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Research article

Exponential stability and numerical simulation of a Bresse-Timoshenko system subject to a neutral delay

Houssem Eddine Khochemane[1,*], Ali Rezaiguia[2,3] and Hasan Nihal Zaidi[2]

[1] Higher Normal School of Technological Education of Skikda, Algeria
[2] Department of Mathematics, Faculty of Science, University of Ha'il, Ha'il 2440, Saudi Arabia
[3] Department of Computer Science and Mathematics, Mouhamed Cherif Messadia University, Souk Ahras, Algeria

* **Correspondence:** Email: khochmanehoussem@hotmail.com, h.Khochemane@enset-skikda.dz.

Abstract: In the present work, we consider a one-dimensional Bresse-Timoshenko system with neutral delay term and a viscous damping acting on vertical displacement of the beam. Under appropriate assumptions on the kernel of this kind of delay and based on the multipliers method, we construct a suitable Lyapunov functional that allows us to establish an exponential decay of the energy in spite of the existence of the delay. Moreover, our result does not depend on any condition on the coefficients of the system. Finally, we present some numerical results to illustrate the theoretical result obtained.

Keywords: Bresse-Timoshenko system; neutral delay; exponential stability; energy method; Lyapunov functional
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1. Introduction

Throughout the years, many works have dealt with beam theories due to their important applications in high technology of flexible structures. Historically, it is well known that one of the oldest beam theories is the Euler-Bernoulli beam theory, which is a simplification of linear isotropic beams. It was first enunciated circa 1750, but it was not applied on a large scale until the development of the Eiffel Tower and the Ferris Wheel in the late 19th century. Following these successful demonstrations, it quickly became a cornerstone of engineering and an enabler of the Second Industrial Revolution. Later on, other beam theories appeared and were considered as improvements of the Euler-Bernoulli theory, such as the Rayleigh beam theory [28] and the Timoshenko beam theory [31]. In [13–15], Elishakoff et al. gave a brief description of the beam model in the one-dimensional case for beam vibrations. The classical Euler-Bernoulli differential equation for free vibration of uniform beams is given by

$$EI\varphi_{xxxx} + \rho A\varphi_{tt} = 0, \tag{1.1}$$

where $\varphi(x, t)$ is the beam deflection from its equilibrium position, E is the modulus of elasticity, I is the moment of inertia of the cross section, ρ is the material density of the beam material, A is the cross-sectional area, x is the axial coordinate, and t is the time. Later, Rayleigh [28] proposed a correction to the Euler-Bernoulli equation (1.1), by taking into account the rotary movements of the beam elements in addition to the translatory ones. From a mathematical modeling point of view for vibrating beams, it is instructive to re-derive briefly this equation. The angle of rotation equals the slope of the deflection curve φ_x, and the corresponding angular acceleration is given by φ_{xtt}. As a consequence, the moment of inertia of the element about an axis through its center of mass equals $\rho I\varphi_{xtt}$. By exploiting this moment and taking into account d'Alembert's principle for dynamic equilibrium [22], we obtain

$$V - M_x + \rho I\varphi_{xtt} = 0, \tag{1.2}$$

where $V(x, t)$ is the shearing force, and $M(x, t)$ is the bending moment. Replacing V from Eq (1.2) in the case of dynamic equilibrium condition for forces in the φ-direction of the transverse vibration, we have

$$V_x = -\rho A\varphi_{tt} = (M_x - \rho I\varphi_{xtt})_x.$$

Physically and from elastic theory, the bending moment M coincides with $EI\varphi_{xx}$, which leads to the final governing equation obtained in the Rayleigh model for the uniform beam oscillations,

$$EI\varphi_{xxxx} + \rho A\varphi_{tt} - \rho I\varphi_{xxtt} = 0, \tag{1.3}$$

which is known as the rotatory inertial equation.

Afterwards, Timoshenko [31] extended Eq (1.3) by incorporating the impact of the shear deformation. In another term, he expressed the slope of the deflection curve in two parts,

$$\varphi_x = -\psi + \beta, \tag{1.4}$$

with ψ as the rotation of the cross-sections with the neglect of the shear deformation and β as the angle associated with the shear deformation at the neutral axis in the same cross-section. On the other hand, according to the mechanics of solids we can write

$$M = EI\psi_x, \tag{1.5}$$

$$V = k_1\beta AG = k_1AG(\varphi_x + \psi), \tag{1.6}$$

where k_1 is the shear coefficient, and G is the shear modulus. The state of dynamic equilibrium of forces in the vertical direction is given by

$$\rho A\varphi_{tt} - V_x = 0. \tag{1.7}$$

Deriving with respect to t Eq (1.4) and substituting it in the dynamic equilibrium equation of motion (1.2), we get

$$V - M_x + \rho I\psi_{tt} = 0. \tag{1.8}$$

The Timoshenko system was obtained by substituting, respectively, (1.6) and (1.5) into (1.7) and (1.8), thus:

$$\begin{cases} -k_1 AG\left(\varphi_x + \psi\right)_x + \rho A\varphi_{tt} = 0, \\ k_1 AG\left(\varphi_x + \psi\right) - EI\psi_{xx} + \rho I\psi_{tt} = 0, \end{cases}$$

where $\rho_1 = \rho A$ represents the mass density, $\rho_2 = \rho I$ is the moment mass inertia, $b = EI$ is the rigidity coefficient (of the cross-section), and $k = k_1 AG$ is the shear modulus of elasticity. Then, the Timoshenko system takes the following form:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) = 0. \end{cases}$$

It should be noted that the mentioned problem plays a crucial role in engineering applications, and for more details on the valuable resources that have been realized regarding the Timoshenko system, we refer the readers to [5–9,31].

For the physical and technical reasons mentioned in [13], Elishakoff proposed a combination of Eq (1.2) which comes from d'Alembert's principle with Eq (1.7) from the Timoshenko hypothesis, resulting in the following coupled system:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + k\left(\varphi_x + \psi\right) = 0. \end{cases}$$

Many investigations have been realized concerning the asymptotic behavior of the solution of the Bresse-Timoshenko system. Among them, we cite the works [18, 19, 34], in which the authors established different types of stability results such as exponential and general decay based on many dissipation terms.

There are also other investigations concerning the delay which appear in many models of mathematics that come from engineering biological science, economics, physiology and epidemiology. Delay effects arise in many applications depending not only on the present state but also on some past occurrences, and they have attracted a lot of attention from researchers in diverse fields of human endeavors, such as mathematics, engineering, science and economics. The presence of delay may be a source of instability of systems which are uniformly asymptotically stable in the absence of delay unless additional control terms have been used (see [10, 11, 17, 25, 26, 33]). Also, the introducing of this complementary may lead to ill-posedness, as shown in many works such as [11, 27] and the references therein. On the other hand, the delay has an important role in the control of PDEs, and this has appeared in a lot of studies (see [1–4, 17, 20, 21, 24–26, 33]). In addition to the well-known discrete delays, there are several others. We are interested here in the neutral delay, where the delay is occurring in the second (highest) derivative. For more details, see previous studies ([12–16,23,32]) and the references therein.

Among the investigations that have been realized concerning the asymptotic behavior with neutral delay, we cite the work of Tatar [30] where he considered the wave equation with neutral delay, and he showed that the solution decays in an exponential manner under some conditions on the kernel of distributed neutral delay.

In [29], Seghour et al. studied the following thermoelastic laminated system with neutral delay:

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x + A w_t = 0, \ x \in (0,1), \ t > 0, \\ I_\rho (3s_{tt} - \psi_{tt}) - G(\psi - w_x) - (3s - \psi) + \mu\theta_x = 0, \ x \in (0,1), \ t > 0, \\ 3I_\rho \left(s_t + \int_0^t h(t-r)\, s_t(r)\, dr \right)_t + 3G(\psi - w_x) + 4\gamma s - 3s_{xx} = 0, \ x \in (0,1), \ t > 0, \\ \theta_t - \kappa\theta_{xx} + \mu(3s - \psi)_{tx} = 0, \ x \in (0,1), \ t > 0, \end{cases}$$

with boundary conditions

$$\begin{cases} \psi(0,t) = s(0,t) = \theta_x(0,t) = w_x(0,t) = 0, \ t \geq 0, \\ \theta(1,t) = w(1,t) = s_x(1,t) = \psi_x(1,t) = 0, \ t \geq 0, \end{cases}$$

and initial data

$$\begin{cases} (w, \psi, s, \theta)(x,0) = (w_0, \psi_0, s_0, \theta_0), \ x \in (0,1), \\ (w_t, \psi_t, s_t)(x,0) = (w_1, \psi_1, s_1), \ x \in (0,1). \end{cases}$$

They showed that the dissipation given by the combination of heat effect and the frictional damping stabilize exponentially the system in the case of equal speeds of wave propagation even if the delays, in general, are of a destructive nature. In the case of non-equal wave speeds and with an additional assumption on the kernel, they proved a polynomial stability.

Motivated by the previous works, in this paper we consider the following Bresse-Timoshenko system subject to a neutral delay:

$$\begin{cases} \rho_1 \left(\varphi_t + \int_0^t h(t-s)\, \varphi_t(s)\, ds \right)_t = k(\varphi_x + \psi)_x - \mu_1 \varphi_t, & \text{in } (0,1) \times (0,+\infty), \\ -\rho_2 \varphi_{ttx} = b\psi_{xx} - k(\varphi_x + \psi), & \text{in } (0,1) \times (0,+\infty), \end{cases} \tag{1.9}$$

with the initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \ \varphi_t(x,0) = \varphi_1(x), & x \in (0,1), \\ \psi(x,0) = \psi_0(x), \ \psi_t(x,0) = \psi_1(x), & x \in (0,1), \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, & t \in (0,+\infty). \end{cases} \tag{1.10}$$

First, we give an existence and uniqueness result of the solution using the Faedo-Galerkin method. Then, based on the energy method and by constructing a suitable Lyapunov functional using the multipliers method as well as under an appropriate assumptions on the kernel of the neutral delay term, we prove that the system is exponentially stable in spite of the existence of the neutral delay irrespective of any stability number. In the absence of neutral delay, there is a similarity with the previous works like [18,19,34] concerning the estimation of the energy terms. In our case and compared to the work of Seghour et al. in [29], we were able to dispense the thermal effect depending on the viscous damping acting on vertical displacement of the beam to control the neutral delay term and to guarantee an exponential stability of the solution irrespective of wave speeds or any other relationship between the system parameters. In other words, the unique dissipation given only by the viscous damping is strong enough to provoke an exponential stability and control the neutral delay. Finally, we present some numerical results using MATLAB software to validate the theoretical result obtained by carrying out a discretization using the classical finite difference method for the spatial and temporal discretization.

This paper is organized as follows: In Section 2, we introduce some assumptions needed in the next sections to prove the main result, and we give a result concerning the well-posedness of problem (1.9)–(1.10). In Section 3, we prove the energy decay of the system. In Section 4, we use the energy method to prove the exponential decay result. In Section 5, some numerical simulations are presented.

2. Preliminaries

In this section we present our assumptions on both kernels and introduce the energy functional and another functional.

We use the standard Lebesgue space $L^2(0,1)$ and the Sobolev space $H_0^1(0,1)$ with their usual scalar products and norms. Also, in what follows we will use the following notations:

$$(h \circ \Psi)(t) = \int_0^t h(t-s)\left(\int_0^1 (\Psi(t) - \Psi(s))^2\, dx\right) ds,\ t \geq 0,$$

and

$$(h * \Psi)(t) = \int_0^1 \int_0^t h(t-s)\Psi(s)^2 ds dx,\ t \geq 0.$$

To achieve our goal, we need to introduce the following hypothesis and assumptions:

(H1) The kernel h is a nonnegative continuously differentiable and summable function satisfying

$$-\mu h(t) \leq h'(t) \leq 0,\ \forall t \geq 0,\ \text{where}\ \mu > 0, \quad \bar{h} = \int_0^\infty h(s)ds < 1.$$

(H2) $\exp(\varsigma t)\, h(t) \in L^1(\mathbb{R}_+)$ for some $\varsigma > 0$.

Note that if $\int_0^{+\infty} e^{\varsigma s} h(s)ds < \infty$, and $\lim\limits_{t \longrightarrow \infty} \exp(\varsigma t)\, h(t) < \infty$, then

$$\int_0^{+\infty} e^{\varsigma s} |h'(s)|\, ds = -\int_0^{+\infty} e^{\varsigma s} h'(s)ds = -e^{\varsigma s} h(s) \mid_0^\infty + \varsigma \int_0^{+\infty} e^{\varsigma s} h(s)ds < \infty.$$

To simplify the calculations, we are obligated to announce this lemma which is usable in the following sections.

Lemma 1 ([29]). *For any function $\Psi \in C^1\left([0,\infty); L^2(0,1)\right)$ and any $h \in C^1([0,\infty))$, we have the following identity:*

$$\begin{aligned} &\int_0^1 \Psi(t)\left(\int_0^t h(t-s)\, \Psi_t(s)ds\right) dx \\ &= -\frac{1}{2}(h' \circ \Psi)(t) + \frac{1}{2}\frac{d}{dt}\int_0^1 \left(\int_0^t h(t-s)\, \Psi^2(s)ds\right) dx \\ &+ \frac{h(t)}{2}\int_0^1 \Psi^2 dx - h(t)\int_0^1 \Psi(0)\Psi(t)\, dx. \end{aligned}$$

For completeness, we state without proof the following global existence and regularity result which can be proved by using the standard Faedo-Galerkin method, for which we refer the reader to [24].

Theorem 1. *Let $(\varphi_0, \varphi_1) \in H_0^1(0,1) \times L^2(0,1)$ and $(\psi_0, \psi_1) \in H_0^1(0,1) \times L^2(0,1)$ be given. Assume that **(H1)–(H2)** are satisfied, and then the problem (1.9)–(1.10) has a unique global (weak) solution satisfying*

$$\varphi, \psi \in C\left(\mathbb{R}_+, H_0^1(0,1)\right) \cap C^1\left(\mathbb{R}_+, L^2(0,1)\right).$$

3. Energy decay

In this section, we use the energy method to study the asymptotic behavior of solutions of the system (1.9)–(1.10). First, we state and prove the following lemma.

Lemma 2. *Let (φ, ψ) be a solution of system (1.9)–(1.10). Then, the energy associated to the system (1.9)–(1.10) is defined by*

$$
\begin{aligned}
E(t) &= \frac{1}{2}\int_0^1 \left(\rho_1\varphi_t^2 + k(\varphi_x+\psi)^2 + \rho_2\varphi_{tx}^2 + b\psi_x^2 + \frac{\rho_1\rho_2}{k}\varphi_{tt}^2\right)dx \\
&+ \rho_1(h*\varphi_t)(t) + \frac{\rho_1\rho_2}{k}(h*\varphi_{tt})(t),
\end{aligned} \tag{3.1}
$$

satisfying

$$
\begin{aligned}
\frac{d}{dt}E(t) \le \frac{\rho_1}{2}(h'\circ\varphi_t)(t) - \frac{\mu_1\rho_2}{k}\int_0^1 \varphi_{tt}^2 dx + \frac{\rho_1\rho_2}{k}(h'\circ\varphi_{tt})(t) \\
-\mu_1\int_0^1 \varphi_t^2\, dx + \zeta h(t),\ \zeta > 0.
\end{aligned} \tag{3.2}
$$

Proof. Multiplying $(1.9)_1$ and $(1.9)_2$, respectively, by φ_t and ψ_t and integrating by parts, we get

$$
\begin{cases}
\dfrac{\rho_1}{2}\dfrac{d}{dt}\int_0^1 \varphi_t^2\, dx + \rho_1\int_0^1 \varphi_t\left(\int_0^t h(t-s)\varphi_t(s)\,ds\right)_t dx \\
= -k\int_0^1 (\varphi_x+\psi)\varphi_{tx}\, dx - \mu_1\int_0^1 \varphi_t^2 dx, \\
\rho_2\int_0^1 \varphi_{tt}\psi_{xt}dx + \dfrac{b}{2}\dfrac{d}{dt}\int_0^1 \psi_x^2 dx = -k\int_0^1 (\varphi_x+\psi)\psi_t dx.
\end{cases} \tag{3.3}
$$

Taking the derivative of $(1.9)_1$ with respect to t, we obtain

$$
\psi_{tx} = \frac{\rho_1}{k}\varphi_{ttt} - \varphi_{txx} + \frac{\rho_1}{k}\left(\int_0^t h(t-s)\varphi_t(s)\,ds\right)_{tt} + \frac{\mu_1}{k}\varphi_{tt},
$$

noting that

$$
\begin{aligned}
&\left(\int_0^t h(t-s)\varphi_t(s)\,ds\right)_{tt} \\
&= \left(\int_0^t h(t-s)\varphi_{tt}(s)\,ds + h(t)\varphi_t(0)\right)_t \\
&= \int_0^t h(t-s)\varphi_{ttt}(s)\,ds + h(t)\varphi_{tt}(0) + h'(t)\varphi_t(0).
\end{aligned}
$$

So,

$$
\psi_{tx} = \frac{\rho_1}{k}\varphi_{ttt} - \varphi_{txx} + \frac{\rho_1}{k}\int_0^t h(t-s)\varphi_{ttt}(s)\,ds
$$

$$+\frac{\rho_1}{k}h(t)\varphi_{tt}(0)+\frac{\rho_1}{k}h'(t)\varphi_t(0)+\frac{\mu_1}{k}\varphi_{tt}. \tag{3.4}$$

Substituting (3.4) in $(3.3)_2$, the system (3.3) becomes

$$\begin{cases} \frac{\rho_1}{2}\frac{d}{dt}\int_0^1 \varphi_t^2\, dx+\rho_1\int_0^1 \varphi_t\left(\int_0^t h(t-s)\varphi_t(s)\,ds\right)_t\, dx \\ =-k\int_0^1(\varphi_x+\psi)\varphi_{tx}dx-\mu_1\int_0^1 \varphi_t^2 dx, \\ \frac{\rho_1\rho_2}{2k}\frac{d}{dt}\int_0^1 \varphi_{tt}^2 dx+\frac{\rho_2}{2}\frac{d}{dt}\int_0^1 \varphi_{tx}^2 dx+\frac{\rho_1\rho_2}{k}\int_0^1 \varphi_{tt}\left(\int_0^t h(t-s)\varphi_{ttt}(s)\,ds\right)dx \\ +\frac{\rho_1\rho_2}{k}h(t)\int_0^1 \varphi_{tt}\varphi_{tt}(0)\,dx+\frac{\rho_1\rho_2}{k}h'(t)\int_0^1 \varphi_{tt}\varphi_t(0)\,dx+\frac{b}{2}\frac{d}{dt}\int_0^1 \psi_x^2 dx \\ =-\frac{\mu_1\rho_2}{k}\int_0^1 \varphi_{tt}^2 dx-k\int_0^1(\varphi_x+\psi)\psi_t dx. \end{cases} \tag{3.5}$$

On the other hand, by applying Lemma 1, we have

$$\begin{aligned} &\rho_1\int_0^1\left(\int_0^t h(t-s)\varphi_t(s)\,ds\right)_t\varphi_t\, dx \\ &=\rho_1 h(t)\int_0^1 \varphi_t\varphi_t(0)\,dx+\rho_1\int_0^1 \varphi_t\left(\int_0^t h(t-s)\varphi_{tt}(s)\,ds\right)dx \\ &=\rho_1 h(t)\int_0^1 \varphi_t\varphi_t(0)\,dx-\frac{\rho_1}{2}(h'\circ\varphi_t)(t)+\frac{\rho_1}{2}\frac{d}{dt}\int_0^1\left(\int_0^t h(t-s)\varphi_t^2(s)\,ds\right)dx \\ &+\frac{\rho_1}{2}h(t)\int_0^1 \varphi_t^2\, dx-\rho_1 h(t)\int_0^1 \varphi_t\varphi_t(0)\,dx, \end{aligned}$$

and

$$\begin{aligned} &\frac{\rho_1\rho_2}{k}\int_0^1 \varphi_{tt}\left(\int_0^t h(t-s)\varphi_{ttt}(s)\,ds\right)dx \\ &=\frac{\rho_1\rho_2}{2k}\frac{d}{dt}\int_0^1\left(\int_0^t h(t-s)\varphi_{tt}^2(s)\,ds\right)dx-\frac{\rho_1\rho_2}{k}h(t)\int_0^1 \varphi_{tt}\varphi_{tt}(0)\,dx \\ &+\frac{\rho_1\rho_2}{k}h(t)\int_0^1 \varphi_{tt}^2 dx-\frac{\rho_1\rho_2}{k}(h'\circ\varphi_{tt})(t). \end{aligned}$$

Therefore, the system (3.5) is equivalent to

$$\begin{aligned} &\frac{d}{2dt}\int_0^1\Big(\rho_1\varphi_t^2+k(\varphi_x+\psi)^2+\rho_2\varphi_{tx}^2+b\psi_x^2+\frac{\rho_1\rho_2}{k}\varphi_{tt}^2 \\ &+\rho_1\int_0^t h(t-s)\varphi_t^2(s)\,ds+\frac{\rho_1\rho_2}{k}\int_0^t h(t-s)\varphi_{tt}^2(s)\,ds\Big)dx \\ &=\frac{\rho_1}{2}(h'\circ\varphi_t)(t)-\frac{\rho_1\rho_2}{k}\left(1+\frac{h(t)}{2}\right)\int_0^1 \varphi_{tt}^2 dx+\frac{\rho_1\rho_2}{k}(h'\circ\varphi_{tt})(t) \\ &-\left(\frac{\rho_1}{2}h(t)+\mu_1\right)\int_0^1 \varphi_t^2\, dx-\frac{\rho_1\rho_2}{k}h(t)\int_0^1 \varphi_{tt}\varphi_t(0)\,dx. \end{aligned}$$

By using Young's inequality and the hypothesis **(H1)**, we obtain

$$
\begin{aligned}
&-\frac{\rho_1\rho_2}{2}h'(t)\int_0^1 \varphi_{tt}\varphi_t(0)\,dx = \frac{\rho_1\rho_2}{2}\mu h(t)\int_0^1 \frac{1}{\sqrt{k}}\varphi_{tt}\sqrt{k}\varphi_t(0)\,dx \\
&\le \frac{\rho_1\rho_2}{2k}\mu h(t)\,\delta_1\int_0^1 \varphi_{tt}^2dx + \frac{\rho_1\rho_2}{2\delta_1}\mu h(t)\,k\int_0^1 \varphi_t^2(0)\,dx,
\end{aligned}
$$

and taking $\delta_1 = \dfrac{1}{2\mu}$

$$
\begin{aligned}
&-\frac{\rho_1\rho_2}{2}h'(t)\int_0^1 \varphi_{tt}\varphi_t(0)\,dx \\
&\le \frac{\rho_1\rho_2}{4k}h(t)\int_0^1 \varphi_{tt}^2dx + \frac{\rho_1\rho_2}{4}\mu^2 h(t)\,k\int_0^1 \varphi_t^2(0)\,dx.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\frac{d}{dt}E(t) \;\le\; & \frac{\rho_1}{2}\left(h'\circ\varphi_t\right)(t) - \frac{\rho_1\rho_2}{k}\int_0^1 \varphi_{tt}^2dx + \frac{\rho_1\rho_2}{k}\left(h'\circ\varphi_{tt}\right)(t) \\
& -\mu_1\int_0^1 \varphi_t^2\,dx + \zeta h(t),
\end{aligned}
$$

where $\zeta = \dfrac{\rho_1\rho_2}{4}\mu^2 k\int_0^1 \varphi_t^2(0)\,dx.$ □

4. Exponential stability of solution

In this section, we establish an exponential decay result of solutions for the considered problem. For that, we need the following lemmas to achieve our goal.

Lemma 3. *Let (φ,ψ) be the solution of (1.9)–(1.10). Then, the functional*

$$
F_1(t) = -\rho_1\int_0^1 \varphi_t\left(\varphi_t + \int_0^t h(t-s)\,\varphi_t(s)\,ds\right)dx - k\int_0^1 \varphi_{tx}\varphi_x dx - \frac{\mu_1}{2}\int_0^1 \varphi_t^2dx,
$$

satisfies the estimate

$$
\begin{aligned}
F_1'(t) \;\le\; & -k\int_0^1 \varphi_{tx}^2dx + \left(\left(3+\frac{h(t)}{2}\right)\rho_1 + \frac{k^2}{4\varepsilon_1}\right)\int_0^1 \varphi_{tt}^2dx + \varepsilon_1\int_0^1 \psi_x^2dx \\
& + \frac{\rho_1}{2}\left(3+h(t)\right)\int_0^1 \varphi_t^2dx + \rho_1 h(t)\int_0^1 \varphi_t^2(0)\,dx + \frac{\rho_1\overline{h}}{2}\left(h*\varphi_t\right)(t) + \rho_1\overline{h}\left(h*\varphi_{tt}\right)(t).
\end{aligned}
$$

Proof. By differentiating $F_1(t)$ with respect to t, using the first equation of (1.9) and integrating by parts, we obtain

$$
F_1'(t) = -2\rho_1\int_0^1 \varphi_{tt}\varphi_t dx + \rho_1\int_0^1 \varphi_{tt}^2dx - k\int_0^1 \varphi_{tt}\psi_x dx - k\int_0^1 \varphi_{tx}^2dx
$$

$$
\begin{aligned}
&-\rho_1 \int_0^1 \varphi_{tt} \int_0^t h(t-s)\varphi_t(s)\,dsdx - \rho_1 \int_0^1 \varphi_t \int_0^t h(t-s)\varphi_{tt}(s)\,dsdx \\
&+\rho_1 \int_0^1 \varphi_{tt} \int_0^t h(t-s)\varphi_{tt}(s)\,dsdx \\
&-\rho_1 h(t) \int_0^1 \varphi_t \varphi_t(0)\,dx + \rho_1 h(t) \int_0^1 \varphi_t(0)\varphi_{tt} dx. \qquad (4.1)
\end{aligned}
$$

By using Young's inequality, we obtain

$$
-2\rho_1 \int_0^1 \varphi_{tt}\varphi_t dx \le \rho_1 \int_0^1 \varphi_{tt}^2 dx + \rho_1 \int_0^1 \varphi_t^2 dx, \qquad (4.2)
$$

$$
-\rho_1 h(t) \int_0^1 \varphi_t\varphi_t(0)\,dx \le \frac{\rho_1}{2} h(t) \int_0^1 \varphi_t^2 dx + \frac{\rho_1}{2} h(t) \int_0^1 \varphi_t^2(0)\,dx, \qquad (4.3)
$$

$$
\rho_1 h(t) \int_0^1 \varphi_t(0)\varphi_{tt} dx \le \frac{\rho_1}{2} h(t) \int_0^1 \varphi_{tt}^2 dx + \frac{\rho_1}{2} h(t) \int_0^1 \varphi_t^2(0)\,dx, \qquad (4.4)
$$

$$
-k \int_0^1 \varphi_{tt}\psi_x dx \le \varepsilon_1 \int_0^1 \psi_x^2 dx + \frac{k^2}{4\varepsilon_1} \int_0^1 \varphi_{tt}^2 dx. \qquad (4.5)
$$

By using Young's and Cauchy Schwarz inequalities, we obtain

$$
-\rho_1 \int_0^1 \varphi_{tt} \int_0^t h(t-s)\varphi_t(s)\,dsdx \le \frac{\rho_1}{2} \int_0^1 \varphi_{tt}^2 dx + \frac{\rho_1 \overline{h}}{2} (h * \varphi_t)(t), \qquad (4.6a)
$$

$$
-\rho_1 \int_0^1 \varphi_t \int_0^t h(t-s)\varphi_{tt}(s)\,dsdx \le \frac{\rho_1}{2} \int_0^1 \varphi_t^2 dx + \frac{\rho_1 \overline{h}}{2} (h * \varphi_{tt})(t), \qquad (4.7)
$$

$$
\rho_1 \int_0^1 \varphi_{tt} \int_0^t h(t-s)\varphi_{tt}(s)\,dsdx \le \frac{\rho_1}{2} \int_0^1 \varphi_{tt}^2 dx + \frac{\rho_1 \overline{h}}{2} (h * \varphi_{tt})(t). \qquad (4.8)
$$

Inserting (4.2)–(4.8) in (4.1), we obtain (4.1). □

Lemma 4. *Let (φ, ψ) be the solution of (1.9)–(1.10). Then, the functional*

$$
F_2(t) := -\rho_2 \int_0^1 \varphi_{tx}\psi dx + \rho_1 \int_0^1 \varphi \left(\varphi_t + \int_0^t h(t-s)\varphi_t(s)\,ds \right) dx, \qquad (4.9)
$$

satisfies the estimate

$$
\begin{aligned}
F_2'(t) \quad \le \quad & -b \int_0^1 \psi_x^2 dx - k \int_0^1 (\varphi_x + \psi)^2 dx + \rho_2 \int_0^1 \varphi_{tx}^2 dx \\
& + \frac{3\rho_1}{2} \int_0^1 \varphi_t^2 dx + \frac{\rho_1 \overline{h}}{2} (h * \varphi_{tt})(t). \qquad (4.10)
\end{aligned}
$$

Proof. By differentiating $F_2(t)$ with respect to t, exploiting (1.9), integrating by parts and using the Timoshenko hypothesis (1.4), we obtain

$$\begin{aligned} F_2'(t) &= -b\int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \varphi_{tx}^2 dx - k\int_0^1 (\varphi_x + \psi)^2 dx \\ &\quad + \rho_1 \int_0^1 \varphi_t^2 dx + \rho_1 \int_0^1 \varphi_t \int_0^t h(t-s)\varphi_t(s)\, ds dx. \end{aligned}$$

By using Young's and Cauchy Schwarz inequalities, we have (4.10). □

Lemma 5. *Let (φ, ψ) be the solution of (1.9)–(1.10). Then, the functional*

$$F_3(t) := e^{-\zeta t}\int_0^1 \int_0^t e^{\zeta s}\widetilde{H}_1(t-s)\varphi_t^2(s)\, ds dx, \tag{4.11}$$

satisfies the following estimate:

$$F_3'(t) = -\zeta F_3(t) + \widetilde{H}_1(0)\int_0^1 \varphi_t^2 dx - (h * \varphi_t)(t), \tag{4.12}$$

where $\widetilde{H}_1(t) = \int_t^\infty h(s) e^{\zeta s} ds$.

Proof. By differentiating $F_3(t)$ with respect to t,

$$\begin{aligned} F_3'(t) &= -\zeta e^{-\zeta t}\int_0^1 \int_0^t e^{\zeta s}\widetilde{H}_1(t-s)\varphi_t^2(s)\, ds dx \\ &\quad + e^{-\zeta t}\int_0^1 \left(\int_0^t e^{\zeta s}\widetilde{H}_1(t-s)\varphi_t^2(s)\, ds\right)_t dx \\ &= -\zeta F_3(t) + e^{-\zeta t}\int_0^1 \left(e^{\zeta t}\widetilde{H}_1(0)\varphi_t^2 - \int_0^t e^{\zeta t} h(t-s)\varphi_t^2(s)\, ds\right) dx \\ &= -\zeta F_3(t) + \widetilde{H}_1(0)\int_0^1 \varphi_t^2 dx - (h * \varphi_t)(t), \end{aligned}$$

which gives (4.12). □

Lemma 6. *Let (φ, ψ) be the solution of (1.9)–(1.10). Then, the functional*

$$F_4(t) := e^{-\zeta t}\int_0^1 \int_0^t e^{\zeta s}\widetilde{H}_1(t-s)\varphi_{tt}^2(s)\, ds dx \tag{4.13}$$

satisfies the estimate

$$F_4'(t) \le -\zeta F_4(t) + \widetilde{H}_1(0)\int_0^1 \varphi_{tt}^2 dx - (h * \varphi_{tt})(t). \tag{4.14}$$

Proof. By differentiating $F_4(t)$ with respect to t, as in the case of $F_3(t)$, we obtain the desired result. □

Next, we define a Lyapunov function $L(t)$ by

$$L(t) := NE(t) + N_1 F_1(t) + N_2 F_2(t) + N_3 (F_3(t) + F_4(t)), \tag{4.15}$$

where N, N_1, N_2 and N_3 are positive constants that will be chosen appropriately later.

Lemma 7. *Let (φ, ψ) be the solution of (1.9)–(1.10). Then, there exist two positive constants κ_1 and κ_2 such that the Lyapunov functional* (4.15) *satisfies*

$$\kappa_1 (E(t) + F_3(t) + F_4(t)) \le L(t) \le \kappa_2 (E(t) + F_3(t) + F_4(t)), \ \forall t \ge 0, \tag{4.16}$$

and

$$L'(t) \le -\beta_1 (E(t) + F_3(t) + F_4(t)) + C_2 h(t), \ \beta_1 > 0. \tag{4.17}$$

Proof. From (4.15), we have

$$\begin{aligned}
&|L(t) - NE(t) - N_3 (F_3(t) + F_4(t))| \\
&= N_1 |F_1(t)| + N_2 |F_2(t)| + N_3 |F_3(t) + F_4(t)| \\
&\le \rho_1 N_1 \int_0^1 \left| \varphi_t \left(\varphi_t + \int_0^t h(t-s) \varphi_t(s) \, ds \right) \right| dx + k N_1 \int_0^1 |\varphi_{tx} \varphi_x| \, dx \\
&+ \frac{\mu_1}{2} N_1 \int_0^1 \varphi_t^2 dx + \rho_2 N_2 \int_0^1 |\varphi_{tx} \psi| \, dx + \rho_1 N_2 \int_0^1 \left| \varphi \left(\varphi_t + \int_0^t h(t-s) \varphi_t(s) \, ds \right) \right| dx.
\end{aligned}$$

By Young's, Cauchy Schwarz and Poincaré's inequalities and with some transformations, we obtain

$$|L(t) - NE(t) - N_3 (F_3(t) + F_4(t))| \le \lambda_1 E(t).$$

Therefore,

$$(N - \lambda_1) E(t) + N_3 (F_3(t) + F_4(t)) \le L(t) \le (N + \lambda_1) E(t) + N_3 (F_3(t) + F_4(t)).$$

By choosing N (depending on N_1, N_2, N_3) sufficiently large, we obtain (4.16) with

$$\begin{aligned}
\kappa_1 &= \min \{N - \lambda_1, N_3\}, \\
\kappa_2 &= \max \{N + \lambda_1, N_3\}.
\end{aligned}$$

Now, by differentiating $L(t)$, recalling (3.2), (4.1), (4.10), (4.12) and (4.14), and setting $\varepsilon_1 = \dfrac{1}{N_1}$, we arrive at

$$\begin{aligned}
L'(t) \ \le \ & -\left[N\mu_1 - N_1 \frac{\rho_1}{2} (3 + h(t)) - \frac{3\rho_1}{4} N_2 - N_3 \widetilde{H}_1(0) \right] \int_0^1 \varphi_t^2 dx \\
& - N_3 \widetilde{H}_1(0) \int_0^1 \psi_t^2 dx - k N_2 \int_0^1 (\varphi_x + \psi)^2 \, dx - (k N_1 - N_2 \rho_2) \int_0^1 \varphi_{tx}^2 dx \\
& - \left[\frac{\rho_1 \rho_2}{k} N - N_1^2 \frac{k^2}{4} - N_3 \widetilde{H}_1(0) \right] \int_0^1 \varphi_{tt}^2 dx
\end{aligned}$$

$$
\begin{aligned}
&-(N_2 b-1)\int_0^1 \psi_x^2 dx-\left(N_3-\frac{\rho_1\overline{h}}{2}N_1\right)(h*\varphi_{tt})(t)\\
&-\left[N_3-\frac{\rho_1\overline{h}}{2}N_1-\frac{\rho_1\overline{h}}{2}N_2\right](h*\varphi_t)(t)+\left(\frac{\rho_1}{2}N\right)(h'\circ\varphi_t)(t)\\
&+\left(\frac{\rho_1\rho_2}{k}N\right)(h'\circ\varphi_{tt})(t)-N_3\zeta\left(F_3(t)+F_4(t)\right). \qquad (4.18)
\end{aligned}
$$

At this point, we need to choose our constants very carefully. First, we choose N_2 large enough such that

$$N_2 b-1>0.$$

Once N_2 is fixed, we take N_1 large enough so that

$$kN_1-N_2\rho_2>0.$$

After that, we pick N_3 large enough such that

$$
\begin{cases}
N_3-\dfrac{\rho_1\overline{h}}{2}N_1-\dfrac{\rho_1\overline{h}}{2}N_2>0,\\
\text{and}\\
N_3-\dfrac{\rho_1\overline{h}}{2}N_1>0.
\end{cases}
$$

Finally, we select N large enough so that

$$
\begin{cases}
N\mu_1-N_1\dfrac{\rho_1}{2}\left(3+h(t)\right)-\dfrac{3\rho_1}{4}N_2-N_3\widetilde{H}_1(0)>0,\\
\text{and}\\
\dfrac{\rho_1\rho_2}{k}N-N_1^2\dfrac{k^2}{4}-N_3\widetilde{H}_1(0)>0.
\end{cases}
$$

With all these choices, we obtain (4.17). □

We are now ready to state and prove the following exponential stability result.

Lemma 8. *Let (φ,ψ) be a solution of (1.9)–(1.10), and assume that* ***(H1)–(H2)*** *hold. Then, there exist two positive constants τ_1 and τ_2 such that*

$$E(t)\le\tau_2 e^{-\tau_1 t},\ \forall t\ge 0. \qquad (4.19)$$

Proof. By using (4.17) and the right side of (4.16), we get

$$L'(t)\le -C_1 L(t)+C_2 h(t), \qquad (4.20)$$

where $C_1=\dfrac{\beta_1}{\kappa_2}>0$.
Multiplying (4.20) by $\exp(C_1 t)$, we obtain

$$\frac{d}{dt}\left(L(t)\exp(C_1 t)\right)\le C_2\exp(C_1 t)\,h(t). \qquad (4.21)$$

Integrating over $(0,T)$ the inequation (4.21) and choosing C_1 smaller than ς, we have

$$\begin{aligned} L(T)\exp\left(C_1T\right) &\leq L(0)+C_2\int_0^T \exp\left(\varsigma t\right)h(t)dt \\ &\leq L(0)+C_2\int_0^\infty \exp\left(\varsigma t\right)h(t)dt. \end{aligned}$$

Thanks to the hypothesis **(H2)**, we can write

$$L(T)\leq C_3\exp\left(-C_1T\right),\ C_3>0,$$

which yields the serial result (4.19), using the fact that $F_3\left(t\right), F_4\left(t\right)$ are positive and the other side of the equivalence relation (4.16) again. The proof is complete. □

5. Numerical approximation

In this section, we will solve numerically the system (1.9)–(1.10) in the one-dimension domain. For that, we use the classic finite difference method for the spatial and temporal discretization. Furthermore, in order to verify the asymptotic behavior of the solution of the discretized problem, we give an example in which the numerical experiment shows that the discrete energy E^n decays exponentially for different choices of the system parameters. Let us introduce the functions $\hat{\varphi}=\varphi_t$, and for any $M,N\in\mathbb{N}$, we introduce the nets

$$\Omega_N=\left\{x_i=i\Delta x,\ i=0,...,N+1 \text{ where } \Delta x=\frac{1}{N+1}\right\},$$

$$\Gamma_M=\left\{t_n=n\Delta t,\ n=0,...,M+1 \text{ where } \Delta t=\frac{T}{M+1}\right\}.$$

Our problem is to find $(\hat{\varphi},\psi)$ satisfying the following numerical scheme:

$$\begin{cases} \frac{\rho_1}{\Delta t}\left(\hat{\varphi}_i^n-\hat{\varphi}_i^{n-1}\right)=\frac{k}{(\Delta x)^2}\left(\varphi_{i+1}^n-2\varphi_i^n+\varphi_{i-1}^n\right)+\frac{k}{2\Delta x}\left(\psi_{i+1}^n-\psi_{i-1}^n\right)-\mu_1\hat{\varphi}_i^n \\ -\frac{\rho_1}{(\Delta t)^2}\int_0^{t_n}h\left(t_n-s\right)\left(\hat{\varphi}_i^{n'}-\hat{\varphi}_i^{n'-1}\right)ds+h\left(t_n\right)\varphi_1\left(x_i\right), \\ \frac{-\rho_2}{2\Delta x\Delta t}\left(\hat{\varphi}_{i+1}^n-\hat{\varphi}_{i-1}^n\right)=\frac{-\rho_2}{2\Delta x\Delta t}\left(\hat{\varphi}_{i+1}^{n-1}-\hat{\varphi}_{i-1}^{n-1}\right)+\frac{b}{(\Delta x)^2}\left(\psi_{i+1}^n-2\psi_i^n+\psi_{i-1}^n\right) \\ -\frac{k}{2\Delta x}\left(\varphi_{i+1}^n-\varphi_{i-1}^n\right)-k\psi_i^n, \end{cases} \tag{5.1}$$

where $s_{n'}=n'\Delta s_{n'}, n'=0,...,M'+1$ with $\Delta s_{n'}=\dfrac{t_n}{M'+1}, \varphi_i^n=\varphi\left(x_i,t_n\right), \hat{\varphi}_i^n=\varphi_t\left(x_i,t_n\right), \psi_i^n=\psi\left(x_i,t_n\right)$, $\hat{\psi}_i^n=\psi_t\left(x_i,t_n\right)$, for all $i=1,...,N$ and $n=1,...,M$. To simplify our numerical calculations in our scheme, we consider the discrete boundary conditions given by

$$\left\{\ \psi_0^n=\psi_{N+1}^n=\varphi_{N+1}^n=\varphi_0^n=0,\right. \tag{5.2}$$

and initial conditions

$$\psi_i^0=\psi_0\left(x_i\right),\ \hat{\psi}_i^0=\psi_1\left(x_i\right),\ \varphi_i^0=\varphi_0\left(x_i\right),\ \hat{\varphi}_i^0=\varphi_1\left(x_i\right), \tag{5.3}$$

where

$$\varphi_i^n = \varphi_i^{n-1} + \Delta t\hat{\varphi}_i^n,\ \psi_i^n = \psi_i^{n-1} + \Delta t\hat{\psi}_i^n,$$

for all $i = 1, ..., N$ and $n = 1, ..., M$.

Note that to find $(\hat{\varphi}, \psi)$, we need to solve two coupled systems of algebraic equations. So, to solve the problem (5.1)–(5.3) at each time step, we propose to consider the following fixed-point algorithm that is stopped when the difference between two successive iterations becomes smaller than a given tolerance ε.

$$\begin{cases} \hat{\varphi}_i^{n,l} = \dfrac{k}{c_1 (\Delta x)^2}\left(\varphi_{i+1}^{n,l-1} - 2\varphi_i^{n,l-1} + \varphi_{i-1}^{n,l-1}\right) + \dfrac{k}{c_1 2\Delta x}\left(\psi_{i+1}^{n,l-1} - \psi_{i-1}^{n,l-1}\right) \\ +\dfrac{\rho_1}{c_1\Delta t}\hat{\varphi}_i^{n-1} - \dfrac{\rho_1}{(\Delta t)^2}\int_0^{t_n} h(t_n - s)\left(\hat{\varphi}_i^{n',l-1} - \hat{\varphi}_i^{n'-1,l-1}\right)ds + h(t_n)\varphi_1(x_i), \\ \dfrac{b}{(\Delta x)^2}\psi_{i+1}^{n,l} - \left(2\dfrac{b}{(\Delta x)^2} + k\right)\psi_i^{n,l} + \dfrac{b}{(\Delta x)^2}\psi_{i-1}^{n,l} = \dfrac{-\rho_2}{2\Delta x\Delta t}\left(\hat{\varphi}_{i+1}^{n,l} - \hat{\varphi}_{i-1}^{n,l}\right) \\ +\dfrac{\rho_2}{2\Delta x\Delta t}\left(\hat{\varphi}_{i+1}^{n-1,l} - \hat{\varphi}_{i-1}^{n-1,l}\right) + \dfrac{k}{2\Delta x}\left(\varphi_{i+1}^{n,l} - \varphi_{i-1}^{n,l}\right), \end{cases} \tag{5.4}$$

with

$$\begin{cases} \varphi_i^{n,0} = \varphi_i^{n-1},\ \psi_i^{n,0} = \psi_i^{n-1},\ \varphi_i^{n,l} = \varphi_i^{n-1} + \Delta t\hat{\varphi}_i^{n,l}, \\ \psi_i^{n,l} = \psi_i^{n-1} + \Delta t\hat{\psi}_i^{n,l}, \end{cases}$$

for all $i = 1, ..., N$ and $n = 1, ..., M$ and $l = 1, 2...$

To approximate the continuous energy (3.1), we use the trapezoidal quadrature formula to compute the integral $I = \int_0^1 f(x)dx$:

$$I_N = \sum_{i=1}^{N} a_i f(x_i) \approx I,$$

where the weights $\{a_i\}_{i=1}^N$ are given by $a_1 = a_N = \dfrac{K}{2}$, for $i = 2, 3, ..., N-1$, $a_i = K$ with $K = \dfrac{1}{N}$.

The same quadrature formula is used to evaluate the integral with respect to s at each time step t_n. Therefore, the discrete energy formula is given by

$$\begin{aligned} E(t_n) \approx J^n = \frac{1}{2}\sum_{i=1}^{N} & a_i[\rho_1 (\hat{\varphi}_i^n)^2 + k((\varphi_x)_i^n + \psi_i^n)^2 + \frac{\rho_2\rho_1}{k}((\hat{\varphi}_t)_i^n)^2 \\ & + \rho_2((\varphi_{tx})_i^n)^2 + b((\psi_x)_i^n)^2] + \rho_1(h_n * \hat{\varphi}_i^n) \\ & + \frac{\rho_2\rho_1}{k}(h_n * (\hat{\varphi}_t)_i^n), \end{aligned} \tag{5.5}$$

with

$$\hat{\varphi}_i^n = \varphi_t(x_i, t_n),\ (\hat{\varphi}_t)_i^n = \frac{1}{\Delta t}\left(\hat{\varphi}_i^{n+1} - \hat{\varphi}_i^n\right),$$

$$(\hat{\varphi}_t)_i^n = \frac{1}{\Delta t}\left(\hat{\varphi}_i^{n+1} - \hat{\varphi}_i^n\right),$$

$$(\varphi_x)_i^n = \frac{\varphi_{i+1}^n - \varphi_{i-1}^n}{2\Delta x},\ (\psi_x)_i^n = \frac{\psi_{i+1}^n - \psi_{i-1}^n}{2\Delta x} \text{ and } (\varphi_{tx})_i^n = \frac{\hat{\varphi}_{i+1}^n - \hat{\varphi}_{i-1}^n}{2\Delta x}.$$

Next, we describe the following numerical example:

Example 1. *For this numerical test, we choose the following different values for the coefficients of the system:*

$$\rho_1 = 1.1,\ \rho_2 = 10,\ k = 0.01,\ \mu_1 = 10,\ b = 1.5.$$

We run our code for the following discretization parameters: $N = 100$, $M = 200$, $T = 1$. We take $\varepsilon = 10^{-5}$. Also, we choose the following initial conditions:

$$\begin{aligned} \varphi_0(x) &= \frac{17}{20}x^2 e^{-2x},\ \varphi_1(x) = \frac{1}{4}\left(x^3 - \frac{2}{3}x^2\right),\ h(t) = \exp(-4t), \\ \psi_0(x) &= x^3(1-x)^2,\ \varphi_2(x) = x^3 - \frac{2}{3}x^2. \end{aligned}$$

Here are the evolution in time of the solutions φ and ψ, the discrete energy and the evolution with respect to x of φ throughout time.

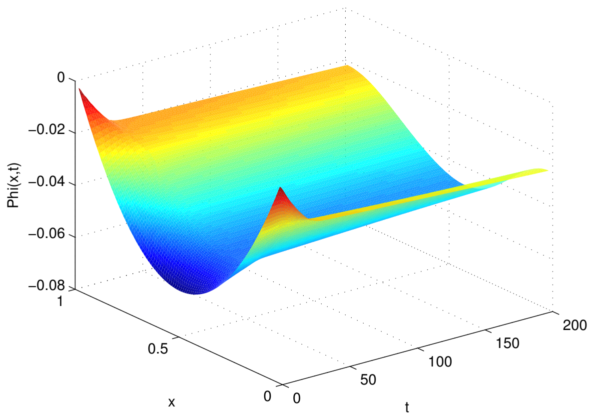

Figure 1. Evolution in time of the function φ.

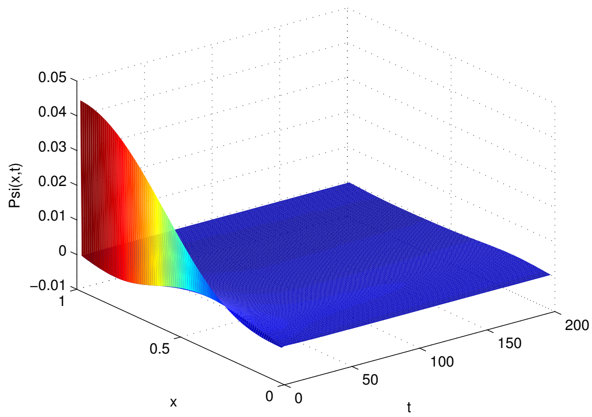

Figure 2. Evolution in time of the function ψ.

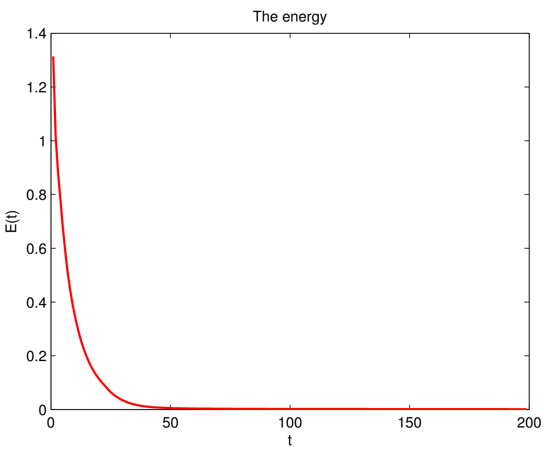

Figure 3. Evolution in time of the discrete energy.

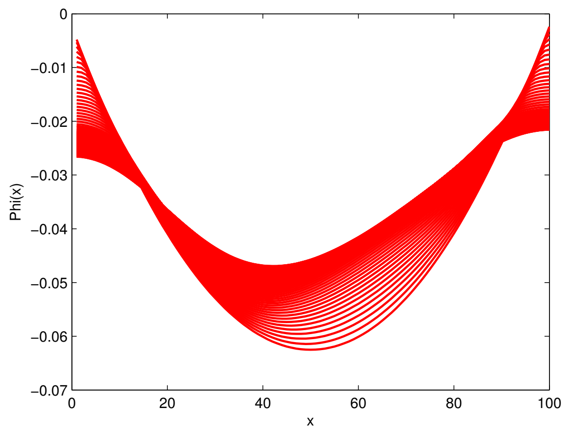

Figure 4. Evolution in x of the function φ throughout time.

In the above numerical example, the graphics presented in Figures 1 and 2 show the evolution in time of the approximation solutions φ and ψ on the interval $[0, T]$, for different choices of the system parameters and of the initial data. Furthermore, Figures 3 and 4 show that the approximate energy (5.5) decays in an exponential manner, which confirms the main theoretical result obtained and the evolution of φ with respect to x throughout time.

6. Conclusions

In this work we investigated the sufficient conditions on the kernel of the neutral delay term to assure the exponential stability of solutions of the Bresse-Timoshenko system subject to this complementary control based on the multipliers technique to construct a suitable Lyapunov functional that allows us to estimate the energy of the considered system. As a future work, we propose to consider the same problem without dissipation due to the frictional damping, and we will search for additional conditions on the kernel of neutral delay term from which the energy can be decreased in an exponential manner.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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