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Research article

A novel approach to study ternary semihypergroups in terms of prime soft hyperideals

Shahida Bashir[1,*], Rabia Mazhar[1], Bander Almutairi[2] and Nauman Riaz Chaudhry[3]

[1] Department of Mathematics, University of Gujrat, Gujrat 50700, Pakistan
[2] Department of Mathematics, College of Science, King Saud University, P.O. Box 2455 Riyadh 11451, Saudi Arabia
[3] Department of Computer Science, University of Gujrat, Gujrat 50700, Pakistan

* **Correspondence:** Email: shahida.bashir@uog.edu.pk; Tel: +923008336240.

Abstract: In this paper, we give the generalized form of soft semihypergroups in ternary structure and have studied it with the help of examples. There are some structures that are not appropriately handled by using the binary operation of the semihypergroup, such as all the sets of non-positive numbers are not closed under binary operation but hold for ternary operation. To deal with this type of problem and handling special type of uncertainty, we study the ternary semihypergroup in terms of prime soft hyperideals. We have introduced prime, strongly prime, semiprime, irreducible and strongly irreducible soft bi-hyperideals in ternary semihypergroups and studied certain properties of these soft bi-hyperideals in ternary semihypergroups. The main advantage of this paper is that we proved that each soft bi-hyperideal of ternary semihypergroup K is strongly prime if it is idempotent and the set of soft bi-hyperideals of K is totally ordered by inclusion.

Keywords: soft subsemihypergroups; prime (strongly prime) soft bi-hyperideals; semiprime soft bi-hyperideals; irreducible (strongly irreducible) soft bi-hyperideals
Mathematics Subject Classification: 20N10, 18B40

1. Introduction

In classical mathematics, all the mathematical formulas and methods are exact which cannot deal with the problems of having uncertainty and incomplete data. Many theories are presented by scientists to tackle such complications such as the vague set theory [1], interval mathematics [2], rough set theory [3,4] and fuzzy set theory [5,6]. According to fuzzy set theory, the problems with uncertainties are solved by use of membership functions. As time passes, many researchers noticed that there are no parameterization tools in the fuzzy set theory. In 1999, Molodtsov gave the idea of soft set to remove this inadequacy [7]. He offered parameters that are helpful to tackle the uncertainties occurring in medical diagnosis and decision making issues. In industrialized countries, the second most reason of cancer death of men is prostate cancer, which depends on elements like age, ethnic background, family cancer history, the level of prostate-specific antigen in blood etc. Many researchers are working to find the risk of prostate cancer with the help of fuzzy set and soft set theories [8].

Zakri et al. [9] have worked to diagnose the educational complications for students with the applications of soft sets and fuzzy sets. They created a survey of dismissed student in Saudi Arabia, Jazan University, Science Department Girls, Mathematic Department from 2009 to 2013, as given in the following table and find the risk of dismissed by using soft set. In Table 1, number of dismissed students is given. Figure 1 shows the graphical representation of this risk.

Table 1. Data of dismissed students.

$Year$	2009	2010	2011	2012	2013
$No.of\ \ dismissed\ \ student$	16	20	33	65	43

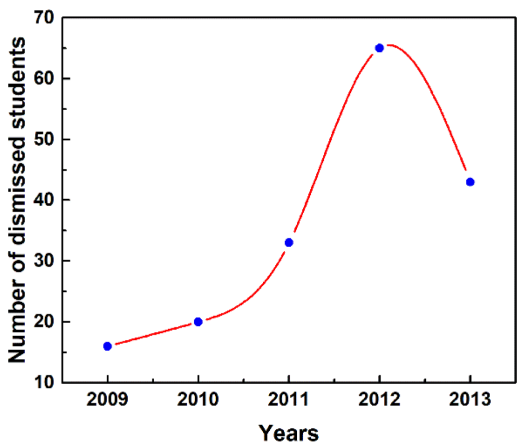

Figure 1. Graph of dismissed students.

1.1. Related works

In 1934, at the 8th Congress of Scandinavian Mathematicians, a French Mathematician Marty gave a theory of algebraic hyperstructure [10]. For more applications and representation of hyperstructures, see [11–13] respectively. In schools, especially at basic level, both academic skills and teacher trainings are not enough to reach required and good consequences unless there is a coperative, pleasent, kind and positive relationship. These associations between students are expected preconditions for organizing interferences focused to get appropriate teaching and learning ability. Consider a set K consisting of students of a specific classroom S. Due to a scientific approach to the relationship between students in a classroom, a final set of relationships R is determined. Initially, several researchers studied social relationships within a school by using a set of binary relations. The most efficient teaching methods within a classroom have also been studied in [14] by using hyperoperations. If n is the number of members of the class, to apply this method it is necessary to propose 2nd meetings for interviews (as for a football league with n teams). This technique is used to get hyperstructures associated with the class and is therefore very significant.

In a book by Corsini and Leoreanu [15], there are a lot of applications of algebraic hyperstructures in the fields of cryptography, geometry, automata, median algebras, relation algebras, artificial intelligence, hypergraphs, binary relations, probabilities, lattices, rough sets, fuzzy sets and codes. Algebraic hyperstructures are studied in many countries of Europe, America and Asia. Also, fuzzy hyperstructure is studied by many researchers [16]. The researchers are attracted to hyperstructures due to their distinctive property that hyperstructure multiplication of any two entries of a set is a set. Usually, the multiplication of any two entries of a set is an entry which belongs to that set. Because of the multi-valued property, hyperstructures are better than the common structures, which give all the possible results of a problem between individuals.

1.2. Innovative contribution

In 1932, ternary algebraic structure was first studied by D. H. Lehmer [17]. To deal with the mathematical frameworks that are not closed under binary operation, we have used ternary operation. For example, $A = \{-i, 0, i\}$ is a semigroup under ternary multiplication. Also Z^+(non-negative integers) is a semigroup because it is closed under binary multiplication while Z^-(negative integers) does not close under binary multiplication, but it is closed under ternary multiplication that results in the formation of a ternary semigroup. An algebraic structure with one associative hyperoperation is called ternary semihypergroup and it is a specific case of an n-ary semihypergroup for n = 3 [18]. A semihypergroup can be reduced to ternary semihypergroup while converse may not be true as a ternary semihypergroup may not be semihypergroup under usual multiplication. Bashir and Du worked on ordered and fuzzy ordered ternary semigroup [19,20]. Continuing this work, Bashir et al. studied bipolar fuzzy ideals of ordered ternary semigroup [21], rough fuzzy ideals of ternary semigroup [22] and bipolar fuzzy ideals of ternary semiring [23].

1.3. Literature review

After introducing the concept of soft set, Molodtsov gave soft sets techniques in 2006 [24]. This theory is studied in different directions by Naz and Shabir [25,26]. Tripathy applied soft sets in game theory [27]. The study of operations of soft set is given by Sezgin et al. [28,29]. Many operations on soft sets are also studied in [30,31]. The comparison of soft sets to fuzzy sets and rough sets is given by Feng et al. [32]. Aktas and Cagman applied soft set on group [33]. Davvaz [34] has done work on soft semihypergroups. Shabir and Kanwal [35] worked on prime bi-ideals of semigroups in 2007. Shabir and Bashir worked on prime ideals in ternary semigroup [36]. Later on, Shabir et al. [37] studied prime fuzzy bi-ideals of semigroup in 2010. Bashir et al. studied prime bi-ideals in ternary semiring [38]. Then, Mehmood [39] studied prime fuzzy bi-hyperideals of a semihypergroup in 2012. Hila and Naka [40–42] worked on hyperideals of ternary semihypergroup. Shabir and Naz [43] presented prime soft bi-hyperideals of semihypergroup. In this paper, we have enhanced the work of [43] and transformed all the definitions, propositions and theorems of [43] in ternary semihypergroups and studied on primeness of soft bi-hyperideals of ternary semihypergroups.

1.4. Organization of the paper

This paper is organized as follows in Figure 2.

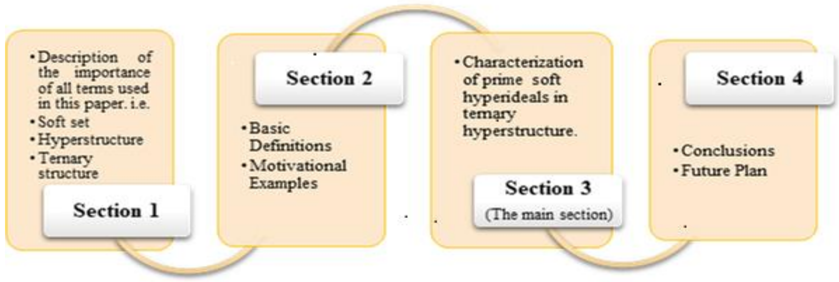

Figure 2. Organization of paper.

The acronyms are listed below in Table 2.

Table 2. List of acronyms.

Acronyms	Representation
TSHG	Ternary subsemihypergroup
SSHG	Soft subsemihypergroup
SBHI	Soft bi-hyperideal
PSBHI	Prime soft bi-hyperideal
SPSBHI	Strongly prime soft bi-hyperideal
SSBHI	Semiprime soft bi-hyperideal
ISBHI	Irreducible soft bi-hyperideal
SISBHI	Strongly irreducible soft bi-hyperideal
iff	If and only if

2. Preliminaries

Some basic definitions and notions are presented here.

Let K be a non-empty set and $P(K)$ be the power set of K. A pair $(K,\circ)$ is called a hypergroupoid if $\circ : K \times K \to P(K)$ is a hyperoperation on K. If A and B are non-empty subsets of K and $x \in K$, then $x \circ A = \{x\} \circ A$, $A \circ x = A \circ \{x\}$ and $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$. Additionally, $(K,\circ)$ is called a semihypergroup if K is hypergroupoid and for all $l, m, n \in K$, $(l \circ m) \circ n = l \circ (m \circ n)$ [40].

The motivating example is as follow: Let K be a semigroup and S be any subsemigroup of K. Then $K/S = \{x * S \mid x \in K\}$ becomes a semihypergroup under the hyperoperation "$\circ$" is defined as $(x * S) \circ (y * S) = \{z * S; z \in (x * S) * (y * S)\}$ for all $x * S, y * S \in K/S$ [44].

By a subset we always mean a non-empty one. A mapping $f : K \times K \times K \to P(K)$ is called a ternary hyperoperation on K if L, M, N are subsets of K, then $f(L, M, N) = \bigcup_{l \in L, m \in M, n \in N} f(l, m, n)$. A non-empty set with ternary hyperoperation $(K,\circ)$ is known as a ternary semihypergroup if for every

$l,m,n,p,q \in K$ we have, $(l \circ m \circ n) \circ p \circ q = l \circ (m \circ n \circ p) \circ q = l \circ m \circ (n \circ p \circ q)$. A subset M of ternary semihypergroup $(K,\circ)$ is said to be a ternary subsemihypergroup (TSHG) of K iff $M \circ M \circ M \subseteq M$. A subset M of ternary semihypergroup K is called a left (lateral, right respectively) hyperideal of K if $K \circ K \circ M \subseteq M$ $(K \circ M \circ K \subseteq M, M \circ K \circ K \subseteq M,$ respectively). Additionally, if M is left, lateral and right hyperideal of K, then M is called an hyperideal of K. In ternary semihypergroup $(K,\circ)$, an element 0 is called a zero element if for all $l,\ m \in K$, $(0 \circ l \circ m) = (l \circ 0 \circ m) = (l \circ m \circ 0) = \{0\}$. An element e in TSHG $(K,\circ)$ is known as a left identity element, if for any $l \in K$, $(e \circ e \circ l) = \{l\}$. Additionally, $e \in K$ is known as an identity element of K if for any $l \in K$, $(l \circ e \circ e) = (e \circ l \circ e) = (e \circ e \circ l) = \{l\}$ [45]. A subsemihypergroup M of K is called a bi-hyperideal of K if $(M \circ K \circ M \circ K \circ M) \subseteq M$ [46].

Throughout, in this paper ternary semihypergroup is represented by K, Universal set is U, powerset of U is $P(U)$ and R,S,T are non-empty subsets of K. Also, $C(U)$ denotes the collection of all soft sets of K over U and $B(U)$ denotes the set of all SBHIs of K over U.

A soft set f_R over U is a function $f_R : K \to P(U)$ such that $f_R(x) = \Phi$ if $x \notin R$ and its representation is given as $f_R = \{(x, f_R(x)) :\ x \in K, f_R(x) \in P(U)\}$. Let $f_R,\ f_S \in C(U)$ and if for all $x \in K$, $f_R(x) \subseteq f_S(x)$ then f_R is a soft subset of f_S and it is denoted by $f_R \widetilde{\subseteq} f_S$. If $f_S \widetilde{\supseteq} f_R$ and $f_S \widetilde{\subseteq} f_R$, then $f_R \cong f_S$. Let $f_R,\ f_S \in C(U)$, then union of f_R and f_S is denoted by $f_R \widetilde{\cup} f_S = f_{R\cup S}$, where $f_{R\cup S}(x) = f_R(x) \cup f_S(x)$ for all $x \in K$. Let $f_R,\ f_S \in C(U)$, then intersection of f_R and f_S is denoted by $f_R \widetilde{\cap} f_S = f_{R\cup S}$ where $f_{R\cap S}(x) = f_R(x) \cap f_S(x)$ for all $x \in K$ [43].

Example 2.1. Consider a soft set f_R, which shows the "attractiveness of cars" for purchase. Suppose that there are five cars in the universal set U, that is $U = \{k_1, k_2, k_3, k_4, k_5\}$ and $R = \{r_1, r_2, r_3, r_4\}$ is a set of decision parameters, where $r_i (i = 1,2,3,4)$ are the parameters *"expensive"*, *"beautiful"*, *"damaged"* and *"cheap"* respectively. Consider a mapping $f : R \to P(U)$. Suppose that $f(r_1) = \{k_1, k_2\}, f(r_2) = \{k_1, k_2, k_5\}, f(r_3) = \{k_3\}, f(r_4) = \{k_3, k_4\}$. The parameterized family $\{f(r_i), i = 1,2,3,4\}$ can be seen as a collection of approximations:

$$f_R = \begin{Bmatrix} \text{expensive cars} = \{k_1, k_2\}, \text{beautiful cars} = \{k_1, k_2, k_5\}, \\ \text{damaged cars} = \{k_3\}, \text{cheap cars} = \{k_3, k_4\} \end{Bmatrix}.$$

Also, this soft set is expressed in a tabular form as given in Table 3.

Table 3. Tabular representation of a soft set f_R.

U	r_1	r_2	r_3	r_4
k_1	1	1	0	0
k_2	1	1	0	0
k_3	0	0	1	1
k_4	0	0	0	1
k_5	0	1	0	0

To store a soft set in a computer, tabular representation is very useful.

Ternary product of any three soft sets is defined as below.

For $f_R,\ g_S,\ h_T \in C(U)$ and $x \in K$, soft ternay product is defined as

$$(f_R * g_S * h_T)(x) = \begin{cases} \bigcup_{x \in l \circ m \circ n} \{f_R(l) \cap g_S(m) \cap h_T(n)\} & \text{if there exist } l,m,n \in K \text{ such that } x \in l \circ m \circ n \\ \Phi & \text{otherwise} \end{cases}.$$

If we define $A_x = \{(l,m,n) \in K \times K \times K : \ x \in l \circ m \circ n\}$, then $f_R * g_S * h_T$ is stated as

$$(f_R * g_S * h_T)(x) = \begin{cases} \bigcup_{x \in l \circ m \circ n} \{f_R(l) \cap g_S(m) \cap h_T(n)\} & \textbf{if } A_x \neq \Phi \\ \Phi & \textbf{if } A_x = \Phi \end{cases}.$$

For each $l,m,n \in K$, $f_R \in C(U)$, is said to be a soft subsemihypergroup (SSHG) of K over U if $\bigcap_{x \in l \circ m \circ n} \{f_R(x)\} \supseteq f_R(l) \cap f_R(m) \cap f_R(n)$. Additionally, $f_R \in C(U)$ is said to be a soft left (lateral, right respectively) hyperideal of K over U if for all $l,m,n \in K$, $f_R(n) \subseteq \bigcap_{x \in l \circ m \circ n} \{f_R(x)\}$ $\left(f_R(m) \subseteq \bigcap_{x \in l \circ m \circ n} \{f_R(x)\}, f_R(l) \subseteq \bigcap_{x \in l \circ m \circ n} \{f_R(x)\} \text{ respectively}\right)$. If f_R is a soft left, lateral and right hyperideal of K over U, then it is called a soft hyperideal of K over U. A soft hyperideal f_R of K over U is said to be a SBHI of K if for all $l,m,n,p,q \in K$, $\bigcap_{x \in l \circ p \circ m \circ q \circ n} \{f_R(x)\} \supseteq f_R(l) \cap f_R(m) \cap f_R(n)$. For a soft set f_R of K over U, upper ζ-inclusion of f_R is defined as $U(f_R, \zeta) = \{x \in K \ : \ f_R(x) \supseteq \zeta\}$. It can be easily seen that $B(U)$ is closed under ternary product and it is closed under arbitrary intersection.

Proposition 2.2. *A soft set f_R is a TSHG of K over U iff $f_R * f_R * f_R \,\tilde{\subseteq}\, f_R$.*

Proof. Straightforward.

3. Prime soft bi-hyperideals

In this part, PSBHIs, SPSBHIs, SSBHIs, ISBHIs, SISBHIs of ternary semihypergroup K over U are studied and characterized ternary semihypergroup under the structure of softness. Also for better understanding, we have given an example of PSBHIs and SPSBHIs. Here, $B_i (\neq \Phi) \subseteq K$ for all i.

Definition 3.1. *A SBHI $f_B \in B(U)$ is called prime if $l_{B_1} * g_{B_2} * h_{B_3} \,\tilde{\subseteq}\, f_B$, implies $l_{B_1} \,\tilde{\subseteq}\, f_B$, $g_{B_2} \,\tilde{\subseteq}\, f_B$ or $h_{B_3} \,\tilde{\subseteq}\, f_B$ for all $l_{B_1}, g_{B_2}, h_{B_3} \in B(U)$.*

Example 3.2. *Let $K = \{r_1, r_2, r_3, r_4\}$ be a ternary semihypergroup under the operation $\circ$ defined as $(r_1 \circ r_2 \circ r_3) = (r_1 \circ r_2) \circ r_3$ and given in Table 4.*

Table 4. Ternary multiplication.

$\circ$	r_1	r_2	r_3	r_4
r_1	$\{r_1\}$	$\{r_1\}$	$\{r_1\}$	$\{r_1\}$
r_2	$\{r_1\}$	$\{r_1\}$	$\{r_1\}$	$\{r_1\}$
r_3	$\{r_1\}$	$\{r_1\}$	$\{r_1, r_2\}$	$\{r_1, r_2\}$
r_4	$\{r_1\}$	$\{r_1\}$	$\{r_1, r_2\}$	$\{r_1, r_2\}$

Here, $A = \{r_1\}$, $B = \{r_1, r_2\}$, $C = \{r_1, r_2, r_3\}$, $D = \{r_1, r_2, r_4\}$ and K are bi-hyperideals of K. Let $U = \{1,2,3\}$ and define $f_A(r_1) = \{1,2\}$, $f_A(r_2) = f_A(r_3) = f_A(r_4) = \Phi$. Then

$$U(f_A, \zeta) = \begin{cases} \{r_1\} & if\, \zeta = \{1\} \\ \{r_1\} & if\, \zeta = \{2\} \\ \Phi & if\, \zeta = \{3\} \\ \{r_1\} & if\, \zeta = \{1,2\} \\ \Phi & if\, \zeta = \{1,3\} \\ \Phi & if\, \zeta = \{2,3\} \\ \Phi & if\, \zeta = \{1,2,3\} \end{cases}.$$

Since $U(f_A,\zeta)$ is bi-hyperideal of K. So, f_A is a SBHI of K. Now, define $g_B(r_1)=\{1,2,3\}$, $g_B(r_2)=\{1,2\}$, $g_B(r_3)=g_B(r_4)=\Phi$. Then

$$U(g_B,\zeta)=\begin{cases}\{r_1,r_2\} & if\ \zeta=\{1\}\\ \{r_1,r_2\} & if\ \zeta=\{2\}\\ \{r_1\} & if\ \zeta=\{3\}\\ \{r_1,r_2\} & if\ \zeta=\{1,2\}\\ \{r_1\} & if\ \zeta=\{1,3\}\\ \{r_1\} & if\ \zeta=\{2,3\}\\ \{r_1\} & if\ \zeta=\{1,2,3\}\end{cases}.$$

So, g_B is a SBHI of K. Now, define $h_C(r_1)=\{1,2\}$, $h_C(r_2)=\{1,2\}$, $h_C(r_3)=\{1\}$, $h_C(r_4)=\Phi$. Then

$$U(h_C,\zeta)=\begin{cases}\{r_1,r_2,r_3\} & if\ \zeta=\{1\}\\ \{r_1,r_2\} & if\ \zeta=\{2\}\\ \Phi & if\ \zeta=\{3\}\\ \{r_1,r_2\} & if\ \zeta=\{1,2\}\\ \Phi & if\ \zeta=\{1,3\}\\ \Phi & if\ \zeta=\{2,3\}\\ \Phi & if\ \zeta=\{1,2,3\}\end{cases}.$$

So, h_C is a SBHI of K. Now, define $j_D(r_1)=\{1,2\}$, $j_D(r_2)=\{1\}$, $j_D(r_3)=\Phi$, $j_D(r_4)=\{1\}$. Then

$$U(j_D,\zeta)=\begin{cases}\{r_1,r_2,r_4\} & if\ \zeta=\{1\}\\ \{r_1\} & if\ \zeta=\{2\}\\ \Phi & if\ \zeta=\{3\}\\ \{r_1\} & if\ \zeta=\{1,2\}\\ \Phi & if\ \zeta=\{1,3\}\\ \Phi & if\ \zeta=\{2,3\}\\ \Phi & if\ \zeta=\{1,2,3\}\end{cases}.$$

So, j_D is a SBHI of K.

It is easy to see that g_B , h_C and j_D are PSBHIs of K while f_A is not PSBHI of K. The reason is that $g_B*h_C*j_D\,\tilde{\subseteq}\,f_A$, but $g_B\,\tilde{\nsubseteq}\,f_A$, $h_C\,\tilde{\nsubseteq}\,f_A$ and $j_D\,\tilde{\nsubseteq}\,f_A$.

Definition 3.3. *A SBHI $f_B\in B(U)$ is called SPSBHI of K over U if $(l_{B_1}*g_{B_2}*h_{B_3})\,\tilde{\cap}\,(g_{B_2}*h_{B_3}*l_{B_1})\,\tilde{\cap}\,(h_{B_3}*l_{B_1}*g_{B_2})\,\tilde{\subseteq}\,f_B$ implies $l_{B_1}\,\tilde{\subseteq}\,f_B$, $g_{B_2}\,\tilde{\subseteq}\,f_B$ or $h_{B_3}\,\tilde{\subseteq}\,f_B$ for all l_{B_1}, g_{B_2} , h_{B_3} $\in B(U)$.*

***Definition* 3.4.** *A SBHI $f_B\in B(U)$ is called semiprime if $g_{B_1}*g_{B_1}*g_{B_1}\,\tilde{\subseteq}\,f_B$ implies $g_{B_1}\,\tilde{\subseteq}\,f_B$ for all $g_{B_1}\in B(U)$.*

Remark 3.5. *For any ternary semihypergroup K over U, every PSBHI is a SSBHI but its counter does not exist.*

Proposition 3.6. *Let $\{f_{R_i}:\ i\in I\}$ be a collection of PSBHIs of K, then $\tilde{\bigcap}_{i\in I} f_{R_i}$ is SSBHI of K.*

Proof. For all $f_{R_i}\in B(U)$ where $i\in I$, $\tilde{\bigcap}_{i\in I} f_{R_i}\in B(U)$. Now let $g_B\in B(U)$ such that $g_B*g_B*g_B\,\tilde{\subseteq}\,\tilde{\bigcap}_{i\in I} f_{R_i}$. Then, $g_B*g_B*g_B\,\tilde{\subseteq}\,f_{R_i}$ for all $i\in I$. As given f_{R_i} is PSBHI of K for all $i\in I$. This *implies* f_{R_i} *is SSBHI of* K *for all* $i \in I$. *So,* $g_B \mathrel{\tilde{\subseteq}} f_{R_i}$ *for all* $i \in I$, *this implies* $g_B \mathrel{\tilde{\subseteq}} \tilde{\bigcap}_{i\in I} f_{R_i}$. *Hence proved.*

Definition 3.7. *A SBHI* $f_R \in B(U)$ *is said to be an irreducible (strongly irreducible) if* $g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3} \cong f_R$, $(g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3} \mathrel{\tilde{\subseteq}} f_R)$ *implies either* $g_{B_1} \cong f_R$ *or* $h_{B_2} \cong f_R$ *or* $l_{B_3} \cong f_R$, $(g_{B_1} \mathrel{\tilde{\subseteq}} f_R$ *or* $h_{B_2} \mathrel{\tilde{\subseteq}} f_R$ *or* $l_{B_3} \mathrel{\tilde{\subseteq}} f_R)$ *respectively, for all* $g_{B_1}, h_{B_2}, l_{B_3} \in B(U)$.

Proposition 3.8. *If* f_R *is a strongly irreducible semiprime-SBHI of* K, *then it is SPSBHI of* K.

Proof. To prove f_R be a S*PSBHI* of K, consider $(g_{B_1} * h_{B_2} * l_{B_3}) \tilde{\cap} (h_{B_2} * l_{B_3} * g_{B_1}) \tilde{\cap} (l_{B_3} * g_{B_1} * h_{B_2}) \mathrel{\tilde{\subseteq}} f_B$ where $g_{B_1}, h_{B_2}, l_{B_3} \in B(U)$. Also, $\left(g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}\right) \in B(U)$.
$\left(g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}\right) * \left(h_{B_2} \tilde{\cap} l_{B_3} \tilde{\cap} g_{B_1}\right) * \left(l_{B_3} \tilde{\cap} g_{B_1} \tilde{\cap} h_{B_2}\right) \mathrel{\tilde{\subseteq}} \left(g_{B_1} * h_{B_2} * l_{B_3}\right)$ and $\left(g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}\right) * \left(h_{B_2} \tilde{\cap} l_{B_3} \tilde{\cap} g_{B_1}\right) * \left(l_{B_3} \tilde{\cap} g_{B_1} \tilde{\cap} h_{B_2}\right) \mathrel{\tilde{\subseteq}} \left(h_{B_2} * l_{B_3} * g_{B_1}\right)$ and $\left(g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}\right) * \left(h_{B_2} \tilde{\cap} l_{B_3} \tilde{\cap} g_{B_1}\right) * \left(l_{B_3} \tilde{\cap} g_{B_1} \tilde{\cap} h_{B_2}\right) \mathrel{\tilde{\subseteq}} \left(l_{B_3} * g_{B_1} * h_{B_2}\right)$.

Thus, $\left(g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}\right) * \left(h_{B_2} \tilde{\cap} l_{B_3} \tilde{\cap} g_{B_1}\right) * \left(l_{B_3} \tilde{\cap} g_{B_1} \tilde{\cap} h_{B_2}\right) \mathrel{\tilde{\subseteq}} \left(g_{B_1} * h_{B_2} * l_{B_3}\right) \tilde{\cap} \left(h_{B_2} * l_{B_3} * g_{B_1}\right) \tilde{\cap} \left(l_{B_3} * g_{B_1} * h_{B_2}\right) \mathrel{\tilde{\subseteq}} f_R$. Since f_R is SSBHI of K, then $\left(g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}\right) \mathrel{\tilde{\subseteq}} f_R$. Also f_R is SISBHI of K, so either $g_{B_1} \mathrel{\tilde{\subseteq}} f_R$ or $h_{B_2} \mathrel{\tilde{\subseteq}} f_R$ or $l_{B_3} \mathrel{\tilde{\subseteq}} f_R$. Thus, proved.

Proposition 3.9. *Let* $f_R \in B(U)$ *with* $f_R(x) = T$ *where* $x \in K$ *and* $T \in P(U)$, *then there exists an ISBHI* l_B *of* K *such that* $f_R \mathrel{\tilde{\subseteq}} l_B$ *and* $l_B(x) = T$.

Proof. Let $X = \{h_S : h_S \in B(U), h_S(x) = T$ where $T \in P(U)$ and $f_R \mathrel{\tilde{\subseteq}} h_S\}$ be a partially ordered set under inclusion. Then $X \neq \Phi$, since $f_R \in X$. Let $V = \left\{h_{S_i} : i \in I\right\}$ be a subset of X which is totally ordered. Consider, for all $a, b, c \in K$ and $x \in l \circ m \circ n$.

$$\bigcap_{x\in l\circ m\circ n} \left(\tilde{\bigcup_{i\in I}} h_{S_i}\right)(x) = \bigcup_{i\in I} \left(\bigcap_{x\in l\circ m\circ n} h_{S_i}(x)\right)$$
$$\supseteq \bigcup_{i\in I} \left(h_{S_i}(l) \cap h_{S_i}(m) \cap h_{S_i}(n)\right)$$
$$= \left\{\bigcup_{i\in I} \left(h_{S_i}(l)\right)\right\} \cap \left\{\bigcup_{i\in I} \left(h_{S_i}(m)\right)\right\} \cap \left\{\bigcup_{i\in I} \left(h_{S_i}(n)\right)\right\}$$
$$= \left(\tilde{\bigcup_{i\in I}} h_{S_i}\right)(l) \cap \left(\tilde{\bigcup_{i\in I}} h_{S_i}\right)(m) \cap \left(\tilde{\bigcup_{i\in I}} h_{S_i}\right)(n).$$

So, $\tilde{\bigcup}_{i\in I} h_{S_i}$ is a TSSHG of K over U.

Now, for all $l, m, n, p, q \in K$ and $y \in l \circ p \circ m \circ q \circ n$.

$$\bigcap_{y\in l\circ p\circ m\circ q\circ n} \left(\tilde{\bigcup_{i\in I}} h_{S_i}\right)(y) = \bigcup_{i\in I} \left(\bigcap_{y\in l\circ p\circ m\circ q\circ n} h_{S_i}(y)\right)$$
$$\supseteq \bigcup_{i\in I} \left(h_{S_i}(l) \cap h_{S_i}(m) \cap h_{S_i}(n)\right)$$
$$= \left\{\bigcup_{i\in I} \left(h_{S_i}(l)\right)\right\} \cap \left\{\bigcup_{i\in I} \left(h_{S_i}(m)\right)\right\} \cap \left\{\bigcup_{i\in I} \left(h_{S_i}(n)\right)\right\}$$
$$= \left(\tilde{\bigcup_{i\in I}} h_{S_i}\right)(l) \cap \left(\tilde{\bigcup_{i\in I}} h_{S_i}\right)(m) \cap \left(\tilde{\bigcup_{i\in I}} h_{S_i}\right)(n).$$

Hence, $\tilde{\bigcup}_{i\in I} h_{S_i}$ is SBHI of K. As $f_R \mathrel{\tilde{\subseteq}} h_{S_i}$ for all $i \in I$, so $f_R \mathrel{\tilde{\subseteq}} \tilde{\bigcup}_{i\in I} h_{S_i}$. Also $(\tilde{\bigcup}_{i\in I} h_{S_i})(x) = \bigcup_{i\in I} (h_{S_i})(x) = T$. Thus, $\tilde{\bigcup}_{i\in I} h_{S_i}$ is the supremum of V. So, there exists a SBHI l_B of K that is maximal, $f_R \mathrel{\tilde{\subseteq}} l_B$ and $l_B(x) = T$. Now we have to prove that l_B is an irreducible. Suppose $l_B \cong$

$d_{B_1} \tilde{\cap} g_{B_2} \tilde{\cap} j_{B_3}$ where d_{B_1}, g_{B_2} and j_{B_3} are SBHIs of K. Then $l_B \tilde{\subseteq} d_{B_1}$ or $l_B \tilde{\subseteq} g_{B_2}$ or $l_B \tilde{\subseteq} j_{B_3}$. We claim that $l_B \cong d_{B_1}$ or $l_B \cong g_{B_2}$ or $l_B \cong j_{B_3}$. Suppose on contrary that $l_B \ncong d_{B_1}$ and $l_B \ncong g_{B_2}$ and $l_B \ncong j_{B_3}$. Since l_B is maximal with respect to the property that $l_B(x) = T$. It follows that $d_{B_1}(x) \neq T, g_{B_2}(x) \neq T$ and $j_{B_3}(x) \neq T$. This implies $T = l_B(x) = d_{B_1} \tilde{\cap} g_{B_2} \tilde{\cap} j_{B_3}(x) \neq T$. Which is a contradiction. Hence either $l_B \cong d_{B_1}$ or $l_B \cong g_{B_2}$ or $l_B \cong j_{B_3}$. Hence proved.

Theorem 3.10. *For a ternary semihypergroup K, there is a correspondence between the following statements.*

(1) $f_B * f_B * f_B \cong f_B$ for all $f_B \in B(U)$.

(2) $\left(g_{B_1} * h_{B_2} * l_{B_3}\right) \tilde{\cap} \left(h_{B_2} * l_{B_3} * g_{B_1}\right) \tilde{\cap} \left(l_{B_3} * g_{B_1} * h_{B_2}\right) \cong g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}$ for all g_{B_1}, h_{B_2} and $l_{B_3} \in B(U)$.

(3) Every $f_B \in B(U)$ is semiprime i.e., $g_{B_1} * g_{B_1} * g_{B_1} \tilde{\subseteq} f_B$ implies $g_{B_1} \tilde{\subseteq} f_B$ for all $g_{B_1} \in B(U)$.

(4) Every proper SBHI *$f_B \in B(U)$ is the intersection of all irreducible semiprime SBHIs of K which are the supersets of f_B.*

Proof. (1) $\Rightarrow$ (2). Let $g_{B_1}, h_{B_2}, l_{B_3} \in B(U)$, then $g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3} \in B(U)$. By supposition

$$(g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}) \cong \left(g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}\right)^3$$

$$\cong \left(g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}\right) * \left(g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}\right) * \left(g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}\right)$$

$$\tilde{\subseteq} \left(g_{B_1} * h_{B_2} * l_{B_3}\right).$$

Similarly, $(g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}) \tilde{\subseteq} \left(h_{B_2} * l_{B_3} * g_{B_1}\right)$ and $(g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}) \tilde{\subseteq} \left(l_{B_3} * g_{B_1} * h_{B_2}\right)$.

So, $\left(g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}\right) \tilde{\subseteq} \left(g_{B_1} * h_{B_2} * l_{B_3}\right) \tilde{\cap} \left(h_{B_2} * l_{B_3} * g_{B_1}\right) \tilde{\cap} \left(l_{B_3} * g_{B_1} * h_{B_2}\right)$.

Conversely, $\left(g_{B_1} * h_{B_2} * l_{B_3}\right), \left(h_{B_2} * l_{B_3} * g_{B_1}\right)$ and $\left(l_{B_3} * g_{B_1} * h_{B_2}\right)$ are SBHIs of K. Also, $\left(g_{B_1} * h_{B_2} * l_{B_3}\right) \tilde{\cap} \left(h_{B_2} * l_{B_3} * g_{B_1}\right) \tilde{\cap} \left(l_{B_3} * g_{B_1} * h_{B_2}\right)$ is a SBHI of K. Then, consider

$$\left(g_{B_1} * h_{B_2} * l_{B_3}\right) \tilde{\cap} \left(h_{B_2} * l_{B_3} * g_{B_1}\right) \tilde{\cap} \left(l_{B_3} * g_{B_1} * h_{B_2}\right)$$

$$\cong \left(\left(g_{B_1} * h_{B_2} * l_{B_3}\right) \tilde{\cap} \left(h_{B_2} * l_{B_3} * g_{B_1}\right) \tilde{\cap} \left(l_{B_3} * g_{B_1} * h_{B_2}\right)\right)^3$$

$$\tilde{\subseteq} \left(g_{B_1} * h_{B_2} * l_{B_3}\right)\left(h_{B_2} * l_{B_3} * g_{B_1}\right)\left(l_{B_3} * g_{B_1} * h_{B_2}\right)$$

$$\tilde{\subseteq} \left(g_{B_1} * g_K * g_K\right)\left(g_K * g_{B_1} * g_K\right)\left(g_K * g_K * g_{B_1}\right)$$

$$\cong g_{B_1} * (g_K * g_K * g_K) * g_{B_1} * (g_K * g_K * g_K) * g_{B_1}$$

$$\tilde{\subseteq} g_{B_1} * g_K * g_{B_1} * g_K * g_{B_1} \tilde{\subseteq} g_{B_1}.$$

Similarly,
$\left(g_{B_1} * h_{B_2} * l_{B_3}\right) \tilde{\cap} \left(h_{B_2} * l_{B_3} * g_{B_1}\right) \tilde{\cap} \left(l_{B_3} * g_{B_1} * h_{B_2}\right) \tilde{\subseteq} h_{B_2}$ and $\left(g_{B_1} * h_{B_2} * l_{B_3}\right) \tilde{\cap} \left(h_{B_2} * l_{B_3} * g_{B_1}\right) \tilde{\cap} \left(l_{B_3} * g_{B_1} * h_{B_2}\right) \tilde{\subseteq} l_{B_3}$.

Thus, $\left(g_{B_1} * h_{B_2} * l_{B_3}\right) \tilde{\cap} \left(h_{B_2} * l_{B_3} * g_{B_1}\right) \tilde{\cap} \left(l_{B_3} * g_{B_1} * h_{B_2}\right) \tilde{\subseteq} g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}$.

Hence, $\left(g_{B_1} * h_{B_2} * l_{B_3}\right) \tilde{\cap} \left(h_{B_2} * l_{B_3} * g_{B_1}\right) \tilde{\cap} \left(l_{B_3} * g_{B_1} * h_{B_2}\right) \cong g_{B_1} \tilde{\cap} h_{B_2} \tilde{\cap} l_{B_3}$.

(2) $\Rightarrow$ (1). Let g_{B_1} be a SBHI of K. Then

$$g_{B_1} \cong g_{B_1} \tilde{\cap} g_{B_1} \tilde{\cap} g_{B_1}$$

$$\cong \left(g_{B_1} * g_{B_1} * g_{B_1}\right) \tilde{\cap} \left(g_{B_1} * g_{B_1} * g_{B_1}\right) \tilde{\cap} \left(g_{B_1} * g_{B_1} * g_{B_1}\right)$$

$$\cong g_{B_1} * g_{B_1} * g_{B_1}.$$

So, g_{B_1} is idempotent.

(1) ⇒ (3). Let $f_B \in B(U)$. To prove f_B is semiprime, consider $g_{B_1} * g_{B_1} * g_{B_1} \tilde{\subseteq} f_B$ for any $g_{B_1} \in B(U)$. Then, by supposition $g_{B_1} \cong g_{B_1} * g_{B_1} * g_{B_1} \tilde{\subseteq} f_B$. Implies, $g_{B_1} \tilde{\subseteq} f_B$. Hence proved.

(3) ⇒ (4). Let $\{f_{B_i} : i \in I\}$ be a family of all ISBHIs of K which are the supersets of f_B. Then, $f_B \tilde{\subseteq} \tilde{\bigcap}_{i\in I} f_{B_i}$. Now, let $x \in K$. Then by Proposition 3.9, there is an ISBHI f_B of K such that $f_B(x) = f_{B_\alpha}(x)$. Thus, $f_{B_\alpha} \in \{f_{B_i} : i \in I\}$. Hence, $\tilde{\bigcap}_{i\in I} f_{B_i} \tilde{\subseteq} f_{B_\alpha}$. So, $\tilde{\bigcap}_{i\in I} f_{B_i}(x) \subseteq f_{B_\alpha}(x) \subseteq f_B(x)$. Thus, $\tilde{\bigcap}_{i\in I} f_{B_i} \tilde{\subseteq} f_B$. Consequently, $\tilde{\bigcap}_{i\in I} f_{B_i} \cong f_B$.

(4) ⇒ (1). By Proposition 2.2, $f_B * f_B * f_B \tilde{\subseteq} f_B$ for all $f_B \in B(U)$. By supposition, $f_B * f_B * f_B \cong \tilde{\bigcap}_{i\in I} f_{B_i}$ where f_{B_i} are irreducible semiprime soft bi-hyperideals of K for all $i \in I$. Thus, $f_B * f_B * f_B \tilde{\subseteq} f_{B_i}$ for all i. Hence, $f_B \tilde{\subseteq} f_{B_i}$ for all i because each f_{B_i} is semiprime. Thus, $f_B \tilde{\subseteq} \tilde{\bigcap}_{i\in I} f_{B_i} \cong f_B * f_B * f_B$. Hence f_B is idempotent.

Theorem 3.11. *Each SBHI $f_K \in B(U)$ is strongly prime iff it is idempotent and $B(U)$ is totally ordered by inclusion.*

Proof. First, suppose that each SBHI of K is strongly prime, then it is semiprime. Thus by Theorem 3.10, each SBHI is idempotent. Let $f_{B_1}, f_{B_2} \in B(U)$, then

$$f_{B_1} \tilde{\cap} f_{B_2} \tilde{\cap} f_k \cong (f_{B_1} * f_{B_2} * f_k)\ \tilde{\cap}\ (f_{B_2} * f_k * f_{B_1})\ \tilde{\cap} (f_k * f_{B_1} * f_{B_2}).$$

Implies

$$(f_{B_1} * f_{B_2} * f_k) \tilde{\cap} (f_{B_2} * f_k * f_{B_1})\ \tilde{\cap} (f_k * f_{B_1} * f_{B_2}) \tilde{\subseteq} (f_{B_1} \tilde{\cap} f_{B_2}).$$

By hypothesis, f_{B_1}, f_{B_2} are SPSBHIs of K, so is $(f_{B_1} \tilde{\cap} f_{B_2})$.

Then $f_{B_1} \tilde{\subseteq} f_{B_1} \tilde{\cap} f_{B_2}$ or $f_{B_2} \tilde{\subseteq} f_{B_1} \tilde{\cap} f_{B_2}$ or $f_k \tilde{\subseteq} f_{B_1} \tilde{\cap} f_{B_2}$. Thus, $f_{B_1} \tilde{\subseteq} f_{B_2}$ or $f_{B_2} \tilde{\subseteq} f_{B_1}$. Since f_{B_1}, f_{B_2} are arbitrary SBHIs of K, so $B(U)$, the set of SBHIs of K is totally ordered by inclusion.

Conversely, let f_B be any SBHI of K and $f_{B_1}, f_{B_2}, f_{B_3} \in B(U)$ such that

$$(f_{B_1} * f_{B_2} * f_{B_3})\ \ \tilde{\cap}\ (f_{B_2} * f_{B_3} * f_{B_1})\ \tilde{\cap}\ \ (f_{B_3} * f_{B_1} * f_{B_2}) \tilde{\subseteq} f_B.$$

By Theorem 3.10, we have

$$f_{B_1} \tilde{\cap} f_{B_2} \tilde{\cap} f_{B_3} \cong (f_{B_1} * f_{B_2} * f_{B_3})\ \ \tilde{\cap}\ (f_{B_2} * f_{B_3} * f_{B_1})\ \tilde{\cap}\ \ (f_{B_3} * f_{B_1} * f_{B_2}) \tilde{\subseteq} f_B.$$

From our supposition, by using the property of inclusion, there are the following six possibilities.
(i) $f_{B_1} \tilde{\subseteq} f_{B_2} \tilde{\subseteq} f_{B_3}$ (ii) $f_{B_1} \tilde{\subseteq} f_{B_3} \tilde{\subseteq} f_{B_2}$ (iii) $f_{B_2} \tilde{\subseteq} f_{B_3} \tilde{\subseteq} f_{B_1}$.
(iv)$f_{B_2} \tilde{\subseteq} f_{B_1} \tilde{\subseteq} f_{B_3}$ (v) $f_{B_3} \tilde{\subseteq} f_{B_1} \tilde{\subseteq} f_{B_2}$ (iv) $f_{B_3} \tilde{\subseteq} f_{B_2} \tilde{\subseteq} f_{B_1}$.
In these cases, we have
(i) $f_{B_1} \tilde{\cap} f_{B_2} \tilde{\cap}\ \ f_{B_3} \cong f_{B_1}$ (ii) $f_{B_1} \tilde{\cap} f_{B_2} \tilde{\cap}\ \ f_{B_3} \cong f_{B_1}$ (iii) $f_{B_1} \tilde{\cap} f_{B_2} \tilde{\cap}\ \ f_{B_3} \cong f_{B_2}$.
(iv) $f_{B_1} \tilde{\cap} f_{B_2} \tilde{\cap}\ \ f_{B_3} \cong f_{B_2}$ (v) $f_{B_1} \tilde{\cap} f_{B_2} \tilde{\cap}\ \ f_{B_3} \cong f_{B_3}$ (iv) $f_{B_1} \tilde{\cap} f_{B_2} \tilde{\cap}\ \ f_{B_3} \cong f_{B_3}$.

Hence, according to each case either $f_{B_1} \tilde{\subseteq} f_{B_2}$ or $f_{B_2} \tilde{\subseteq} f_{B_1}$ or $f_{B_3} \tilde{\subseteq} f_B$ Hence, f_B is strongly prime.

Theorem 3.12. *Let $B(U)$ be a totally ordered set under inclusion, then each* SBHI *of K is idempotent iff it is prime.*

Proof. Suppose that any $f_B \in\ B(U)$ is idempotent and let $f_{B_1}, f_{B_2}, f_{B_3} \in B(U)$ such that

$$f_{B_1} * f_{B_2} * f_{B_3} \tilde{\subseteq} f_B.$$

From our supposition, by using the property of inclusion, there are the following six possibilities.
(i) $f_{B_1} \widetilde{\subseteq} f_{B_2} \widetilde{\subseteq} f_{B_3}$ (ii) $f_{B_1} \widetilde{\subseteq} f_{B_3} \widetilde{\subseteq} f_{B_2}$ (iii) $f_{B_2} \widetilde{\subseteq} f_{B_3} \widetilde{\subseteq} f_{B_1}$.
(v) $f_{B_2} \widetilde{\subseteq} f_{B_1} \widetilde{\subseteq} f_{B_3}$ (v) $f_{B_3} \widetilde{\subseteq} f_{B_1} \widetilde{\subseteq} f_{B_2}$ (iv) $f_{B_3} \widetilde{\subseteq} f_{B_2} \widetilde{\subseteq} f_{B_1}$.
From (i) and (ii), we have
$f_{B_1} = f_{B_1} * f_{B_1} * f_{B_1} \widetilde{\subseteq} f_{B_1} * f_{B_2} * f_{B_3} \widetilde{\subseteq} f_B$ implies $f_{B_1} \widetilde{\subseteq} f_B$ as f_{B_1} is idempotent.
Similarly, for other choices we have $f_{B_2} \widetilde{\subseteq} f_B$ or $f_{B_3} \widetilde{\subseteq} f_B$. So, f_B is prime.

Conversely, suppose that each SBHI of K is prime, so by Remark 3.5, it is semiprime. Thus by Theorem 3.10, each SBHI of K is idempotent.

Theorem 3.13. *Let $B(U)$ be a totally ordered set under inclusion, then each* PSBHI *of K is equivalent to strongly prime.*

Proof. Suppose f_B is a PSBHI of K. To prove that f_B is a SPSBHI of K, let $f_{B_1}, f_{B_2}, f_{B_3} \in B(U)$ such that

$$(f_{B_1} * f_{B_2} * f_{B_3}) \ \widetilde{\cap} \ (f_{B_2} * f_{B_3} * f_{B_1}) \ \widetilde{\cap} \ (f_{B_3} * f_{B_1} * f_{B_2}) \widetilde{\subseteq} f_B.$$

From our supposition, by using the property of inclusion, there are the following six possibilities.
(i) $f_{B_1} \widetilde{\subseteq} f_{B_2} \widetilde{\subseteq} f_{B_3}$ (ii) $f_{B_1} \widetilde{\subseteq} f_{B_3} \widetilde{\subseteq} f_{B_2}$ (iii) $f_{B_2} \widetilde{\subseteq} f_{B_3} \widetilde{\subseteq} f_{B_1}$.
(vi)$f_{B_2} \widetilde{\subseteq} f_{B_1} \widetilde{\subseteq} f_{B_3}$ (v) $f_{B_3} \widetilde{\subseteq} f_{B_1} \widetilde{\subseteq} f_{B_2}$ (iv) $f_{B_3} \widetilde{\subseteq} f_{B_2} \widetilde{\subseteq} f_{B_1}$.
From (i) and (ii) we have,
$f_{B_1}^3 \cong f_{B_1}^3 \ \widetilde{\cap} \ f_{B_1}^3 \ \widetilde{\cap} \ f_{B_1}^3 \widetilde{\subseteq} \left(f_{B_1} * f_{B_2} * f_{B_3}\right) \widetilde{\cap} \ (f_{B_2} * f_{B_3} * f_{B_1}) \ \widetilde{\cap} \ (f_{B_3} * f_{B_1} * f_{B_2}) \ \widetilde{\subseteq} \ f_B$.
Implies $f_{B_1} \widetilde{\subseteq} f_{\mathrm{B}}$. Similarly, we have $f_{B_2} \widetilde{\subseteq} f_{\mathrm{B}}$ or $f_{B_1} \widetilde{\subseteq} f_{\mathrm{B}}$. So f_B is strongly prime.

Conversely, suppose f_B is a SPSBHI of K over U. Now we have to prove that f_B is PSBHI of K. For this consider,
$f_{B_1} * f_{B_2} * f_{B_3} \widetilde{\subseteq} \ f_B$. Implies
$\left(f_{B_1} * f_{B_2} * f_{B_3}\right) \widetilde{\cap} \ (f_{B_2} * f_{B_3} * f_{B_1}) \ \widetilde{\cap} \ (f_{B_3} * f_{B_1} * f_{B_2}) \widetilde{\subseteq} f_{B_1} * f_{B_2} * f_{B_3} \widetilde{\subseteq} f_B$.
Implies either $f_{B_1} \widetilde{\subseteq} f_B$ or $f_{B_2} \widetilde{\subseteq} f_{\mathrm{B}}$ or $f_{B_3} \widetilde{\subseteq} f_{\mathrm{B}}$. As f_B is SPSBHI of K. This shows that f_B is a PSBHI of K. Thus every SPSBHI of K is a PSBHI of K.

Theorem 3.14. *Let $g_B \in B(U)$, then the following statements are equivalent.*
(1) *$B(U)$ is totally ordered by inclusion.*
(2) *g_B is strongly irreducible.*
(3) *g_B is irreducible.*

Proof. $(1) \Rightarrow (2)$. To prove that g_B is a strongly irreducible, let $g_{B_1}, g_{B_2}, g_{B_3} \in B(U)$ such that

$$g_{B_1} \ \widetilde{\cap} \ g_{B_2} \ \widetilde{\cap} \ g_{B_3} \widetilde{\subseteq} g_B.$$

By our supposition, $g_{B_1} \ \widetilde{\cap} \ g_{B_2} \ \widetilde{\cap} \ g_{B_3} \cong g_{B_1}$ or g_{B_2} or g_{B_3}. Thus, either $g_{B_1} \widetilde{\subseteq} g_B$ or $g_{B_2} \widetilde{\subseteq} g_{\mathrm{B}}$ or $g_{B_3} \widetilde{\subseteq} g_{\mathrm{B}}$. So g_B is strongly irreducible.

$(2) \Rightarrow (3)$. To show that g_B is an irreducible, let $g_{B_1}, g_{B_2}, g_{B_3}$ be any three SBHIs of K such that $g_{B_1} \ \widetilde{\cap} \ g_{B_2} \ \widetilde{\cap} \ g_{B_3} \widetilde{\subseteq} g_B$. Implies $g_{\mathrm{B}} \widetilde{\subseteq} g_{B_1}$ or $g_{\mathrm{B}} \widetilde{\subseteq} g_{B_2}$ or $g_{\mathrm{B}} \widetilde{\subseteq} g_{B_3}$.

And by hypothesis we have, $g_{B_1} \widetilde{\subseteq} g_B$ or $g_{B_2} \widetilde{\subseteq} g_{\mathrm{B}}$ or $g_{B_3} \widetilde{\subseteq} g_{\mathrm{B}}$. Hence either, $g_{B_1} \cong g_B$ or $g_{B_2} \cong g_B$ or $g_{B_3} \cong g_B$. Thus, g_B is irreducible. Hence, each SBHI of K is irreducible.

$(3) \Rightarrow (1)$. Let $g_{B_1}, g_{B_2} \in B(U)$. Then, $g_{B_1} \widetilde{\cap} \, g_{B_2} \in B(U)$ and so is irreducible. Consider, $g_{B_1} \ \widetilde{\cap} \ g_{B_2} \ \widetilde{\cap} \ g_K \cong g_{B_1} \ \widetilde{\cap} \ g_{B_2}$, implies $g_{B_1} \cong g_{B_1} \ \widetilde{\cap} \ g_{B_2}$ or $g_{B_2} \cong g_{B_1} \ \widetilde{\cap} \ g_{B_2}$ or $g_K \cong g_{B_1} \ \widetilde{\cap} \ g_{B_2}$. Implies either $g_{B_1} \widetilde{\subseteq} g_{B_2}$ or $g_{B_2} \widetilde{\subseteq} g_{B_1}$. Hence, $B(U)$ is totally ordered by inclusion.

4. Conclusions

Hyperstructures are better than the common structures because of the multi-valued property, which give all the possible results of a problem between individuals. In this section, we describe how this research work is better and related to previous work. In [43], Naz studied the PSBHIs of semihypergroups. By extending the work of [43], we worked on ternary semihypergroups. We introduced PSBHIs in ternary semihypergroups. This technique is more useful than [43] because there are many algebraic structures that are not closed under binary multiplication but closed under ternary multiplication, such as Z^- (set of negative integers), Q^- (set of negative rational numbers) and R^- (set of negative real numbers). To remove this difficulty, we studied the ternary operation, and have generalized all results in ternary semihypergroup. Hence, the technique used in this paper is more general than previous.

Molodtsov initiated the idea of soft set theory for solving the problems with uncertainty. In this paper, we use the concept of soft set theory on ternary algebraic structure. We generalized the work of [43] to ternary framework. Many related theorems, propositions and examples are discussed here with ternary hyperoperation. We generalize the ternary semihypergroups by the characterizations of PSBHIs. The main advantage of this paper is that we have proved with ternary operation that each SBHI of K is strongly prime iff, it is idempotent and the set of SBHIs of K is totally ordered by inclusion.

If data is incomplete and uncertain, the above technique is not appropriate. For this, we will use parameterization tool with fuzzy set and bipolar fuzzy set. In the future, based on these results, we will apply soft sets to bipolar fuzzy hyperideals in ternary semihypergroups and extend it to the structure of soft semihyperrings and soft ternary semihyperrings in a similar way.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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