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Research article

The Pontryagin dual of the multiplicative group of positive rational numbers

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Abstract: The Pontryagin dual of the multiplicative group of positive rational numbers is constructed. Then we study its topological generators and representations.

Keywords: Pontryagin dual; topological generator; Hilbert space; representation
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1. Introduction

We are inspired by Alain Connes' pioneering and excellent work [1], where he realized just those imaginary parts of zeros lying on $\mathrm{Re}(s) = \frac{1}{2}$ of L-function to be the spectrums of the differential operator on a Hilbert space. Meanwhile, in loc.cit., he proposed a global trace formula in noncommutative geometry, which is equivalent to the Riemann hypothesis of the L-function $L(\chi, s)$. Nevertheless the trace formula may be hard to prove.

When we try to understand his work, we find a key thought is to deal with the action of the group $\mathbb{R}_+^\times$, i.e., the positive real numbers. Since the positive rational number group $\mathbb{Q}_+^\times$ is dense in $\mathbb{R}_+^\times$, the action of $\mathbb{R}_+^\times$ can be reflected by $\mathbb{Q}_+^\times$. On the other hand, $\mathbb{Q}_+^\times \cong \oplus_p p^{\mathbb{Z}}$ is the expression of the fundamental theorem of arithmetic by group language. Hence, it is an important object for both group theory and arithmetic. The Pontryagin dual $\widehat{\mathbb{Q}_+^\times}$ of $\mathbb{Q}_+^\times$ is a very interesting space and the Hilbert function space on $\widehat{\mathbb{Q}_+^\times}$ can give some information about L-function.

First, we calculate the Pontryagin dual group of the multiplicative group of positive rational numbers. Then we get some topological generators in the dual group. Next, we consider the Hilbert space of the dual group, in which we find a function attached to the Dirichlet L-function. For simplicity, we only consider the rational field. The idea can be generalized for general number fields, but it may be more complicated.

2. The dual of $\mathbb{Q}_+^\times$

Denote $\mathbb{R}_+^\times$ the multiplicative group of positive real numbers and $\mathbb{Q}_+^\times$ the positive rational numbers. Then $\mathbb{Q}_+^\times$ is dense in $\mathbb{R}_+^\times$ under the usual topology. Since $\mathbb{Q}_+^\times \cong \oplus_p p^{\mathbb{Z}}$, where $p^{\mathbb{Z}} = \{p^n : n \in \mathbb{Z}\}$, the set $\{p : p \text{ is prime}\}$ is the set of topological generators of $\mathbb{R}_+^\times$.

The subgroup $p^{\mathbb{Z}}$ is discrete in $\mathbb{R}_+^\times$. This fact can be obtained from the topological isomorphism

$$\log : \mathbb{R}_+^\times \to \mathbb{R}, \quad x \mapsto \log x.$$

Since $\log(p^{\mathbb{Z}}) = \mathbb{Z}\log p$, we have $\mathbb{R}_+/p^{\mathbb{Z}} \cong \mathbb{R}/\mathbb{Z}\log p$.

Let p, q be different primes. Then the group $\langle p, q\rangle := \{p^m q^n : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}_+^\times$. This can be deduced by the following proposition.

Proposition 2.1. *Let $a, b \in \mathbb{R}$ with $b \neq 0$, $\frac{a}{b} \notin \mathbb{Q}$. Then $\mathbb{Z}a + \mathbb{Z}b = \{ma + nb : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.*

Proof. Consider the map

$$f : \mathbb{Z}a + \mathbb{Z}b \to \mathbb{R}_{\geq 0}, \quad x \mapsto |x|,$$

where $\mathbb{R}_{\geq 0}$ is the set of the nonnegative real numbers. If $f(\mathbb{Z}a + \mathbb{Z}b) \setminus \{0\}$ has a least absolute value, then $\mathbb{Z}a + \mathbb{Z}b$ is discrete in $\mathbb{R}$. But any discrete subgroup of $\mathbb{R}$ is of the form $\mathbb{Z}x$, $x \in \mathbb{R}$. Since $\frac{a}{b} \notin \mathbb{Q}$, the free abelian group $\mathbb{Z}a + \mathbb{Z}b$ is of rank 2. Hence, $\mathbb{Z}a + \mathbb{Z}b \neq \mathbb{Z}x$ for any $x \in \mathbb{R}$. This means $\mathbb{Z}a + \mathbb{Z}b$ is dense in $\mathbb{R}$. □

Corollary 2.2. *Let p, q be different primes. Then the group $\langle p, q\rangle$ is dense in $\mathbb{R}_+^\times$.*

Proof. Under the isomorphism $\log : \mathbb{R}_+^\times \to \mathbb{R}$, we have

$$\log\langle p, q\rangle \cong \mathbb{Z}\log p + \mathbb{Z}\log q.$$

Since $\log p/\log q \notin \mathbb{Q}$, we have that $\mathbb{Z}\log p + \mathbb{Z}\log q$ is dense in $\mathbb{R}$ by Proposition 2.1. Hence $\langle p, q\rangle$ is dense in $\mathbb{R}_+^\times$. □

Corollary 2.3. *The cyclic group $\langle e^{2\pi i\theta}\rangle = \{e^{2\pi in\theta} : n \in \mathbb{Z}\}$ is dense in $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ for those $\theta \notin \mathbb{Q}$.*

Proof. Consider the homomorphism

$$f_\theta : \mathbb{R} \to S^1, \quad x \mapsto e^{2\pi i\theta x}.$$

Then the kernel of f_θ is $\mathbb{Z}\frac{1}{\theta}$. Since $\theta \notin \mathbb{Q}$, the group $\mathbb{Z} + \mathbb{Z}\frac{1}{\theta}$ is dense in $\mathbb{R}$ by Proposition 2.1. Hence $\langle e^{2\pi i\theta}\rangle$ is dense in S^1. □

Remark 2.4 (Kronecker's approximation theorem). There is a general form of Kronecker's approximation theorem, that is, the real numbers $1, a_1, \cdots, a_n$ are linearly independent over $\mathbb{Q}$ if and only if the set $\{(ma_1, \cdots, ma_n) : m \in \mathbb{Z}\}$ is dense in the torus $\mathbb{R}^n/\mathbb{Z}^n$. One can see [2] for details or see Corollary 3.4.

Let G be a topological group. We denote by $\widehat{G}$ the Pontryagin dual of G. The topological group $\mathbb{R}_+^\times$ is topologically isomorphic to $\mathbb{R}$. Since $\mathbb{R}$ is self-dual, so is $\mathbb{R}_+^\times$. The group $p^{\mathbb{Z}}$ is the discrete topology. Under the commutative diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{e^x} & \mathbb{R}_+^\times \\ \uparrow & & \uparrow \\ \mathbb{Z}\log p & \xrightarrow{\cong} & p^{\mathbb{Z}} \end{array}$$

we have $\widehat{p^{\mathbb{Z}}} \cong \mathbb{R}_+^\times/p^{\mathbb{Z}}$. Further, the dual of $\mathbb{Q}_+^\times$ with the discrete topology is the compact group $\prod_p \mathbb{R}_+^\times/p^{\mathbb{Z}}$. This can be obtained by the following well known theorem (see [3, Theorem 23.21], which is also a special case of [4, Proposition 1.1.13])

Theorem 2.5. *Let $\{G_i\}_i$ be a set of compact groups. Then*

$$\widehat{\prod_i G_i} \cong \bigoplus_i \widehat{G_i}.$$

Corollary 2.6. *Let $\mathbb{Q}_+^\times$ be given by the discrete topology. Then we have*

$$\widehat{\mathbb{Q}_+^\times} \cong \prod_p \mathbb{R}_+^\times/p^{\mathbb{Z}} \cong \prod_p \mathbb{R}/\mathbb{Z}\log p \cong \prod_p \mathbb{R}/\mathbb{Z}\frac{1}{\log p}.$$

Proof. Since $\widehat{p^{\mathbb{Z}}} \cong \mathbb{R}_+^\times/p^{\mathbb{Z}}$, we have that $\widehat{\mathbb{R}_+^\times/p^{\mathbb{Z}}} \cong p^{\mathbb{Z}}$ by the property of Pontryagin dual. Moreover,

$$\widehat{\prod_p \mathbb{R}_+^\times/p^{\mathbb{Z}}} \cong \bigoplus_p \widehat{\mathbb{R}_+^\times/p^{\mathbb{Z}}} \cong \bigoplus_p p^{\mathbb{Z}} \cong \mathbb{Q}_+^\times.$$

Hence, we have $\widehat{\mathbb{Q}_+^\times} \cong \prod_p \mathbb{R}_+^\times/p^{\mathbb{Z}}$. The second isomorphism in this corollary is deduced locally by the isomorphisms

$$\mathbb{R}_+^\times/p^{\mathbb{Z}} \xrightarrow{\log} \mathbb{R}/\mathbb{Z}\log p.$$

The third isomorphism in this corollary is deduced locally by the following isomorphism restricting on the group $\mathbb{Z}\log p$

$$\mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto \frac{x}{(\log p)^2}.$$

□

3. The topological generators of compact groups

The inverse limit of $\mathbb{Z}/m\mathbb{Z}$ with respect to the natural maps is $\hat{\mathbb{Z}} := \varprojlim_m \mathbb{Z}/m\mathbb{Z} = \prod_p \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the p-adic integers. We know that $\mathbb{Z}$ is dense in $\hat{\mathbb{Z}}$ and a topological generator is 1 in $\hat{\mathbb{Z}}$.

There is a canonical homomorphism from $\mathbb{Z}$ to $\mathbb{R}/\mathbb{Z}\log p$ defined by

$$\tau :\ \mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}\log p \xrightarrow{\cong} S^1,\ \ m \longmapsto e^{2\pi i m/\log p}.$$

Then $\mathbb{Z}$ is dense in $\mathbb{R}/\mathbb{Z}\log p$ by Corollary 2.3, this is because $\log p \notin \mathbb{Q}$. Similarly, for the map

$$\tau' : \mathbb{Z} \longrightarrow \mathbb{R}\Big/\mathbb{Z}\frac{1}{\log p} \xrightarrow{\cong} S^1, \ \ m \longmapsto e^{2\pi i m \log p},$$

$\mathbb{Z}$ is dense in $\mathbb{R}/\mathbb{Z}\frac{1}{\log p}$. In general, we have the following theorem.

Theorem 3.1. *For $r \in \mathbb{Q} \setminus \{0\}$, let τ_r be the canonical homomorphism*

$$\tau_r : \mathbb{Z} \longrightarrow \prod_p \mathbb{R}\Big/\mathbb{Z}\frac{1}{\log p}, \ \ m \longmapsto (rm)_p,$$

where $(rm)_p$ means that the p-th component is $rm \in \mathbb{R}\big/\mathbb{Z}\frac{1}{\log p}$. Then $\tau_r(\mathbb{Z})$ is dense in $\prod_p \mathbb{R}/\mathbb{Z}\frac{1}{\log p}$. Hence, any nonzero rational number r is a topological generator of $\prod_p \mathbb{R}/\mathbb{Z}\frac{1}{\log p}$.

Proof. It is easy to see that τ_r is injective. Let V be the closure of $\tau_r(\mathbb{Z})$ in $\prod_p \mathbb{R}/\mathbb{Z}\frac{1}{\log p}$. If $V \neq \prod_p \mathbb{R}/\mathbb{Z}\frac{1}{\log p}$, then $\prod_p \mathbb{R}/\mathbb{Z}\frac{1}{\log p} \setminus V$ is a nonempty open set. Let J be a finite set of primes such that for each $p \in J$, there is a nonempty open set U_p satisfying $\left(\prod_{p\in J} U_p \times \prod_{p\notin J} \mathbb{R}/\mathbb{Z}\frac{1}{\log p}\right) \cap V = \emptyset$. Consider the group $\prod_{p\in J} \mathbb{R}/\mathbb{Z}\frac{1}{\log p}$ and its open set $\prod_{p\in J} U_p$. Let V_J be the closure of $\tau_r(\mathbb{Z})$ in $\prod_{p\in J} \mathbb{R}/\mathbb{Z}\frac{1}{\log p}$. Then we have $V_J \cap \prod_{p\in J} U_p = \emptyset$. This means that $\tau_r(\mathbb{Z})$ is not dense in $\prod_{p\in J} \mathbb{R}/\mathbb{Z}\frac{1}{\log p}$, which is a contradiction by the following claim.

Claim: $\tau_r(\mathbb{Z})$ is dense in $\prod_{p\in J} \mathbb{R}/\mathbb{Z}\frac{1}{\log p}$ for each finite set J of primes.

Consider the isomorphism

$$f : \mathbb{R}^n \to \mathbb{R}^n, \ \ (a_1, \cdots, a_n) \mapsto (a_1 \log p_1, \cdots, a_n \log p_n).$$

Then we have $f(\prod_{i=1}^n \mathbb{Z}\frac{1}{\log p_i}) = \prod_{i=1}^n \mathbb{Z}$. Also, $f(1, \cdots, 1) = (\log p_1, \cdots, \log p_n)$. By Kronecker's approximation theorem(see Remark 2.4), we just need to show that the numbers $1, \log p_1, \cdots, \log p_n$ are linearly independent over $\mathbb{Q}$. Suppose that there is

$$a_0 + a_1 \log p_1 + \cdots + a_n \log p_n = 0,$$

where $a_i \in \mathbb{Q}$. Multiplying the above equation by a suitable integer, we can assume $a_i \in \mathbb{Z}$. Then we have

$$\log(p_1^{a_1} \cdots p_n^{a_n}) = -a_0.$$

Taking exponential function on both sides, we have

$$p_1^{a_1} \cdots p_n^{a_n} = e^{-a_0}.$$

Since we are assuming that $a_0 \in \mathbb{Z}$, e^{-a_0} is rational only when $a_0 = 0$. Then by the fundamental theorem of arithmetic, we have $a_1 = \cdots = a_n = 0$, that is, the real numbers $1, \log p_1, \cdots, \log p_n$ are linearly independent over $\mathbb{Q}$. Hence, $\tau_r(\mathbb{Z})$ is dense in $\prod_{p\in J} \mathbb{R}/\mathbb{Z}\frac{1}{\log p}$ for each finite set J of primes. □

This theorem shows that the group $\prod_p \mathbb{R}/\mathbb{Z}\frac{1}{\log p}$ is similar to the group $\prod_p \mathbb{Z}_p$. Both have a topological generator, but their topologies are very different. $\prod_p \mathbb{R}/\mathbb{Z}\frac{1}{\log p}$ is a torus with rank ∞, which is a compact connected Hausdorff space, while $\prod_p \mathbb{Z}_p$ is a compact totally disconnected Hausdorff space.

Corollary 3.2. *A topological generator of* $\prod_p \mathbb{R}/\mathbb{Z}\log p$ *is* $(\cdots,(\log p)^2,\cdots)$.

Proof. This follows from Theorem 3.1 and the isomorphism

$$\prod_p \mathbb{R}\Big/\mathbb{Z}\frac{1}{\log p} \longrightarrow \prod_p \mathbb{R}/\mathbb{Z}\log p, \quad (x_p)_p \longmapsto (x_p(\log p)^2)_p.$$

□

The following theorem is very useful.

Theorem 3.3. *Let* G *be a compact group and* $\widehat{G}$ *be its character group. Then* $g \in G$ *is a topological generator if and only if* $\chi(g) \neq 1$ *for each nontrivial character* $\chi \in \widehat{G}$.

Proof. See [5, Page 66, Paragraph 3], or [6, Theorem I]. □

In fact, the above result is equivalent to Kronecker's approximation theorem in the torus case.

Corollary 3.4. *The real numbers* $1, a_1, \cdots, a_n$ *are linearly independent over* $\mathbb{Q}$ *if and only if the set* $\{(ma_1,\cdots,ma_n): m \in \mathbb{Z}\}$ *is dense in the torus* $\mathbb{R}^n/\mathbb{Z}^n$.

Proof. The character group of $\mathbb{R}^n/\mathbb{Z}^n$ is $\mathbb{Z}^n$. Let $\chi \in \widehat{\mathbb{R}^n/\mathbb{Z}^n}$. Then χ can be expressed by

$$\chi(t_1,\ldots,t_n) = \prod_{j=1}^{n} e^{2\pi i m_j t_j} = e^{2\pi i \sum_{j=1}^n m_j t_j},$$

where $m_j \in \mathbb{Z}, t_j \in \mathbb{R}/\mathbb{Z}$. We can view this χ as the vector $m_\chi = (m_1,\cdots,m_n)$. Then by Theorem 3.3, $(a_1,\cdots,a_n)$ is a topological generator of $\mathbb{R}^n/\mathbb{Z}^n$ if and only if

$$\chi(a_1,\cdots,a_n) = e^{2\pi i \sum_{j=1}^n m_j a_j} \neq 1$$

for every nontrivial character χ.

Let $(a_1,\cdots,a_n)$ be a topological generator of $\mathbb{R}^n/\mathbb{Z}^n$. Suppose $m_0 + \sum_{j=1}^n m_j a_j = 0$, where $m_j \in \mathbb{Q}$. Moreover, we can assume $m_j \in \mathbb{Z}$. If $m_1 = \cdots = m_n = 0$ does not hold, then we get a nontrivial character $m_\chi = (m_1,\cdots,m_n)$. For this nontrivial character χ, we have $\chi(a_1,\cdots,a_n) = 1$, which is a contradiction. Hence, $m_1 = \cdots = m_n = 0$, implying that $m_0 = 0$. This means $1, a_1,\cdots,a_n$ are linearly independent over $\mathbb{Q}$.

Let $1, a_1,\cdots,a_n$ be linearly independent over $\mathbb{Q}$. If $(a_1,\cdots,a_n)$ is not a topological generator of $\mathbb{R}^n/\mathbb{Z}^n$, then there exists a nontrivial character χ, i.e., $m_\chi = (m_1,\cdots,m_n) \neq 0$ such that

$$\chi(a_1,\cdots,a_n) = e^{2\pi i \sum_{j=1}^n m_j a_j} = 1.$$

Then we have $\sum_{j=1}^n m_j a_j \in \mathbb{Z}$. Hence there exists some $m \in \mathbb{Z}$ such that $m + \sum_{j=1}^n m_j a_j = 0$. Since $m_1,\cdots,m_n$ are not all 0, this means $1, a_1,\cdots,a_n$ are not linearly independent over $\mathbb{Q}$, which is a contradiction. Therefore, $(a_1,\cdots,a_n)$ is a topological generator of $\mathbb{R}^n/\mathbb{Z}^n$. □

Lemma 3.5. *Let* J *be a finite set of primes,* $k_p \in \mathbb{N}$, *and* $n_p \in \mathbb{Z}$ *such that* $n_p = 0$ *or* $(p, n_p) = 1$. *If*

$$\sum_{p\in J} n_p/p^{k_p} = n$$

for some $n \in \mathbb{Z}$, *then we have* $n_p = 0$ *for all* $p \in J$.

Proof. Consider the equation

$$1 = e^{2\pi i \sum_{p\in J} n_p/p^{k_p}} = \prod_{p\in J} e^{2\pi i n_p/p^{k_p}}.$$

Take $q \in J$ and let

$$m_q = \prod_{p\in J, p\neq q} p^{k_p}.$$

Then we have

$$1^{m_q} = e^{2\pi i n_q m_q/q^{k_q}}.$$

Since $(m_q, q) = 1$, we have $n_q = 0$ for all $q \in J$. □

Remark 3.6. Another proof of Lemma 3.5 suggested by one of reviewers is as following: Multiply $\prod_{p\in J} p^{k_p}$ on the equation $\sum_{p\in J} n_p/p^{k_p} = n$. Then one has $\sum_{p\in J} n_p m_p = n\prod_{p\in J} p^{k_p}$, where $m_p = \frac{1}{p^{k_p}}\prod_{p\in J} p^{k_p}$. Modulo p for this equation, one has $n_p m_p \equiv 0 \mod p$. Since $(m_p, p) = 1$, we have $n_p = 0$ for all $p \in J$ by the condition on n_p.

Denote

$$\mathbb{T}_p = \mathbb{R}\Big/\mathbb{Z}\frac{1}{\log p}.$$

Then we have the following corollary.

Corollary 3.7. *Consider the compact group*

$$\prod_p \mathbb{T}_p \times \mathbb{Z}_p.$$

Then the image of the diagonal map

$$\mathbb{Z} \longrightarrow \prod_p \mathbb{T}_p \times \mathbb{Z}_p, \quad n \longmapsto (\cdots, n, n, \cdots)$$

is dense in $\prod_p \mathbb{T}_p \times \mathbb{Z}_p$.

Proof. The character group of $\mathbb{T}_p$ (resp. $\mathbb{Z}_p$) is $\mathbb{Z}\frac{1}{\log p}$ (resp. $\mathbb{Z}(p^\infty)$, see [3, (25.2)], which is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$. Here $\mathbb{Q}_p$ is the p-adic field). A character χ_1 of $\mathbb{T}_p$ can be expressed by $\chi_1(t) = e^{2\pi i m t \log p}$, where $m \in \mathbb{Z}, t \in \mathbb{T}_p$. However, a character χ_2 of $\mathbb{Z}_p$ can be expressed by $\chi_2(t) = e^{2\pi i m\tilde{t}/p^k}$, where $m \in \mathbb{Z}, k \in \mathbb{N}, t \in \mathbb{Z}_p, \tilde{t} \in \mathbb{Z}$ such that $\tilde{t} \equiv t \mod p^k$, $(m, p) = 1$ or $m = 0$. Hence, by Theorem 2.5, any character χ of $\prod_p \mathbb{T}_p \times \mathbb{Z}_p$ can be written as follows

$$\chi((t_p)_p, (s_p)_p) = \prod_{p\in J} e^{2\pi i m_p t_p \log p} e^{2\pi i n_p \tilde{s}_p/p^{k_p}},$$

where J is some finite set of primes, $m_p, n_p \in \mathbb{Z}, k_p \in \mathbb{N}, t_p \in \mathbb{T}_p, s_p \in \mathbb{Z}_p, \tilde{s}_p \in \mathbb{Z}$ such that $\tilde{s}_p \equiv s_p \mod p^{k_p}$.

Supposing

$$1 = \chi((1, 1\cdots)) = e^{2\pi i\left(\sum\limits_{p\in J} m_p \log p + \sum\limits_{p\in J} n_p/p^{k_p}\right)},$$

we obtain

$$\sum_{p\in J} m_p \log p + \sum_{p\in J} n_p/p^{k_p} \in \mathbb{Z}.$$

So, for some $n \in \mathbb{Z}$, there is

$$\log(\prod_{p\in J} p^{m_p}) = n - \sum_{p\in J} n_p/p^{k_p},$$

that is,

$$\prod_{p\in J} p^{m_p} = e^{n-\sum\limits_{p\in J} n_p/p^{k_p}}. \tag{3.1}$$

Since e is a transcendental number, the right side of the above equation is a rational number only if

$$n - \sum_{p\in J} n_p/p^{k_p} = 0.$$

From Lemma 3.5, we have $n_p = 0$ for all $p \in J$.

Furthermore, by (3.1), we obtain

$$\prod_{p\in J} p^{m_p} = 1.$$

The fundamental theorem of arithmetic implies that $m_p = 0,\ \forall p \in J$.

This indicates that χ is a trivial character. Hence, by Theorem 3.3, the image of $\mathbb{Z}$ is dense in $\prod_p \mathbb{T}_p \times \mathbb{Z}_p$. □

4. The representation of compact abelian groups

Let G be a compact abelian Lie group. All the irreducible representations of G are of dimension 1. Let $L^2(G)$ be the Hilbert space of G, which consists of square integrable complex functions. Let $\widehat{G}$ denote the character group of G. Each function $f \in L^2(G)$ can be uniquely expressed by the series

$$f = \sum_{\varphi\in\widehat{G}} c_\varphi \cdot \varphi,$$

where $c_\varphi \in \mathbb{C}$ (see [7, §17.B]).

For $G = \mathbb{R}/\mathbb{Z}$, the character group $\widehat{G} \cong \mathbb{Z}$, because the characters $\varphi_n : G \to S^1$ are of the form

$$\varphi_n : \mathbb{R}/\mathbb{Z} \to S^1, \ \ \theta \mapsto e^{2\pi i n\theta}.$$

For $G = \mathbb{R}/\mathbb{Z}\log p$, the characters of G are

$$\varphi_{n_p,p} : \mathbb{R}/\mathbb{Z}\log p \to S^1, \ \ \theta \mapsto e^{\frac{2\pi i n_p\theta}{\log p}}. \tag{4.1}$$

Hence, for $G = \prod_p \mathbb{R}/\mathbb{Z}\log p$, the character of G is of the form

$$\varphi : \prod_p \mathbb{R}/\mathbb{Z}\log p \to S^1, \ \ (\theta_p)_p \mapsto \prod_p e^{\frac{2\pi i n_p\theta_p}{\log p}} = e^{2\pi i \sum_p \frac{n_p\theta_p}{\log p}}, \tag{4.2}$$

where $n_p = 0$ for almost all p by Theorem 2.5. The character of $\prod_p \mathbb{R}/\mathbb{Z}\log p$ is φ as in Eq (4.2). All the characters $\{\varphi\}$ form a complete orthogonal base of $L^2(\prod_p \mathbb{R}/\mathbb{Z}\log p)$. Then for each integrable function $f \in L^2(\prod_p \mathbb{R}/\mathbb{Z}\log p)$, we obtain its Fourier series (see [8, §17])

$$f\left((\theta_p)_p\right) = \sum_{\varphi\in\widehat{G}} c_\varphi e^{2\pi i \sum_p \frac{n_{p,\varphi}\theta_p}{\log p}} \tag{4.3}$$

where $G = \prod_p \mathbb{R}/\mathbb{Z}\log p$, $c_\varphi = \int_G f(g) e^{-2\pi i \sum_p \frac{n_{p,\varphi}\theta_p}{\log p}} dg$, the Haar measure dg of G such that $\int_G dg = 1$. An integrable function $f \in L^2(\prod_p \mathbb{R}/\mathbb{Z}\log p)$ is uniquely determined by its Fourier series. Furthermore, for each $f \in L^2(\prod_p \mathbb{R}/\mathbb{Z}\log p)$, we have the Parseval equation (see [8, §17])

$$\int_G |f(g)|^2 dg = \sum_{\varphi\in\widehat{G}} |c_\varphi|^2,$$

where $G = \prod_p \mathbb{R}/\mathbb{Z}\log p$.

Take a Dirichlet character $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \longrightarrow S^1$. The L-function of χ is

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}},$$

which is absolutely convergent on $\mathrm{Re}(s) > 1$. For p dividing m, we set $\chi(p) = 0$. So, taking the logarithm, we have

$$\log L(\chi, s) = -\sum_p \log(1-\chi(p)p^{-s}) = \sum_p \sum_{n=1}^{\infty} \frac{\chi(p^n)}{np^{ns}}, \quad \mathrm{Re}(s) > 1, \tag{4.4}$$

where we take a branch of the logarithm such that $\log(\mathbb{R}_+^\times) = \mathbb{R}$.

From (4.4), we have

$$-\frac{L'(\chi, s)}{L(\chi, s)} = \sum_p \sum_{n=1}^{\infty} \frac{\chi(p^n)\log p}{p^{ns}}, \quad \mathrm{Re}(s) > 1.$$

Theorem 4.1. *For each $s \in \mathbb{C}$ with $Re(s) > \frac{1}{2}$, the value $\log L(\chi, s)$ corresponds to a function*

$$f_{s,\chi}(\theta_p)_p = \sum_p \sum_{n=1}^{\infty} \frac{\chi(p^n)\sqrt{\log p}}{np^{ns}} \frac{e^{\frac{2\pi i n\theta_p}{\log p}}}{\sqrt{\log p}} \in L^2\left(\prod_p \mathbb{R}/\mathbb{Z}\log p\right).$$

Furthermore, $f_{s,\chi}(0) = \log L(\chi, s)$ for $Re(s) > 1$.

Proof. Consider the Eq (4.3). For each character $\varphi_{n,p}$ in Eq (4.1), which can be seen as the character of $\prod_p \mathbb{R}/\mathbb{Z}\log p$ through the projection $\prod_p \mathbb{R}/\mathbb{Z}\log p \to \mathbb{R}/\mathbb{Z}\log p$, we take $c_{p,n} = \frac{\chi(p^n)\sqrt{\log p}}{np^{ns}}$; for the other remaining character φ of $\prod_p \mathbb{R}/\mathbb{Z}\log p$, we let $c_\varphi = 0$. Then we have

$$\sum_p \sum_{n=1}^{\infty} \frac{\chi(p^n)\sqrt{\log p}}{np^{ns}} \overline{\left(\frac{\chi(p^n)\sqrt{\log p}}{np^{ns}}\right)} \le \sum_p \sum_{n=1}^{\infty} \frac{\log p}{p^{2n\mathrm{Re}(s)}} = -\frac{\zeta'(2\mathrm{Re}(s))}{\zeta(2\mathrm{Re}(s))},$$

which is absolutely convergent when $\mathrm{Re}(s) > 1/2$. Hence, we have an integrable function

$$f_{s,\chi} = \sum_p \sum_{n=1}^{\infty} \frac{\chi(p^n)\sqrt{\log p}}{np^{ns}} \frac{e^{\frac{2\pi in\theta_p}{\log p}}}{\sqrt{\log p}} \in L^2\left(\prod_p \mathbb{R}/\mathbb{Z}\log p\right).$$

It is easy to see $f_{s,\chi}(0) = \log L(\chi, s)$ for $\mathrm{Re}(s) > 1$. □

5. Conclusions

We calculate the Pontryagin dual of the multiplicative group of positive rational numbers under the discrete topology. We find some canonical topological generators in the dual group. Besides, we find some canonical generator in other compact group. We study the representation of the dual group the multiplicative group of positive rational numbers. Furthermore, in the Hilbert function space of the dual group we find a function attached to the Dirichlet L-function , which may be a new way to understand the Dirichlet L-function.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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