AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 20395–20414.
DOI: 10.3934/math.20231040
Received: 22 March 2023
Revised: 24 May 2023
Accepted: 29 May 2023
Published: 25 June 2023

*Research article*

# Power-barrier option pricing formulas in uncertain financial market with floating interest rate

**Hua Zhao**[1], **Yue Xin**[2,*], **Jinwu Gao**[3] **and Yin Gao**[4]

[1] School of Business Administration, Chongqing Technology and Business University, Chongqing, 400067, China
[2] School of Mathematics, Renmin University of China, Beijing 100872, China
[3] School of Economics, Ocean University of China, Qingdao 266100, China
[4] College of Science, Beijing Forestry University, Beijing 100083, China

* **Correspondence:** Email: 2019000138@ruc.edu.cn.

**Abstract:** Power-barrier option is a typical exotic option formed by attaching some restrictions to the power option, where the power option evolves from standard European option with the strike price and underlying good price attached to some power. Compared with the ordinary options, power-barrier option can provide investors with stable leverage and premium income. Therefore, power-barrier option is more favored by investors. This paper mainly discusses the pricing problems of power-barrier option in uncertain financial market. The fluctuation of stock price is regarded as an uncertain process and the interest rate is floating. The uncertain differential equation is invoked to simulate this fluctuation in an uncertain environment. Then, the clear pricing formulas of power-barrier option are given. Finally, the corresponding numerical examples and a real data example are put forward to illustrate the method.

**Keywords:** power option; barrier option; floating interest rate; uncertain differential equation; option pricing
**Mathematics Subject Classification:** 91G30, 34H05, 91G80

## 1. Introduction

Power option evolves from standard European option. The main feature of power option is to add some power to the strike price and the underlying good price to form a non-linear payoff. Certainly from a practical point of view, the non-linear payoff of power option provides huge leverage and interesting hedging for fund management. Compared with the ordinary options, power option can serve to hedge risks and bring more premium income to investors. Therefore, power option is the most favored by investors among all options. There are many examples of power options [1,2], for instance, German Bankers Trust has issued a power option with the power of order 2, known as the parabola option. In order to meet the different needs of investors, power-barrier option appears in market. Power-barrier option is a typical exotic option formed by adding some restrictions to the power option, mainly including knock-in power option and knock-out power option. When the underlying good price arrives at the preset barrier level during the option's life, the option becomes an ordinary power option or invalid. These interesting restrictions essentially limit the stock price within the life of the option, which can mitigate the investment risks.

Option pricing problem is the most concerning issue in modern financial market. Because uncertain factors dominate the financial market, stochastic differential equations (SDEs) derived by Wiener process can be used for option pricing. Black and Scholes [3] proposed the European option pricing formula. Inspired by Black-Scholes formula, Margrabe [4] extended it to American option, which provided a basis for the option pricing theory in modern financial market. As a polynomial option, power option provides market participants with great flexibility. Kim et al. [5] gave a semi-analytical solution for power option based on the Heston stochastic volatility model. Macovschi and Quittard-Pinon [1] appropriately deconstructed the complex polynomial options into several simple power options, and derived a closed formula for the power option. Pasricha and Goel [6] assumed that the asset price was driven by a jump-diffusion process and presented the valuation of a power exchange option. It is generally believed the stock price obeys a stochastic process, and the SDEs are used for option pricing. However, one of the main contributions of Kahneman and Tversky [7] in their research is the discovery of probability distortions that the deciders usually make decisions based on the nonlinear transformations of probability measures. Moreover, Liu [8] proposed a paradox about stochastic financial theory, that is, the stock price can be infinite, which is impossible in the real stock market.

Actually, the fluctuation of the stock price is not completely random, it is often influenced by the investors' belief degree, because investors often make choices according to the market information they have mastered. Motivated by this, Liu [9] proposed an uncertain process, which is virtually a series of uncertain variables that change over time to depict the dynamic uncertain environment in the financial market. Moreover, the uncertain differential equations (UDEs) driven by a Liu process are used to model the stock price, then European option pricing formula is rendered. This work was extended to American option by Chen [10]. Compared with ordinary options, exotic options have great flexibility to adapt to the various needs of investors. A substantial body of researchers investigated the pricing formulas of exotic options [11]. Zhang and Liu [12] investigated Asian option pricing whose returns are related to the geometric average value of the underlying good in the option's life. Gao et al. [13] gave the Lookback option pricing formula whose returns depend on the optimal price value of the underlying good during the option's life. In addition to the above options, barrier option is also a special exotic option, which becomes effective or invalid when the price of the underlying good arrives at the predetermined barrier level. As a barrier option, the European barrier option was first investigated by Yao and Qin [14] under the uncertain environment. Inspired by this, the American barrier option [15, 16], Asian barrier option [17] and barrier Lookback option [18] were constantly explored by researchers. However, these options cannot provide investors with huge leverage and effective risk hedging. This paper first discusses the power-barrier option in uncertain financial market. Whether from a practical or theoretical perspective, the research on power-barrier option has great significance.

In this paper, the UDEs are invoked to model the fluctuation of the stock price. Then, the power-barrier option pricing formulas are given, including knock-in power option and knock-out power option. Knock-in power option indicates that the option takes effect when the underlying good price arrives at the preset barrier level. While, knock-out power option indicates that when the underlying good price penetrates the preset barrier level, the option becomes invalid and worthless. As far as we know, the formulas we raise are fresh. This paper is arranged as follows. Section 2 mainly discusses the knock-in power option and gives the corresponding pricing formulas. Section 3 mainly discusses the knock-out power option and gives the corresponding pricing formulas. For each of the above options, the corresponding numerical examples are put forward in Section 4. Section 5 gives a real data analysis by using Tencent's stock data and one-year Chinese treasury yield. Section 6 makes a concise conclusion.

## 2. Knock-in power option

This section discusses a special barrier option, namely knock-in power option. Generally, this option indicates that the power option will only take effect when the stock price arrives the barrier level $D$. The main elements of the knock-in power option are the expiration time $T$, the strike prices $H$ and the barrier level $D$. The stock price $Y_t$ follows the UDE.

$$dY_t = v_1 Y_t dt + v_2 Y_t dC_{1t} \tag{2.1}$$

where $C_{1t}$ is a Liu process, $v_1$ is the log-drift and $v_2$ is the log-diffusion. The solution of the model (2.1) is

$$Y_t = Y_0 \exp(v_1 t + v_2 C_{1t}).$$

The inverse uncertainty distribution (IUD) of $Y_t$ is

$$\phi_t^{-1}(\alpha) = Y_0 \exp\left(v_1 t + \frac{\sqrt{3} v_2 t}{\pi} \ln \frac{\alpha}{1-\alpha}\right),\ Y_0 \geq 0,\ v_2 > 0.$$

Most studies [9,14,15,17] believe that interest rate is fixed $u_t = u$. However, as an important tool of economic market, interest rate fluctuates frequently due to human activities and transactions. To better describe the real market environment, we assume that the interest rate is floating, that is

$$u_t = u + v_3 \frac{dC_{2t}}{dt}, \tag{2.2}$$

where $v_3$ is a constant and $C_{2t}$ is a Liu process. Hence, we get the discount rate

$$\exp\left(-\int_0^T u_t\, dt\right) = \exp(-uT - v_3 C_{2T}).$$

To better describe the barrier option, an indicator function $\chi_K$ is

$$\chi_K(a) = \begin{cases} 1, & if\, a \geq K \\ 0, & if\, a < K \end{cases}$$

where $K$ is a positive constant number.

*2.1. Knock-in power option pricing formulas*

An up-and-in power call (UIPC) option refers to setting a barrier level $D$ higher than the initial stock price. During the option's life, when the stock price fluctuates upwards and reaches the preset barrier value, the option takes effect and becomes an ordinary power option. Consider a UIPC option with order $m$ and stock price $Y_t$. At the initial time 0, the investor buys the option contract from the issuer at the price of $F_c^{ui}$. The payoff that the option brings to invertor at time $T$ is

$$\chi_D\left(\sup_{0\le t\le T} Y_t\right)(Y_T^m - H^m)^+ .$$

Taking the discount rate into account, the payoff at initial time 0 is

$$W_c = \exp(-uT - v_3 C_{2T})\chi_D\left(\sup_{0\le t\le T} Y_t\right)(Y_T^m - H^m)^+ .$$

The equal expected income of investor and issuer prompts both parties to trade. Thus, the UIPC option price $F_c^{ui}$ is

$$F_c^{ui} = E\left[\exp(-uT - v_3 C_{2T})\chi_D\left(\sup_{0\le t\le T} Y_t\right)(Y_T^m - H^m)^+\right]. \tag{2.3}$$

A down-and-in power put (DIPP) option refers to setting a barrier level $D$ lower than the initial stock price. During the option's life, when the stock price fluctuates downwards and reaches the preset barrier value, the option takes effect and becomes an ordinary power option. Consider a DIPP option with order $m$ and stock price $Y_t$. At the initial time 0, the investor buys the option contract from the issuer at the price of $F_p^{di}$. The payoff that option brings to invertor at time $T$ is

$$\left(1 - \chi_D\left(\inf_{0\le t\le T} Y_t\right)\right)(H^m - Y_T^m)^+.$$

Taking the discount rate into account, the payoff at initial time 0 is

$$W_p = \exp(-uT - v_3 C_{2T})\left(1 - \chi_D\left(\inf_{0\le t\le T} Y_t\right)\right)(H^m - Y_T^m)^+.$$

The equal expected income of investor and issuer prompts both parties to trade. Thus, the DIPP option price $F_p^{di}$ is

$$F_p^{di} = E\left[\exp(-uT - v_3 C_{2T})\left(1 - \chi_D\left(\inf_{0\le t\le T} Y_t\right)\right)(H^m - Y_T^m)^+\right]. \tag{2.4}$$

*2.2. Some theorems*

We give two theorems to compute the price of the options mentioned in Section 2.1.

**Theorem 2.1.** *Let a UIPC option for the models* (2.1) *and* (2.2) *with order m have a barrier level D, a strike price H and an exercise data T. The option price is*

$$F_c^{ui} = \int_{a_0}^{1} \exp\left(-uT - v_3\Psi^{-1}(1-\alpha)\right)((Y_T^\alpha)^m - H^m)^+ \, d\alpha$$

*where*

$$a_0 = \left(1 + \exp\left(\frac{\pi(v_1 T + \ln Y_0 - \ln D)}{\sqrt{3} v_2 T}\right)\right)^{-1},$$

$$Y_T^\alpha = Y_0 \exp\left(v_1 T + \frac{\sqrt{3} v_2 T}{\pi} \ln \frac{\alpha}{1-\alpha}\right),$$

*and*

$$\Psi^{-1}(1-\alpha) = \frac{\sqrt{3} T}{\pi} \ln \frac{1-\alpha}{\alpha}.$$

*Proof.* At first, let's show that the uncertain variable

$$W_c = \exp(-uT - v_3 C_{2T}) \chi_D \left(\sup_{0 \le t \le T} Y_t\right) (Y_T^m - H^m)^+$$

has an IUD

$$W_c^\alpha = \exp\left(-uT - v_3 \Psi^{-1}(1-\alpha)\right) \chi_D \left(\sup_{0 \le t \le T} Y_t^\alpha\right) ((Y_T^\alpha)^m - H^m)^+$$

where

$$Y_t^\alpha = Y_0 \exp\left(v_1 t + \frac{\sqrt{3} v_2 t}{\pi} \ln \frac{\alpha}{1-\alpha}\right)$$

and

$$\Psi^{-1}(1-\alpha) = \frac{\sqrt{3} T}{\pi} \ln \frac{1-\alpha}{\alpha}.$$

On the one hand, assume

$$Y_t(\gamma) \le Y_t^\alpha, \ \forall t \in [0, T]$$

and

$$C_{2T}(\gamma) \ge \frac{\sqrt{3} T}{\pi} \ln \frac{1-\alpha}{\alpha}$$

for some $\gamma \in \Gamma$. Then we have

$$\chi_D \left(\sup_{0 \le t \le T} Y_t(\gamma)\right) (Y_T^m(\gamma) - H^m)^+ \le \chi_D \left(\sup_{0 \le t \le T} Y_t^\alpha\right) ((Y_T^\alpha)^m - H^m)^+$$

and

$$\exp\left(-uT - v_3 C_{2T}(\gamma)\right) \le \exp\left(-uT - v_3 \Psi^{-1}(1-\alpha)\right).$$

Hence, we have

$$\{W_c \le W_c^\alpha\} \supset \left\{C_{2T} \ge \Psi^{-1}(1-\alpha)\right\} \cap \left\{Y_t \le Y_t^\alpha, \ \forall t \in [0, T]\right\}.$$

Based on Theorem A.1 and the independent of $C_{2T}$ and $Y_t$, we get

$$\begin{aligned}
&M\{W_c \le W_c^\alpha\} \\
&\ge M\left\{\left\{C_{2T} \ge \Psi^{-1}(1-\alpha)\right\} \cap \left\{Y_t \le Y_t^\alpha, \ \forall t \in [0, T]\right\}\right\} \\
&= M\left\{C_{2T} \ge \Psi^{-1}(1-\alpha)\right\} \wedge M\left\{Y_t \le Y_t^\alpha, \ \forall t \in [0, T]\right\} \\
&= \alpha.
\end{aligned}$$

On the other hand, assume

$$Y_t(\gamma) > Y_t^\alpha,\ \forall t \in [0, T]$$

and

$$C_{2T}(\gamma) < \frac{\sqrt{3}T}{\pi} \ln \frac{1-\alpha}{\alpha}$$

for some $\gamma \in \Gamma$. Then we have

$$\chi_D\left(\sup_{0\le t\le T} Y_t(\gamma)\right)(Y_T^m(\gamma) - H^m)^+ > \chi_D\left(\sup_{0\le t\le T} Y_t^\alpha\right)((Y_T^\alpha)^m - H^m)^+$$

and

$$\exp\left(-uT - v_3 C_{2T}(\gamma)\right) > \exp\left(-uT - v_3\Psi^{-1}(1-\alpha)\right).$$

Hence, we have

$$\{W_c > W_c^\alpha\} \supset \left\{C_{2T} < \Psi^{-1}(1-\alpha)\right\} \cap \{Y_t > Y_t^\alpha,\ \forall t \in [0, T]\}.$$

Based on Theorem A.1 and the independent of $C_{2T}$ and $Y_t$, we get

$$\begin{aligned}
&M\{W_c > W_c^\alpha\}\\
&\ge M\left\{\left\{C_{2T} < \Psi^{-1}(1-\alpha)\right\} \cap \{Y_t > Y_t^\alpha,\ \forall t \in [0, T]\}\right\}\\
&= M\left\{C_{2T} < \Psi^{-1}(1-\alpha)\right\} \wedge M\{Y_t > Y_t^\alpha,\ \forall t \in [0, T]\}\\
&= 1 - \alpha.
\end{aligned}$$

Based on the duality axiom, we have

$$M\{W_c \le W_c^\alpha\} + M\{W_c > W_c^\alpha\} = 1.$$

Thus

$$M\{W_c \le W_c^\alpha\} = \alpha$$

which means the IUD of $W_c$ is

$$\exp\left(-uT - v_3\Psi^{-1}(1-\alpha)\right)\chi_D\left(\sup_{0\le t\le T} Y_t^\alpha\right)((Y_T^\alpha)^m - H^m)^+.$$

When

$$\sup_{0\le t\le T} Y_t^\alpha \ge D$$

we have

$$\alpha \ge \left(1 + \exp\left(\frac{\pi(v_1 T + \ln Y_0 - \ln D)}{\sqrt{3}v_2 t}\right)\right)^{-1} = a_0.$$

Therefore, the price is

$$\begin{aligned}
F_c^{ui} &= \int_0^1 \exp\left(-uT - v_3\Psi^{-1}(1-\alpha)\right)\chi_D\left(\sup_{0\le t\le T} Y_t^\alpha\right)((Y_T^\alpha)^m - H^m)^+\, d\alpha\\
&= \int_{a_0}^1 \exp\left(-uT - v_3\Psi^{-1}(1-\alpha)\right)((Y_T^\alpha)^m - H^m)^+\, d\alpha.
\end{aligned}$$

The theorem is thus proved.

**Theorem 2.2.** *Let a DIPP option for the models* (2.1) *and* (2.2) *with the order m have a barrier level D, a strike price H and an exercise data T. The option price is*

$$F_p^{di} = \int_{a_0}^{1} \exp\left(-uT - v_3\Psi(1-\alpha)\right)\left(H^m - (Y_T^{1-\alpha})^m\right)^+ \, d\alpha$$

*where*

$$a_0 = \left(1 + \exp\left(\frac{\pi(\ln D - \ln Y_0 - v_1 T)}{\sqrt{3}v_2 T}\right)\right)^{-1},$$

$$Y_T^{1-\alpha} = Y_0 \exp\left(v_1 T + \frac{\sqrt{3}v_2 T}{\pi}\ln\frac{1-\alpha}{\alpha}\right),$$

*and*

$$\Psi^{-1}(1-\alpha) = \frac{\sqrt{3}T}{\pi}\ln\frac{1-\alpha}{\alpha}.$$

*Proof.* At first, let's show that the uncertain variable

$$W_p = \exp(-uT - v_3 C_{2T})\left(1 - \chi_D\left(\inf_{0\le t\le T} Y_t\right)\right)(H^m - Y_T^m)^+$$

has an IUD

$$W_p^\alpha = \exp\left(-uT - v_3\Psi^{-1}(1-\alpha)\right)\left(1 - \chi_D\left(\inf_{0\le t\le T} Y_t^{1-\alpha}\right)\right)(H^m - (Y_T^{1-\alpha})^m)^+$$

where

$$Y_t^{1-\alpha} = Y_0 \exp\left(v_1 t + \frac{\sqrt{3}v_2 t}{\pi}\ln\frac{1-\alpha}{\alpha}\right)$$

and

$$\Psi^{-1}(1-\alpha) = \frac{\sqrt{3}T}{\pi}\ln\frac{1-\alpha}{\alpha}.$$

On the one hand, assume

$$Y_t \ge Y_t^{1-\alpha},\ \forall t \in [0,T]$$

and

$$C_{2T}(\gamma) \ge \frac{\sqrt{3}T}{\pi}\ln\frac{1-\alpha}{\alpha}$$

for some $\gamma \in \Gamma$. Then we have

$$\left(1 - \chi_D\left(\inf_{0\le t\le T} Y_t\right)\right)(H^m - Y_T^m)^+ \le \left(1 - \chi_D\left(\inf_{0\le t\le T} Y_t^{1-\alpha}\right)\right)(H^m - (Y_T^{1-\alpha})^m)^+$$

and

$$\exp\left(-uT - v_3 C_{2T}(\gamma)\right) \le \exp\left(-uT - v_3\Psi^{-1}(1-\alpha)\right).$$

Hence, we have

$$\left\{W_p \le W_p^\alpha\right\} \supset \left\{C_{2T} \ge \Psi^{-1}(1-\alpha)\right\} \wedge \left\{Y_t \ge Y_t^{1-\alpha}\right\}.$$

Based on Theorem A.1 and the independent of $C_{2T}$ and $Y_t$, we get

$$\begin{aligned}
&M\{W_c \geq W_c^{\alpha}\} \\
&\geq M\left\{\left\{C_{2T} \geq \Psi^{-1}(1-\alpha)\right\} \cap \left\{Y_t \geq Y_t^{1-\alpha},\ \forall t \in [0,T]\right\}\right\} \\
&= M\left\{C_{2T} \geq \Psi^{-1}(1-\alpha)\right\} \wedge M\left\{Y_t \geq Y_t^{1-\alpha},\ \forall t \in [0,T]\right\} \\
&= \alpha.
\end{aligned}$$

On the other hand, assume

$$Y_t(\gamma) < Y_t^{1-\alpha},\ \forall t \in [0,T]$$

and

$$C_{2T}(\gamma) < \frac{\sqrt{3}T}{\pi} \ln \frac{1-\alpha}{\alpha}$$

for some $\gamma \in \Gamma$. Then we have

$$\left(1 - \chi_D\left(\inf_{0\leq t\leq T} Y_t\right)\right)(H^m - Y_T^m)^+ > \left(1 - \chi_D\left(\inf_{0\leq t\leq T} Y_t^{1-\alpha}\right)\right)(H^m - (Y_T^{1-\alpha})^m)^+$$

and

$$\exp\left(-uT - v_3 C_{2T}(\gamma)\right) > \exp\left(-uT - v_3 \Psi^{-1}(1-\alpha)\right).$$

Hence, we have

$$\left\{W_p > W_p^{\alpha}\right\} \supset \left\{C_{2T} < \Psi^{-1}(1-\alpha)\right\} \wedge \left\{Y_t < Y_t^{1-\alpha}\right\}.$$

Based on Theorem A.1 and the independent of $C_{2T}$ and $Y_t$, we get

$$\begin{aligned}
&M\{W_p > W_p^{\alpha}\} \\
&\geq M\left\{\left\{C_{2T} < \Psi^{-1}(1-\alpha)\right\} \cap \left\{Y_t < Y_t^{1-\alpha},\ \forall t \in [0,T]\right\}\right\} \\
&= M\left\{\left\{C_{2T} < \Psi^{-1}(1-\alpha)\right\} \wedge M\left\{Y_t < Y_t^{1-\alpha},\ \forall t \in [0,T]\right\}\right\} \\
&= 1-\alpha.
\end{aligned}$$

Based on the duality axiom, we have

$$M\{W_p \leq W_p^{\alpha}\} + M\{W_p > W_p^{\alpha}\} = 1.$$

Thus

$$M\{W_p \leq W_p^{\alpha}\} = \alpha$$

which means the IUD of $W_p$ is

$$\exp\left(-uT - v_3\Psi^{-1}(1-\alpha)\right)\left(1 - \chi_D\left(\inf_{0\leq t\leq T} Y_t^{1-\alpha}\right)\right)(H^m - (Y_T^{1-\alpha})^m)^+.$$

When

$$\inf_{0\leq t\leq T} Y_t^{1-\alpha} < D$$

we have

$$\alpha > \left(1 + \exp\left(\frac{\pi(\ln D - \ln Y_0 - v_1 T)}{\sqrt{3}v_2 t}\right)\right)^{-1} = a_0.$$

Therefore, the price is

$$\begin{aligned} F_p^{di} &= \int_0^1 \exp\left(-uT - v_3\Psi(1-\alpha)\right)\left(1 - \chi_D\left(\sup_{0\le t\le T} Y_t^{1-\alpha}\right)\right)\left(H^m - (Y_T^{1-\alpha})^m\right)^+ \, d\alpha \\ &= \int_{a_0}^1 \exp\left(-uT - v_3\Psi(1-\alpha)\right)\left(H^m - (Y_T^{1-\alpha})^m\right)^+ \, d\alpha. \end{aligned}$$

The theorem is thus proved.

## 3. Knock-out power option

Generally, the knock-out power option limits the stock price to the trigger point $D$. During the option's life, if the stock price exceeds the preset barrier value, the option becomes invalid. The main elements of the knock-out option are the expiration time $T$, the strike prices $H$, and the barrier level $D$. The stock price $Y_t$ and interest rate $u_t$ follow the models (2.1) and (2.2), respectively.

### *3.1. Knock-out power option price formulas*

A down-and-out power call (DOPC) option indicates that the stock price is always higher than the preset barrier level $D$. Once the stock price falls below the preset value, the option becomes invalid. Consider a DOPC option with order $m$ and stock price $Y_t$. At the initial time 0, the investor buys the option contract from the issuer at the price $F_c^{do}$. The payoff that option brings to invertor at time $T$ is

$$\chi_D\left(\inf_{0\le t\le T} Y_t\right)(Y_T^m - H^m)^+.$$

Taking the discount rate into account, the payoff at initial time 0 is

$$W_c = \exp(-uT - v_3 C_{2T})\chi_D\left(\inf_{0\le t\le T} Y_t\right)(Y_T^m - H^m)^+.$$

The equal expected income of investor and issuer prompts both parties to trade. Thus, the option price $F_c^{do}$ is

$$F_c^{do} = E\left[\exp(-uT - v_3 C_{2T})\chi_D\left(\inf_{0\le t\le T} Y_t\right)(Y_T^m - H^m)^+\right]. \tag{3.1}$$

An up-and-out power put (UOPP) option indicates that the stock price is always lower than the preset barrier level $D$. Once the stock price penetrates the preset barrier value, the option becomes invalid. Consider a UOPP option with order $m$ and stock price $Y_t$. At the initial time 0, the investor buys the option contract from the issuer at the price of $F_p^{uo}$. The payoff that option brings to invertor at time $T$ is

$$\left(1 - \chi_D\left(\sup_{0\le t\le T} Y_t\right)\right)(H^m - Y_T^m)^+ .$$

Taking the discount rate into account, the payoff at initial time 0 is

$$W_c = \exp(-uT - v_3 C_{2T})\left(1 - \chi_D\left(\sup_{0\le t\le T} Y_t\right)\right)(H^m - Y_T^m)^+ .$$

The equal expected income of investor and issuer prompts both parties to trade. Thus, the option price $F_c^{do}$ is

$$F_c^{do} = E\left[\exp(-uT - v_3 C_{2T})\left(1 - \chi_D\left(\sup_{0\le t\le T} Y_t\right)\right)(H^m - Y_T^m)^+\right]. \tag{3.2}$$

*3.2. Some theorems*

We give two theorems to compute the price of the options mentioned in Section 3.1

**Theorem 3.1.** *Let a DOPC option for the models* (2.1) *and* (2.2) *with order m have a barrier level D, a strike price H and an exercise data T. The option price is*

$$F_c^{do} = \exp(-v_1 T) \int_{a_0}^{1} \exp(-uT - v_3\Psi^{-1}(1-\alpha))((Y_T^\alpha)^m - H^m)^+$$

*where*

$$a_0 = \left(1 + \exp\left(\frac{\pi(v_1 T + \ln Y_0 - \ln D)}{\sqrt{3}v_2 t}\right)\right)^{-1},$$

$$Y_T^\alpha = Y_0 \exp\left(v_1 T + \frac{\sqrt{3}v_2 T}{\pi}\ln\frac{\alpha}{1-\alpha}\right),$$

*and*

$$\Psi^{-1}(1-\alpha) = \frac{\sqrt{3}T}{\pi}\ln\frac{1-\alpha}{\alpha}.$$

*Proof.* At first, let's show that the uncertain variable

$$W_c = \exp(-uT - v_3 C_{2T})\chi_D\left(\inf_{0\le t\le T} Y_t\right)(Y_T^m - H^m)^+$$

has an IUD

$$W_c^\alpha = \exp(-uT - v_3\Psi^{-1}(1-\alpha))\chi_D\left(\inf_{0\le t\le T} Y_t^\alpha\right)((Y_T^\alpha)^m - H^m)^+$$

where

$$Y_t^\alpha = Y_0 \exp\left(v_1 t + \frac{\sqrt{3}v_2 t}{\pi}\ln\frac{\alpha}{1-\alpha}\right)$$

and

$$\Psi^{-1}(1-\alpha) = \frac{\sqrt{3}T}{\pi}\ln\frac{1-\alpha}{\alpha}.$$

On the one hand, assume

$$Y_t(\gamma) \le Y_t^\alpha,\ \forall t \in [0,T]$$

and

$$C_{2T}(\gamma) \ge \frac{\sqrt{3}T}{\pi}\ln\frac{1-\alpha}{\alpha}$$

for some $\gamma \in \Gamma$. Then we have

$$\chi_D\left(\inf_{0\le t\le T} Y_t\right)(Y_T^m - H^m)^+ \le \chi_D\left(\inf_{0\le t\le T} Y_t^\alpha\right)((Y_T^\alpha)^m - H^m)^+$$

and

$$\exp(-uT - v_3 C_{2T}) \le \exp(-uT - v_3\Psi^{-1}(1-\alpha)).$$

Hence, we have

$$\{W_c \le W_c^\alpha\} \supset \left\{C_{2T} \ge \Psi^{-1}(1-\alpha)\right\} \wedge \{Y_t \le Y_t^\alpha\}.$$

Based on Theorem A.1 and the independent of $C_{2T}$ and $Y_t$, we get

$$\begin{aligned}
&M\{W_c \le W_c^\alpha\}\\
&\ge M\left\{\left\{C_{2T} \ge \Psi^{-1}(1-\alpha)\right\} \cap \{Y_t < Y_t^\alpha,\ \forall t \in [0,T]\}\right\}\\
&= M\left\{C_{2T} < \Psi^{-1}(1-\alpha)\right\} \wedge M\{Y_t < Y_t^\alpha,\ \forall t \in [0,T]\}\\
&= \alpha.
\end{aligned}$$

On the other hand, assume

$$Y_t(\gamma) > Y_t^\alpha,\ \forall t \in [0,T]$$

and

$$C_{2T}(\gamma) < \frac{\sqrt{3}T}{\pi} \ln \frac{1-\alpha}{\alpha}$$

for some $\gamma \in \Gamma$. Then we have

$$\chi_D\left(\inf_{0\le t\le T} Y_t\right)(Y_T^m - H^m)^+ > \chi_D\left(\inf_{0\le t\le T} Y_t^\alpha\right)((Y_T^\alpha)^m - H^m)^+$$

and

$$\exp(-uT - v_3 C_{2T}) > \exp(-uT - v_3\Psi^{-1}(1-\alpha)).$$

Hence, we have

$$\{W_c > W_c^\alpha\} \supset \left\{C_{2T} < \Psi^{-1}(1-\alpha)\right\} \wedge \{Y_t > Y_t^\alpha\}.$$

Based on Theorem A.1 and the independent of $C_{2T}$ and $Y_t$, we get

$$\begin{aligned}
&M\{W_c > W_c^\alpha\}\\
&\ge M\left\{\left\{C_{2T} < \Psi^{-1}(1-\alpha)\right\} \cap \{Y_t > Y_t^\alpha,\ \forall t \in [0,T]\}\right\}\\
&= M\left\{\left\{C_{2T} < \Psi^{-1}(1-\alpha)\right\} \wedge M\{Y_t > Y_t^\alpha,\ \forall t \in [0,T]\}\right\}\\
&= 1-\alpha.
\end{aligned}$$

Based on the duality axiom, we have

$$M\{W_c \le W_c^\alpha\} + M\{W_c > W_c^\alpha\} = 1.$$

Thus

$$M\{W_c \le W_c^\alpha\} = \alpha$$

which means the IUD of

$$\exp(-uT - v_3 C_{2T})\chi_D\left(\inf_{0\le t\le T} Y_t\right)(Y_T^m - H^m)^+$$

is

$$\exp(-uT - v_3\Psi^{-1}(1-\alpha))\chi_D\left(\inf_{0\le t\le T} Y_t^\alpha\right)((Y_T^\alpha)^m - H^m)^+.$$

When

$$\inf_{0\le t\le T} Y_t^\alpha \ge D$$

we have

$$\alpha \geq \left(1 + \exp\left(\frac{\pi(v_1 T + \ln Y_0 - \ln D)}{\sqrt{3}v_2 t}\right)\right)^{-1} = a_0.$$

Therefore, the price is

$$\begin{aligned} F_c^{do} &= \int_0^1 \chi_D\left(\inf_{0\leq t\leq T} Y_t^\alpha\right)\exp(-uT - v_3\Psi^{-1}(1-\alpha))((Y_T^\alpha)^m - H^m)^+ \\ &= \int_{a_0}^1 \exp(-uT - v_3\Psi^{-1}(1-\alpha))((Y_T^\alpha)^m - H^m)^+. \end{aligned}$$

The theorem is thus proved.

**Theorem 3.2.** *Let a UOPP option for the models* (2.1) *and* (2.2) *with order m have a barrier level D, a strike price H and an exercise data T. The option price is*

$$F_p^{uo} = \int_{a_0}^1 \exp(-uT - v_3\Psi^{-1}(1-\alpha))\left(H^m - (Y_T^{1-\alpha})^m\right)^+ d\alpha$$

*where*

$$a_0 = \left(1 + \exp\left(\frac{\pi(\ln D - \ln Y_0 - v_1 T)}{\sqrt{3}v_2 t}\right)\right)^{-1},$$

$$Y_T^{1-\alpha} = Y_0 \exp\left(v_1 T + \frac{\sqrt{3}v_2 T}{\pi}\ln\frac{1-\alpha}{\alpha}\right),$$

*and*

$$\Psi^{-1}(1-\alpha) = \frac{\sqrt{3}T}{\pi}\ln\frac{1-\alpha}{\alpha}.$$

*Proof.* At first, let's show that the uncertain variable

$$W_p = \exp\left(-uT - v_3 C_{2T}\right)\left(1 - \chi_D\left(\sup_{0\leq t\leq T} Y_t\right)\right)(H^m - Y_T^m)^+$$

has an IUD

$$W_p^\alpha = \exp\left(-uT - v_3\Psi^{-1}(1-\alpha)\right)\left(1 - \chi_D\left(\sup_{0\leq t\leq T} Y_t^{1-\alpha}\right)\right)\left(H^m - (Y_T^{1-\alpha})^m\right)^+$$

where

$$Y_t^{1-\alpha} = Y_0 \exp\left(v_1 t + \frac{\sqrt{3}v_2 t}{\pi}\ln\frac{1-\alpha}{\alpha}\right)$$

and

$$\Psi^{-1}(1-\alpha) = \frac{\sqrt{3}T}{\pi}\ln\frac{1-\alpha}{\alpha}.$$

On the one hand, assume

$$Y_t(\gamma) \geq Y_t^{1-\alpha},\ \forall t \in [0, T]$$

and

$$C_{2T}(\gamma) \geq \Psi^{-1}(1-\alpha)$$

for some $\gamma \in \Gamma$. Then, we have

$$\left(1-\chi_D\left(\sup_{0\leq t\leq T} Y_t\right)\right)(H^m - Y_T^m)^+ \leq \left(1-\chi_D\left(\sup_{0\leq t\leq T} Y_t^{1-\alpha}\right)\right)\left(H^m - (Y_T^{1-\alpha})^m\right)^+$$

and

$$\exp(-uT - v_3 C_{2T}) \leq \exp\left(-uT - v_3 \Psi^{-1}_{1-\alpha}\right).$$

Based on the Theorem A.1 and the independent of $C_{2T}$ and $Y_t$, we get

$$\begin{aligned}
&M\{W_p \leq W_p^\alpha\} \\
&\geq M\left\{\left\{C_{2T} \geq \Psi^{-1}(1-\alpha)\right\} \cap \left\{Y_t \geq Y_t^{1-\alpha},\ \forall t \in [0,T]\right\}\right\} \\
&= M\left\{C_{2T} \geq \Psi^{-1}(1-\alpha)\right\} \wedge M\left\{Y_t \geq Y_t^{1-\alpha},\ \forall t \in [0,T]\right\} \\
&= \alpha.
\end{aligned}$$

On the other hand, assume

$$Y_t(\gamma) < Y_t^{1-\alpha},\ \forall t \in [0,T]$$

and

$$C_{2T}(\gamma) < \Psi^{-1}(1-\alpha)$$

for some $\gamma \in \Gamma$. Then, we have

$$\left(1-\chi_D\left(\sup_{0\leq t\leq T} Y_t\right)\right)(H^m - Y_T^m)^+ > \left(1-\chi_D\left(\sup_{0\leq t\leq T} Y_t^{1-\alpha}\right)\right)\left(H^m - (Y_T^{1-\alpha})^m\right)^+$$

and

$$\exp(-uT - v_3 C_{2T}) > \exp(-uT - v_3 \Psi^{-1}(1-\alpha).$$

Based on the Theorem A.1 and the independent of $C_{2T}$ and $Y_t$, we get

$$\begin{aligned}
&M\{W_p > W_p^\alpha\} \\
&\geq M\left\{\left\{C_{2T} \geq \Psi^{-1}(1-\alpha)\right\} \cap \left\{Y_t \geq Y_t^{1-\alpha},\ \forall t \in [0,T]\right\}\right\} \\
&= M\left\{C_{2T} < \Psi^{-1}(1-\alpha)\right\} \wedge M\left\{Y_t < Y_t^{1-\alpha},\ \forall t \in [0,T]\right\} \\
&= 1-\alpha.
\end{aligned}$$

Based on the duality axiom, we have

$$M\{W_p \leq W_p^\alpha\} + M\{W_p > W_p^\alpha\} = 1.$$

Thus

$$M\{W_p \leq W_p^\alpha\} = \alpha$$

which means the IUD of

$$\exp(-uT - v_3 C_{2T})\left(1-\chi_D\left(\sup_{0\leq t\leq T} Y_t\right)\right)(H^m - Y_T^m)^+$$

is

$$\exp\left(-uT - v_3\Psi^{-1}(1-\alpha)\right)\left(1-\chi_D\left(\sup_{0\le t\le T} Y_t^{1-\alpha}\right)\right)\left(H^m - (Y_T^{1-\alpha})^m\right)^+.$$

When

$$\sup_{0\le t\le T} Y_t^{1-\alpha} < D$$

we have

$$\alpha > \left(1+\exp\left(\frac{\pi(\ln D - \ln Y_0 - v_1 T)}{\sqrt{3}v_2 t}\right)\right)^{-1} = a_0.$$

Therefore, the price is

$$\begin{aligned} F_p^{uo} &= \int_0^1 \exp(-uT - v_3\Psi^{-1}(1-\alpha))\left(1-\chi_D\left(\sup_{0\le t\le T} Y_t^{1-\alpha}\right)\right)\left(H^m - (Y_T^{1-\alpha})^m\right)^+ d\alpha \\ &= \int_{a_0}^1 \exp(-uT - v_3\Psi^{-1}(1-\alpha))\left(H^m - (Y_T^{1-\alpha})^m\right)^+ d\alpha. \end{aligned}$$

The theorem is thus proved.

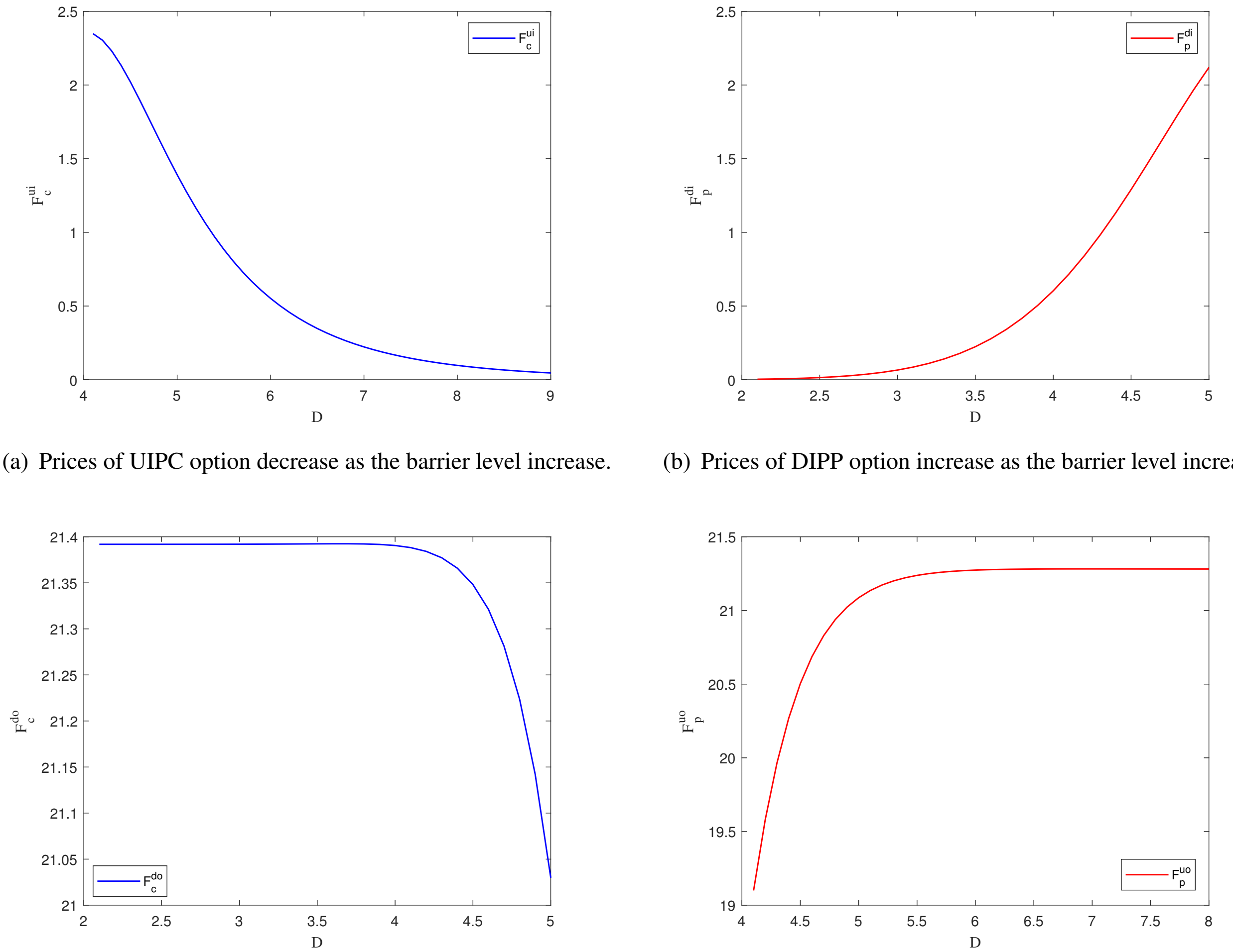
## 4. Numerical examples

In this section, we design four sets of numerical examples corresponding to the above four cases to illustrate how to price options. We set different model parameters for four options, namely the UIPC option, DIPP option, DOPC option and UOPP option, and give the corresponding option prices, as shown in Table 1. For example, the first column in the Table 1 indicates that we set the parameters of models (2.1) and (2.2) as $v_1 = 0.05, v_2 = 0.04, v_3 = 0.01, u = 0.02, Y_0 = 3, H = 4, D = 5, T = 5$, and the price of UIPC option is $F_c^{ui} = 1.3901$.

**Table 1.** Model parameters and the corresponding option price.

| | UIPC | DIPP | DOPC | UOPP |
|---|---|---|---|---|
| $v_1$ | 0.05 | 0.05 | 0.04 | 0.03 |
| $v_2$ | 0.04 | 0.04 | 0.03 | 0.04 |
| $v_3$ | 0.01 | 0.01 | 0.015 | 0.02 |
| $u$ | 0.02 | 0.02 | 0.02 | 0.02 |
| $Y_0$ | 3 | 4.5 | 5 | 3 |
| $H$ | 4 | 5.5 | 4 | 6 |
| $D$ | 5 | 3.5 | 3 | 5 |
| $T$ | 5 | 5 | 5 | 5 |
| $F$ | 1.3901 | 0.2240 | 21.3919 | 21.0866 |

Figure 1 shows the relationship between the option price and barrier level when other factors are fixed. Take Figure 1(a) as an example, when the barrier level varies from 4.1 dollars to 9 dollars, the corresponding UIPC option price varies from 2.348 dollars to 0.0457 dollars. The prices of the UIPC option and DOPC option decrease as the barrier level increase, which is consistent with the constraints imposed by the barrier price on options. UIPC option takes effect when the stock price fluctuates upwards and reaches the present barrier value, which means the higher the barrier level, the harder it is for the option to take effect, and the lower the option price. While, the DOPC option becomes invalid when the stock price falls below the present barrier level, which indicates the higher the barrier level, the easier it is for the option to become invalid, and the lower the option price. However, the prices of the DIPP option and UOPP option increase as the barrier level increase, which is consistent with the constraints imposed by the barrier price on options. DIPP option takes effect when the stock price fluctuates downwards and reaches the present barrier value, which means the higher the barrier level, the easier it is for the option to take effect, and the higher the option price. While, the UOPP option becomes invalid when the stock price penetrates the present barrier level, which indicates the higher the barrier level, the harder it is for the option to become invalid, and the higher the option price.




(a) Prices of UIPC option decrease as the barrier level increase.

(b) Prices of DIPP option increase as the barrier level increase.

(c) Prices of DOPC option decrease as the barrier level increase.

(d) Prices of UOPP option increase as the barrier level increase.

**Figure 1.** The relationship between option price and barrier level in Example 4.

## 5. Real data analysis

In this section, we illustrate our approach with real financial data. We choose the closing stock price of Tencent and One-year Chinese treasury yield from Sept. 9 to Oct. 26 2021 as research object, which are available at https://cn.investing.com/markets/ and http://www.chinamoney. com.cn/chinese/, respectively. The data are shown in Figure 2.

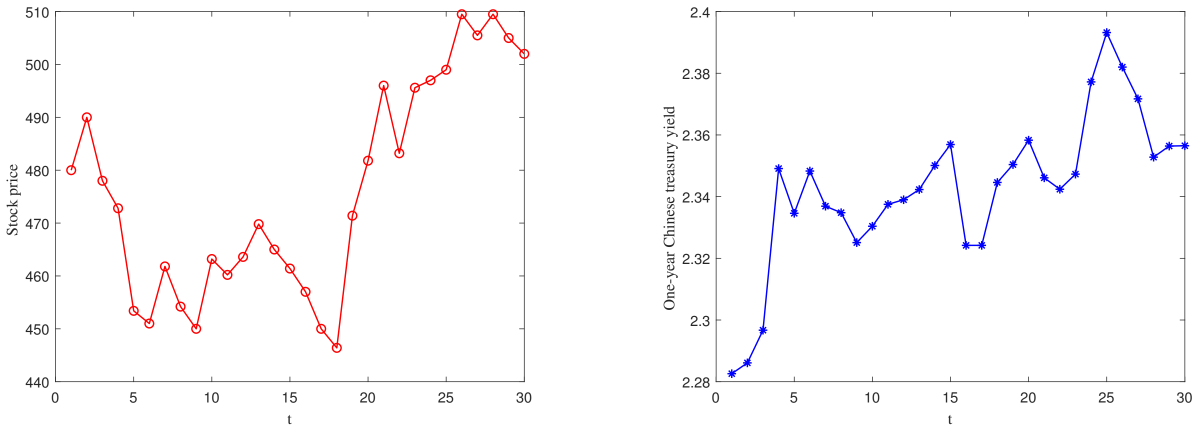



**Figure 2.** The stock price of Tencent and one-year Chinese treasury yield from Sept. 9 to Oct. 26 2021.

First, we use the method based on residuals [19] to estimate the parameters of model (2.1), the estimation is

$$v_1 = 0.0019, \qquad v_2 = 0.0234.$$

It follows from Liu and Liu [19] that the $i$-th ($i = 2, 3, \cdots, 30$) residual of model (2.1) is

$$\epsilon_i = \left(1 + \exp\left(\frac{\pi(\ln x_{t_{i-1}} + v_1(t_i - t_{i-1}) - \ln x_{t_i})}{\sqrt{3}v_2(t_i - t_{i-1})}\right)\right)^{-1},$$

which can be regarded as a sample of linear uncertainty distribution $L(0, 1)$, i.e.,

$$\epsilon_2, \epsilon_3, \cdots, \epsilon_{30} \sim L(0, 1).$$

The uncertain hypothesis test proposed by Ye and Liu [20] proved that testing whether the model fits the data is equivalent to testing whether the residuals obey a linear uncertainty distribution $L(0, 1)$. Given the significance level $\alpha = 0.05$, the test is

$$H = \Big\{(\epsilon_2, \epsilon_3, \cdots, \epsilon_{30}) : \text{there are at least 2 of indexed } t's \text{ with } 2 \leq t \leq 30$$

$$\text{such that } \epsilon_i < 0.025 \text{ or } \epsilon_i > 0.975\Big\}.$$

It can be seen from Figure 3 only $\epsilon_{19} \notin [0.025, 0.975]$, we have $(\epsilon_2, \epsilon_3, \cdots, \epsilon_{30}) \notin H$. Thus the model is a goof fit to the observed data.

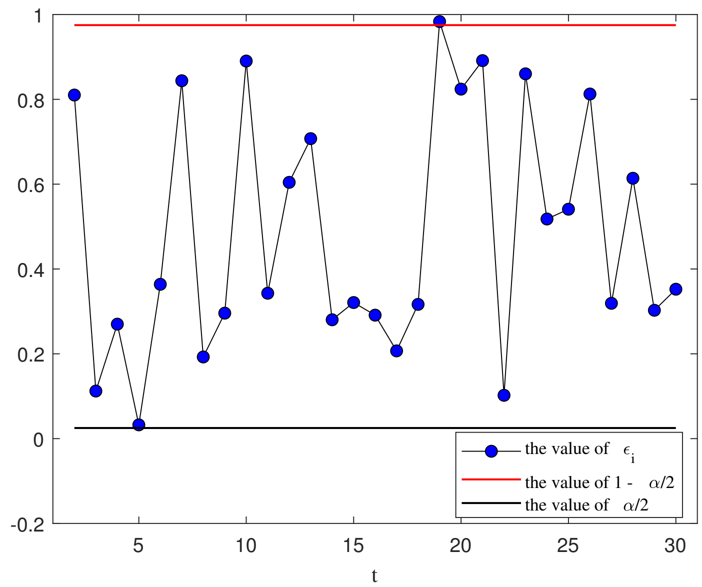



**Figure 3.** The value of $\epsilon_i$ for model (2.1) as well as the value of $\alpha/2$ and $1 - \alpha/2$ for $i = 2, 3, \cdots, 30$, respectively.

Then, we use the method of moments [21] to estimate the parameters of model (2.2), the estimation is

$$u = 0.0234, \qquad v_3 = 0.0002.$$

Thus, we obtain the uncertain stock model

$$\begin{cases} dY_t = 0.0019Y_t dt + 0.0234Y_t dC_{1t} \\ u_t = 0.0234 + 0.0002\frac{dC_t}{dt}. \end{cases}$$

Based on the above model, we assume that the initial stock price $Y_0 = 480$, which is the stock price on Oct. 9 and the expiration date $T = 15$. For the four options introduce in Sections 3 and 4, we give the option prices under different strike prices and barrier levels, see Table 2.

**Table 2.** Prices of four option

| | UIPC | DIPP | DOPC | UOPP |
|---|---|---|---|---|
| $H$ | 490 | 500 | 490 | 510 |
| $D$ | 510 | 470 | 500 | 470 |
| $F$ | 78070 | 32760 | 78130 | 35880 |

## 6. Conclusions

The paper mainly discussed the pricing formulas of power-barrier option in uncertain financial market. The stock price was assumed to obey an uncertain process and interest rate was floating. The UDE was invoked to model the fluctuation of the stock price. Some power-barrier option pricing formulas were given, which were knock-in power option and knock-out power option. Moreover, the corresponding numerical examples and a real data example were presented to demonstrate how to get the price of the option.

An interesting future research direction is to consider other uncertain differential equations, e.g., uncertain Ornstein–Uhlenbeck model, uncertain mean-reverting model, and uncertain exponential Ornstein–Uhlenbeck model, to describe stock prices and give pricing formulas of power-barrier option. It is also signification to explore more barrier option pricing formulations in the uncertain financial market.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The authors have no relevant financial or non-financial interests to disclose.

## Data availability

All data analyzed during this study are included in Figure 2, which are available at https://cn.investing. com/markets/ and http://www. chinamoney. com.cn/chinese/, respectively.

## References


1. S. Macovschi, F. Quittard-Pinon, On the pricing of power and other polynomial options, *J. Deriv.*, **13** (2006), 61–71. https://doi.org/10.3905/jod.2006.635421
2. R. C. Heynen, H. M. Kat, Pricing and hedging power options, *Financ. Eng. Jpn. Mark.*, **3** (1996), 253–261. https://doi.org/10.1007/BF02425804
3. F. Black, M. Scholes, The pricing of options and corporate liabilities, *J. Polit. Econ.*, **81** (1973), 637–654. https://doi.org/10.1086/260062
4. W. Margrabe, The value of an option to exchange one asset for another, *J. Finance*, **33** (1978), 177–186. https://doi.org/10.2307/2326358
5. J. Kim, B. Kim, K. S. Moon, I. S. Wee, Valuation of power options under Heston's stochastic volatility model, *J. Econ. Dyn. Control*, **36** (2012), 1796–1813. https://doi.org/10.1016/j.jedc.2012.05.005
6. P. Pasricha, A. Goel, Pricing vulnerable power exchange options in an intensity based framework, *J. Comput. Appl. Math.*, **355** (2019), 106–115. https://doi.org/10.1016/j.cam.2019.01.019
7. D. Kahneman, A. Tversky, Prospect theory: an analysis of decision under risk, *Econometrica*, **47** (1979), 263–292. https://doi.org/10.2307/1914185
8. B. Liu, Toward uncertain finance theory, *J. Uncertainty Anal. Appl.*, **1** (2013), 1. https://doi.org/10.1186/2195-5468-1-1

9. B. Liu, Some research problems in uncertainty theory, *J. Uncertain Syst.*, **3** (2009), 3–10.

10. X. Chen, American option pricing formula for uncertain financial market, *Int. J. Oper. Res.*, **8** (2011), 27–32.

11. Z. Zhang, W. Liu, Y. Sheng, Valuation of power option for uncertain financial market, *Appl. Math. Comput.*, **286** (2016), 257–264. https://doi.org/10.1016/j.amc.2016.04.032

12. Z. Zhang, W. Liu, Geometric average asian option pricing for uncertain financial market, *J. Uncertain Syst.*, **8** (2014), 317–320.

13. Y. Gao, X. Yang, Z. Fu, Lookback option pricing problem of uncertain exponential Ornstein-Uhlenbeck model, *Soft Comput.*, **22** (2018), 5647–5654. https://doi.org/10.1007/s00500-017-2558-y

14. K. Yao, Z. Qin, Barrier option pricing formulas of an uncertain stock model, *Fuzzy Optim. Decis. Making*, **20** (2021), 81–100. https://doi.org/10.1007/s10700-020-09333-w

15. R. Gao, K. Liu, Z. Li, R. Lv, American barrier option pricing formulas for stock model in uncertain environment, *IEEE Access*, **7** (2019), 97846–97856. https://doi.org/10.1109/ACCESS.2019.2928029

16. R. Gao, K. Liu, Z. Li, L. Lang, American barrier option pricing formulas for currency model in uncertain environment, *J. Syst. Sci. Complexity*, **35** (2022), 283–312. https://doi.org/10.1007/s11424-021-0039-y

17. X. Yang, Z. Zhang, X. Gao, Asian-barrier option pricing formulas of uncertain financial market, *Chaos Solition Fract.*, **123** (2019), 79–86. https://doi.org/10.1016/j.chaos.2019.03.037

18. Y. Gao, L. Jia, Pricing formulas of barrier-lookback option in uncertain financial markets, *Chaos Solitons Fract.*, **147** (2021), 110896. https://doi.org/10.1016/j.chaos.2021.110986

19. Y. Liu, B. Liu, Residual analysis and parameter estimation of uncertain different equations, *Fuzzy Optim. Decis. Making*, **21** (2022), 513–530. https://doi.org/10.1007/s10700-021-09379-4

20. T. Ye, B. Liu, Uncertain hypothesis test for uncertain differential equations, *Fuzzy Optim. Decis. Making*, **22** (2023), 195–211. https://doi.org/10.1007/s10700-022-09389-w

21. K. Yao, B. Liu, Parameter estimation in uncertain differential equations, *Fuzzy Optim. Decis. Making*, **19** (2020), 1–12. https://doi.org/10.1007/s10700-019-09310-y

22. B. Liu, *Uncertainty theory*, 2 Eds., Springer Berlin, 2007. https://doi.org/10.1007/978-3-540-73165-8

23. K. Yao, X. Chen, A numerical method for solving uncertain differential equations, *J. Intell. Fuzzy Syst.*, **25** (2013), 825–832. https://doi.org/10.3233/IFS-120688

## Appendix

In this section, some basic knowledge in uncertainty theory applied in this paper is introduced.

**Definition A.1.** *(Liu [9]) An uncertain process $C_t$ is said to be a Liu process if*
*(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,*
*(ii) $C_t$ has stationary and independent increments,*
*(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance $t^2$.*

**Theorem A.1.** *(Liu [22]) Let $M$ be an uncertain measure, and $\Gamma_1$ and $\Gamma_2$ be two events. If $\Gamma_1 \subset \Gamma_2$, then we have*

$$M\{\Gamma_1\} \leq M\{\Gamma_2\}.$$

**Theorem A.2.** *(Yao and Chen [23]) Let $\alpha$ $(0 < \alpha < 1)$ be a real number. The UDE*

$$dY_t = g(t, Y_t)dt + h(t, Y_t)dC_t$$

*has a solution $Y_t$, and the corresponding ordinary differential equation*

$$dY_t^\alpha = g(t, Y_t^\alpha)dt + |h(t, Y_t^\alpha)|\psi^{-1}(\alpha)dt$$

*has the solution $\alpha$-path $Y_t^\alpha$, where*

$$\psi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

*Then*

$$M\{Y_t \leq Y_t^\alpha,\ \forall t\} = \alpha, \quad M\{Y_t > Y_t^\alpha,\ \forall t\} = 1 - \alpha.$$

*Thus, $Y_t$ has an IUD*

$$\phi_t^{-1}(\alpha) = Y_t^\alpha.$$