7$, $\beta_k = (3-k)/4$, $\rho_k = (3k+46)/50$, $\psi_k(t) = 1/(k+2) + \sin((k+2)t/((k+3)t-k+5))$, $\sigma_k = (3k+2)/(5-k)$, $\nu_k = (2k+3)/8$, $t_k = k/2$, $k = 0,1,2$, $T = 3/2$, $\mu_i = (4i+3)/(12-2i)$, $\eta_i = (2i+2)/5$, $\lambda_l = (2l+2)/(7-2l)$, $\theta_l = (2l+3)/4$, $\xi_l = (3l+2)/6$, $i = 0,1,2$, $l = 0,1,2$ and $A = e$. Thanks to the given data, we can compute that $\Lambda \approx 1.319519900$, $\Omega_1 \approx 0.226529808$, $\Omega_2 \approx 0.656891205$, $\Omega_3 \approx 1.166135348$, $\Omega_4 \approx 0.688903756$, $\Omega_5 \approx 0.228769110$, $\Omega_6 \approx 6.890783193$, $\Omega_7 \approx 5.410714286$, $\Omega_8 \approx 0.341866237$, and $\Omega_9 \approx 0.707853736$. The following functions will be considered for theoretical confirmation:

$$\begin{aligned} f(t,u,v,w) &= \frac{\ln(2t+3)}{\cos(\pi t)+3} + \Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\left(\frac{3e^{-5t}}{(t+2)^2+1}\cdot\frac{|u|}{5|u|+2} + \frac{5-2\sin(t)}{5e^t}\cdot\frac{|v|}{4|u|+3}\right. \\ &\quad \left. + \frac{3\cos(2t)}{7+\tan(t+\pi)}\cdot\frac{|w|}{2|w|+1}\right), \\ \phi_k(u(t_k)) &= \frac{1}{8t_k}\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)u(t_k) + e^{t_k}, \quad \phi_k^*(u(t_k)) = \frac{2t_k}{10t_k+30}\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)u(t_k) + \ln(t_k+1). \end{aligned}$$

For any u_i, v_i, $w_i \in R$, $i = 1,2$, and $t \in [0,3/2]$, it follows that

$$\begin{aligned} &|f(t,u_1,u_2,u_3) - f(t,v_1,v_2,v_3)| \leq \Psi_{\psi_k}^{2-\gamma_k}(t,t_k)\left(\frac{3}{10}|u_1-v_1| + \frac{2}{5}|u_2-v_2| + \frac{3}{7}|u_3-v_3|\right), \\ &|\phi_k(u)-\phi_k(v)| \leq \frac{1}{4}\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)|u-v|, \quad |\phi_k^*(u)-\phi_k(v)| \leq \frac{2}{35}\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)|u-v|. \end{aligned}$$

It is easy to see that the conditions (H_1) and (H_2) are fulfilled under $L_1 = 3/10$, $L_2 = 2/5$, $L_3 = 3/7$, $I_1 = 1/4$ and, $I_2 = 2/35$. Then we have $\Delta_1 \approx 0.449865758$ and $\Delta_2 \approx 0.278219606$, which implies that $\Delta_1 + \Delta_2 \approx 0.728085364 < 1$. Since all the conditions of Theorem 3.1 are satisfied, the considered problem (5.1) has a unique solution on $[0, 3/2]$. Furthermore, thanks of (4.10), we get

$$\mathfrak{C}_F := \frac{\Omega_1(\Omega_2 + \Omega_4 + \Omega_6) + \Omega_3 + \Omega_5 + m\Psi_*^{\gamma_m}}{1 - (\Delta_1 + \Delta_2)} \approx 17.965178470 > 0.$$

Therefore, the considered problem (5.1) is UH stable on $[0, T]$. By setting $\chi(\epsilon) = \mathfrak{C}_F \epsilon$ via $\chi(0) = 0$, we obtain from Corollary 4.1 that the considered problem (5.1) is GUH stable on $[0, 3/2]$. Moreover, if we put $\chi(t) = \Psi_{\psi_k}^{\frac{3}{\rho_k}}(t, t_k)$ into (U_1), we have

$${}_{\rho_k}I_{t_k}^{\alpha_k;\psi_k}\chi(t) = \frac{\Gamma_{\rho_k}(3 + \rho_k)\Psi_{\psi_k}^{\frac{\alpha_k}{\rho_k}}(t, t_k)}{\Gamma_{\rho_k}(3 + \rho_k + \alpha_k)}\Psi_{\psi_k}^{\frac{3}{\rho_k}}(t, t_k) \leq \frac{\Gamma_{\rho_k}(3 + \rho_k)\Psi_{\psi_k}^{\frac{\alpha_k}{\rho_k}}(t, t_k)}{\Gamma_{\rho_k}(3 + \rho_k + \alpha_k)}\chi(t).$$

Then, we have

$$\mathfrak{C}_\chi = \max_{k \in \{0,1,2\}} \left\{ \frac{\Gamma_{\rho_k}(3 + \rho_k)\Psi_{\psi_k}^{\frac{\alpha_k}{\rho_k}}(t, t_k)}{\Gamma_{\rho_k}(3 + \rho_k + \alpha_k)} \right\} \approx 0.017240540.$$

By applying (4.14), one has

$$\mathfrak{C}_{F,\chi_F} := \frac{\Omega_1[(\Omega_4 + \Omega_7 + \Omega_8)\mathfrak{C}_\chi + \Omega_4 + \Omega_8] + (m\Psi_*^{\gamma_m} + \Omega_9)\mathfrak{C}_\chi + m\Psi_*^{\gamma_m} + \Omega_5}{1 - (\Delta_1 + \Delta_2)} \approx 7.913856366.$$

Then, by Theorem 4.2, the considered problem (5.1) is UHR stable on $[0, 3/2]$. Finally, if we set $\chi(\epsilon) = \mathfrak{C}_{F,\chi_F}\epsilon$ via $\chi(0) = 0$ and $\epsilon = 1$ in Corollary 4.2, we obtain that the considered problem (5.1) is GUHR stable with respect to (δ, χ) on $[0, 3/2]$. In addition, we will present the graphical relations between $\Delta_1 + \Delta_2$, α_k, and $\beta_k \in [0, 1]$ for $k = 0, 1, 2$ in Figure 1, while Table 1 shows the relationship between $\alpha_k, \beta_k, \Lambda, \Omega_i$, $i = 1, 2, \ldots, 6$, and $\Delta_1 + \Delta_2 < 1$.

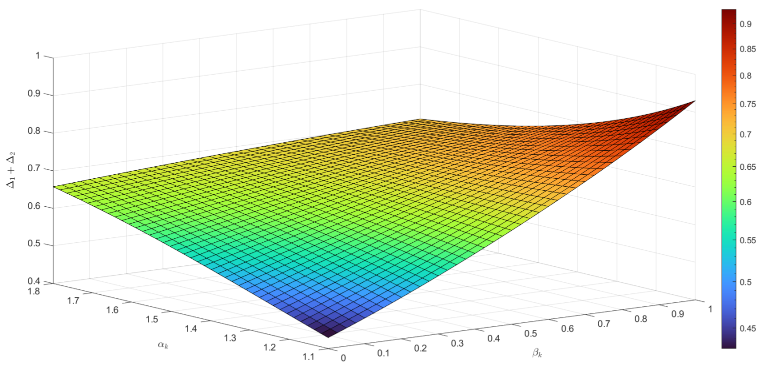

Figure 1. The condition $\Delta_1 + \Delta_2$ of Example 5.1 under $\alpha_k \in (1, 2]$ and $\beta_k \in [0, 1]$ for $k = 0, 1, 2$.

Table 1. The relationship between α_k, β_k, Λ, Ω_i, $i = 1, 2, \ldots, 6$, and $\Delta_1 + \Delta_2 < 1$.

α_k	β_k	Λ	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	$\Delta_1 + \Delta_2 < 1$
1.10	0.00	2.89747	0.02794	0.64651	1.02427	3.17599	0.13384	6.36489	0.42946
1.17	0.10	3.12101	0.04342	0.69646	1.03920	2.46685	0.16217	12.51157	0.49162
1.24	0.20	2.73831	0.06766	0.65209	1.05567	1.83607	0.18540	12.78250	0.54965
1.31	0.30	2.23210	0.10206	0.56559	1.07170	1.35043	0.20338	11.13307	0.60005
1.38	0.40	1.77633	0.14772	0.46890	1.08537	0.99976	0.21660	9.16775	0.64098
1.45	0.50	1.41494	0.20455	0.37834	1.09517	0.75403	0.22583	7.43935	0.67190
1.52	0.60	1.14457	0.27068	0.30046	1.10018	0.58419	0.23191	6.07673	0.69323
1.59	0.70	0.94888	0.34221	0.23655	1.10018	0.46769	0.23567	5.05917	0.70601
1.66	0.80	0.81105	0.41335	0.18549	1.09556	0.38856	0.23781	4.32682	0.71166
1.73	0.90	0.71760	0.47711	0.14527	1.08715	0.33603	0.23891	3.82072	0.71175
1.80	1.00	0.65893	0.52647	0.11380	1.07611	0.30315	0.239388	3.49485	0.70783

Example 5.2. *Consider the following impulsive problem of the form:*

$$\begin{cases} {}^{H}_{\frac{3k+46}{50}}\mathfrak{D}^{\frac{e^{k-1}+2}{e^{k-1}+1}, \frac{3-k}{4}; \psi_k}_{t_k^+} u(t) = f\left(t, u(t), {}_{\frac{3k+46}{50}}I^{\frac{3k+2}{5-k}; \psi_k}_{t_k} u(t), {}_{\frac{3k+46}{50}}I^{\frac{2k+3}{8}; \psi_k}_{t_k} u(t)\right), \ t \neq t_k, \ k = 0, 1, 2, \\ {}_{\frac{3k+46}{50}}I^{\frac{3k+46}{50}(2-\gamma_k); \psi_k}_{t_k^+} u(t_k^+) - {}_{\frac{3k+43}{50}}I^{\frac{3k+43}{50}(2-\gamma_{k-1}); \psi_{k-1}}_{t_{k-1}^+} u(t_k^-) = \phi_k(u(t_k)), \quad k = 1, 2, \\ {}^{RL}_{\frac{3k+46}{50}}\mathfrak{D}^{\frac{3k+46}{50}(\gamma_k - 1); \psi_k}_{t_k^+} u(t_k^+) - {}^{RL}_{\frac{3k+43}{50}}\mathfrak{D}^{\frac{3k+43}{50}(\gamma_{k-1} - 1); \psi_{k-1}}_{t_{k-1}^+} u(t_k^-) = \phi_k^*(u(t_k)), \quad k = 1, 2, \\ u(0) = 0, \quad \sum_{i=0}^{2} \left(\frac{4i+3}{12-2i}\right) u\left(\frac{2i+2}{5}\right) + \sum_{l=0}^{1} \left(\frac{2l+2}{7-2l}\right) {}_{\frac{3l+46}{50}}I^{\frac{2l+3}{4}; \psi_l}_{t_l} u\left(\frac{3l+2}{6}\right) = e. \end{cases} \tag{5.2}$$

From the considered problem (5.1), we set $\alpha_k = (e^{k-1}+2)/(e^{k-1}+1)$, $\beta_k = (3-k)/4$, $\rho_k = (3k+46)/50$, $\psi_k(t) = (t^{t^2-k+2})/(t+2k+10)$, $\sigma_k = (3k+2)/(5-k)$, $\nu_k = (2k+3)/8$, $t_k = k/2$, $k = 0, 1, 2$, $T = 3/2$, $\mu_i = (4i+3)/(12-2i)$, $\eta_i = (2i+2)/5$, $\lambda_l = (2l+2)/(7-2l)$, $\theta_l = (2l+3)/4$, $\xi_l = (3l+2)/6$, $i = 0, 1, 2$, $l = 0, 1$ and $A = e$. Thanks to the given data, we can compute that $\Lambda \approx 0.812170396$, $\Omega_1 \approx 0.167309417$, $\Omega_2 \approx 0.079568573$, $\Omega_3 \approx 1.027705947$, $\Omega_4 \approx 0.923594556$, $\Omega_5 \approx 0.229753595$, $\Omega_6 \approx 14.887728830$. The following functions will be considered for theoretical confirmation:

$$\begin{aligned} f(t, u, v, w) &= \frac{(3t+5)\Psi^{2-\gamma_k}_{\psi_k}(t, t_k)}{t^2+5t+25}\left(\frac{2|u|^2+7|u|}{|u|+3} - 1\right) \\ &\quad + \Psi^{2-\gamma_k}_{\psi_k}(t, t_k)\left(\frac{\sin(t)}{2e^t} \cdot \frac{|v|}{|v|+2} + \frac{\ln(t+8)}{\cos^2(t)+3} \cdot \frac{|w|}{3|w|+1}\right), \\ \phi_k(u(t_k)) &= \frac{e^{t_k-1}}{2t_k}\Psi^{2-\gamma_k}_{\psi_k}(t, t_k)\frac{|u(t_k)|}{3|u(t_k)|+5}, \qquad \phi_k^*(u(t_k)) = \frac{\cos(\pi t_k)}{4t_k+3}\Psi^{2-\gamma_k}_{\psi_k}(t, t_k)\frac{|u(t_k)|}{|u(t_k)|+2}. \end{aligned}$$

For any $u, v, w \in R$, and $t \in [0, 3/2]$, it follows that

$$|f(t, u, v, w)| \leq \frac{(3t+5)}{t^2+5t+25}\left(2\Psi^{2-\gamma_k}_{\psi_k}(t, t_k)|u| + 1\right) + \Psi^{2-\gamma_k}_{\psi_k}(t, t_k)\left(\frac{\sin(t)|v|}{4} + \frac{|w|\ln(t+8)}{3}\right),$$

$$|\phi_k(u)| \leq \frac{1}{3}\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)|u|, \qquad |\phi_k^*(u)| \leq \frac{1}{5}\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)|u|,$$
$$|\phi_k(u)-\phi_k(v)| \leq \frac{1}{5}\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)|u-v|, \qquad |\phi_k^*(u)-\phi_k(v)| \leq \frac{1}{10}\Psi_{\psi_k}^{2-\gamma_k}(t,t_k)|u-v|.$$

By applying (H_3)–(H_5) with $\Theta(u) = 2|u|+1$, we obtain that $M_1 = 0.296890807$, $M_2 = 0.178134484$, $g_1^* = 0.273381295$, $g_2^* = 0.249373747$, $g_3^* = 0.750430599$, $\Xi_1 = 1/5$, $\Xi_2 = 1/10$, and

$$[(\Omega_5+\Omega_1\Omega_4)\Xi_1 + (m\Psi_*^{\gamma_m}+\Omega_1\Omega_6)\Xi_2] \approx 0.421759604 < 1.$$

By using (3.30) and (H_6), we have that

$$\mathfrak{C}^* := (\Omega_5+\Omega_1\Omega_4)M_1 + (m\Psi_*^{\sigma_m}+\Omega_1\Omega_6)M_2 + \Omega_1|A| \approx 1.081294628.$$

Then,

$$\sup_{\Re_2\in(0,\infty)} \frac{\Re_2}{g_1^*\Theta(\Re_2)(\Omega_3+\Omega_1\Omega_2)+\mathfrak{C}^*} \approx 1.756882645,$$
$$\frac{1}{1-[g_2^*\Psi_*^{\sigma_m}+g_3^*\Psi_*^{\nu_m}](\Omega_3+\Omega_1\Omega_2)} \approx 1.123309473,$$

which yields $\Re_2 > 4.041270259$. Since all the conditions of Theorem 3.2 are satisfied, the considered problem (5.2) has a solution on $[0,3/2]$.

Example 5.3. *Consider the following the impulsive (ρ_k,ψ_k)-HFP-IDE-MIBCs of the form:*

$$\begin{cases} {}^{H}_{\frac{k+18}{20}}\mathfrak{D}_{t_k^+}^{\alpha_k,\frac{4-k}{5};\psi_k}u(t) = 2, \ t\neq t_k, \ k=0,1,2, \\ {}_{\frac{k+18}{20}}I_{t_k^+}^{\frac{k+18}{20}(2-\gamma_k);\psi_k}u(t_k^+) - {}_{\frac{k+17}{20}}I_{t_{k-1}^+}^{\frac{k+17}{20}(2-\gamma_{k-1});\psi_{k-1}}u(t_k^-) = -\dfrac{k+1}{k+2}, \quad k=1,2, \\ {}^{RL}_{\frac{k+18}{20}}\mathfrak{D}_{t_k^+}^{\frac{k+18}{20}(\gamma_k-1);\psi_k}u(t_k^+) - {}^{RL}_{\frac{k+17}{20}}\mathfrak{D}_{t_{k-1}^+}^{\frac{k+17}{20}(\gamma_{k-1}-1);\psi_{k-1}}u(t_k^-) = (-1)^k\dfrac{3}{2}, \quad k=1,2, \\ u(0)=0, \quad \displaystyle\sum_{i=0}^{3}\left(\frac{4i+3}{12-2i}\right)u\left(\frac{2i+2}{5}\right) + \sum_{l=0}^{3}\left(\frac{2l+2}{7-2l}\right){}_{\frac{l+18}{20}}I_{t_l}^{\frac{2l+3}{4};\psi_l}u\left(\frac{3l+2}{6}\right) = e. \end{cases} \tag{5.3}$$

Form the considered problem (5.3), we set $\alpha_k \in \{(2\pi+k-2)/4, 1+\sqrt{(k+1)/(k+6)}, (e^{k-1}+2)/(e^{k-1}+1), \ln(k+4)\}$, $\beta_k = (4-k)/5$, $\rho_k = (k+18)/20$, $\psi_k(t) \in \{1/(k+2)+\sin((k+2)t/((k+3)t-k+5)), (k+4)/2-\arccos((t^2+kt-2)/10), (t^{t^2-k+3})/(t+2k+8), 2-(\ln[(k+2)t+3k+3])/(\ln[(k+1)t+2k+2])\}$, $t_k = k/2$, $k=0,1,2,3$, $T=2$, $\phi_k(u(t_k)) = -(k+1)/(k+2)$, $\phi_k^*(u(t_k)) = (-1)^k(3/2)$, $k=1,2,3$, $\mu_i = (4i+3)/(12-2i)$, $\eta_i = (2i+2)/5$, $\lambda_l = (2l+2)/(7-2l)$, $\theta_l = (2l+3)/4$, $\xi_l = (3l+2)/6$, $i=0,1,2,3$, $l=0,1,2,3$, and $A=e$. Thanks to the given data, we can compute that $\Lambda \approx 1.3195199$. By using Lemma 2.6 with $f(t,u(t),{}_{\rho_k}I_{t_k}^{\sigma_k;\psi_k}u(t),{}_{\rho_k}I_{t_k}^{\nu_k;\psi_k}u(t)) = 2$, the solution of the considered problem (5.3) can be written as

$$u(t) = \left\{\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Lambda\Gamma_{\rho_k}(\rho_k\gamma_k)} + \frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Lambda\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\sum_{j=0}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\right\}\Bigg\{e$$

$$
\begin{aligned}
&-\Bigg(\sum_{i=0}^{4}\left(\frac{4i+3}{12-2i}\right)\frac{\Psi_{\psi_i}^{\gamma_i-1}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i\gamma_i)}\sum_{j=0}^{i-1}\left(\frac{2\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j-\rho_j(\gamma_j-1)+\rho_j)}-\frac{j+2}{j+3}\right)\\
&+\sum_{i=0}^{4}\left(\frac{4i+3}{12-2i}\right)\frac{\Psi_{\psi_i}^{\gamma_i-2}(\eta_i,t_i)}{\Gamma_{\rho_i}(\rho_i(\gamma_i-1))}\Bigg[\sum_{j=0}^{i-1}\left(\frac{2\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j+\rho_j(2-\gamma_j)+\rho_j)}+(-1)^{j+1}\frac{3}{2}\right)\\
&+\sum_{j=1}^{i-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left(\frac{2\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r-\rho_r(\gamma_r-1)+\rho_r)}-\frac{r+2}{r+3}\right)\Bigg]+\sum_{i=0}^{4}\left(\frac{4i+3}{6-i}\right)\frac{\Psi_{\psi_i}^{\frac{\alpha_i}{\rho_i}}(\eta_i,t_i)}{\Gamma_{\rho_i}(\alpha_i+\rho_i)}\\
&+\sum_{l=0}^{3}\left(\frac{4l+4}{7-2l}\right)\frac{\Psi_{\psi_l}^{\frac{\alpha_l+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\alpha_l+\theta_l+\rho_l)}+\sum_{l=0}^{3}\left(\frac{2l+2}{7-2l}\right)\frac{\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-1)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l\gamma_l+\theta_l)}\sum_{j=0}^{l-1}\left(\frac{2\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j-\rho_j(\gamma_j-1)+\rho_j)}-\frac{j+2}{j+3}\right)\\
&+\sum_{l=0}^{3}\left(\frac{2l+2}{7-2l}\right)\frac{\Psi_{\psi_l}^{\frac{\rho_l(\gamma_l-2)+\theta_l}{\rho_l}}(\xi_l,t_l)}{\Gamma_{\rho_l}(\rho_l(\gamma_l-1)+\theta_l)}\Bigg[\sum_{j=0}^{l-1}\left(\frac{2\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j+\rho_j(2-\gamma_j)+\rho_j)}+(-1)^{j+1}\frac{3}{2}\right)\\
&+\sum_{j=1}^{l-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left(\frac{2\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r-\rho_r(\gamma_r-1)+\rho_r)}-\frac{r+2}{r+3}\right)\Bigg]\Bigg)\Bigg\}\\
&+\frac{2\Psi_{\psi_k}^{\frac{\alpha_k}{\rho_k}}(t,t_k)}{\Gamma_{\rho_k}(\alpha_k+\rho_k)}+\frac{\Psi_{\psi_k}^{\gamma_k-1}(t,t_k)}{\Gamma_{\rho_k}(\rho_k\gamma_k)}\sum_{j=0}^{k-1}\left(\frac{2\Psi_{\psi_j}^{\frac{\alpha_j-\rho_j(\gamma_j-1)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j-\rho_j(\gamma_j-1)+\rho_j)}-\frac{j+2}{j+3}\right)\\
&+\frac{\Psi_{\psi_k}^{\gamma_k-2}(t,t_k)}{\Gamma_{\rho_k}(\rho_k(\gamma_k-1))}\Bigg[\sum_{j=0}^{k-1}\left(\frac{2\Psi_{\psi_j}^{\frac{\alpha_j+\rho_j(2-\gamma_j)}{\rho_j}}(t_{j+1},t_j)}{\Gamma_{\rho_j}(\alpha_j+\rho_j(2-\gamma_j)+\rho_j)}+(-1)^{j+1}\frac{3}{2}\right)\\
&+\sum_{j=1}^{k-1}\frac{\Psi_{\psi_j}(t_{j+1},t_j)}{\rho_j}\sum_{r=0}^{j-1}\left(\frac{2\Psi_{\psi_r}^{\frac{\alpha_r-\rho_r(\gamma_r-1)}{\rho_r}}(t_{r+1},t_r)}{\Gamma_{\rho_r}(\alpha_r-\rho_r(\gamma_r-1)+\rho_r)}-\frac{r+2}{r+3}\right)\Bigg].
\end{aligned}
\tag{5.4}
$$

Hence, the solution of the considered problem (5.3) is divided into three cases.

Case I. If we set $\alpha_k \in \{\pi/2 + (k-2)/4,\ 1+\sqrt{(k+1)/(k+6)},\ (e^{k-1}+2)/(e^{k-1}+1),\ \ln(k+4)\}$ and $\psi_k(t) = 1/(k+2) + \sin(((k+2)t)/((k+3)t + (5-k)))$ for $k = 0,1,2,3$, then the solution of the considered problem (5.3) is displayed in Figure 2.

Case II. If we set $\alpha_k = \pi/2 + (k-2)/4$ and $\psi_k(t) \in \{1/(k+2) + \sin(((k+2)t)/((k+3)t-k+5)), (k+4)/(2) - \arccos((t^2+kt-2)/(10)), (t^{t^2-k+3})/(t+2k+8), 2-(\ln[(k+2)t+3k+3])/(\ln[(k+1)t+2k+2])\}$ for $k = 0,1,2,3$, then the solution of the considered problem (5.3) is displayed in Figure 3.

Case III. If we set $\alpha_k \in \{\pi/2 + (k-2)/4,\ 1+\sqrt{(k+1)/(k+6)},\ (e^{k-1}+2)/(e^{k-1}+1),\ \ln(k+4)\}$ and $\psi_k(t) \in \{1/(k+2) + \sin(((k+2)t)/((k+3)t-k+5)), (k+4)/(2) - \arccos((t^2+kt-2)/(10)), (t^{t^2-k+3})/(t+2k+8), 2-(\ln[(k+2)t+3k+3])/(\ln[(k+1)t+2k+2])\}$ for $k = 0,1,2,3$, then the solution of the considered problem (5.3) is displayed in Figure 4.

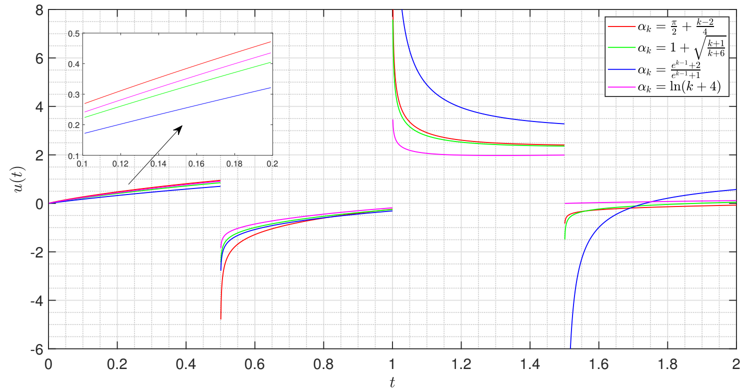

Figure 2. The solution of Example (5.3) via $\alpha_k \in \{\frac{\pi}{2} + \frac{k-2}{4}, 1 + \sqrt{\frac{k+1}{k+6}}, \frac{e^{k-1}+2}{e^{k-1}+1}, \ln(k+4)\}$ and $\psi_k(t) = \frac{1}{k+2} + \sin(\frac{(k+2)t}{(k+3)t+(5-k)})$ for $k = 0, 1, 2, 3$.

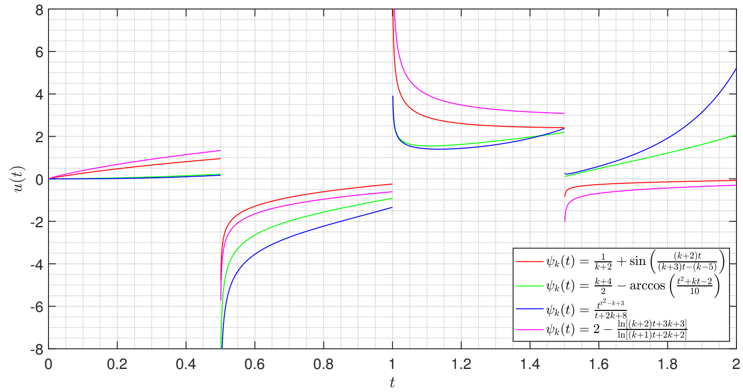

Figure 3. The solution of Example (5.3) via $\alpha_k = \frac{\pi}{2} + \frac{k-2}{4}$ and $\psi_k(t) \in \{\frac{1}{k+2} + \sin(\frac{(k+2)t}{(k+3)t-k+5}), \frac{k+4}{2} - \arccos(\frac{t^2+kt-2}{10}), \frac{t^{t^2-k+3}}{t+2k+8}, 2 - \frac{\ln[(k+2)t+3k+3]}{\ln[(k+1)t+2k+2]}\}$ for $k = 0, 1, 2, 3$.

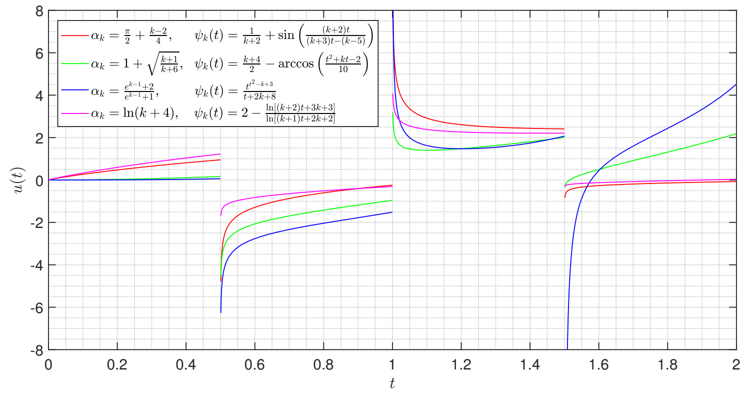

Figure 4. The solution of Example (5.3) via $\alpha_k \in \{\frac{\pi}{2} + \frac{k-2}{4}, 1 + \sqrt{\frac{k+1}{k+6}}, \frac{e^{k-1}+2}{e^{k-1}+1}, \ln(k+4)\}$ and $\psi_k(t) \in \{\frac{1}{k+2} + \sin(\frac{(k+2)t}{(k+3)t-k+5}), \frac{k+4}{2} - \arccos(\frac{t^2+kt-2}{10}), \frac{t^{t^2-k+3}}{t+2k+8}, 2 - \frac{\ln[(k+2)t+3k+3]}{\ln[(k+1)t+2k+2]}\}$ for $k = 0, 1, 2, 3$.

6. Conclusions

In this paper, we have investigated existence theory and stability results for a class of nonlinear impulsive boundary value problem of fractional integro-differential equations supplemented with mixed nonlocal multi-point and multi-term integral boundary conditions in the context of the (ρ_k, ψ_k)-Hilfer fractional derivatve. Firstly, the solution to the linear variant impulsive considered problem was introduced in terms of a Volterra integral equation. The uniqueness result was proved by using Banach's fixed point theorem, while the existence result was established by means of a fixed point theorem due to O'Regan. In addition, a variety of Ulam's stability such as UH, GUH, UHR and GUHR stability were studied by applying nonlinear functional analysis technique. Finally, three examples illustrating the results are also provided to confirm the correctness of the theoretical results. The novelty of our results is not only finding a distinctive qualitative theory for this problem within the given frame but also addressing some new, interesting exceptional cases for various values of the parameters related to the considered problem. For example,

- (i) If we set $\lambda_l = 0$ for all $l = 0, 1, \ldots, n$, then the considered problem (1.4) reduces to BVP for nonlinear impulsive (ρ_k, ψ_k)-Hilfer-FIDEs under nonlocal multi-point boundary conditions: $u(0) = 0$, $\sum_{i=0}^{m} \mu_i u(\eta_i) = A$.
- (ii) If we set $\mu_i = 0$ for all $i = 0, 1, \ldots, m$, then the considered problem (1.4) reduces to BVP for nonlinear impulsive (ρ_k, ψ_k)-Hilfer-FIDEs under nonlocal multi-term integral boundary conditions: $u(0) = 0$, $\sum_{l=0}^{n} \lambda_{l\,\rho_l} I_{t_l}^{\theta_l;\psi_l} u(\xi_l) = A$.

This research would provide a significant contribution to the literature on the qualitative theory, which might involve the growth of the idea introduced in this field as well as the possibility for further generalizations in a wide range of exclusive outputs for applications and theories. One proposal is that future studies explore the existence and uniqueness of solutions for additional forms of nonlinear differential-integral equations in the setting of other fractional operators with varied boundary conditions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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