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Research article

Strong convergence to fixed points of an evolution subfamily

Gul Rahmat[1], **Tariq Shah**[1], **Muhammad Sarwar**[2,*], **Saber Mansour**[3] **and Hassen Aydi**[4,5,6,*]

[1] Department of Mathematics, Islamia College University Peshawar, Peshawar, Pakistan; gulrahmat@icp.edu.pk; tariqshahmaths@gmail.com

[2] Department of Mathematics, University of Malakand, Chakdara Dir(L), Khyber Pakhtunkhwa, Pakistan

[3] Department of Mathematics, Umm Al-Qura University, Faculty of Applied sciences, P. O. Box 14035, Holly Makkah 21955, Saudi Arabia; samansour@uqu.edu.sa

[4] Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia

[5] China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

[6] Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa

* **Correspondence:** Email: sarwarswati@gmail.com, hassen.aydi@isima.rnu.tn.

Abstract: In this manuscript, we give strong convergence results for a fixed point of a subfamily of an evolution family on a convex and closed subset $\mathcal{D}$ of a Banach space B. An example is also provided which shows the applications of evolution families and our main results. At the end, an open problem is given.

Keywords: non-expansive mappings; strong convergence; evolution family; Banach space
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1. Introduction

The idea of a fixed point was started in 19th century and different mathematicians, like Schauder, Tarski, Brouwer [1–3] and others worked on it in 20th century. The presence for common fixed points of different families with nonexpansive and contractive mappings in Hilbert spaces as well as in Banach spaces was the exhaustive topic of research since the early 1960s as explored by many researchers like Banach, Brouwer and Browder etc. Latterly, Khamsi and Kozlowski [4, 5] proved results in modular function spaces for common fixed points of nonexpansive, asymptotically nonexpansive and contractive mappings. The theory of a fixed point has a substantial position in the fields of analysis, geometry, engineering, topology, optimization theory, etc. For some latest algorithms developed in the fields of optimisation and inverse problems, we refer to [6, 7]. For more detailed study of fixed point and applications, see [8–22] and the references there in.

The concept of fixed points of one parameter semigroups of linear operators on a Banach space was originated from 19th century from the remarkable work of Hille-Yosida in 1948. Now-a-days, it has a lot of applications in many fields such as stochastic processes and differential equations. Semigroups have a monumental position in the fields of functional analysis, quantum mechanics, control theory, functional equations and integro-differential equations. Semigroups also play a significant role in mathematics and application fields. For example, in the field of dynamical systems, the state space will be defined by the vector function space and the system of an evolution function of the dynamical system will be represented by the map $(\mathfrak{h})\mathfrak{k} \to \mathsf{T}(\mathfrak{h})\mathfrak{k}$. For related study, we refer to [23].

Browder [24] gave a result for the fixed point of nonexpansive mappings in a Banach space. Suzuki [25] proved a result for strong convergence of a fixed point in a Hilbert space. Reich [26] gave a result for a weak convergence in a Banach space. Similarly, Ishikawa [27] presented a result for common fixed points of nonexpansive mappings in a Banach space. Reich and Shoikhet also proved some results about fixed points in non-linear semigroups, see [20]. Nevanlinna and Reich gave a result for strong convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces, see [28, 29]. There are different results on strong convergence of a fixed point of semigroups and there are sets of common fixed points of semigroups by the intersection of two operators from the family. These results are much significant in the field of fixed point theory. In a recent time, different mathematicians are working to generalize such type of results for a subfamily of an evolution family, see [30–32].

The fixed point of a periodic evolution subfamily was discussed in [30] by Rahmat et al. They gave a result for finding common fixed points of the evolution subfamily with the help of a strongly converging sequence. The method applied in [30] is successfully useful for showing the presence of a fixed point of an evolution subfamily. The purpose of this work is to show the existence of a fixed point of an evolution subfamily with the help of a sequence acting on a Banach space.

Definition 1.1. *Let* $\mathfrak{v}\colon \mathsf{A} \to \mathsf{A}$ *be a self-mapping. A point* $\mathsf{r} \in \mathsf{A}$ *is a fixed point of* $\mathfrak{v}$ *if* $\mathfrak{v}(\mathsf{r}) = \mathsf{r}$.

The idea of semigroups is originated from the solution of the Cuachy differnetial equations of the form:

$$\begin{cases} \dot{\Lambda}(\mathfrak{b}) = \mathcal{K}(\Lambda(\mathfrak{b})), \ \mathfrak{b} \geq 0, \\ \Lambda(0) = \Lambda_0, \end{cases}$$

where $\mathcal{K}$ is a linear operator.

Definition 1.2. *A family* $\mathbf{Y} = \{\mathrm{Y}(\mathfrak{a}); \mathfrak{a} \geq 0\}$ *of bounded linear operators is a semigroup if the following conditions hold:*
(i) $\mathrm{Y}(0) = I$.
(ii) $\mathrm{Y}(\mathsf{j} + \mathsf{k}) = \mathrm{Y}(\mathsf{j})\mathrm{Y}(\mathsf{k}), \ \forall \mathsf{j}, \mathsf{k} \geq 0$.

When $\mathcal{K} = \mathcal{K}(\mathrm{t})$, then such a system is called a non-autonomous system. The result of this system produces the idea of an evolution family.

Definition 1.3. *A family* $\mathbf{E} = \{\mathsf{E}(\mathfrak{u}, \mathfrak{g}); \mathfrak{u} \geq \mathfrak{g} \geq 0\}$ *of bounded linear operators is said to be an evolution family if the following conditions hold:*
(i) $\mathsf{E}(\mathfrak{p}, \mathfrak{p}) = \mathtt{I}, \ \forall \mathfrak{p} \geq 0.$
(ii) $\mathsf{E}(\mathfrak{j}, \mathfrak{q})\mathsf{E}(\mathfrak{q}, \mathfrak{b}) = \mathsf{E}(\mathfrak{j}, \mathfrak{b}), \ \forall \mathfrak{j} \geq \mathfrak{q} \geq \mathfrak{b} \geq 0.$

Remark 1.4. *If the evolution family is periodic of each number* $\mathfrak{r} \geq 0$*, then it forms a semigroup. If we take* $\mathsf{E}(\mathfrak{c}, 0) = \mathtt{Y}(\mathfrak{c})$*, then*
$(a1)$ $\mathtt{Y}(0) = \mathsf{E}(0, 0) = I$.
$(a2)$ $\mathtt{Y}(\mathfrak{c} + \mathfrak{y}) = \mathsf{E}(\mathfrak{c} + \mathfrak{y}, 0) = \mathsf{E}(\mathfrak{c} + \mathfrak{y}, \mathfrak{y})\mathsf{E}(\mathfrak{y}, 0) = \mathsf{E}(\mathfrak{c}, 0)\mathsf{E}(\mathfrak{y}, 0) = \mathtt{Y}(\mathfrak{c})\mathtt{Y}(\mathfrak{y})$, *which shows that a periodic evolution family of each positive period, is a semigroup.*

Similarly, if we take $\mathtt{Y}(\mathfrak{r} - \mathfrak{d}) = \mathsf{E}(\mathfrak{r}, \mathfrak{d})$*, then*
$(b1)$ $\mathsf{E}(\mathfrak{r}, \mathfrak{r}) = \mathtt{Y}(0) = I$.
$(b2)$ $\mathsf{E}(\mathfrak{r}, \mathfrak{b}) = \mathsf{E}(\mathfrak{r}, \mathfrak{d})\mathsf{E}(\mathfrak{d}, \mathfrak{b}) = \mathtt{Y}(\mathfrak{r} - \mathfrak{d})\mathtt{Y}(\mathfrak{d} - \mathfrak{b}) = \mathtt{Y}(\mathfrak{r} - \mathfrak{b})$, *which shows that a semigroup is an evolution family.*

Remark 1.5. *A semigroup is an evolution family, but the converse is not true. In fact, the converse holds if the evolution family is periodic of every number* $\mathfrak{s} \geq 0$.

Remark 1.6. *[33] Let* $0 \leq \mathsf{s} \leq 1$ *and* $\mathsf{b}, \mathsf{k} \in \mathcal{H}$*, then the following equality holds:*

$$||\mathsf{sb} + (1 - \mathsf{s})\mathsf{k}||^2 = \mathsf{s}||\mathsf{b}||^2 + (1 - \mathsf{s})||\mathsf{k}||^2 - \mathsf{s}(1 - \mathsf{s})||\mathsf{b} - \mathsf{k}||^2.$$

In this work, we will generalize results from [33] for an evolution subfamily and also give some other results for an evolution subfamily.

2. Preliminaries

First, denote the set of real numbers and natural numbers by $\mathbb{R}$ and $\mathbb{N}$, respectively. We denote the family of semigroups by $\mathbf{Y}$, evolution family by $\mathbf{E}$ and evolution subfamily by $\mathbf{G}$. By B, $\mathcal{H}$ and $\mathcal{D}$, we will indicate a Banach space, Hilbert space and a convex closed set, respectively. We use $\rightarrow$ for a strong and $\rightharpoonup$ for a weak convergence. The set of fixed points of

$$\mathbf{G} = \{\mathsf{G}(\mathfrak{s}, 0); \mathfrak{s} \geq 0\}$$

is denoted by

$$F(\mathbf{G}) = \cap_{\mathfrak{s} \geq 0} F(\mathsf{G}(\mathfrak{s}, 0)).$$

We generalize results of semigroups from [33] for an evolution subfamily $\mathbf{G}$ in a Banach space. These types of families are not semigroups. The following example illustrates this fact and gives the difference between them.

Example 2.1. *As*

$$\mathbf{E} = \{\mathsf{E}(\mathfrak{h}, \mathfrak{r}) = \frac{\mathfrak{h} + 1}{\mathfrak{r} + 1}; \mathfrak{h} \geq \mathfrak{r} \geq 0\}$$

is an evolution family because it satisfies both conditions of an evolution family.

If we take $\mathfrak{r} = 0$*, that is,* $\{\mathsf{E}(\mathfrak{h}, 0)\} = \mathbf{G}$*, then it becomes a subfamily of* $\mathbf{E}$ *and it is not a semigroup.*

Suzuki proved the following result in [25]:

Theorem 2.2. *Consider a family*

$$\mathbf{Y} = \{\mathrm{Y}(\mathfrak{i}), \mathfrak{i} \geq 0\}$$

of strongly continuous non-expansive operators on $\mathcal{D}$ *(where* $\mathcal{D}$ *is a subset of a Hilbert space* $\mathcal{H}$*) such that* $F(\mathbf{Y}) \neq \emptyset$*. Take two sequences* $\{\gamma_m\}$ *and* $\{\mathfrak{q}_m\}$ *in* $\mathbb{R}$ *with*

$$\lim_{m\to\infty} \mathfrak{q}_m = \lim_{m\to\infty} \frac{\gamma_m}{\mathfrak{q}_m} = 0,$$

$\mathfrak{q}_m > 0$ *and* $\gamma_m \in (0,1)$*. Let* $\mathfrak{b} \in \mathcal{D}$ *be fixed and* $\{\mathfrak{k}_m\}$ *be a sequence in* $\mathcal{D}$ *such that*

$$\mathfrak{k}_m = \gamma_m \mathfrak{b} + (1-\gamma_m)\mathrm{Y}(\mathfrak{q}_m)\mathfrak{k}_m, \quad \forall m \in \mathbb{N},$$

then $\{\mathfrak{k}_m\} \to \mathfrak{h} \in F(\mathbf{Y})$.

The following result was given by Shimizu and Takahashi [15] in 1998:

Theorem 2.3. *Take a family*

$$\mathbf{Y} = \{\mathrm{Y}(\mathfrak{i}), \mathfrak{i} \geq 0\}$$

of operators which are non-expansive and strongly continuous on $\mathcal{D} \subset \mathcal{H}$ *such that* $F(\mathbf{Y}) \neq \emptyset$*. Take two sequences* $\{\zeta_m\}$ *and* $\{\lambda_m\}$ *in* $\mathbb{R}$ *with*

$$\lim_{m\to\infty} \zeta_m = 0, \quad \lim_{m\to\infty} \lambda_m = \infty,$$

where $\zeta_m \in (0,1)$ *and* $\lambda_m > 0$*. Let* $\mathfrak{c} \in \mathcal{D}$ *be fixed and* $\{\mathfrak{g}_m\} \in \mathcal{D}$ *be a sequence such that*

$$\mathfrak{g}_m = \zeta_m \mathfrak{c} + (1-\zeta_m)\frac{1}{\lambda_m}\int_0^{\lambda_m} \mathrm{Y}(\mathfrak{s})d\mathfrak{s}$$

for all $m \in \mathbb{N}$*. Then* $\{\mathfrak{g}_m\} \to \mathfrak{a} \in F(\mathbf{Y})$.

Motivated from above results, we take an implicit iteration for $\mathbf{G} = \{\mathsf{G}(\mathfrak{b},0), \mathfrak{b} \geq 0\}$ of nonexpansive mappings, given as:

$$\begin{cases} \tau_m = \gamma_m \tau_{m-1} + (1-\gamma_m)\mathsf{G}(\zeta_m, 0)\tau_m, \ m \geq 1, \\ \tau_0 \in \mathcal{D}. \end{cases} \tag{2.1}$$

We present some results for convergence of Eq (2.1) in a Banach space and a Hilbert space for a nonexpansive evolution subfamily.

3. Main results

We start with the following lemmas:

Lemma 3.1. *Consider an evolution family* $\mathbf{E}$ *and a subfamily* $\mathbf{G} = \{\mathsf{G}(\mathfrak{c},); \mathfrak{c} \geq 0\}$ *of* $\mathbf{E}$ *with period* $\mathbf{r} \in \mathbb{R}^+$*, then*

$$\bigcap_{\mathfrak{c}\geq 0} F(\mathsf{G}(\mathfrak{c},0)) = \bigcap_{0\leq \mathfrak{c}\leq \mathbf{r}} F(\mathsf{G}(\mathfrak{c},0)).$$

Proof. As it is obviously true that

$$\bigcap_{\mathfrak{c}\geq 0} F(\mathsf{G}(\mathfrak{c},0)) \subset \bigcap_{0\leq \mathfrak{c}\leq \mathbf{r}} F(\mathsf{G}(\mathfrak{c},0)),$$

we are proving the other part, i.e.,

$$\bigcap_{0\leq \mathfrak{c}\leq \mathbf{r}} F(\mathsf{G}(\mathfrak{c},0)) \subset \bigcap_{\mathfrak{c}\geq 0} F(\mathsf{G}(\mathfrak{c},0)).$$

Take a real number

$$\mathfrak{k} \in \bigcap_{0\leq \mathfrak{c}\leq \mathbf{r}} F(\mathsf{G}(\mathfrak{c},0)),$$

then

$$\mathsf{G}(\mathfrak{c},0)\mathfrak{k} = \mathfrak{k},\ \forall 0 \leq \mathfrak{c} \leq \mathbf{r}.$$

As we know that any real number $\mathfrak{c} \geq 0$ is written in the form of $\mathfrak{c} = \mathsf{m}\mathbf{r} + \varepsilon$, for some $\mathsf{m} \in \mathbb{Z}^+$ and $0 \leq \varepsilon \leq \mathbf{r}$, consider

$$\begin{aligned}\mathsf{G}(\mathfrak{c},0)\mathfrak{k} &= \mathsf{G}(\mathsf{m}\mathbf{r}+\varepsilon,0)\mathfrak{k}\\ &= \mathsf{G}(\mathsf{m}\mathbf{r}+\varepsilon,\mathsf{m}\mathbf{r})\mathsf{G}(\mathsf{m}\mathbf{r},0)\mathfrak{k}\\ &= \mathsf{G}(\varepsilon,0)\mathsf{G}^{\mathsf{m}}(\mathbf{r},0)\mathfrak{k}\\ &= \mathsf{G}(\varepsilon,0)\mathfrak{k}\\ &= \mathfrak{k}.\end{aligned}$$

This shows that

$$\bigcap_{0\leq \mathfrak{c}\leq \mathbf{r}} F(\mathsf{G}(\mathfrak{c},0)) \subset \bigcap_{\mathfrak{c}\geq 0} F(\mathsf{G}(\mathfrak{c},0)).$$

Hence, we conclude that

$$\bigcap_{\mathfrak{c}\geq 0} F(\mathsf{G}(\mathfrak{c},0)) = \bigcap_{0\leq \mathfrak{c}\leq \mathbf{r}} F(\mathsf{G}(\mathfrak{c},0)).$$

This completes the proof. □

Lemma 3.2. *If* $\mathbf{Y} = \{\Upsilon(\beta);\beta \geq 0\}$ *is a semigroup on a Hilbert space* $\mathcal{H}$, *then*

$$F(\mathbf{Y}) = \bigcap_{\beta\geq 0} F(\Upsilon(\beta)) = \bigcap_{0\leq\beta\leq 1} F(\Upsilon(\beta)).$$

Proof. Since

$$\bigcap_{\beta\geq 0} F(\Upsilon(\beta)) \subset \bigcap_{0\leq\beta\leq 1} F(\Upsilon(\beta)),$$

we only prove the other part, that is,

$$\bigcap_{0\leq\beta\leq 1} F(\Upsilon(\beta)) \subset \bigcap_{\beta\geq 0} F(\Upsilon(\beta)).$$

Take a real number u such that

$$\mathsf{u} \in \bigcap_{0\leq\beta\leq 1} F(\Upsilon(\beta)),$$

then we have $\Upsilon(\mathsf{u}) = \mathsf{u}$, for every $\beta \in [0, 1]$. Since $\beta \in [0, 1]$, we can write it as $\beta = \mathsf{n} + \varrho$, where $\mathsf{n} \in \mathbb{Z}^+$ and $0 \leq \varrho \leq 1$. Therefore, we have

$$\begin{aligned}\Upsilon(\beta)\mathsf{u} &= \Upsilon(\mathsf{n} + \varrho)\mathsf{u}\\ &= \Upsilon(\mathsf{n})\Upsilon(\varrho)\mathsf{u}\\ &= \Upsilon^{\mathsf{n}}(1)\Upsilon(\varrho)\mathsf{u}\\ &= \Upsilon^{\mathsf{n}}(1)\mathsf{u}\\ &= \mathsf{u}.\end{aligned}$$

This shows that $\mathsf{u} \in \bigcap_{\beta\geq 0} F(\Upsilon(\beta))$. It implies that

$$\bigcap_{0\leq\beta\leq 1} F(\Upsilon(\beta)) \subset \bigcap_{\beta\geq 0} F(\Upsilon(\beta)).$$

Thus,

$$\bigcap_{\beta\geq 0} F(\Upsilon(\beta)) = \bigcap_{0\leq\beta\leq 1} F(\Upsilon(\beta)).$$

This completes the proof. □

Now, we give a result for a weak convergence of a sequence in a Hilbert space.

Theorem 3.3. *Let* $\mathbf{G} = \{\mathsf{G}(a, 0)\}$ *be a subfamily of* $\mathbf{E}$ *of strongly continuous nonexpansive operators on* $\mathcal{D}$ *and* $\mathsf{F}(\mathbf{G}) \neq \emptyset$, *where* $\mathcal{D}$ *is a subset of* $\mathcal{H}$. *Take two sequences* $\{\gamma_m\}$ *and* $\{\zeta_m\}$ *in* $\mathbb{R}$ *such that*

$$\{\gamma_m\} \subset (0, \mathsf{c}] \subset (0, 1), \quad \zeta_m > 0,$$

$$\liminf_{m\to\infty} \zeta_m = 0, \quad \limsup_{m\to\infty} \zeta_m > 0,$$

and

$$\lim_{m\to\infty} (\zeta_{m+1} - \zeta_m) = 0.$$

Then

$$\tau_m = \gamma_m \tau_{m-1} + (1 - \gamma_m)\mathsf{G}(\zeta_m, 0)\tau_m \rightharpoonup \tau,$$

where $\tau \in \mathsf{F}(\mathbf{G})$.

Proof. Claim (i). For any $\mathsf{z} \in \mathsf{F}(\mathbf{G})$, $\lim_{m\to\infty} \|\tau_m - \mathsf{z}\|$ exists. In fact,

$$\begin{aligned}\|\tau_m - \mathsf{z}\| &= \|\gamma_m(\tau_{m-1} - \mathsf{z}) + (1 - \gamma_m)(\mathsf{G}(\zeta_m, 0)\tau_m - \mathsf{z})\|\\ &\leq \gamma_m\|\tau_{m-1} - \mathsf{z}\| + (1 - \gamma_m)\|\mathsf{G}(\zeta_m, 0)\tau_m - \mathsf{z}\|\\ &\leq \gamma_m\|\tau_{m-1} - \mathsf{z}\| + (1 - \gamma_m)\|\tau_{m-1} - \mathsf{z}\|, \quad \forall m \geq 1.\end{aligned}$$

Thus, we have

$$\|\tau_m - \mathsf{z}\| \leq \|\tau_{m-1} - \mathsf{z}\|, \quad \forall m \geq 1.$$

This shows that $\lim_{m\to\infty} \|\tau_m - \mathsf{z}\|$ exists. Therefore, the sequence $\{\tau_m\}$ is bounded.
Claim (ii).

$$\lim_{m\to\infty} \|\mathsf{G}(\zeta_m, 0)\tau_m - \tau_m\| = 0.$$

From Remark 1.6, we have

$$\begin{aligned}\|\tau_m - \mathsf{z}\|^2 &= \|\gamma_m(\tau_{m-1} - \mathsf{z}) + (1-\gamma_m)(\mathsf{G}(\zeta_m, 0)\tau_m - \mathsf{z})\|^2\\ &= \gamma_m\|\tau_{m-1} - \mathsf{z}\|^2 + (1-\gamma_m)\|\mathsf{G}(\zeta_m, 0)\tau_m - \mathsf{z}\|^2\\ &- \gamma_m(1-\gamma_m)\|\tau_{m-1} - \mathsf{G}(\zeta_m, 0)\tau_m\|^2\\ &\leq \gamma_m\|\tau_{m-1} - \mathsf{z}\|^2 + (1-\gamma_m)\|\tau_m - \mathsf{z}\|^2\\ &- \gamma_m(1-\gamma_m)\|\tau_{m-1} - \mathsf{G}(\zeta_m, 0)\tau_m\|^2.\end{aligned}$$

Thus, we have

$$\|\tau_m - \mathsf{z}\|^2 \leq \|\tau_{m-1} - \mathsf{z}\|^2 - (1-\gamma_m)\|\tau_{m-1} - \mathsf{G}(\zeta_m, 0)\tau_m\|^2, \quad \forall m \geq 1.$$

As we know that $\{\gamma_m\} \subset (0, \mathsf{c}] \subset (0, 1)$, so we have

$$(1-\mathsf{c})\|\tau_{m-1} - \mathsf{G}(\zeta_m, 0)\tau_m\|^2 \leq \|\tau_{m-1} - \mathsf{z}\|^2 - \|\tau_m - \mathsf{z}\|^2, \tag{3.1}$$

i.e.,

$$(1-\mathsf{c})\limsup_{m\to\infty}\|\tau_{m-1} - \mathsf{G}(\zeta_m, 0)\tau_m\|^2 \leq \limsup_{m\to\infty}\|\tau_{m-1} - \mathsf{z}\|^2 - \|\tau_m - \mathsf{z}\|^2 = 0.$$

Therefore,

$$\lim_{m\to\infty}\|\tau_{m-1} - \mathsf{G}(\zeta_m, 0)\tau_m\| = 0.$$

On the other hand,

$$\lim_{m\to\infty}\|\tau_m - \mathsf{G}(\zeta_m, 0)\tau_m\| = \lim_{m\to\infty}\gamma_{m-1}\|\tau_{m-1} - \mathsf{G}(\zeta_m, 0)\tau_m\| = 0.$$

Claim (iii).

$$\{\tau_m\} \rightharpoonup \tau, \quad \text{where } \tau \in \mathsf{F}(\mathbf{G}).$$

As $\{\tau_m\}$ is bounded, take a subsequence $\{\omega_{m_i}\}$ of $\{\tau_m\}$ such that $\{\omega_{m_i}\} \rightharpoonup \tau$. Let $\omega_{m_i} = \mathsf{h_i}$, $\gamma_{m_i} = \xi_i$ and $\zeta_{m_i} = \mathsf{v_i}$. From [34], we have

$$\lim_{i\to\infty}\mathsf{v}_i = \lim_{i\to\infty}\frac{\|\mathsf{h_i} - \mathsf{G}(\mathsf{v}_i, 0)\mathsf{h_i}\|}{\mathsf{v}_i} = 0.$$

Now, we will show that $\mathsf{G}(\zeta, 0)\tau = \tau$.

We have

$$\begin{aligned}\|\mathsf{h_i} - \mathsf{G}(\zeta, 0)\tau\| &\leq \sum_{\mathsf{a}=0}^{[\frac{\zeta}{\mathsf{v_i}}]-1}\|G((\mathsf{a}+1)\mathsf{v_i}, 0)\mathsf{h_i} - G(\mathsf{a}\mathsf{v_i}, 0)\mathsf{h_i}\| + \|G([\frac{\zeta}{\mathsf{v_i}}]\mathsf{v_i}, 0)\mathsf{h_i}\\ &- G(\frac{\zeta}{\mathsf{v_i}}\mathsf{v_i}, 0)\tau\| + \|G(\frac{\zeta}{\mathsf{v_i}}\mathsf{v_i}, 0)\tau - G(\zeta, 0)\|\\ &\leq \frac{\zeta}{\mathsf{v_i}}\|G(\mathsf{v_i}, 0)\mathsf{h_i} - \mathsf{h_i}\| + \|\mathsf{h_i} - \tau\| + \|G(\zeta - [\frac{\zeta}{\mathsf{v_i}}]\mathsf{v_i}, 0)\tau - \tau\|\\ &\leq \zeta\frac{\|G(\mathsf{v_i}, 0)\mathsf{h_i} - \mathsf{h_i}\|}{\mathsf{v_i}} + \|\mathsf{h_i} - \tau\| + \max_{0\leq \mathsf{v}\leq \mathsf{v_i}}\|G(\mathsf{v}, 0)\tau - \tau\|, \quad \forall i \in \mathbb{N}.\end{aligned}$$

Thus, we get

$$\limsup_{i\to\infty} \|\mathsf{h_i} - \mathsf{G}(\zeta, 0)\tau\| \leq \limsup_{i\to\infty} \|\tau_i - \tau\|.$$

Hence, $\mathsf{G}(\zeta, 0)\tau = \tau$ by using Opial's condition. Therefore, $\tau \in F(\mathbf{G})$. Now, we need to show that $\{\tau_m\} \rightharpoonup \tau$. For this, take a subsequence $\{\eta_{m_j}\}$ of $\{\tau_m\}$ such that $\eta_{m_j} \rightharpoonup \mathsf{u}$ and $\mathsf{u} \neq \tau$. By above method, we can show that $\mathsf{u} \in F(\mathbf{G})$. Since both limits $\lim_{m\to\infty} \|\tau_m - \tau\|$ and $\lim_{m\to\infty} \|\tau_m - \mathsf{u}\|$ exist, we can write

$$\begin{aligned}
\lim_{m\to\infty} \|\tau_m - \tau\| &= \limsup_{i\to\infty} \|\omega_{m_i} - \tau\| < \limsup_{i\to\infty} \|\omega_{m_i} - \mathsf{u}\| \\
&= \lim_{m\to\infty} \|\tau_m - \mathsf{u}\| \\
&= \limsup_{j\to\infty} \|\eta_{m_j} - \mathsf{u}\| < \limsup_{j\to\infty} \|\eta_{m_j} - \tau\| \\
&= \lim_{m\to\infty} \|\tau_m - \tau\|.
\end{aligned}$$

It shows that $\mathsf{u} = \tau$, which is a contradiction. Thus, $\tau_m \rightharpoonup \tau$.

This completes the proof. □

Now, we will provide a theorem in a Banach space for a weak convergence.

Theorem 3.4. *Consider a reflexive Banach space* B *in* $\mathbb{R}$ *with Opial's property and a subset* $\mathcal{D}$ *of* B. *Let* $\mathbf{G} = \{G(\mathsf{a}, 0)\}$ *be a subfamily of* $\mathbf{E}$ *of nonexpansive and strongly continuous mappings such that* $F(\mathbf{G}) \neq \emptyset$. *Take two sequences* $\{\gamma_m\}$ *and* $\{\zeta_m\}$ *such that* $\gamma_m \subset (0, 1)$, $\zeta_m > 0$ *and*

$$\lim_{m\to\infty} \zeta_m = \lim_{m\to\infty} \frac{\gamma_m}{\zeta_m} = 0.$$

Then

$$\tau_m = \gamma_m \tau_{m-1} + (1 - \gamma_m)\mathsf{G}(\zeta_m, 0)\tau_m \to \tau \in F(\mathbf{G}).$$

Proof. Claim 1. As

$$\lim_{m\to\infty} \zeta_m = \lim_{m\to\infty} \frac{\gamma_m}{\zeta_m} = 0,$$

then we have $\lim_{m\to\infty} \gamma_m = 0$. This shows that there exists a positive integer p, for all $k \in \mathbb{N}$ so that $\gamma_m \subset (0, \mathsf{c}] \subset (0, 1)$.

From Theorem 3.3, $\lim_{m\to\infty} \|\tau_m - \mathsf{z}\|$ exists for each $\mathsf{z} \in F(\mathbf{G})$.

Claim 2. $\{\mathsf{G}(\zeta_m, 0)\tau_m\}$ is bounded. From (2.1), we have

$$\begin{aligned}
\|\mathsf{G}(\zeta_m, 0)\tau_m\}\| &= \|\frac{1}{1-\gamma_m}\tau_m - \frac{\gamma_m}{1-\gamma_m}\tau_{m-1}\| \\
&\leq \frac{1}{1-\gamma_m}\|\tau_m\| + \frac{\gamma_m}{1-\gamma_m}\|\tau_{m-1}\| \\
&\leq \frac{1}{\mathsf{c}}\|\tau_m\| + \frac{\mathsf{c}}{1-\mathsf{c}}\|\tau_{m-1}\|,
\end{aligned}$$

which shows that $\{\mathsf{G}(\zeta_m, 0)\tau_m\}$ is bounded.

Claim 3. $\{\tau_m\} \rightharpoonup \tau$.

As $\{\tau m\}$ is bounded, take a sub-sequence $\{\omega_{m_l}\}$ of $\{\tau_m\}$ such that $\omega_{m_l} \rightharpoonup \tau$. Let $\omega_{m_l} = \mathsf{b}_\mathsf{l}$, $\gamma_{m_l} = \rho_l$ and $\zeta_{m_l} = \mathsf{y}_\mathsf{l}$, $l \in \mathbb{N}$. Let $\zeta > 0$ be fixed, then

$$\begin{aligned}\|\mathsf{b}_\mathsf{l} - \mathsf{G}(\zeta, 0)\tau\| \leq & \sum_{\mathsf{a}=0}^{[\frac{\zeta}{\mathsf{y}_\mathsf{l}}]-1} \|G((\mathsf{a}+1)\mathsf{y}_\mathsf{l}, 0)\mathsf{b}_\mathsf{l} - G(\mathsf{a}\mathsf{y}_\mathsf{l}, 0)\mathsf{b}_\mathsf{l}\| + \|G([\frac{\zeta}{\mathsf{y}_\mathsf{l}}]\mathsf{y}_\mathsf{l}, 0)\mathsf{b}_\mathsf{l} \\ & - G(\frac{\zeta}{\mathsf{y}_\mathsf{l}}\mathsf{y}_\mathsf{l}, 0)\tau\| + \|G(\frac{\zeta}{\mathsf{y}_\mathsf{l}}\mathsf{y}_\mathsf{l}, 0)\tau - G(\zeta, 0)\| \\ & \leq \frac{\zeta}{\mathsf{y}_\mathsf{l}}\|G(\mathsf{y}_\mathsf{l}, 0)\mathsf{b}_\mathsf{l} - \mathsf{b}_\mathsf{l}\| + \|\mathsf{b}_\mathsf{l} - \tau\| + \|G(\zeta - [\frac{\zeta}{\mathsf{y}_\mathsf{l}}]\mathsf{y}_\mathsf{l}, 0)\tau - \tau\| \\ & \leq \zeta\frac{\|G(\mathsf{y}_\mathsf{l}, 0)\mathsf{b}_\mathsf{l} - \mathsf{b}_\mathsf{l}\|}{\mathsf{y}_\mathsf{l}} + \|\mathsf{b}_\mathsf{l} - \tau\| + \max_{0 \leq \mathsf{y} \leq \mathsf{y}_\mathsf{l}} \|G(\mathsf{y}, 0)\tau - \tau\|, \quad \forall l \in \mathbb{N}.\end{aligned}$$

Thus, we have

$$\limsup_{l\to\infty} \|\mathsf{b}_\mathsf{l} - \mathsf{G}(\zeta, 0)\tau\| \leq \limsup_{l\to\infty} \|\mathsf{b}_\mathsf{l} - \tau\|.$$

Therefore,

$$G(\zeta, 0)\tau = \tau \in F(\mathbf{G})$$

by using Opial's property. By same method given in Theorem 3.3, we can prove that $\{\tau_m\} \rightharpoonup \tau$.

This completes the proof. □

Theorem 3.5. *Consider a real reflexive Banach space* B *with Opial's property and a subset* $\mathcal{D}$ *of* B. *Let* $\mathbf{G} = \{G(\mathsf{a}, 0)\}$ *be a subfamily of* $\mathbf{E}$ *of strongly continuous nonexpansive mappings such that* $F(\mathbf{G}) \neq \emptyset$. *Take two sequences* $\{\gamma_m\}$ *and* $\{\zeta_m\}$ *such that* $\gamma_m \subset (0, 1)$, $\zeta_m > 0$ *and*

$$\lim_{m\to\infty} \zeta_m = \lim_{m\to\infty} \frac{\gamma_m}{\zeta_m} = 0.$$

Then

$$\tau_m = \gamma_m\tau_{m-1} + (1 - \gamma_m)\mathsf{G}(\zeta_m, 0)\tau_m \to \tau \in F(\mathbf{G}).$$

Proof. Claim 1. For any $\mathsf{z} \in F(\mathbf{G})$, $\lim_{m\to\infty} \|\tau_m - \mathsf{z}\|$ exists.
Claim 2.

$$\|\mathsf{G}(\zeta_m, 0)\tau_m - \tau_m\| \to 0 \text{ as } m \to \infty. \tag{3.2}$$

As from Theorem 3.4, $\{\mathsf{G}(\zeta_m, 0)\tau_m\}$ is bounded. Also, from (1.4), we have

$$\|\tau_m - \mathsf{G}(\zeta_m, 0)\tau_m\| = \gamma_m\|\tau_{m-1} - \mathsf{G}(\zeta_m, 0)\tau_m\| \to 0 \text{ as } m \to \infty.$$

Therefore,

$$\|\mathsf{G}(\zeta_m, 0)\tau_m - \tau_m\| \to 0 \text{ as } m \to \infty.$$

Claim 3. For any $\zeta > 0$,

$$\lim_{m\to\infty} \|\mathsf{G}(\zeta_m, 0)\tau_m - \tau_m\| = 0.$$

In fact, we have

$$\|\tau_m - \mathsf{G}(\zeta_m, 0)\tau_m\| \leq \sum_{\mathsf{b}=0}^{\frac{\zeta}{\zeta_m}-1} \|\mathsf{G}((\mathsf{b}+1)\zeta_m, 0)\tau_m - \mathsf{G}(\mathsf{b}\zeta_m, 0)\tau_m\|$$
$$+ \|\mathsf{G}((\frac{\zeta}{\zeta_m})\zeta_m, 0)\tau_m - \mathsf{G}(\zeta, 0)\tau_m\|$$
$$\leq \frac{\zeta}{\zeta_m}\|\mathsf{G}(\zeta_m, 0)\tau_m - \tau_m\| + \|\mathsf{G}((\zeta - \frac{\zeta}{\zeta_m})\zeta_m, 0)\tau_m - \tau_m\|$$
$$\leq \zeta\frac{\gamma_m}{\zeta_m}\|\tau_{m-1} - \mathsf{G}(\zeta_m, 0)\| + \max_{s\in[0,\zeta_m]}\{\|\mathsf{G}(\mathfrak{v}, 0)\tau_m - \tau_m\|\},\ \forall\ m \in \mathbb{N}.$$

Thus, from this equation and Eq (3.2), we get

$$\lim_{m\to\infty} \|\mathsf{G}(\zeta_m, 0)\tau_m - \tau_m\| = 0.$$

Claim 4. Now, we will show that $\{\tau_m\} \to \tau \in F(\mathbf{G})$.

Since $\{\tau_m\}$ is bounded, it must have a convergent sub-sequence $\{\mu_{m_k}\}$ such that $\mu_{m_k} \to \tau$. From Claim 3, we have

$$\|\tau - \mathsf{G}(\zeta, 0)\tau\| = \lim_{k\to\infty} \|\mu_{m_k} - G(\zeta, 0)\mu_{m_k}\| = 0.$$

Thus, $\tau \in F(\mathbf{G})$. Hence, we have

$$\lim_{m\to\infty} \|\tau_m - \tau\| = \lim_{k\to\infty} \|\mu_{m_k} - \tau\| = 0.$$

This completes the proof. □

4. An example and open problem

Example 4.1. *Consider the Hilbert space* $\mathcal{H} = \mathsf{L}^2([0,\pi], \mathbb{C})$ *and let* $\mathbf{T} = \{\mathsf{T}(\mathsf{a}); \mathsf{a} \geq 0\}$ *be a semigroup such that*

$$(\mathsf{T}(\mathsf{a})\mathfrak{u})(\mathtt{t}) = \frac{2}{\pi}\sum_{\mathsf{m}=1}^{\infty} \mathsf{e}^{-\mathsf{am}^2}\mathsf{w}_\mathsf{m}(\mathfrak{u})\,sin\mathsf{m}\mathtt{t}, \quad \mathtt{t} \in [0,\pi],\ \mathsf{a} \geq 0.$$

Here,

$$\mathsf{w}_\mathsf{m}(\mathfrak{u}) = \int_0^\pi \mathfrak{k}(\mathsf{a})\,sin(\mathsf{ma})\mathsf{ds}.$$

Surely, it is nonexpansive and strongly continuous semigroup in this Hilbert space. The linear operator Λ *generates this semigroup such that* $\Lambda\mathfrak{u} = \ddot{\mathfrak{u}}$. *Let for all* $\mathfrak{k} \in \mathcal{H}$*, the set* $\mathcal{M}(\Lambda)$ *represent the maximal domain of* Λ *such that* $\mathfrak{u}$ *and* $\ddot{\mathfrak{u}}$ *must be continuous. Also,* $\mathfrak{u}(0) = 0 = \mathfrak{u}(\pi)$. *Now, consider the non-autonomous Cauchy problem:*

$$\begin{cases} \frac{\partial \mathsf{h}(\mathtt{t},\varepsilon)}{\partial \mathtt{t}} = & \mathfrak{g}(\mathtt{t})\frac{\partial^2\mathsf{h}(\mathtt{t},\varepsilon)}{\partial^2\varepsilon},\ \ \mathtt{t} > 0,\ 0 \leq \varepsilon \leq \pi, \\ \mathsf{h}(0,\varepsilon) = & \mathfrak{b}(\varepsilon), \\ \mathsf{h}(\mathtt{t},0) = & \mathsf{h}(\mathtt{t},\pi) = 0,\ \ \mathtt{t} \geq 0, \end{cases}$$

where $\mathfrak{b}(.) \in \mathcal{H}$ *and* $\mathfrak{g}$*:* $\mathbb{R}^+ \to [1, \infty)$ *are nonexpansive functions on* $\mathbb{R}^+$*. This function* $\mathfrak{g}$ *is periodic, i.e.,* $\mathfrak{g}(\mathsf{j} + \mathsf{p}) = \mathfrak{g}(\mathsf{j})$ *for every* $\mathsf{j} \in \mathbb{R}^+$ *and for some* $\mathsf{p} \geq 1$*. Take the function*

$$\mathsf{K}(\mathtt{t}) = \int_0^{\mathtt{t}} \mathfrak{g}(\mathtt{t})d\mathtt{t},$$

then the property of evolution equations will be satisfied by the solution $\mathfrak{k}(.)$ *of the above non-autonomous Cauchy problem. Therefore,*

$$\mathfrak{k}(\mathtt{t}) = \mathsf{A}(\mathtt{t}, \mathfrak{h})\mathfrak{k}(\mathfrak{h}),$$

where

$$\mathsf{A}(\mathtt{t}, \mathfrak{h}) = \mathsf{T}(\mathsf{K}(\mathtt{t}) - \mathsf{K}(\mathfrak{h})),$$

see Example 2.9b [35].

As the function $\mathtt{t} \to \mathsf{e}^{\mathfrak{r}\mathtt{t}}\|\mathsf{h}(\mathtt{t})\|$ *is bounded for any* $\mathfrak{r} \geq 0$ *on the set of non-negative real numbers, we have*

$$\begin{aligned}
\int_0^\infty \|\mathsf{A}(\mathtt{t}, 0)\mathfrak{u}\|^2 d\mathtt{t} &= \frac{2}{\pi}\int_0^\infty \sum_{\mathfrak{r}=g}^{\infty} \mathsf{w_r}^2(\mathfrak{u})\mathsf{e}^{-2\mathfrak{r}^2\mathsf{K}(\mathtt{t})}d\mathtt{t} \\
&= \frac{2}{\pi}\sum_{\mathfrak{r}=g}^{\infty} \mathsf{w_r}^2(\mathfrak{u})\int_0^\infty \mathsf{e}^{-2\mathfrak{r}^2\mathsf{K}(\mathtt{t})}d\mathtt{t} \\
&= \|\mathfrak{u}\|_2^2 \int_0^\infty \mathsf{e}^{-2\mathfrak{r}^2\mathsf{K}(\mathtt{t})}d\mathtt{t} \\
&\leq \|\mathfrak{u}\|_2^2 \int_0^\infty \mathsf{e}^{-2\mathsf{K}(\mathtt{t})}d\mathtt{t}.
\end{aligned}$$

On the other side, we have

$$\begin{aligned}
\int_0^\infty \mathsf{e}^{-2\mathsf{K}(\mathtt{t})}d\mathtt{t} &= \sum_{\mathsf{b}=0}^{\infty}\int_{\mathsf{bc}}^{(\mathsf{b}+1)\mathsf{c}} \mathsf{e}^{-2\mathsf{K}(\mathtt{t})}d\mathtt{t} \\
&= \sum_{\mathsf{b}=0}^{\infty}\int_0^{\mathsf{c}} \mathsf{e}^{-2\mathsf{K}(\mathsf{bc}+\beta)}d\beta \\
&= \sum_{\mathsf{b}=0}^{\infty} \mathsf{e}^{-2\mathsf{bK}(\mathsf{c})}\int_0^{\mathsf{c}} \mathsf{e}^{-2\mathsf{K}(\beta)}d\beta \\
&\leq \mathsf{c}\sum_{\mathsf{b}=0}^{\infty} \mathsf{e}^{-2\mathsf{bK}(\mathsf{c})} \\
&= \mathsf{c}\frac{\mathsf{e}^{2\mathsf{K}(\mathsf{c})}}{\mathsf{e}^{2\mathsf{K}(\mathsf{c})} - 1} \\
&= \mathsf{W}.
\end{aligned}$$

Therefore, we have

$$\int_0^\infty \|\mathsf{A}(\mathtt{t}, 0)\mathfrak{u}\|^2 d\mathtt{t} \leq \mathsf{W}\|\mathfrak{u}\|_2^2.$$

By using Theorem 3.2 from [36], we have $\mathfrak{a}(\mathsf{A}) \leq \frac{-1}{2\mathsf{C}}$*, where* $\mathfrak{a}(\mathsf{A})$ *is the growth bound of the family* A *and* $\mathsf{C} \geq 1$*. For more details, see [36].*

This shows that the evolution family on the Hilbert space $\mathcal{H}$ *is nonexpansive, so Theorem 3.5 can be applied to such evolution families and will be helpful in finding its solution and uniqueness.*

Example 4.2. *Let*

$$E(t, s) = \frac{t+1}{s+1}$$

be an evolution family on the space l_3*, then clearly the space* l_3 *is not a Hilbert space, but it is reflexive. If we take its subfamily* $\mathsf{G}(t, 0) = t + 1$ *then we still can apply our results to this subfamily. Let* $\gamma_m = \frac{1}{m^2}$ *and* $\zeta_m = \frac{1}{m}$*, then clearly*

$$\lim_{m\to\infty} \frac{\gamma_m}{\zeta_m} = 0,$$

so by Theorem 3.5 we have the sequence of iteration

$$\tau_m = \frac{1}{m^2}\tau_{m-1} + (1 - \frac{1}{m^2})\mathsf{G}(\frac{1}{m}, 0)\tau_m \to 0 \in F(\mathbf{G}),$$

where 0 *is the unique fixed point of the subfamily* G*.*

Open problem. We have an open problem for the readers that whether Lemmas 3.1 and 3.2 and Theorem 3.5 can be generalized to all periodic and non-periodic evolution families?

5. Conclusions

The idea of semigroupos arise from the solution of autonomous abstract Cauchy problem while the idea of evolution family arise from the solution of non-autonomous abstrct Cauchy problem, which is more genreal than the semigroups. In [33], the strong convergence theorms for fixed points for nonexpansive semigroups on Hilbert spaces are proved. We generalized the results to a subfamily of an evolution family on a Hilbert space. These results may be come a gateway for many researchers to extends these ideas to the whole evolution family rather than the subfamily in future. Also these results are helpfull for the mathematician and others to use for existence and uniqeness of solution of non-autonomous abstarct Cauchy problems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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