AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 20415–20436.
DOI:10.3934/math.20231041
Received: 13 March 2023
Revised: 05 May 2023
Accepted: 09 May 2023
Published: 25 June 2023

Research article

A new approach to the study of fixed points based on soft rough covering graphs

Imran Shahzad Khan[1], **Nasir Shah**[2], **Abdullah Shoaib**[1], **Poom Kumam**[3] **and Kanokwan Sitthithakerngkiet**[4,*]

[1] Department of Mathematics and Statistics, Riphah International University, Islamabad, Pakistan

[2] Department of Mathematics, Islamabad Model College for Girls, F-6/2, Islamabad, Pakistan

[3] Center of Excellence in Theoretical and Computational Science (TaCS-CoE) & KMUTT Fixed Point Research Laboratory, Departments of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Thung Khru, Bangkok 10140, Thailand

[4] Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok (KMUTNB), Wongsawang, Bangsue, Bangkok 10800, Thailand

* **Correspondence:** Email: kanokwan.s@sci.kmutnb.ac.th.

Abstract: Mathematical approaches to structure model problems have a significant role in expanding our knowledge in our routine life circumstances. To put them into practice, the right formulation, method, systematic representation, and formulation are needed. The purpose of introducing soft graphs is to discretize these fundamental mathematical ideas, which are inherently continuous, and to provide new tools for applying mathematical analysis technology to real-world applications including imperfect and inexact data or uncertainty. Soft rough covering models (briefly, $\mathcal{SRC}$-Models), a novel theory that addresses uncertainty. In this present paper, we have introduced two new concepts $\mathcal{L}$i-soft rough covering graphs ($\mathcal{L}$i-$\mathcal{SRCG}$s) and the concept of fixed point of such graphs. Furthermore, we looked into a some algebras that dealt with the fixed points of $\mathcal{L}$i-$\mathcal{SRCG}$s. Applications of the algebraic structures available in covering soft sets to soft graphs may reveal new facets of graph theory.

Keywords: $\mathcal{L}$i-soft rough covering sets; $\mathcal{L}$i-soft rough covering graphs; soft neighborhoods; lattices; fixed points; double stone algebra
Mathematics Subject Classification: 05C72, 05C76, 05C85

1. Introduction

Traditional mathematical methods are frequently unable to address a wide range of complicated problems that arise in a variety of fields, including economics, engineering, sociology, medicine, environmental science, and many others. This is because many problems are inherently vague and uncertain. Uncertainty related issues and situations are dealt with using an antiquated and powerful instrument of probability theory, which is only appropriate when the occurrence of events is solely determined by chance. Along with probability theory, additional well-known theories have been created to address uncertainty, including fuzzy sets, intuitionistic fuzzy sets, rough sets, soft sets, and combinations of these theories.

The idea of fuzzy sets, which Lotfi. A. Zadeh [1] introduced in 1965, has shown to be a highly effective theoretical solution to ambiguity and vagueness. This idea is founded on fuzzy membership functions, which depict an element's membership in a set. By focusing on the fuzzy information granularity, fuzzy set theory can be utilized to obtain, simulate, and even explain the fuzziness in many practical materials of information. Almost all areas of mathematics, medicine, engineering, and other fields have had concepts reinterpreted using fuzzy sets as a result of their extensive uses.

Pawlak [2] proposed the theory of rough sets in the early 1980's which is another nice mathematical tool dealing the set's approximation for dealing with the uncertainty of imprecise data and vagueness. Given the facts at hand, this idea is the best replacement for fuzzy set theory and tolerance theory. Rough sets and fuzzy set theory are two distinct approaches used to address the ambiguity, imprecision, and haziness of real-world situations. Each of these theories has its inherent limitations. Many applications together with machine learning, data mining, knowledge discovery and pattern recognition can be found in [3–8]. The fundamental idea of traditional rough set theory is the lower/upper approximation concept, which is typically based on equivalence relations and partitions. Sometimes dealing with a real-world issue while constrained by an equivalence relation is challenging. In order to tackle such kind of situations, the concept of rough set has been extended to the notion of covering rough sets introduced in [9–15] which are an important generalizations of classical and traditional rough sets by relaxing partition of universe to covering. A more all-encompassing notion used to handle the attribute subset is covering, which is a method to expand any partition. Covering-based rough sets are more rational and logical than traditional rough sets for addressing uncertainty related issues, and this theory has attracted significant attention and produced numerous useful research outcomes.

In 1999, Molodtsov [16] introduced the concept of soft sets which is a very new and effective mathematical technique for handling uncertainties. It is a set connected with parameters and has been used in many different contexts. Several methods have been developed for addressing the imprecision, uncertainty, and ambiguity of real-world situations including fuzzy set theory, the rough, soft sets and blend of these theories. Each of these theories comes with certain built in drawbacks. [14–21] has several uses, including machine learning, data mining, pattern detection, and knowledge discovery.

In order to establish an applicable mathematical systems for covering-based rough set and promote its applications in various fields of life, it has been linked with some other theories like fuzzy set theory, soft set theory, neutrosophic set theory, graph theory and blend of theories [7, 14, 21–24]. The notion of a family of fuzzy complementary β-neighborhood and thus four types of covering-basedoptimistic (pessimistic) multigranulation fuzzy rough sets models are presented in [25]. Also, four new kinds of covering-based M-optimistic (pessimistic) multigranulation fuzzy rough sets models are constructed. Some characterizations of these models and its relation with Zhan's model are studied.

Lattice theory and partial order play an important role in many fields of engineering and computer science and they have many applications in distributed computing, that is, vector clocks and global predicate detection, concurrency theory, occurrence nets and pomsets, programming language semantics (fixed-point semantics), and data mining [26]. They are also useful in other disciplines of mathematics such as combinatorics, group theory and number theory. Many authors have combined the rough set theory and lattice theory, and some useful results have been obtained. Based on the existing works about the connection of rough sets and lattice theory, Chen et al. [4] used the notion of covering to define the approximation operators on a completely distributive lattice and set up a unified framework for generalizations of rough sets. Shah et al. [27] discussed another approach to roughness of soft graphs with applications in decision making, see also [19, 24, 28–31]. Rough approximation models via graphs based on neighborhood systems can be seen in [32]. Some applications of soft graphs linked with rough sets and soft sets can be seen in [27, 33–36]. Further, the concept of dual hesitant fuzzy graphs (DHFGs) proposed by [37], where a two-stage MADM approach is constructed by means of DHFGs for addressing complicated MADM situations with correlations and prioritization relationships. He et al. [38] revised the "tight" bounds for path-factor critical avoidable graphs. Since the avoidable graph is a special case of deleted graphs, a link is established between the path-factor critical graph and the path-factor deleted graph. In [34], Praba defined a novel rough set called minimal soft rough set by using minimal soft description of the objects. They also analyzed the relation between modified soft rough set and minimal soft rough set. They proposed a lattice structure on minimal soft rough sets. Uncertainty measures associated with neighborhood based soft covering rough graphs such as roughness measure, entropy measure and granularity are proposed in [36]. Atef and Nada [39] introduced the concept of the complementary fuzzy soft neighborhood as a generalization of Zhan's method, which increases the lower approximation and decreases the upper approximation. As a result, three new types of soft fuzzy rough covering models are constructed. These constructions' properties are discussed. They define three categories of fuzzy soft measure degrees in light of these results. A decision-making algorithm is then described based on the suggested operations, and its performance is illustrated with a numerical example. Further the relationships among these three models and Zhan's model are presented [39, 40]. $\mathcal{L}$i and Zhu [11] introduced the lattice structures of fixed points of the lower approximations of two types of covering-based rough sets in which they discussed that under what conditions two partially ordered sets are some lattice structures. They defined two types of sets called the fixed point set of neighborhoods and the fixed point set of covering, respectively. Fixed points of covering upper and lower approximation operators are introduced in [41] in which by using some results about the Feynman paths, they have shown that the family of all fixed points of covering upper and lower approximation operators is an atomic frame and a complete lattice, respectively.

Z-soft rough covering models introduced by Zhan et al. [42] are important generalizations of classical rough set theory to deal with data structure and more complex problems of the real world. It can be seen in [43] that the CSR approach uses property soft neighborhoods to establish models that successfully grow the lower approximation and lower the upper approximation and study the relationships between these models and some of the topological properties based on the CSR approach. In order to solve MGDM problems, they finally developed an algorithm for the presented model. Different kinds of uncertainty measures related to $\mathcal{L}$i-soft rough covering sets and their limitations are presented in [44]. The concept of fixed point sets by using Z-soft rough covering models is introduced by Imran et al. [45], where they have discussed different algebraic structures along with their limitations connecting both lattices and Z-soft rough covering models. The purpose of introducing soft graphs is to discretize these fundamental mathematical ideas, which are inherently continuous, and to provide new tools for applying mathematical analysis technology to real-world applications including imperfect and inexact data or uncertainty. $\mathcal{L}$i et al. presented the idea of soft rough covering models (briefly, $\mathcal{SRC}$-Models) , a novel theory that addresses uncertainty. Two new notions have been introduced in the current paper. $\mathcal{L}$i-soft rough covering graphs ($\mathcal{L}$i-$\mathcal{SRCG}$s) and the notion of fixed point sets, also known as $\mathcal{L}$i-$\mathcal{SRCFP}$ sets of such graphs. Several types of approximation operators and their related properties are discussed using $\mathcal{L}$i-$\mathcal{SRCG}$s . We also investigated a few algebras that dealt with the fixed points of $\mathcal{L}$i-$\mathcal{SRCG}$s . Applications of the algebraic structures available in covering soft sets to soft graphs may open up new areas of graph theory. We go over the prerequisites for the family of $\mathcal{L}$i-$\mathcal{SRCFP}$ sets acquiring lattice structure, distributive lattice, complete lattice, and some algebra pertaining to soft graphs. This paper is organized as follows.

The basics of rough sets, covering soft sets, $\mathcal{L}$i-soft rough covering models, lattices and soft graphs are reviewed in Section 2 of this article. In Section 3, we define the concept of $\mathcal{L}$i-$\mathcal{SRCG}$s , $\mathcal{L}$i-soft reduct of covering soft sets, and their attributes based on $\mathcal{L}$i-$\mathcal{SRCG}$s . Further, we investigate the conditions in which the soft neighborhood $\mathcal{L}$i-$\mathcal{SRCFP}$ sets transform into particular lattice structures. In Section 4, we explore the idea of soft graphs'$\mathcal{L}$i-soft rough covering fixed point sets ($\mathcal{L}$i-$\mathcal{SRCFP}$ sets). We have talked about the idea of bounds for any two $\mathcal{L}$i-$\mathcal{SRCG}$s elements. Several algebraic structures related to $\mathcal{L}$i-$\mathcal{SRCG}$s are also addressed. In Section 5, we finally put our paper to conclusion.

2. Preliminaries

This section provides a succinct review of certain essential concepts, results, and core ideas that will be useful in comprehending the remaining chapters of this thesis. The universe V is assumed to be a non-void finite set throughout this article, along with the void (empty) set $\emptyset$ and R, the parameter's set.

Definition 2.1. *A family C of non-void subsets of U is said to be a covering of U if $\bigcup C = U$. Also, for a subset Y of U, the sets*

(i) given below, denoted by $L_C(\mathcal{Y})$ and $H_c(\mathcal{Y})$, are called respectively, the second type covering lower and upper approximation of Y,

$$F_{\mathfrak{L}}(\mathcal{Y}) = \cup\{K \in C : K \subseteq Y\},$$
$$F_{\mathcal{H}}(\mathcal{Y}) = \cup\{K \in C : K \cap Y \neq \emptyset\},$$

(ii) given below, denoted by $F_{\mathfrak{L}}(\mathcal{Y})$ and $F_{\mathcal{H}}(\mathcal{Y})$, are called respectively, covering lower and upper approximations of sixth type of Y

$$L_C(\mathcal{Y}) = \{x \in U : N(x) \subseteq Y\},$$
$$H_c(\mathcal{Y}) = \{x \in U : N(x) \cap Y \neq \emptyset\},$$

where $N(x) = \bigcap\{Q \in C : x \in Q\}$ is called neighborhood of x with respect to C.

Proposition 2.1. *For any subset Y of U, the following laws always hold true:*

(i) $F_{\mathfrak{L}}(\emptyset) = \emptyset$ and $L_C(\emptyset) = \emptyset$
(ii) $F_{\mathfrak{L}}(\mathcal{U}) = U$ and $L_C(\mathcal{U}) = U$

(iii) $F_{\mathfrak{L}}(\mathcal{Y}) \subseteq Y$ *and* $L_C(\mathcal{Y}) \subseteq Y$ *(iv)* $F_{\mathfrak{L}}(\mathcal{F}_{\mathfrak{L}}(\mathcal{Y})) = F_{\mathfrak{L}}(\mathcal{Y})$ *and* $L_C(\mathcal{L}_C(\mathcal{Y})) = L_C(\mathcal{Y})$ *(v)* $Y \subseteq X$ *implies* $F_{\mathfrak{L}}(\mathcal{Y}) \subseteq F_{\mathfrak{L}}(\mathcal{X})$ *and* $L_C(\mathcal{Y}) \subseteq L_C(\mathcal{X})$ *(vi) for all* $K \in C$, $F_{\mathfrak{L}}(\mathcal{K}) = (\mathcal{K})$ *and* $L_C(\mathcal{K}) = L_C(\mathcal{K})$.

Definition 2.2. *Let* $\tau : R \to P(\mathcal{U})$ *is a set valued mapping, then the ordered pair* $T = (\pi, \Re)$ *is called a soft set over U. In this case, the pair* $D = (\mathcal{U}, \mathfrak{T})$, *is called soft approximation space (briefly,* $\mathcal{SAS}$*).*

Definition 2.3. *A soft set* $(\tau, \Re)$ *is called covering soft set (briefly,* $\mathcal{CSS}$*)if*

(i) it is full, that is, if $\bigcup_{\sigma\in\Re} \tau(\sigma) = U$ *and,*
(ii) for every $\sigma \in R$, $\tau(\sigma) \neq \emptyset$.

In such case, the ordered pair $T = (\mathcal{U}, \mathbb{C}_{\mathcal{V}})$ *is called* $\mathcal{SCAS}$ *(soft covering approximation space). Then, for W⊆ U, the following two sets:*

$$\begin{aligned} \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) &= \cup\{\tau(\sigma) \in \mathbb{C}_{\mathcal{V}} : \tau(\sigma) \subseteq \mathcal{W}\} \text{ and} \\ \overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) &= \cup\{\tau(\sigma) \in \mathbb{C}_{\mathcal{V}} : \tau(\sigma) \cap \mathcal{W} \neq \emptyset\} \end{aligned}$$

In the above sets the operators $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W})$ *and* $\overline{\mathcal{F}_\tau}(\mathcal{W})$ *are called respectively,* $\mathcal{L}$i$-\mathcal{SCLA}$ *operator and* $\mathcal{L}$i$-\mathcal{SCUA}$ *operator.*

Definition 2.4. *A partial ordered set (or briefly, a poset) is an ordered pair* $(\mathfrak{L}, \preceq)$, *consisting of a non-void* $\mathfrak{L}$ *and a partial order* $\preceq$ *on* $\mathfrak{L}$.

Definition 2.5. *A lattice is a poset* $(\mathfrak{L}, \preceq)$ *in which* $a \wedge b = \inf(a, b)$ *and* $a \vee b = \sup(a, b)$ *exist for any pair of elements a and b of* $\mathfrak{L}$*. Also,* $\mathfrak{L}$ *is said to have a lower bound* 0 *if and only for any t in* $\mathfrak{L}$ *we have,* $0 \preceq t$*. Analogously,* $\mathfrak{L}$ *is said to have an upper bound* 1 *if and only for any t in* $\mathfrak{L}$ *we have,* $t \preceq 1$*. A lattice* $(\mathfrak{L}, \preceq)$ *is bounded if and only if has both* 0 *and* 1.

Definition 2.6. *A lattice* $(\mathfrak{L}, \preceq)$ *is said to be distributed if and only if for any a, b,t in* $\mathfrak{L}$ *we have,* $a \wedge (b \vee t) = (a \wedge b) \vee (a \wedge t)$ *and* $a \vee (b \wedge t) = (a \vee b) \wedge (a \vee t)$*. Otherwise,* $(\mathfrak{L}, \preceq)$ *is said to be non distributive.*

Definition 2.7. *Let* 0 *be a lower bound and 1 be an upper bound in a lattice* $(\mathfrak{L}, \preceq)$*. An element p in* $\mathfrak{L}$ *is said to be*

(i) join irreducible if $p = a \vee b$ *implies* $p = a$ *or* $p = b$.
(ii) complement of t if $p \vee t = 1$ *and* $p \wedge t = 0$.
(iii) pseudocomplement of $t \in \mathfrak{L}$*, if* $p \wedge t = 0$ *and for all* $r \in \mathfrak{L}$, $t \wedge r = 0$ *implies* $r \preceq p$.
(iv) dual pseudocomplement of $t \in \mathfrak{L}$*, if* $p \vee t = 1$ *and for all* $r \in \mathfrak{L}$, $t \vee r = 1$ *implies* $p \preceq r$.

Definition 2.8. *A lattice* $(\mathfrak{L}, \preceq)$ *is said to be*

(i) complemented if it is (a) bounded and (b) every element of $\mathfrak{L}$ *has a complement ;*
(ii) pseudocomplemented (briefly, pseud^{Cd}*) if every member of* $\mathfrak{L}$ *has pseudocomplement ;*
(iii) a Stone algebra if it is a lattice which is (a) distributive (b) pseud^{Cd} *and (c) satisfying the identity* $p^* \vee p^{**} = 1$, *for all* $p \in \mathfrak{L}$, *(where* p^**is pseudocomplemented of p) ;*
(iv) a dual pseudocomplemented (briefly, D-pseud^{Cd}*) if each of its member has dual pseudocomplement;*

(v) *a dual Stone algebra, if the lattice meets the conditions of being (a) distributive (b) D-pseud^{Cd}, and (c) satisfying the identity (dual Stone identity) $p^* \wedge p^{**} = 1$, for all $p \in \mathfrak{L}$;*
(vi) *a double p-algebra if it is simultaneously (a) pseud^{Cd} and (b) D-pseud^{Cd} ;*
(vii) *a double Stone algebra, if it is (a) Stone algebra and (b) dual Stone algebra.*

Definition 2.9. *Let $\Omega = (\mathcal{V},\mathcal{E})$ be a graph. Then, a quadruple $\Theta = (\Omega, \delta, \gamma, \mathfrak{R})$ is called a soft graph, provided*

(i) *$\delta : \mathfrak{R} \rightarrow \mathfrak{P}(\mathcal{V})$ is a soft set over vertex $\mathcal{V}$;*
(ii) *$\gamma : \mathfrak{R} \rightarrow \mathfrak{P}(\mathcal{E})$ is a soft set over $\mathcal{E}$ and for every $\sigma \in \mathfrak{R}$, the pair $H = (\delta(\sigma), \gamma(\sigma))$ represents a subgraph of Ω.*

Further, if Θ is a soft graph such that

(a) $\bigcup\limits_{\sigma\in\mathfrak{R}} \delta(\sigma) = \mathcal{V}$, *then Θ is called full soft vertex graph;*
(b) $\bigcup\limits_{\sigma\in\mathfrak{R}} \gamma(\sigma) = \mathcal{E}$, *then Θ is called full soft vertex graph ;*
(c) $\left(\bigcup\limits_{\sigma\in\mathfrak{R}} \delta(\sigma), \bigcup\limits_{\sigma\in\mathfrak{R}} \gamma(\sigma)\right) = (\mathcal{V}, \mathcal{E})$,*then Θ is called full soft graph.*

3. $\mathcal{L}$i-soft rough covering graphs and their fixed points

$\mathcal{L}$i-SRC sets are significant mathematical tools for addressing challenges in real world that involve uncertainty. Another useful tool for displaying information through matrices, relations, and diagrams is graph theory, which has obvious applications. This section focuses on the description of a novel blend of $\mathcal{L}$i-soft rough covering sets and graphs, called $\mathcal{L}$i-soft rough covering graphs ($\mathcal{L}$i-$\mathcal{SRCG}$s). This approach will assist us discuss the idea of uncertainty in these concepts as well as improve the application of $\mathcal{L}$i-$\mathcal{SRC}$ sets and soft graphs. It is shown that the family of $\mathcal{L}$i-$\mathcal{SRCFP}$ sets is a lattice. We have also proposed some basic properties and related examples in details.

Definition 3.1. *Let Θ be a $\mathcal{CSV}$ -Graph and $Q = (V, \mathbb{C}_{\mathcal{V}})$ be a $\mathcal{SVCAS}$. Then, for any $W \subseteq V$, $\mathcal{SVCL}$ and $\mathcal{SVCU}$ approximation operators are respectively, defined as:*

$$\begin{aligned} \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) &= \cup\{\tau(\sigma) \in \mathbb{C}_{\mathcal{V}} : \tau(\sigma) \subseteq \mathcal{W}\} \text{ and} \\ \overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) &= \cup\{\tau(\sigma) \in \mathbb{C}_{\mathcal{V}} : \tau(\sigma) \cap \mathcal{W} \neq \emptyset\}. \end{aligned}$$

In the above sets, the operators $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W})$ and $\overline{\mathcal{F}_{\tau}}(\mathcal{W})$ are called respectively, $\mathcal{L}$i-$\mathcal{SVCL}$ operator and $\mathcal{L}$i-$\mathcal{SVCU}$ operator. In case, $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) = \overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W})$, then W is called $\mathcal{L}$i -soft vertex covering definable, where the graph $G_Q := (\mathcal{V}, \mathcal{E})$ is called $\mathcal{L}$i-$\mathcal{SVC}$ definable. But on the other hand, if $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) \neq \overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W})$, then the set W is called $\mathcal{L}$i-$\mathcal{SRVC}$ set and the graph G_Q is called $\mathcal{L}$i-$\mathcal{SRVC}$

Example 3.1. *Suppose $\Omega = (V, \mathcal{E})$ is a graph given in Figure 1 below, with $V = \{x_1, x_2, ..., x_7\}$, $\mathcal{E} = \{e_1, e_2, ..., e_{10}\}$,*

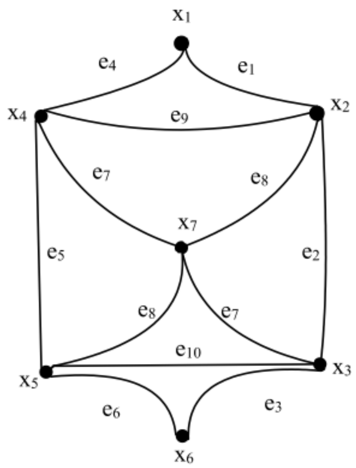

Figure 1. A simple graph $\Omega = (V, \mathcal{E})$ having vertex set V and edge set $\mathcal{E}$ which can be used to find the $\mathcal{SVCAS}$.

Let $Q = (\mathcal{V}, \mathfrak{C}_{\mathcal{V}})$ *be a* $\mathcal{SVCAS}$*, where* $C_{\mathcal{V}} = (\delta, \mathfrak{R})$*,* $R = \{\sigma_1, \sigma_2, ..., \sigma_6\}$ *is parameters set and* $\delta : R \to P(\mathcal{V})$ *is a set valued mapping, presented in Table* 1 *such that* $\delta(\sigma_1) = \{x_3, x_4, x_5\}$, $\delta(\delta_2) = \{x_1, x_2\}$, $\delta(\sigma_3) = \{x_3, x_5, x_6\}$, $\delta(\sigma_4) = \{x_3, x_4\}$, $\delta(\sigma_5) = \{x_1, x_2, x_3\}$, $\delta(\sigma_6) = \{x_1, x_2, x_5\}$*,*

Table 1. Tabular representation of soft set $(\delta, \mathfrak{R})$.

$\mathfrak{R} \setminus V$	x_1	x_2	x_3	x_4	x_5	x_6	x_7
σ_1	0	0	1	1	1	0	0
σ_2	1	1	0	0	0	0	0
σ_3	0	0	1	0	1	1	1
σ_4	0	0	1	1	0	0	0
σ_5	1	1	1	0	0	0	0
σ_6	1	1	0	0	1	0	0

Then, clearly, the pair $Q = (\mathcal{V}, C_{\mathcal{V}})$ *is a* $\mathcal{SVCAS}$*. Suppose we have a subset* $W = \{x_1, x_2, x_4\}$ *of vertex set V. Then,* $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W})$ *and* $\overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W})$ *can be calculated in the following manners:*

$$\begin{aligned} \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) &= \cup\{\tau(\sigma) \in \mathbb{C}_{\mathcal{V}} : \tau(\sigma) \subseteq \mathcal{W}\} = \{x_1, x_2\} \text{ and} \\ \overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) &= \cup\{\tau(\sigma) \in \mathbb{C}_{\mathcal{V}} : \tau(\sigma) \cap \mathcal{W} \neq \emptyset\} = \{x_1, x_2, x_3, x_4, x_5\}. \end{aligned}$$

Since $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) \neq \overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W})$*, so W is a* $\mathcal{SVCR}$ *and* $\Delta_{Q} := (V, E)$ *is* $\mathcal{L}$i-$\mathcal{SRVC}$ *-Graph with*

$$\underline{\Delta}_Q = \left(\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}), E\right) = (\{x_1, x_2\}, \{e_1, e_2, ..., e_{10}\}),$$

$$\overline{\Delta}_Q = \left(\overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}), E\right) = (\{x_1, x_2, x_3, x_4\}, \{e_1, ..., e_{10}\}) \text{ and}$$

Note if $W = \{x_1, x_2\} \subseteq V$, *then* $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) = \overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) = \{x_1, x_2\}$ *showing that W is a* $\mathcal{L}$i-$\mathcal{SVC}$ *definable set* $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) = \overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) = \{x_1, x_2\}$. *Also,*

$$\underline{\Delta}_{Q} = \left(\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}), \mathcal{E}\right) = (\{x_1, x_2\}, \{e_1, e_2, ..., e_{10}\}) = \overline{\Delta}_{Q} = \left(\overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}), \mathcal{E}\right)$$

Definition 3.2. *A full soft edge graph* Θ, *such that* $\gamma(\sigma) \neq \emptyset$ *for all* $\sigma \in R$, *is called* $\mathcal{CSE}$ *Graph. In this case,* $D = (\mathcal{E}, C_{\mathcal{E}})$ *is called* $\mathcal{SECAS}$.

Definition 3.3. *Let* $D = (E, C_{\mathcal{E}})$ *be a* $\mathcal{SECAS}$ *then, the sets*

$$\begin{aligned} \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N}) &= \cup\{\gamma(\sigma) \in \mathbb{C}_{\mathcal{E}} : \gamma(\sigma) \subseteq \mathcal{N}\} \text{ and} \\ \overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N}) &= \cup\{\gamma(\sigma) \in \mathbb{C}_{\mathcal{E}} : \gamma(\sigma) \cap \mathcal{N} \neq \emptyset\}, \ \mathcal{N} \subseteq \mathcal{E}, \end{aligned}$$

are called the $\mathcal{L}$i-$\mathcal{SECL}$ *and* $\mathcal{L}$i-$\mathcal{SECU}$ *approximations of* $\mathcal{N}$, *respectively.*

Also, if $\overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N}) = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N})$, *where* $\mathcal{N}$ *is a subset of E. Then,* $\mathcal{N}$ *is called* $\mathcal{L}$i-$\mathcal{SEC}$ *definable set and the graph* $\Delta_Q := (\mathcal{V}, \mathcal{E})$ *is called* $\mathcal{L}$i-$\mathcal{SEC}$ *definable. The graph* Δ_Q *is called* $\mathcal{L}$i-$\mathcal{SECAS}$ *Graph only if* $\overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N}) \neq \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N})$. *In this case, the subset N of* $\mathcal{E}$ *is called* $\mathcal{L}$i-$\mathcal{SECR}$ *set.*

Example 3.2. *Continued from Example* 3.2, *if* $Q = (E, C_{\mathcal{E}})$ *represents a* $\mathcal{SECAS}$ *with* $\gamma(\sigma_1) = \{e_1, e_3, e_5, e_{10}\}$, $\gamma(\sigma_2) = \{e_4\}$, $\gamma(\sigma_3) = \{e_4, e_5, e_6\}$, $\gamma(\sigma_4) = \{e_1, e_2, e_5, e_6, e_{10}\}$, $\gamma(\sigma_5) = \{e_1, e_3\}$ *and* $\gamma(\sigma_6) = \{e_6, e_7, e_9\}$, *see Table 2 below.*

Table 2. A table for soft set.

$\Re \setminus E$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
σ_1	1	0	1	0	1	0	0	0	0	1
σ_2	0	0	0	1	0	0	0	0	0	0
σ_3	0	0	0	1	1	1	0	0	0	0
σ_4	1	1	0	0	1	1	0	0	0	1
σ_5	1	0	1	0	0	0	0	0	0	0
σ_6	0	0	0	0	0	1	1	0	1	0

Let $\mathcal{N} = \{e_1, e_3, e_4, e_{10}\} \subseteq \mathcal{E}$. *Then*

$$\begin{aligned} \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N}) &= \cup\{\gamma(\sigma) \in \mathbb{C}_{\mathcal{E}} : \gamma(\sigma) \subseteq \mathcal{N}\} = \{e_1, e_3, e_4\} \text{ and} \\ \overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N}) &= \cup\{\gamma(\sigma) \in \mathbb{C}_{\mathcal{E}} : \gamma(\sigma) \cap \mathcal{N} \neq \emptyset\} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_{10}\}. \end{aligned}$$

Here, $\mathcal{N}$ *is a* $\mathcal{L}$i-$\mathcal{SREC}$ *set and* $\Delta_{\mathcal{D}} := (V, E)$ *is* $\mathcal{L}$i-$\mathcal{SRECG}$ *because* $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N}) \neq \overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N})$ *such that*

$$\underline{G}_{\mathcal{D}} = \left(V, \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N})\right) = (\{x_1, x_2, ..., x_7\}, \{e_1, e_3, e_4\}) \text{ and}$$

$$\overline{G}_{\mathcal{D}} = \left(V, \overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N})\right) = (\{x_1, x_2, ..., x_7\}, \{e_1, .., e_6, e_{10}\}).$$

Definition 3.4. *Suppose Θ is a soft graph. Then, Θ is called*

(i) *$\mathcal{L}$i-soft covering definable if $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) = \overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W})$ and $\overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N}) = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N})$,*

(ii) *$\mathcal{L}$i-soft rough covering graph (briefly, $\mathcal{L}$i-$\mathcal{SCR}$ $\mathcal{G}$raph) if $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) \neq \overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W})$ and $\overline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N}) \neq \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{E}}}(\mathcal{N})$, where $\mathcal{W} \subseteq \mathcal{V}$ and $\mathcal{N} \subseteq \mathcal{E}$.*

Definition 3.5. *Let Θ be a $\mathcal{CS}$E-$\mathcal{G}$raph such that $Q = (\mathcal{V}, \mathfrak{C}_{\mathcal{V}})$ is a $\mathcal{SVCAS}$ and $\delta(\sigma) \in \mathbb{C}_v$. Then, $\delta(\sigma)$ is called $\mathcal{L}$i-soft union reducible element (briefly, $\mathcal{L}$i-$\mathcal{SU}$red element) if $\delta(\sigma_i)$ is the union of some $\delta(\sigma_j) \in_{\mathbb{C}_{\mathcal{V}}} -\{\delta(\sigma_i)\}$ for $i \neq j$. Any other element which is not $\mathcal{L}$i-$\mathcal{SU}$red element, is called $\mathcal{L}$i-soft union irreducible element ($\mathcal{L}$i-$\mathcal{SU}$irred element). If every $\delta(\sigma_i) \in \mathbb{C}_{\mathcal{V}}$ is $\mathcal{L}$i-$\mathcal{SU}$irred element, then $\mathbb{C}_{\mathcal{V}}$ is called $\mathcal{L}$i-soft irreducible, otherwise $\mathbb{C}_{\mathcal{V}}$ is called $\mathcal{L}$i-soft reducible.*

It can be seen that if $\delta(\sigma_i)$ is a $\mathcal{L}$i-$\mathcal{SU}$red element of $\mathfrak{C}_V$, then $\mathfrak{C}_V - \{\delta(\sigma_i)\}$ is still a covering soft set over the universe set V.

Example 3.3. *Consider a $\mathcal{CSV}$-$\mathcal{G}$raph $\Theta = (\Omega, \delta, \gamma, \Re)$ where, $\mathcal{V}$ = a finite universe (vertex set) = $\{r_1, r_2, r_3, r_4\}$ and $\Re = \{\sigma_g, \sigma_y, \sigma_v, \sigma_w\}$ such that $(\delta, \Re)$ is $\mathcal{CSS}$ over $\mathcal{V}$, see below in Table 3, so that $\delta(\sigma_g) = \{r_1\}$, $\delta(\sigma_y) = \{r_2, r_3\}$, $\delta(\sigma_v) = \{r_1, r_2, r_3\}$, $\delta(\sigma_w) = \{r_2, r_4\}$ and $\mathbb{C}_{\mathcal{V}} = \{\delta(\sigma_g), \delta(\sigma_y), \delta(\sigma_v), \delta(\sigma_w)\} = \{\{r_1\}, \{r_2, r_3\}, \{r_1, r_2, r_3\}, \{r_2, r_4\}\}$.*

Table 3. Tabular representation of soft set $(\delta, \Re)$.

$\Re \setminus \mathcal{V}$	r_1	r_2	r_3	r_4
σ_g	1	0	0	0
σ_y	0	1	1	0
σ_v	1	1	1	0
σ_w	0	1	0	1

Clearly, $\delta(\sigma_v) = \{r_1, r_2, r_3\} = \delta(\sigma_g) \cup \delta(\sigma_y)$, where $\delta(\sigma_g)$, $\delta(\sigma_y) \in \mathbb{C}_{\mathcal{V}} - \{\delta(\sigma_3)\}$ showing that $\delta(\sigma_v)$ is a $\mathcal{L}$i-$\mathcal{SU}$red element in $\mathbb{C}_{\mathcal{V}}$. The elements $\delta(\sigma_g)$, $\delta(\sigma_y)$ and $\delta(\sigma_w)$ are $\mathcal{L}$i-$\mathcal{SU}$irred elements in $\mathbb{C}_{\mathcal{V}}$.

Definition 3.6. *Let $\mathcal{CSV}$-Graph be $\Theta = (\Omega, \delta, \gamma, \Re)$ and $Q = (\mathcal{V}, \mathfrak{C}_{\mathcal{V}})$, be a $\mathcal{SVCAS}$. Then, family $\mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ of subsets of $\mathcal{V}$, defined by $\mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}}) = \{\mathcal{W} \in P(\mathcal{V}) : \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) = \mathcal{W}\}$, is called $\mathcal{L}$i-soft rough vertex covering fixed point set (briefly $\mathcal{L}$i-$\mathcal{SRVCFP}$ set) induced by $\mathbb{C}_{\mathcal{V}}$.*

Example 3.4. *Consider a $\mathcal{CSV}$-Graph $\Theta = (\Omega, \delta, \gamma, \Re)$ as shown in Figures 2 and 3, where $\Omega = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{x_1, x_2, x_3, x_4\}$, $\mathcal{E} = \{e_1, e_2, , , , e_8\}$ and parameters set $\Re = \{\sigma_g, \sigma_y, \sigma_v, \sigma_w\}$, $(\delta, \Re)$ is a $\mathcal{CSS}$ over $\mathcal{V}$, below in Table 4 , so that $\delta(\sigma_g) = \{x_1, x_2\}$, $\delta(\sigma_y) = \{x_3\}$, $\delta(\sigma_v) = \{x_1, x_2, x_3\}$, $\delta(\sigma_w) = \mathcal{V}$.*

Table 4. Tabular representation of soft set $(\delta, \Re)$.

$\Re \setminus \mathcal{V}$	x_1	x_2	x_3	x_4
σ_g	1	1	0	0
σ_y	1	1	1	0
σ_v	0	0	1	0
σ_w	1	1	1	1

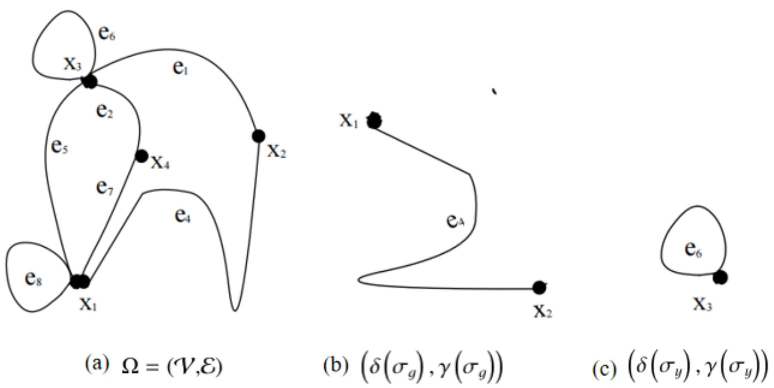

Figure 2. Simple graph $\Omega = (\mathcal{V},\mathcal{E})$ and two of its subgraphs corresponding to first two parameters.

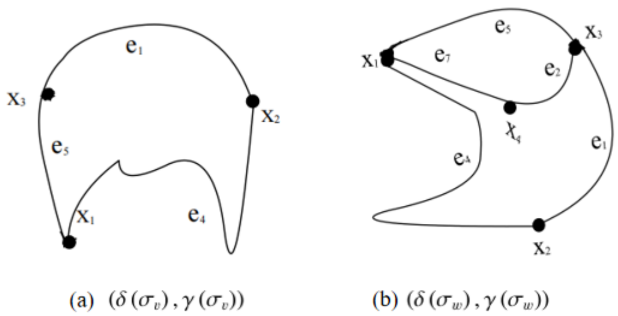

Figure 3. The rest of two subgraphs corresponding to last two parameters.

Here, $\mathbb{C}_{\mathcal{V}} = \{\{x_1, x_2\}, \{x_3\}, \{x_1, x_2, x_3\}, , \{x_1, x_2, x_3, x_4\}\}$. Let $\mathcal{W} = \{x_1, x_2\} \subseteq \mathcal{V}$, then $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) = \cup\{\delta(\sigma) \in_{\mathbb{C}_{\mathcal{V}}} : \delta(\sigma) \subseteq \mathcal{W}\} = \{r_1, r_2\} = \mathcal{W}$. That is, $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) = \mathcal{W}$ In this case, $\mathcal{W}$ is a member of $\mathcal{L}$i-$\mathcal{SRCFP}$ set or $\mathcal{W} \in \mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}})$.

Similarly, when $\mathcal{W} = \{r_1, r_2, r_3\}$ then $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) = \mathcal{W}$, showing that $\mathcal{W} \in \mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}})$.

Proposition 3.1. *Suppose* $\Theta = (\Omega, \delta, \gamma, \mathcal{R})$ *is a* $\mathcal{CSV}$*-*$\mathcal{G}$*raph and* $\delta(\sigma)$ *is a* $\mathcal{L}$*i-*$\mathcal{SU}$*irred element of* $\mathbb{C}_{\mathcal{V}}$*, then* $\mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}}) = \mathfrak{F}_{fix}((\mathbb{C}_{\mathcal{V}}) - \{\delta(\sigma)\})$.

Proof. By definition, $\mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}}) = \{\mathcal{W} \in P(\mathcal{V}) : \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) = \mathcal{W}\}$ and $\mathfrak{F}_{fix}((\mathbb{C}_{\mathcal{V}}) - \{\delta(\sigma)\}) =$

$\{\mathcal{W} \in P(\mathcal{V}) : \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}-\{\delta(\sigma)\}}(\mathcal{W}) = \mathcal{W}\}$. But $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}-\{\delta(\sigma)\}}(\mathcal{W}) = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W})$ for any $\mathcal{W} \in P(\mathcal{V})$. Therefore, $\mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}}) = \mathfrak{F}_{fix}((\mathbb{C}_{\mathcal{V}}) - \{\delta(\sigma)\})$. □

Now, the following proposition shows that $\mathcal{L}$i-$\mathcal{SRVCFP}$ set with respect to soft covering $\mathbb{C}_{\mathcal{V}}$ is the same as that of one, induced by the reduction of the $\mathcal{CSS}$ $\mathbb{C}_{\mathcal{V}}$.

Definition 3.7. *If every $\delta(\sigma_i) \in \mathbb{C}_{\mathcal{V}}$ is $\mathcal{L}$i-$\mathcal{SU}$irred element, then $\mathbb{C}_{\mathcal{V}}$ is called $\mathcal{L}$i-soft irreducible, otherwise $\mathbb{C}_{\mathcal{V}}$ is called $\mathcal{L}$i-soft reducible. It can be seen that if $\delta(\sigma_i)$ is a $\mathcal{L}$i-SUred element of $\mathbb{C}_{\mathcal{V}}$, then $\mathbb{C}_{\mathcal{V}} - \{\delta(\sigma_i)\}$ is still a CSS over the universe set V. For $\mathbb{C}_{\mathcal{V}}$, the newly $\mathcal{L}$i-soft union irreducible covering soft set with respect to the above reduction, is called $\mathcal{L}$i-soft reduct of $\mathbb{C}_{\mathcal{V}}$, and is denoted by $\mathcal{L}$i-$S_{redct}(\mathbb{C}_{\mathcal{V}})$.*

Proposition 3.2. *Let $\Theta = (\Omega, \delta, \gamma, \mathcal{R})$ be a $\mathcal{CSV}$-Graph and let $\mathbb{C}_{\mathcal{V}}$ is a covering soft set over V, then $F_{fix}(\mathbb{C}_{\mathcal{V}}) = F_{fix}(\mathcal{L}\text{i-}\mathcal{S}_{redct}(\mathbb{C}_{\mathcal{V}}))$.*

Proof. By definition,

$$\begin{aligned} \mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}}) &= \{\mathcal{W} \in P(\mathcal{V}) : \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) = \mathcal{W}\} \text{ and} \\ \mathfrak{F}_{fix}(\mathcal{L}\text{i-}\mathcal{S}_{redct}(\mathbb{C}_{\mathcal{V}})) &= \{\mathcal{W} \in P(\mathcal{V}) : \underline{\mathcal{F}}_{\mathcal{L}\text{i-}\mathcal{S}_{redct}(\mathbb{C}_{\mathcal{V}})}(\mathcal{W}) = \mathcal{W}\}. \end{aligned}$$

Since, $\underline{\mathcal{F}}_{\mathcal{L}\text{i-}\mathcal{S}_{redct}(\mathbb{C}_{\mathcal{V}})}(\mathcal{W}) = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W})$, for any $\mathcal{W} \in P(\mathcal{V})$. Thus, $\mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}}) = \mathfrak{F}_{fix}(\mathcal{L}\text{i-}\mathcal{S}_{redct}(\mathbb{C}_{\mathcal{V}}))$. □

The $\mathcal{L}$i-$\mathcal{SRCFP}$- set induced by $\mathbb{C}_{\mathcal{V}}$ together with the set inclusion, $\left(\mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \subseteq\right)$, is a POS. Actually, we see that whether this POS is a lattice or not. In the theorem given below, we show that $F_{fix}(\mathbb{C}_{\mathcal{V}})$ ($\mathcal{L}$i-$\mathcal{SRCFP}$- set) is a lattice and for any two members of this lattice. And for such lattice, we have find its lub and glb.

Proposition 3.3. *Let $Q = (\mathcal{V}, \mathfrak{C}_{\mathcal{V}})$, be a $\mathcal{SVCAS}$ for $\mathcal{CSV}$-Graph $\Theta = (\Omega, \delta, \gamma, \mathfrak{R})$. Suppose, W, $Y \in F_{fix}(\mathbb{C}_{\mathcal{V}})$. Then, $\left(\mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \subseteq\right)$ is a lattice, where $W \vee Y = W \cup Y$ and $W \wedge Y = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W} \cap \mathcal{Y})$.*

Proof. We have to prove only that $\mathcal{W} \cup \mathcal{Y} \in \mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ as well as $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W} \cap \mathcal{Y}) \in \mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ for any $\mathcal{W}$, $\mathcal{Y} \in \mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. If possible, suppose $\mathcal{W} \cup \mathcal{Y} \notin \mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}})$, then there exists $r \in \mathcal{W} \cup \mathcal{Y}$ such that for all $\delta(\sigma) \in_{\mathbb{C}_{\mathcal{V}}}$ and $x \in \delta(\sigma)$ implies $\delta(\sigma) \nsubseteq \mathcal{W} \cup \mathcal{Y}$. Since $r \in \mathcal{W} \cup \mathcal{Y}$, so $r \in \mathcal{W}$ or $r \in \mathcal{Y}$. Hence $\delta(\sigma) \nsubseteq \mathcal{W}$ or $\delta(\sigma) \nsubseteq \mathcal{Y}$, that is, $r \notin \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W})$ or $r \notin \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{Y})$, which gives a contradictory with the fact $\mathcal{W}$, $\mathcal{Y} \in \mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. Therefore, $\mathcal{W} \cup \mathcal{Y} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. Now, for any $\mathcal{W}$, $\mathcal{Y} \in \mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}})$, if $\mathcal{W} \cap \mathcal{Y} \subseteq \mathcal{V}$ then by using the fact $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}\left(\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(M)\right) = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(M)$, we get $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}\left(\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W} \cap \mathcal{Y})\right) = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W} \cap \mathcal{Y})$. This implies that $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W} \cap \mathcal{Y}) \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ for any $\mathcal{W}, \mathcal{Y} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. Thus, $\left(\mathfrak{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \subseteq\right)$ is lattice. □

The above theorem shows that the $\mathcal{L}$i-$\mathcal{SRCFP}$- set having the relation of set inclusion is a POS and for any two members of the $\mathcal{L}$i-$\mathcal{SRCFP}$- set, the lub is the join of these two members, while the glb is the lower approximation of the intersection of these two members. Actually, $\left(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \wedge, \vee\right)$ is defined from the view point of algebra and $\left(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \subseteq\right)$ is defined from the viewpoint of partially ordered set. Further, $\left(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \wedge, \vee\right)$ and $\left(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \subseteq\right)$ are both lattices.

Remark 3.1. *In above Proposition, we have seen that* $\left(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \subseteq\right)$ *is a lattice such that the sets U and* $\emptyset$ *are respectively, the greatest and least of* $\left(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \subseteq\right)$*. But we know a lattice having least and greatest elements is bounded. Therefore,* $\left(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \subseteq\right)$ *is a lattice which is bounded.*

Therefore, in the assertion that follows, we shall demonstrate that any $\mathcal{L}$i-$\mathcal{SU}irred$ element of $\mathbb{C}_{\mathcal{V}}$ is also a Join $\mathcal{L}$i-$\mathcal{SU}irred$ element of $\mathbb{C}_{\mathcal{V}}$, and any $\mathcal{L}$i-$\mathcal{SU}red$ element of $\mathbb{C}_{\mathcal{V}}$ is a Join $\mathcal{L}$i-$\mathcal{SU}red$ element of $F_{fix}(\mathbb{C}_{\mathcal{V}})$.

Proposition 3.4. *Let* $\Theta = (\Omega, \delta, \gamma, \Re)$ *be a CSV-Graph and* $Q = (\mathcal{V}, \mathfrak{C}_{\mathcal{V}})$*, be a SVCAS such that* $\delta(\sigma_i) \in \mathbb{C}_{\mathcal{V}}$*. Then,*

(i) the set $\delta(\sigma_i)$ *is a Join* $\mathcal{L}$i-$\mathcal{SU}$*irred element of* $F_{fix}(\mathbb{C}_{\mathcal{V}})$*, if* $\delta(\sigma_i)$ *is a* $\mathcal{L}$i-$\mathcal{SU}$*irred element of* $\mathbb{C}_{\mathcal{V}}$*.*

(ii) the set $\delta(\sigma_i)$ *is Join* $\mathcal{L}$i-$\mathcal{SU}$*red element of* $F_{fix}(\mathbb{C}_{\mathcal{V}})$*, if* $\delta(\sigma_i)$ *is a Join* $\mathcal{L}$i-$\mathcal{SU}$*red element of* $\mathbb{C}_{\mathcal{V}}$ *for parameters* σ_i*.*

Proof. Since for any parameter $\sigma_i \in \mathfrak{Q}$ and $\delta(\sigma_i) \in \mathbb{C}_{\mathcal{V}}$, $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\delta(\sigma_i)) = \delta(\sigma_i)$. This shows $\delta(\sigma_i) \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ as $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}) = \{\mathcal{W} \in P(\mathcal{V}) : \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) = \mathcal{W}\}$.

(i) Suppose there are elements $\mathcal{Y}$, Z in $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ such that $\delta(\sigma_i) = \mathcal{Y} \cup Z$, then there exists some elements $\delta\left(\sigma_j\right), (j \in I)$ and $\delta(\sigma_k), (k \in K)$ in $\mathbb{C}_{\mathcal{V}}$ such that $\delta\left(\sigma_j\right) \subseteq \mathcal{Y}$, $\delta(\sigma_k) \subseteq Z$ and

$$\begin{aligned} \mathcal{Y} &= \bigcup\{\delta\left(\sigma_j\right) \in \mathbb{C}_{\mathcal{V}} : \delta\left(\sigma_j\right) \subseteq \mathcal{Y}\ (j \in I)\}, \\ Z &= \bigcup\{\delta(\sigma_k) \in \mathbb{C}_{\mathcal{V}} : \delta(\sigma_k) \subseteq Z\ (k \in K)\}, \\ \text{where } I, K &\subseteq \{g, y, ..., \left|\mathbb{C}_{\mathcal{V}}\right|\}. \end{aligned}$$

Therefore, $\delta(\sigma_i) = \left(\bigcup_{\delta(\sigma_j)\subseteq\mathcal{Y}\ (j\in I)} \delta\left(\sigma_j\right)\right) \cup \left(\bigcup_{\delta(\sigma_k)\subseteq Z\ (k\in K)} \delta(\sigma_k)\right)$. Since $\delta(\sigma_i)$ is a $\mathcal{L}$i-$\mathcal{SU}irred$ element of $\mathbb{C}_{\mathcal{V}}$, so there exists $t \in I \cup K$ so that $\delta(\sigma_i) = \delta(\sigma_t)$. But $\delta(\sigma_i) = \mathcal{Y} \cup Z$ implies $\mathcal{Y} \subseteq \delta(\sigma_i)$ and $Z \subseteq \delta(\sigma_i)$. If $\delta(\sigma_t) \subseteq \mathcal{Y}$, then $\delta(\sigma_i) = \mathcal{Y}$. If $\delta(\sigma_t) \subseteq Z$ $\delta(\sigma_i) = Z$. This indicates that $\delta(\sigma_i) = \mathcal{Y}$ or $\delta(\sigma_i) = Z$. Thus, $\delta(\sigma_i)$ is a Join $\mathcal{L}$i-$\mathcal{SU}irred$ element of $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. Hence, the set $\delta(\sigma_i)$ is a *Join* $\mathcal{L}$i-$\mathcal{SU}irred$ element of $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$, if $\delta(\sigma_i)$ is a $\mathcal{L}$i-$\mathcal{SU}irred$ element of $\mathbb{C}_{\mathcal{V}}$.

(ii) Suppose $\delta(\sigma_i)$ is a $\mathcal{L}$i-$\mathcal{SU}red$ element of $\mathbb{C}_{\mathcal{V}}$ for parameters $\sigma_i \in \mathfrak{Q}$. Then there are some elements in $\mathbb{C}_{\mathcal{V}} - \{\delta(\sigma_i)\}$ in such away that $\delta(\sigma_i)$ is the union of those elements. In other words, there are some elements $\delta\left(\sigma_j\right), (j \in I)$ such that $\delta(\sigma_i) = \bigcup_{j\in I} \delta\left(\sigma_j\right)$, where $I \subseteq S = \{g, y, ..., \left|\mathbb{C}_{\mathcal{V}}\right|\}$.

Since, for any $H \subseteq S$, we have $\delta(\sigma_k)_{(k\in H)} \subseteq \bigcup_{j\in H} \delta\left(\sigma_j\right)$. Then,

$$\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}\left(\bigcup_{j\in H} \delta\left(\sigma_j\right)\right) = \bigcup\{\delta(\sigma_k)_{(k\in H)} : \delta(\sigma_k) \subseteq \bigcup_{j\in H} \delta\left(\sigma_j\right)\} = \bigcup_{j\in H} \delta\left(\sigma_j\right),$$

Therefore, for any K, T $\subseteq$ I,

$$\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}\left(\bigcup_{k\in K} \delta(\sigma_k)\right) = \bigcup_{k\in K} \delta(\sigma_k),$$

and

$$\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}\left(\bigcup_{t\in T}\delta(\sigma_t)\right) = \bigcup_{t\in T}\delta(\sigma_t).$$

Hence,we have $\bigcup_{k\in K}\delta(\sigma_k) \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ and $\bigcup_{t\in T}\delta(\sigma_t) \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. As a result, K_1 and $T_1 \subseteq I$ exists such that $\bigcup_{j\in I}\delta\left(\sigma_j\right) = \delta(\sigma_i) = \left(\bigcup_{k\in K_1}\delta(\sigma_k)\right)\cup = \left(\bigcup_{t\in T_1}\delta(\sigma_t)\right)$. This shows that $\delta(\sigma_i)$ is a Join $\mathcal{L}$i-$\mathcal{SU}red$ element for $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. Thus, the set $\delta(\sigma_i)$ is a Join $\mathcal{L}$i-$\mathcal{SU}red$ element of $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$, if $\delta(\sigma_i)$ is a $\mathcal{L}$i-$\mathcal{SU}red$ element of $\mathbb{C}_{\mathcal{V}}$ for parameters σ_i. □

Proposition 3.5. *Let $Q = (\mathcal{V}, \mathfrak{C}_{\mathcal{V}})$, be a SVCAS of CSV-Graph $\Theta = (\Omega, \delta, \gamma, \Re)$. Then, for any $r \in V$, $N_{\mathfrak{B}}(r)$ is a Join $\mathcal{L}$i-$\mathcal{SU}irred$ element of the lattice $\left(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \subseteq\right)$.*

Proof. Suppose $\left(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \subseteq\right)$ is a lattice and there exist $\mathcal{W}, \mathcal{Y} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ such that $N_{\mathfrak{B}}(r) = \mathcal{W} \cup \mathcal{Y}$. Since $r \in N(r)$, $r \in \mathcal{W} \cup \mathcal{Y}$. Therefore, $r \in \mathcal{W}$ or $r \in \mathcal{Y}$. Furthermore, as $\mathcal{W}, \mathcal{Y} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$, then $N_{\mathfrak{B}}(r) \subseteq \mathcal{W} \subseteq \mathcal{W} \cup \mathcal{Y} = N_{\mathfrak{B}}(r)$ or $N_{\mathfrak{B}}(r) \subseteq \mathcal{Y} \subseteq \mathcal{W} \cup \mathcal{Y} = N_{\mathfrak{B}}(r)$ Therefore, $N_{\mathfrak{B}}(r) = \mathcal{W}$ or $N_{\mathfrak{B}}(r) = \mathcal{Y}$. Thus $N_{\mathfrak{B}}(r)$ is a Join $\mathcal{L}$i-$\mathcal{SU}irred$ element of the lattice $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$, for every $r \in \mathcal{V}$. □

We have already shown that the set $\left(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \subseteq\right)$ is a lattice if and only if $W \vee Y = W \cup Y$ and $W \wedge Y = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W} \cap \mathcal{Y})$. Now in the following we show that $\left(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \subseteq\right)$, is a complete lattice.

Proposition 3.6. *Let $\Theta = (\Omega, \delta, \gamma, \Re)$ be a $\mathcal{CSV}$-Graph and $Q = (\mathcal{V}, \mathfrak{C}_{\mathcal{V}})$, be a $\mathcal{SVCAS}$. Then, the lattice $\left(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \subseteq\right)$ is complete.*

Proof. For any $\mathcal{G} \subseteq \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$, we need to prove that $\wedge\mathcal{G} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ and $\vee\mathcal{G} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. Equivalently, we have to prove that $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\cap\mathcal{G}) \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ and $\cup\mathcal{G} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. Since $\mathcal{G} \subseteq \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ then $\cap\mathcal{G} \subseteq \mathcal{V}$. According to the fact that $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}\left(\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\cap\mathcal{G})\right) = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\cap\mathcal{G})$, that is, $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\cap\mathcal{G}) \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. If $\cup\mathcal{G} \notin \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ then there exists $t \in \cup\mathcal{G}$ such that $\delta(\sigma_i) \subsetneq \cup\mathcal{G}$, for any $\delta(\sigma_i) \in \mathbb{C}_{\mathcal{V}}$ and $t \in \delta(\sigma_i)$. Hence, there exists an element $\mathcal{W}$ in $\mathcal{G}$ so that $t \in \mathcal{W}$ and $\delta(\sigma_i) \subsetneq \mathcal{W}$, for any $\delta(\sigma_i) \in \mathbb{C}_{\mathcal{V}}$ and $t \in \delta(\sigma_i)$. So, $t \notin \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W})$ that is, $\mathcal{W} \notin \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. Which is contradictory with the fact that $\mathcal{W} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. Hence $\cup\mathcal{G} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. Therefore, $\left(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \subseteq\right)$ is a complete lattice. □

Note that the $\mathcal{L}$i-$\mathcal{SRCFP}$-set induced by $\mathbb{C}_{\mathcal{V}}$ need not be always a distributive lattice. The example that follows will support our assertion.

Example 3.5. *Suppose we have a $\mathcal{CSV}$-Graph $\Theta = (\Omega, \delta, \gamma, \Re)$ such that $Q = (\mathcal{V}, \mathfrak{C}_{\mathcal{V}})$, is a $\mathcal{SVCAS}$ where $V = \{x_1, x_2, x_3, x_4\}$ is a vertex set and $R = \left\{\sigma_g, \sigma_y, \sigma_v\right\}$, set representing all parameters $\sigma_g, \sigma_y, \sigma_v, \sigma_w$ as shown in Table 5, with $\delta\left(\sigma_g\right) = \{x_1, x_2\}$, $\delta\left(\sigma_y\right) = \{x_2, x_3\}$, $\delta(\sigma_v) = \{x_1, x_3, x_4\}$. Also,$\mathbb{C}_{\mathcal{V}} = \{\delta\left(\sigma_g\right), \delta\left(\sigma_y\right), \delta(\sigma_v)\} = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_3, x_4\}\}$.*

Table 5. A Table for soft set $(\delta, \Re)$.

$\Re \diagdown \mathcal{V}$	x_1	x_2	x_3	x_4
σ_g	1	1	0	0
σ_y	0	1	1	0
σ_v	1	0	1	1

Then, by using the expression $F_{fix}(\mathbb{C}_{\mathcal{V}}) = \{ W \in P(\mathcal{V}) : \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) = W\}$, we obtain $F_{fix}(\mathbb{C}_{\mathcal{V}}) = \{\emptyset, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \{x_2, x_3\}, \{x_1, x_3, x_4\}, V\}$. Also, $\{x_1, x_2, x_3\} \wedge (\{x_1, x_3, x_4\} \vee \{x_1, x_2\}) = \{x_1, x_2, x_3\}$, but $(\{x_1, x_3, x_4\} \wedge \{x_1, x_2, x_3\}) \vee (\{x_1, x_2, x_3\} \wedge \{x_1, x_2\}) = \{x_1, x_2\}$. Which shows that $F_{fix}(\mathbb{C}_{\mathcal{V}}) = \{\emptyset, \{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_3, x_4\}, \{x_1, x_2, x_3\}, V\}$ is not a distributive lattice. In other words, $\mathcal{L}$i-$\mathcal{SRCFP}$-set, that is not distributive.

In the statement that follows, we investigate the requirement for the $\mathcal{L}$i-$\mathcal{SRCFP}$ set to transform into a distributive lattice.

Proposition 3.7. *Let Θ be a $\mathcal{CSV}$-Graph and $Q = (\mathcal{V}, \mathfrak{C}_{\mathcal{V}})$, be a $\mathcal{SVCAS}$. If $\mathbb{C}_{\mathcal{V}}$ is soft unary, then $\left(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \subseteq\right)$ is a distributive lattice.*

Proof. Suppose $\mathcal{W}, \mathcal{Y}$ and $\mathcal{Z} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ where $\mathcal{W}, \mathcal{Y}, Z \subseteq \mathcal{V}$. then for parameters $\sigma_i, \sigma_j, \sigma_k$ there are some elements $\delta(\sigma_i), (i \in I), \delta\left(\sigma_j\right), (j \in I)$ and $\delta(\sigma_k), (k \in K)$ in $\mathbb{C}_{\mathcal{V}}$ such that $\delta(\sigma_i) \subseteq \mathcal{W}$, $\delta\left(\sigma_j\right) \subseteq \mathcal{Y}$, $\delta(\sigma_k) \subseteq Z$ with

$$\begin{aligned}
\mathcal{W} &= \bigcup\{\delta(\sigma_i) \in (\mathbb{C}_{\mathcal{V}}) : \delta(\sigma_i) \subseteq \mathcal{W},\ (i \in I)\},\\
\mathcal{Y} &= \bigcup\{\delta(\sigma_J) \in (\mathbb{C}_{\mathcal{V}}) : \delta(\sigma_J) \subseteq \mathcal{Y},\ (j \in J)\},\\
\mathcal{Z} &= \bigcup\{\delta(\sigma_k) \in (\mathbb{C}_{\mathcal{V}}) : \delta(\sigma_k) \subseteq \mathcal{Z},\ (k \in K)\},
\end{aligned}$$

where I, J, K $\subseteq\{g, y, ..., \left|\mathbb{C}_{\mathcal{V}}\right|\}$.

It can easily be seen that

$$\begin{aligned}
\mathcal{W} \wedge (\mathcal{Y} \vee Z) &= \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W} \cap (\mathcal{Y} \cup Z)) = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}((\mathcal{W} \cap \mathcal{Y}) \cup (\mathcal{W} \cap Z))\\
&= \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}\left(\begin{array}{c}\left(\bigcup_{\delta(\sigma_i) \subseteq \mathcal{W}(i \in I)} \delta(\sigma_i) \cap \bigcup_{\delta(\sigma_j) \subseteq \mathcal{Y}(j \in J)} \delta\left(\sigma_j\right)\right) \cup \\ \left(\bigcup_{\delta(\sigma_i) \subseteq \mathcal{W}(i \in I)} \delta(\sigma_i) \cap \bigcup_{\delta(\sigma_k) \subseteq \mathcal{Z}(k \in K)} \delta(\sigma_k)\right)\end{array}\right)\\
&= \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}\left(\begin{array}{c}\left(\bigcup_{\substack{\delta(\sigma_i) \subseteq \mathcal{W}(i \in I)\\ \delta(\sigma_j) \subseteq \mathcal{Y}(j \in J)}} \{\delta(\sigma_i) \cap \delta\left(\sigma_j\right)\}\right) \cup \\ \left(\bigcup_{\substack{\delta(\sigma_i) \subseteq \mathcal{W}(i \in I)\\ \delta(\sigma_k) \subseteq \mathcal{Z}(k \in K)}} \{\delta(\sigma_i) \cap \delta(\sigma_k)\}\right)\end{array}\right).
\end{aligned}$$

Moreover,

$$\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W} \cap \mathcal{Y}) \cup \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W} \cap Z) = (\mathcal{W} \wedge \mathcal{Y}) \vee (\mathcal{W} \wedge Z)$$

$$= \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}} \left(\bigcup_{\substack{\delta(\sigma_i)\subseteq \mathcal{W}(i\in I)\\ \delta(\sigma_j)\subseteq \mathcal{Y}(j\in J)}} \{\delta(\sigma_i) \cap \delta(\sigma_j)\} \right)$$

$$\cup \ \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}} \left(\bigcup_{\substack{\delta(\sigma_i)\subseteq \mathcal{W}(i\in I)\\ \delta(\sigma_k)\subseteq \mathcal{Z}(k\in K)}} \{\delta(\sigma_i) \cap \delta(\sigma_k)\} \right)$$

Since, $\mathbb{C}_{\mathcal{V}}$ is unary, then for every $x \in \mathcal{V}$, $|\text{Mdes}_{\mathcal{T}}(x)| = 1$. We suppose, $\text{Mdes}_{\mathcal{T}}(x) = \{\delta(\sigma_x)\}$ for any $x \in \mathcal{V}$. Then, $\delta(\sigma_i) \cap \delta(\sigma_j)$ is the union of finite elements in $\mathbb{C}_{\mathcal{V}}$. Hence,

$$\begin{aligned}
\delta(\sigma_i) \cap \delta(\sigma_j) &= \bigcup_{x\in\delta(\sigma_i)\cap\delta(\sigma_j)} \delta(\sigma_x). \text{ Therefore, } \mathcal{W} \wedge (\mathcal{Y} \vee Z) = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}} (\mathcal{W} \cap (\mathcal{Y} \cup Z))) \\
&= \bigcup_{x\in \left(\bigcup_{\substack{\delta(\sigma_i)\subseteq \mathcal{W}(i\in I)\\ \delta(\sigma_j)\subseteq \mathcal{Y}(j\in J)}} \{\delta(\sigma_i)\cap\delta(\sigma_j)\}\right) \cup \left(\bigcup_{\substack{\delta(\sigma_i)\subseteq \mathcal{W}(i\in I)\\ \delta(\sigma_k)\subseteq \mathcal{Z}(k\in K)}} \{\delta(\sigma_i)\cap\delta(\sigma_k)\}\right)} \delta(\sigma_x) \\
&= \left(\bigcup_{y\in \left(\bigcup_{\substack{\delta(\sigma_i)\subseteq \mathcal{W}(i\in I)\\ \delta(\sigma_j)\subseteq \mathcal{Y}(j\in J)}} \{\delta(\sigma_i)\cap\delta(\sigma_j)\}\right)} \delta(\sigma_y) \right) \cup \left(\bigcup_{z\in \left(\bigcup_{\substack{\delta(\sigma_i)\subseteq \mathcal{W}(i\in I)\\ \delta(\sigma_k)\subseteq \mathcal{Z}(k\in K)}} \{\delta(\sigma_i)\cap\delta(\sigma_k)\}\right)} \delta(\sigma_z) \right) \\
&= (\mathcal{W} \wedge \mathcal{Y}) \vee (\mathcal{W} \wedge \mathcal{Z}).
\end{aligned}$$

□

Hence, $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ is a distributive lattice.

Proposition 3.8. *Let $Q = (\mathcal{V}, \mathfrak{C}_{\mathcal{V}})$, be a $\mathcal{SVCAS}$ such that $\mathbb{C}_{\mathcal{V}}$ is soft unary. Then, $P\cap Q \in F_{fix}(\mathbb{C}_{\mathcal{V}})$, for any $P, Q \in F_{fix}(\mathbb{C}_{\mathcal{V}})$.*

Proof. Let $p \in \mathcal{P}\cap \mathcal{Q}$ then $p \in \mathcal{P}$ and $p \in \mathcal{Q}$. Since, $\mathbb{C}_{\mathcal{V}}$ is soft unary so, for every $x \in \mathcal{V}$, $|\text{Mdes}_{\mathcal{T}}(x)| = 1$. We suppose, $\text{Mdes}_{\mathcal{T}}(x) = \{\delta(\sigma_x)\}$ for any $x \in \mathcal{V}$. As, $\mathcal{P}, \mathcal{Q} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$, then $\delta(\sigma_y) \subseteq \mathcal{P}$ and $\delta(\sigma_y) \subseteq \mathcal{Q}$, that is, $\delta(\sigma_y) \subseteq \mathcal{P}\cap\mathcal{Q}$. Therefore,

$$\begin{aligned}
\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}} (\mathcal{P} \cap \mathcal{Q}) &= \bigcup \{\delta(\sigma_i) \in (\mathbb{C}_{\mathcal{V}}) : \delta(\sigma_i) \subseteq \mathcal{P} \cap \mathcal{Q}\} \\
&= \bigcup \{\delta(\sigma_y) : y \in \mathcal{P} \cap \mathcal{Q}\} = \mathcal{P} \cap \mathcal{Q}
\end{aligned}$$

showing that $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}} (\mathcal{P} \cap \mathcal{Q}) = \mathcal{P} \cap \mathcal{Q}$.

Thus, $\mathcal{P}\cap\mathcal{Q}\in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$, that is, an intersection of any two members of $(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}))$ $\mathcal{L}$i-$\mathcal{SRCFP}$ set induced by $\mathbb{C}_{\mathcal{V}}$, a soft unary covering, is closed. □

4. Algebraic Structures related to$\mathcal{L}$i-Soft Rough Covering Graphs

In this section, we will prove $\mathcal{L}$i-$\mathcal{SRCFP}$ set, with respect to a soft unary covering $\mathbb{C}_{\mathcal{V}}$ over the vertex set V is a $pseud^{Cd}$ lattice and a D- $pseud^{Cd}$ lattice. It means any element of $\mathcal{L}$i-$\mathcal{SRCFP}$ set has both a pseudocomplement and a D- pseudocomplement. Also, we will see that for any member of $\mathcal{L}$i-$SRCFP$ set, its $pseud^{Cmt}$(pseudocomplement) represents the SLA of its complement and D- $pseud^{Cmt}$ represents the union of all $Join$ $\mathcal{L}$i-$\mathcal{SU}irred$ elements containing the element in its complement. We also discuss some algebras connected with $\mathcal{L}$i-$\mathcal{SRCFP}$ sets.

Proposition 4.1. *Let $T = (\mathcal{V}, \mathbb{C}_{\mathcal{V}})$ be a $\mathcal{SCAS}$ such that $\mathbb{C}_{\mathcal{V}}$ is soft unary. Then,*
(i) $F_{fix}(\mathbb{C}_{\mathcal{V}})$ is a $pseud^{Cd}$ lattice such that $W^* = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{W})$, where $W \in F_{fix}(\mathbb{C}_{\mathcal{V}})$;
(ii) $F_{fix}(\mathbb{C}_{\mathcal{V}})$ is a D- $pseud^{Cd}$ lattice, and $W^+ = \bigcup_{x\in\sim\mathcal{W}(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})))} \delta(\sigma_i)$, for any $W \in F_{fix}(\mathbb{C}_{\mathcal{V}})$, parameter $(\sigma_i) \in Q$, where $\sim W$ is the complement of W in V and $J(F_{fix}(\mathbb{C}_{\mathcal{V}}))$ denotes all Join Li-SUirred elements in $F_{fix}(\mathbb{C}_{\mathcal{V}})$.

Proof. (i) Suppose $\mathcal{W} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$, then we have $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}\left(\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{W})\right) = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{W})$, showing that $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{W}) \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. But $\left(\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{W})\right) \subseteq (\sim \mathcal{W})$, so $\mathcal{W} \cap \left(\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{W})\right) = \emptyset$. Therefore, $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}\left(\mathcal{W} \cap \left(\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{W})\right)\right) = \emptyset$. Now it is needed to prove that $\mathcal{Y} \subseteq \left(\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{W})\right)$ if $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W} \cap \mathcal{Y}) = \emptyset$ for any $\mathcal{Y} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$, $\mathcal{W} \cap \mathcal{Y} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ for $\mathcal{W}, \mathcal{Y} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. Hence, $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W} \cap \mathcal{Y}) = (\mathcal{W} \cap \mathcal{Y})$. Further, if $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W} \cap \mathcal{Y}) = \emptyset$ then $\mathcal{W} \cap \mathcal{Y} = \emptyset$. Therefore, for any $\mathcal{Y} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ if $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W} \cap \mathcal{Y}) = (\mathcal{W} \cap \mathcal{Y})$ then $(\mathcal{W} \cap \mathcal{Y}) = \emptyset$. As $(\mathcal{W} \cap \mathcal{Y}) = \emptyset$ so $\mathcal{Y} \subseteq (\sim \mathcal{W})$. Which shows $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{Y}) \subseteq \left(\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{W})\right)$. But $\mathcal{Y} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ gives $\mathcal{Y} = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{Y})$ and so $\mathcal{Y} = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{Y}) \subseteq \left(\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{W})\right)$. Thus, $\mathcal{W}^* = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{W})$ for any $\mathcal{W} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. That is, $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ is a $pseud^{Cd}$ lattice.
(ii) For any $\mathcal{W} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$,

$$\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}\left(\bigcup_{x\in\sim\mathcal{W}(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})))} \delta(\sigma_i)\right) = \bigcup_{x\in\sim\mathcal{W}(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})))} \delta(\sigma_i).$$

So,

$$\bigcup_{x\in\sim\mathcal{W}(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})))} \delta(\sigma_i) \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \text{ for any } \mathcal{W} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}).$$

Further it is easy to show that $\mathcal{W} \cup \left(\bigcup_{x\in\sim\mathcal{W}(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})))} \delta(\sigma_i)\right) = \mathcal{V}$. now we need only to show that for any

$$\mathcal{Y} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}), \text{if } \mathcal{W} \cup \mathcal{Y} = \mathcal{V}, \text{then } \bigcup_{x\in\sim\mathcal{W}(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})))} \delta(\sigma_i) \subseteq \mathcal{Y}.$$

The following two cases serve as evidence for it.

Case 1 : If $\bigcup_{x\in\sim\mathcal{W}\left(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}))\right)} \delta(\sigma_i) =\sim \mathcal{W}$, then $\bigcup_{x\in\sim\mathcal{W}\left(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}))\right)} \delta(\sigma_i) \subseteq \mathcal{Y}$.

Case 2 : If $\sim \mathcal{W} \subset \bigcup_{x\in\sim\mathcal{W}\left(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}))\right)} \delta(\sigma_i)$, then $\sim \mathcal{W} \subset \mathcal{Y}$. If $\sim \mathcal{W} = \mathcal{Y}$, then $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{W}) = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{Y}) = \mathcal{Y} = \sim \mathcal{W}$. Since,

$$\begin{aligned}
\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{W}) &= \cup\{\delta(\sigma_i) \in \mathbb{C}_{\mathcal{V}} : \delta(\sigma_i) \subseteq \sim \mathcal{W}\} \\
&= \cup\{\delta(\sigma_x) \in \mathrm{Mdes}_{\mathcal{T}}(x) : x \in Mdes_{\mathcal{T}}(x)\} \\
&= \cup\{\delta(\sigma_i) \in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})) : x \in\sim \mathcal{W} \wedge x \in \delta(\sigma_{\rangle})\} \\
&= \bigcup_{x\in\sim\mathcal{W}\left(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}))\right)} \delta(\sigma_i), \quad \bigcup_{x\in\sim\mathcal{W}\left(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}))\right)} \delta(\sigma_i) =\sim \mathcal{W},
\end{aligned}$$

which is contradictory with $\sim \mathcal{W} \subset \bigcup_{x\in\sim\mathcal{W}\left(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}))\right)} \delta(\sigma_i)$.

Suppose $\mathcal{Y} \subset \bigcup_{x\in\sim\mathcal{W}\left(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}))\right)} \delta(\sigma_i)$, then there will exist $y \in \bigcup_{x\in\sim\mathcal{W}\left(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}))\right)} \delta(\sigma_i)$ such that $y \notin \mathcal{Y}$. So, $y \notin\sim \mathcal{W}$ showing that there exists $z \in\sim \mathcal{W}$ such that $y \in \delta(\sigma_i)$ for any $\delta(\sigma_i) \in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ and $z \in \delta(\sigma_i)$. Since, $\sim \mathcal{W}$ is properly contained in $\mathcal{Y}$ and $z \in \mathcal{Y}$, so, $\delta(\sigma_i) \subsetneq \mathcal{Y}$ for any $\delta(\sigma_i) \in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}))$ and $z \in \delta(\sigma_i)$. That is, $z \notin \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{Y})$. Equivalently, we can say that $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{Y}) \neq \mathcal{Y}$, which is a contradiction to the fact that $\mathcal{Y} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. Hence, $\bigcup_{x\in\sim\mathcal{W}\left(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}))\right)} \delta(\sigma_i) \subseteq \mathcal{Y}$. As a Consequence, $\mathcal{W}^{+} = \bigcup_{x\in\sim\mathcal{W}\left(x\in\delta(\sigma_i)\in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}))\right)} \delta(\sigma_i)$ for any $\mathcal{W} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$, that is, $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ is a D-$pseud^{Cd}$ lattice. Thus, we have seen that $\mathcal{L}$i-$\mathcal{SRCFP}$- set induced by any soft unary covering $\mathbb{C}_{\mathcal{V}}$ over $\mathcal{V}$ represents a $pseud^{Cd}$ lattice and also a D- $pseud^{Cd}$ lattice. This shows that $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ is a double p-algebra. □

Remark 4.1. *In general, no soft unary covering by the* $\mathcal{L}$i-$\mathcal{SRCFP}$ *set will generate a Stone algebra or a dual Stone algebra.*

Example 4.1. *Let* $\Theta = (\Omega, \delta, \gamma, \mathfrak{R})$ *represents a* $\mathcal{CSV}$*-Graph and* $Q = (\mathcal{V}, \mathfrak{C}_{\mathcal{V}})$*, be a* $\mathcal{SVCAS}$*, where* $V = \{x_1, x_2, x_3, x_4\}$ *is a vertex set and* $R = \left\{\sigma_g, \sigma_y, \sigma_v, \sigma_w\right\}$*, set representing all parameters* $\sigma_g, \sigma_y, \sigma_v, \sigma_w$ *as shown in Table 6, with* $\delta\left(\sigma_g\right) = \{x_3\}$, $\delta\left(\sigma_y\right) = \{x_1\}$, $\delta(\sigma_v) = \{x_1, x_3, x_4\}$, $\delta(\sigma_w) = \{x_2, x_3\}$.

Table 6. Table of soft set $(\delta, \mathfrak{R})$.

$\mathfrak{R} \setminus \mathcal{V}$	x_1	x_2	x_3	x_4
σ_g	1	1	1	0
σ_y	1	0	0	0
σ_v	1	0	1	1
σ_w	0	1	1	0

Let

$$\mathbb{C}_{\mathcal{V}} = \{\{x_3\}, \{x_1\}, \{x_1, x_3, x_4\}, \{x_2, x_3\}\}.$$

Then,

$$\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}) = \{\emptyset, \{x_1\}, \{x_3\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_3, x_4\}, \{x_1, x_2, x_3\}, \mathcal{V}\}.$$

Let

$$\mathcal{M} = \{x_3\} \text{ and } N = \{x_2, x_3\}.$$

Then,

$$\mathcal{M}^* = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{M}) = \{x_1\},$$

since,

$$\mathcal{M}^* = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{M}) = \{x_1\}, \text{ so } \mathcal{M}^{**} = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{M}^*) = \{x_2, x_3\}.$$

That is, $\mathcal{M}^* \cup \mathcal{M}^{**} \neq \mathcal{V}$. Therefore, $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ is not a Stone algebra.

Also,

$$\mathcal{N}^+ = \bigcup_{x \in \sim\mathcal{N}\left(x \in \delta(\sigma_i) \in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}))\right)} \delta(\sigma_i) = \{x_1, x_3, x_4\},$$

$$\mathcal{N}^{++} = \bigcup_{x \in \sim\mathcal{N}^+\left(x \in \delta(\sigma_i) \in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}))\right)} \delta(\sigma_i) = \{x_2, x_3\},$$

showing that

$$\mathcal{N}^+ \cap \mathcal{N}^{++} \neq \emptyset.$$

Thus, $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ is not a dual Stone algebra.

The purpose of the following claim is to investigate the circumstances and conditions under which the $\mathcal{L}$i-$\mathcal{SRCFP}$-set, which is caused by any soft covering, yields a double Stone algebra and a boolean lattice, respectively.

Proposition 4.2. *Suppose a $\mathcal{CSV}$-Graph is represented by Θ having a $\mathcal{SVCAS}$, denoted by $Q = (\mathcal{V}, \mathfrak{C}_{\mathcal{V}})$. If Li-$S_{redct}(\mathbb{C}_{\mathcal{V}})$ is a partition of V, then $F_{fix}(\mathbb{C}_{\mathcal{V}})$ represents a boolean lattice.*

Proof. We need first to show that if$\mathcal{L}$i-$\mathcal{S}_{redct}(\mathbb{C}_{\mathcal{V}})$ is a partition, then $\mathbb{C}_{\mathcal{V}}$ is a soft unary covering. We assume on contrary that $\mathbb{C}_{\mathcal{V}}$ is not soft unary covering. Therefore, there will be an element $x \in \mathcal{V}$ such that $|\text{Mdes}_{\mathcal{T}}(x)| \gneqq 1$. So, we have parameters $\sigma_1, \sigma_2 \in \mathfrak{R}$ with $\delta(\sigma_1), \delta(\sigma_2) \in \mathcal{L}$i-$\mathcal{S}_{redct}(\mathbb{C}_{\mathcal{V}})$ so that $\delta(\sigma_1), \delta(\sigma_2) \in \text{Mdes}_{\mathcal{T}}(x)$, that is, $x \in \delta(\sigma_1) \cap \delta(\sigma_2)$, which is contradictory with the fact that $\mathcal{L}$i-$\mathcal{S}_{redct}(\mathbb{C}_{\mathcal{V}})$ is a partition of $\mathcal{V}$. Therefore, one can see that $\mathbb{C}_{\mathcal{V}}$ is a soft unary covering. But, $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ is distributive lattice. Moreover, $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ represents a bounded lattice. Now, we will have to show only that $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ is a lattice which is complemented. Equivalently, we have to prove that $\sim \mathcal{W} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ for any $\mathcal{W} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. If we assume that $\sim \mathcal{W} \notin \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$, then there is an element y in $\sim \mathcal{W}$ such that for $\sigma_i \in \mathfrak{R}$, $\delta(\sigma_i) \subsetneq \sim \mathcal{W}$. Since, $\mathbb{C}_{\mathcal{V}}$ is soft unary then for every vertex $x \in \mathcal{V}$ we have $Mdes_{\mathcal{T}}(x) = \{\delta(\sigma_x)\}$. So, $\delta\left(\sigma_y\right) \subsetneq \sim \mathcal{W}$. As a result, there is $x \in \mathcal{W}$ such that $x \in \delta\left(\sigma_y\right)$ exists. But it is given $\mathcal{L}$i-$\mathcal{S}_{redct}(\mathbb{C}_{\mathcal{V}})$ is partitioning the universe $\mathcal{V}$, so we have $\delta\left(\sigma_y\right) = \delta(\sigma_x)$. Thus, $\delta(\sigma_x) \subsetneq \mathcal{W}$, i.e, $x \notin \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W})$. In other words, $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) \neq \mathcal{W}$, which is contradictory with $\mathcal{W} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. Therefore, $\sim \mathcal{W} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$, showing that $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ is a complemented lattice. Hence, as a consequence, $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ is a boolean lattice. □

Proposition 4.3. *Suppose a $\mathcal{CSV}$-Graph is represented by Θ having a $\mathcal{SVCAS}$, denoted by $Q = (\mathcal{V}, \mathfrak{C}_{\mathcal{V}})$. If Li-$S_{redct}(\mathbb{C}_{\mathcal{V}})$ partitioning the vertex set V, then $F_{fix}(\mathbb{C}_{\mathcal{V}})$ is a double Stone algebra.*

Proof. For any $\mathcal{W} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$, we prove $\mathcal{W}^* =\sim \mathcal{W} = \mathcal{W}^+$. We know that $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ is a boolean lattice, therefore $\sim \mathcal{W} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ that is, $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{W}) =\sim \mathcal{W}$. So, $\mathcal{W}^* = \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\sim \mathcal{W}) =\sim \mathcal{W}$. For any $y \in\sim \mathcal{W}$, if $\delta\left(\sigma_y\right) \in Mdes_{\mathcal{T}}(y)$, then $\delta\left(\sigma_y\right)$ is a *Join* $\mathcal{L}$i-$\mathcal{SU}red$ element of $\mathbb{C}_{\mathcal{V}}$. Using Proposition 4.1, $\delta\left(\sigma_y\right) \in J(\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}}))$. Assume that $x \in \mathcal{W}$ exists such that $x \in \delta\left(\sigma_y\right)$. Since $\mathcal{L}$i-$\mathcal{S}_{redct}(\mathbb{C}_{\mathcal{V}})$, being a partition of V, leads to the fact that $\delta\left(\sigma_y\right) \in Mdes_{\mathcal{T}}(x)$. In the light of this, we conclude that, $\delta\left(\sigma_y\right) \subsetneq \mathcal{W}$, that is, $x \notin \underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W})$. So, $\underline{\mathcal{F}}_{\mathbb{C}_{\mathcal{V}}}(\mathcal{W}) \neq \mathcal{W}$ or $\mathcal{W} \notin \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. Which is a contradiction because $\mathcal{W} \in \mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$. Therefore,

$$\mathcal{W}^+ = \bigcup_{x\in\sim\mathcal{W}\left(x\in\delta(\sigma_i)\in J\left(\mathcal{F}_{fix}\left(\mathbb{C}_{\mathcal{V}}\right)\right)\right)} \delta(\sigma_i) = \bigcup_{x\in\sim\mathcal{W}\delta(\sigma_i)\in Mdes_{\mathcal{T}}(y)} \delta(\sigma_i) =\sim \mathcal{W}$$

Similarly, it is easy to prove that $\mathcal{W}^{**} =\sim\sim \mathcal{W} = \mathcal{W} = \mathcal{W}^{++}$. Therefore, $\mathcal{W}^* \cup \mathcal{W}^{**} = \mathcal{V}$, $\mathcal{W}^+ \cap \mathcal{W}^{++} = \emptyset$, demonstrating that $\mathcal{F}_{fix}(\mathbb{C}_{\mathcal{V}})$ is both a dual Stone algebra and a Stone algebra. □

5. Conclusions and future works

Soft set theory and rough set theory are two newer tools to discuss uncertainty. Soft graph theory is a nice way to depict certain information. In order to discuss uncertainty in soft graphs, a possible amalgamation of three different concepts, that is, rough sets, soft sets and graphs is discussed. In this paper, we have introduced two new concepts $\mathcal{L}$i-soft rough covering graphs ($\mathcal{L}$i-$\mathcal{SRCG}$s) and the concept of fixed point sets,called $\mathcal{L}$i-$\mathcal{SRCFP}$ set, induced by soft covering $\mathbb{C}_{\mathcal{V}}$ of such graphs.$\mathcal{L}$i-$\mathcal{SRCG}$s are used to discuss various kinds of approximation operators and the properties associated with them. We review the conditions for the family of $\mathcal{L}$i-$\mathcal{SRCFP}$ sets to become a lattice structure, distributive lattice and complete lattice. It is shown that the $\mathcal{L}$i-$\mathcal{SRCFP}$ set is both a double Stone algebra and a boolean lattice. Furthermore, we looked into some algebras that dealt with the fixed points of $\mathcal{L}$i-$\mathcal{SRCG}$s . Applications of the algebraic structures available in covering soft sets to soft graphs may reveal new facets of graph theory. This work shows a novel approach for dealing with fixed points based on soft rough covering graphs. The future work will be focused on
(*i*) constructing the fixed points sets based on some other kind of soft rough graphs;
(*ii*) constructing the fixed points sets based on multi-granular soft rough covering sets;
(*iii*) constructing the fixed points sets based on upper approximation operators;
(*iv*) comparison between the proposed study and the study to be used in (*iii*);
(*v*) developing the family of $\mathcal{SRCFP}$ *sets* based on soft neighborhood (of elements of universe set) and studying the conditions that these sets become some lattice structure. Further, some algebra can be discussed related to $\mathcal{SRCFP}$ *sets* based on soft neighborhoods.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT.

This research was funded by National Science, Research and Innovation Fund (NSRF), and King Mongkut's University of Technology North Bangkok with Contract no. KMUTNB-FF-66-05.

Conflict of interest

The authors declare that they do not have any competing interests.

References

1. L. A. Zadeh, Fuzzy sets, *Inform. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
2. Z. Pawlak, Rough sets, *Int. J. Comput. Inform. Sci.*, **11** (1982), 341–356. https://doi.org/10.1007/BF01001956
3. Z. Bonikowski, E. Bryniariski, V. W. Skardowska, Extension and intensions in the rough set theory, *Inform. Sci.*, **107** (1998), 149–167. https://doi.org/10.1016/S0020-0255(97)10046-9
4. D. G. Chen, W. X. Zhang, D. Yeung, E. C. C. Tsang, Rough approximations on a complete completely distributive lattice with applications to generalized rough sets, *Inform. Sci.*, **176** (2006), 1829–1848. https://doi.org/10.1016/j.ins.2005.05.009
5. T. Y. Lin, Y. Y. Yao, L. A. Zadeh, *Rough sets, granular computing and data mining*, Heidelberg: Physica-Verlag, 2001.
6. J. Y. Liang, K. S. Chin, C. Y. Dang, R. C. M. Yam, A new method for measuring uncertainty and fuzziness in rough set theory, *Int. J. Gen. Syst.*, **31** (2002), 331–342. https://doi.org/10.1080/0308107021000013635
7. G. L. Liu, Y. Sai, A comparison of two types of rough sets induced by coverings, *Int. J. Approx. Reason.*, **50** (2009), 521–528. https://doi.org/10.1016/j.ijar.2008.11.001
8. Z. Pawlak, *Rough sets: Theoretical aspects of reasoning about data*, Springer Science Business Media, 1991.
9. Y. F. Liu, W. Zhu, Characteristic of partition-circuit matroid through approximation number, In: *2012 IEEE International conference on granular computing*, 2012, 314–319. https://doi.org/10.1109/GrC.2012.6468668
10. Y. Liu, W. Zhu, Y. Zhang, Relationship between partition matroid and rough set through k-rank matroid, *J. Inform. Comput. Sci.*, **8** (2012), 2151–2163.
11. Q. Li, W. Zhu, Lattice structures of fixed points of the lower approximations of two types of covering-based rough sets, *arXiv preprint*, 2012, 109–145. https://doi.org/10.48550/arXiv.1209.5569
12. S. P. Wang, Q. X. Zhu, W. Zhu, F. Min, Quantitative analysis for covering-based rough sets using the upper approximation number, *Inform. Sci.*, **220** (2012), 483–491. https://doi.org/10.1016/j.ins.2012.07.030

13. S. Yüksel, Z. G. Ergül, N. Tozlu, Soft covering based rough sets and their application, *Sci. World J.*, **2014** (2014), 970893. https://doi.org/10.1155/2014/970893
14. J. M. Zhan, M. I. Ali, N. Mehmood, On a novel uncertain soft set model: Z-soft fuzzy rough set model and corresponding decision making methods, *Appl. Soft Comput.*, **56** (2017), 446–457. https://doi.org/10.1016/j.asoc.2017.03.038
15. H. Zhang, J. M. Zhan, Rough soft lattice implication algebras and corresponding decision making methods, *Int. J. Mach. Learn. Cyb.*, **8** (2017), 1301–1308. https://doi.org/10.1007/s13042-016-0502-6
16. D. Molodtsov, Soft set theory-first results, *Comput. Math. Appl.*, **37** (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5
17. M. I. Ali, F. Feng, X. Y. Liu, W. K. Min, M. Shabir, On some new operations in soft set theory, *Comput. Math. Appl.*, **57** (2009), 1547–1553. https://doi.org/10.1016/j.camwa.2008.11.009
18. P. K. Maji, R. Biswas, A. R. Roy, Soft set theory, *Comput. Math. Appl.*, **45** (2003), 555–562. https://doi.org/10.1016/S0898-1221(03)00016-6
19. P. K. Maji, A. R. Roy, R. Biswas, An application of soft sets in a decision making problem, *Comput. Math. Appl.*, **44** (2002), 1077–1083.
20. D. Molodtsov, *The theory of soft sets (in Russian)*, Moscow: URSS Publishers, 2004. https://doi.org/10.1016/S0898-1221(99)00056-5
21. J. M. Zhan, J. C. R Alcantud, A novel type of soft rough covering sets and its application to multicriteria group decision making, *Artif. Intell. Rev.*, **52** (2019), 2381–2410. https://doi.org/10.1007/s10462-018-9617-3
22. J. C. R. Alcantud, J. M. Zhan, Multi-granular soft rough covering sets, *Soft Comput.*, **24** (2020), 9391–9402. https://doi.org/10.1007/s00500-020-04987-5
23. E. Bryniarski, A calculus of rough sets of the first order, *Bull. Pol. Acad. Sci. Math.*, **37** (1989), 71–77.
24. F. Feng, Soft rough sets applied to multicriteria group decision making, *Ann. Fuzzy Math. Inf.*, **2** (2011), 69–80.
25. M. Atef, A. El F. El Atik, Some extensions of covering-based multigranulation fuzzy rough sets from new perspectives, *Soft Comput.*, **25** (2021), 6633–6651 https://doi.org/10.1007/s00500-021-05659-8 .
26. G. Birkhoff, *Lattice theory*, American Mathematical Society, 1940.
27. N. Shah, N. Rehman, M. Shabir, M. I. Ali, Another approach to roughness of soft graphs with applications in decision making, *Symmetry*, **10** (2018), 145. https://doi.org/10.3390/sym10050145
28. F. Fatimah, D. Rosadi, R. B. F. Hakim, J. C. R. Alcantud, N-soft sets and their decision making algorithms, *Soft Comput.*, **22** (2018), 3829–3842. https://doi.org/10.1007/s00500-017-2838-6
29. F. Feng, X. Y. Liu, V. L. Fotea, Y. B. Jun, Soft sets and soft rough sets, *Inform. Sci.*, **181** (2011), 1125–1137. https://doi.org/10.1016/j.ins.2010.11.004
30. W. Zhu, F. Y. Wang, Reduction and axiomization of covering generalized rough sets, *Inform. Sci.*, **152** (2003), 217–230. https://doi.org/10.1016/S0020-0255(03)00056-2
31. W. Zhu, Relationship among basic concepts in covering-based rough sets, *Inform. Sci.*, **179** (2009), 2478–2486. https://doi.org/10.1016/j.ins.2009.02.013

32. A. El F. El Atik, A. Nawar, M. Atef, Rough approximation models via graphs based on neighborhood systems, *Granular Comput.*, **6** (2021), 1025–1035. https://doi.org/10.1007/s41066-020-00245-z
33. M. Akram, S. Nawaz, Operations on soft graphs, *Fuzzy Inform. Eng.*, **7** (2015), 423–449. https://doi.org/10.1016/j.fiae.2015.11.003
34. B. Praba, G. Gomathi, M. Aparajitha, A lattice structure on minimal soft rough sets and its applications, *New Math. Nat. Comput.*, **16** (2020), 255–269. https://doi.org/10.1142/S1793005720500155
35. N. Shah, M. I. Ali, M. Shabir, A. Abbas, N. Rehman, Uncertainty measure of Z-soft covering rough models based on a knowledge granulation, *J. Intell. Fuzzy Syst.*, **38** (2020), 1637–1647. https://doi.org/10.3233/JIFS-182708
36. N. Rehman, N. Shah, M. I. Ali, C. Park, Uncertainty measurement for neighborhood based soft covering rough graphs with applications, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A*, **113** (2019), 2515–2535. https://doi.org/10.1007/s13398-019-00632-5
37. W. H. Bai, J. J. Ding, C. Zhang, Dual hesitant fuzzy graphs with applications to multi-attribute decision making, *Int. J. Cognitive Comput. Eng.*, **1** (2020), 18–26. https://doi.org/10.1016/j.ijcce.2020.09.002
38. Z. He, L. Liang , H. M. Baskonus, W Gao, Remarks on path-factor critical avoidable graphs, *Int. J. Cogn. Comput. Eng.*, **4** (2023), 65–70. https://doi.org/10.1016/j.ijcce.2023.02.004
39. M. Atef, S. I. Nada, On three types of soft fuzzy coverings based rough sets, *Math. Comput. Simul.*, **185** (2021), 452–467. https://doi.org/10.1016/j.matcom.2020.12.023
40. M. Atef, S. Nada, A. Gumaei, A. S. Nawar, On three types of soft rough covering-based fuzzy sets, *J. Math.*, **2021** (2021), 6677298. https://doi.org/10.1155/2021/6677298
41. A. A. Estaji, M. Vatandoost, R. Pourkhandani, Fixed points of covering upper and lower approximation operators, *Soft Comput.*, **23** (2019), 11447–11460. https://doi.org/10.1007/s00500-019-04113-0
42. J. M. Zhan, K. Y. Zhu, A novel soft rough fuzzy set: Z-soft rough fuzzy ideals of hemirings and corresponding decision making, *Soft Comput.*, **21** (2017), 1923–1936. https://doi.org/10.1007/s00500-016-2119-9
43. M. Atef, S. Nada, A. Nawar, Covering soft rough sets and its topological properties with application, *Soft Comput.*, **271** (2023), 4451–4461. https://doi.org/10.1007/s00500-023-07812-x
44. Z. Li, N. Xie, G. Wen, Soft coverings and their parameter reductions, *Appl. Soft Comput.*, **31** (2015), 48–60. https://doi.org/10.1016/j.asoc.2015.02.027
45. I. S. Khan, C. Park, A. Shoaib, N. Shah, A study of fixed point sets based on Z-soft rough covering models, *AIMS Mathematics*, **7** (2022), 13278–13291. https://doi.org/10.3934/math.2022733