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Research article

On Ulam stability of a second order linear difference equation

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Abstract: In this paper we obtain some Ulam stability results for the second order and the third order linear difference equation with nonconstant coefficients in a Banach space. The main idea of the approach is to decompose the second order linear difference equation in a Riccati difference equation and a first order difference equation. In this way we extend some results for linear difference equations with constant coefficients and for linear difference equations with periodic coefficients.

Keywords: Ulam stability; approximate solution; linear difference equation; Riccati difference equation; nonconstant coefficients
Mathematics Subject Classification: Primary 39A30, Secondary 39B82

1. Introduction

The problem of Ulam stability can be formulated for various functional equations. The starting point of Ulam stability theory was a problem formulated by Ulam in 1940 in a talk at the University of Wisconsin-Madison concerning the approximate solutions of group homomorphisms. Generally, we say that an equation is Ulam stable if for every approximate solution of the equation there exists an exact solution of the equation near it (see the papers [10, 16, 18, 23, 25]). For more details on the Ulam stability on the functional equations, see the monographs [17, 19]. The problem can be also formulated for difference equations. Since a discrete dynamical system is described by a difference equation, this type of stability is related to the notion of perturbation of such a system. The Ulam stability of difference equations was intensively studied in the recent years (see for more details [10]).

Recently many papers on Hyers-Ulam stability of difference equations are devoted to the relation of it with hyperbolicity and the exponential dichotomy. Remark here the papers by D. Dragičević concerning some nonautonomous and nonlinear difference equations [12–14]. D. R. Anderson and M. Onitsuka gave interesting results on the influence of stepsize in Hyers-Ulam stability of the first order difference equations and on the best constant of some second order linear difference equations with constant coefficients [1, 20]. Recall also the results given by A. R. Baias, J. Brzdek, D. Popa et al. in the characterization of Ulam stability and on the best constant for various linear and nonlinear difference equations [2–5, 7, 11, 15].

Let $\mathbb{K}$ be either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers and X be a Banach space over $\mathbb{K}$. Consider the difference equation

$$x_{n+p} = f_n(x_n, x_{n+1}, \ldots, x_{n+p-1}),\ n \in \mathbb{N},\ p \in \mathbb{N}^*, \tag{1.1}$$

where $f_n : X^p \to X$ and $x_0, x_1, \ldots, x_{p-1} \in X$.

Definition 1.1. The Eq (1.1) is called *Ulam stable* if there exists a constant $L \geq 0$ such that for every $\varepsilon > 0$ and every sequence $(x_n)_{n\geq 0}$ in X satisfying

$$\|x_{n+p} - f_n(x_n, x_{n+1}, \ldots, x_{n+p-1})\| \leq \varepsilon,\ n \in \mathbb{N}, \tag{1.2}$$

there exists a sequence $(y_n)_{n\geq 0}$ in X such that

$$y_{n+p} = f_n(y_n, y_{n+1}, \ldots, y_{n+p-1}),\ n \in \mathbb{N} \tag{1.3}$$

and

$$\|x_n - y_n\| \leq L\varepsilon,\ n \in \mathbb{N}. \tag{1.4}$$

Remark 1.2. If in Definition 1.1, ε is replaced by a sequence of positive numbers $(\varepsilon_n)_{n\geq 0}$ and $L\varepsilon$ by a sequence $(\delta_n)_{n\geq 0}$ depending on $(\varepsilon_n)_{n\geq 0}$, then we get the notion of generalized Ulam stability. The number L is called an Ulam constant of the Eq (1.1). Denote by L_R the infimum of all Ulam constants of the Eq (1.1).

In particular, for $p = 1$, we consider the linear difference equation

$$x_{n+1} = a_n x_n + b_n,\ n \geq 0, \tag{1.5}$$

with $(a_n)_{n\geq 0}$ a sequence in $\mathbb{K}$ and $(b_n)_{n\geq 0}$ a sequence in X.

The following result concerning the generalized Ulam stability of (1.5) can be found in [21].

Theorem 1.3. *Let $(\varepsilon_n)_{n\geq 0}$ be a sequence of positive numbers such that*

$$\limsup \frac{\varepsilon_n}{\varepsilon_{n-1}|a_n|} < 1 \text{ or } \liminf \frac{\varepsilon_n}{\varepsilon_{n-1}|a_n|} > 1. \tag{1.6}$$

Then there exists $L \geq 0$ such that for every sequence $(x_n)_{n\geq 0}$ in X satisfying

$$\|x_{n+1} - a_n x_n - b_n\| \leq \varepsilon_n,\ n \geq 0, \tag{1.7}$$

there exists a sequence $(y_n)_{n\geq 0}$ in X such that

$$y_{n+1} = a_n y_n + b_n,\ n \geq 0 \tag{1.8}$$

and

$$\|x_n - y_n\| \leq L\varepsilon_{n-1},\ n \geq 1. \tag{1.9}$$

Remark 1.4. If in Theorem 1.3, we take $\varepsilon_n = \varepsilon,\ n \geq 0$, then the condition (1.6) becomes

$$\limsup \frac{1}{|a_n|} < 1 \text{ or } \liminf \frac{1}{|a_n|} > 1. \tag{1.10}$$

On the other hand it was proved that if $\lim_{n\to\infty} |a_n| = 1$, then (1.5) is not stable (see [8]).

So, we obtain the following result.

Theorem 1.5. *Suppose that $\lim_{n\to\infty} |a_n|$ exists. Then the Eq (1.5) is Ulam stable if and only if $\lim_{n\to\infty} |a_n| \neq 1$.*

2. Second order equations

Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be sequences in $\mathbb{K}$ and $(c_n)_{n\geq 0}$ a sequence in X. In what follows we deal with Ulam stability of the second order linear difference equation

$$x_{n+2} = a_n x_{n+1} + b_n x_n + c_n,\ n \geq 0, \tag{2.1}$$

where $x_0, x_1 \in X$.

The following results will be useful in the sequel.

Lemma 2.1. *Suppose that $(x_n)_{n\geq 0}$ satisfies (2.1) and let $(u_n)_{n\geq 0}$ be a sequence in $\mathbb{K}$ defined by the Riccati difference equation*

$$u_{n+1} = a_n + \frac{b_n}{u_n},\ n \geq 0,\ u_0 \in \mathbb{K}. \tag{2.2}$$

If $(z_n)_{n\geq 0}$ is given by the relation

$$z_n = x_{n+1} - u_n x_n,\ n \geq 0, \tag{2.3}$$

then

$$z_{n+1} = (a_n - u_{n+1}) z_n + c_n,\ n \geq 0. \tag{2.4}$$

Proof. From (2.1) it follows that

$$\begin{aligned} x_{n+2} - u_{n+1}x_{n+1} &= a_n x_{n+1} - u_{n+1}x_{n+1} + b_n x_n + c_n \\ &= (a_n - u_{n+1})x_{n+1} + b_n x_n + c_n \\ &= (a_n - u_{n+1})(x_{n+1} + \frac{b_n}{a_n - u_{n+1}} x_n) + c_n \\ &= (a_n - u_{n+1})(x_{n+1} - \frac{u_n}{b_n} b_n x_n) + c_n \\ &= (a_n - u_{n+1})(x_{n+1} - u_n x_n) + c_n \\ &= (a_n - u_{n+1}) z_n + c_n,\ n \geq 0. \end{aligned}$$

□

Lemma 2.2. *Let $(A_n)_{n\geq 0}$, $A_n = \begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix}$, be a sequence of matrices with entries in $\mathbb{K}$ and $(v_n)_{n\geq 0}$ be the sequence defined by the difference equation*

$$v_{n+1} = \frac{p_n v_n + q_n}{r_n v_n + s_n},\ n \geq 0,\ x_0 \in \mathbb{K}.$$

Then

$$v_n = \frac{\alpha_n v_0 + \beta_n}{\gamma_n v_0 + \delta_n},\ n \geq 1,$$

where

$$A_{n-1} \cdot \ldots \cdot A_0 = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix},\ n \geq 1.$$

Proof. The proof may be established by using mathematical induction on n. □

Remark 2.3. In particular, for the sequence $(u_n)_{n\geq 0}$ given by (2.1) we get

$$u_n = \frac{\alpha_n u_0 + \beta_n}{\alpha_{n-1} u_0 + \beta_{n-1}},\ n \geq 1,$$

where $A_n = \begin{pmatrix} a_n & b_n \\ 1 & 0 \end{pmatrix}$, $n \geq 0$, and

$$A_{n-1} \cdot \ldots \cdot A_0 = \begin{pmatrix} \alpha_n & \beta_n \\ \alpha_{n-1} & \beta_{n-1} \end{pmatrix},\ n \geq 1.$$

Lemma 2.4. *If the equation* $x_{n+1} - u_n x_n - z_n = 0$ *is Ulam stable with the constant* L_1 *and the Eq* (2.4) *is Ulam stable with the constant* L_2*, then the Eq* (2.1) *is Ulam stable with the constant* $L_1 L_2$.

Proof. Let $\varepsilon > 0$ and let $(x_n)_{n\geq 0}$ be a sequence in X such that

$$\|x_{n+2} - a_n x_{n+1} - b_n x_n - c_n\| \leq \varepsilon,\ n \geq 0.$$

Put $z_n = x_{n+1} - u_n x_n$, where $(u_n)_{n\geq 0}$ satisfies relation (2.2). Then

$$\|z_{n+1} - (a_n - u_{n+1}) z_n - c_n\| \leq \varepsilon,\ n \geq 0,$$

so according to the stability of the Eq (2.4), there exists $(w_n)_{n\geq 0}$,

$$w_{n+1} = (a_n - u_{n+1}) w_n + c_n,\ n \geq 0, \tag{2.5}$$

such that

$$\|z_n - w_n\| \leq L_2 \varepsilon,\ n \geq 0.$$

Taking account of (2.3) we get

$$\|x_{n+1} - u_n x_n - w_n\| \leq L_2 \varepsilon,\ n \geq 0.$$

Now, since the Eq (2.3) is Ulam stable, it follows that there exists a sequence $(y_n)_{n\geq 0}$, satisfying the relation

$$y_{n+1} = u_n y_n + w_n,\ n \geq 0, \tag{2.6}$$

such that

$$\|x_n - y_n\| \leq L_1 L_2 \varepsilon,\ n \geq 0. \tag{2.7}$$

To complete the proof it remains to show that $(w_n)_{n\geq 0}$ satisfies the Eq (2.1). For this we replace $(y_n)_{n\geq 0}$ from (2.6) to (2.5) and we obtain

$$y_{n+2} - u_{n+1} y_{n+1} = (a_n - u_{n+1})(y_{n+1} - u_n y_n) + c_n.$$

Finally, taking account of the relation (2.2), we get

$$y_{n+2} = a_n y_{n+1} + b_n y_n + c_n,\ n \geq 0.$$

The theorem is proved. □

The main result on the stability of the Eq (2.1) is given in the next theorem.

Theorem 2.5. *Suppose that for the sequence* $(u_n)_{n\geq 0}$ *given by* (2.2), $\lim_{n\to\infty} |u_n|$, $\lim_{n\to\infty} |a_n - u_{n+1}|$ *exist and:*

1) $\lim_{n\to\infty} |u_n| \neq 1$*;*

2) $\lim_{n\to\infty} |a_n - u_{n+1}| \neq 1$.

Then the Eq (2.1) *is Ulam stable.*

In the following theorem we present a nonstability result for the Eq (2.1).

Theorem 2.6. *If there exists* $u_0 \in \mathbb{K}$ *such that* $\lim_{n\to\infty} |a_n - u_{n+1}| = 1$ *and* $(u_n)_{n\geq 0}$ *is bounded, then the Eq* (2.1) *is not Ulam stable.*

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} |a_n - u_{n+1}| = 1$, from Theorem 1.5 it follows that the equation

$$z_{n+1} = (a_n - u_{n+1})z_n + c_n$$

is not Ulam stable, i.e., there exists a sequence $(\bar{z}_n)_{n\geq 0}$ in X, satisfying the inequality

$$\|\bar{z}_{n+1} - (a_n - u_{n+1})\bar{z}_n - c_n\| \leq \varepsilon,\ n \geq 0, \tag{2.8}$$

such that for every sequence $(\bar{y}_n)_{n\geq 0}$ with

$$\bar{y}_{n+1} = (a_n - u_{n+1})\bar{y}_n + c_n,\ n \geq 0, \tag{2.9}$$

we have

$$\sup_{n\geq 0} \|\bar{y}_n - \bar{z}_n\| = \infty. \tag{2.10}$$

Let $(x_n)_{n\geq 0}$ be a sequence in X defined by the relation

$$x_{n+1} - u_n x_n = \bar{z}_n,\ n \geq 0 \tag{2.11}$$

(it suffices to take $x_0 = 0$ in order to determine $(x_n)_{n\geq 0}$ step by step).

The inequality (2.8) implies that the sequence $(x_n)_{n\geq 0}$ satisfies

$$\|x_{n+2} - a_n x_{n+1} - b_n x_n - c_n\| \leq \varepsilon,\ n \geq 0. \tag{2.12}$$

Let now $(y_n)_{n\geq 0}$ be an arbitrary sequence defined by

$$y_{n+2} = a_n y_{n+1} + b_n y_n + c_n$$

and $(\bar{y}_n)_{n\geq 0}$ be the sequence given by

$$\bar{y}_n = y_{n+1} - u_n y_n,\ n \geq 0. \tag{2.13}$$

Then the relations (2.9) and (2.10) hold.

Finally, we have to prove that $\sup_{n\geq 0} \|\bar{y}_n - \bar{z}_n\| = \infty$. Suppose the contrary. Then there exists $M > 0$ such that

$$\|x_n - y_n\| \leq M,\ n \geq 0.$$

From (2.11) and (2.13) it follows that

$$\begin{aligned}\|\bar{y}_n - \bar{z}_n\| &= \|y_{n+1} - u_n y_n - x_{n+1} + u_n x_n\| \\ &\le \|y_{n+1} - x_{n+1}\| + |u_n| \cdot \|y_n - x_n\| \\ &\le (1 + |u_n|) \cdot M,\end{aligned}$$

for every $n \ge 0$, which contradicts relation (2.10), if we take also into account that $(u_n)_{n\ge 0}$ is bounded. □

The following examples illustrate our theoretical results.

Example 2.7. Suppose that X is a Banach space over $\mathbb{C}$. The linear recurrence

$$x_{n+2} = -2e^{in\frac{\pi}{2}} x_{n+1} + i(-1)^n x_n,\ n \ge 0,\ x_0 \in X,$$

is not Ulam stable.

Indeed, in this case $u_{n+1} = -2e^{in\frac{\pi}{2}} + \frac{i(-1)^n}{u_n}$. Further, taking $u_0 = i$, one can show that $(u_n)_{n\ge 0}$ is bounded and $\lim\limits_{n\to\infty} |a_n - u_{n+1}| = 1$, since $u_n = ie^{in\frac{\pi}{2}}$, $n \ge 0$ (induction on n).

Example 2.8. Let X be a Banach space over $\mathbb{R}$. The linear recurrence

$$x_{n+2} = -2\frac{2n^2 - n - 2}{(2n+1)(2n+3)} x_{n+1} + \frac{1}{2n+3} x_n,\ n \ge 0,\ x_0, x_1 \in X,$$

is not Ulam stable.

Indeed, if we take $u_0 = -1$, then $u_n = \frac{1}{2n-1}$, $n \ge 1$ and hence $(u_n)_{n\ge 0}$ is bounded with $\lim\limits_{n\to\infty} |a_n - u_{n+1}| = 1$.

Example 2.9. Suppose that X is a Banach space over $\mathbb{C}$. The linear recurrence

$$x_{n+2} = (-1)^{n+1} x_n,\ n \ge 0,\ x_0 \in X,$$

is not Ulam stable.

Indeed, if we take $u_0 = e^{\frac{\pi i}{4}}$, then $u_n = e^{\frac{2n+1}{4}\pi i}$, $n \ge 1$ and hence $(u_n)_{n\ge 0}$ is bounded with $\lim\limits_{n\to\infty} |a_n - u_{n+1}| = 1$.

We present and discuss finally some particular cases and we give an example of Ulam stability for (2.1). We also mention that the following result contains in particular the case of stability for the linear difference equation with constant coefficients proved in [9, 10].

Corollary 2.10. *Let $a, b \in \mathbb{K}$ and $c \in X$. The linear difference equation of the second order with constant coefficients*

$$x_{n+2} = a x_{n+1} + b x_n + c,\ n \ge 0 \tag{2.14}$$

is Ulam stable if and only if none of the roots of the characteristic equation $\lambda^2 - a\lambda - b = 0$ lie on the unit circle.

Proof. If for all $n \ge 0$, $a_n = a$ and $b_n = b$ in (2.1), then $u_{n+1} = a + \frac{b}{u_n}$, $n \ge 0$, and by Remark 2.3,

$$u_n = \frac{\alpha_n u_0 + \beta_n}{\alpha_{n-1} u_0 + \beta_{n-1}},\ n \ge 1,$$

where $A = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ and

$$A^n = \begin{pmatrix} \alpha_n & \beta_n \\ \alpha_{n-1} & \beta_{n-1} \end{pmatrix},\ n \geq 1.$$

Suppose in what follows that the eigenvalues λ_1 and λ_2 of A, i.e. the roots of the characteristic equation $\lambda^2 - a\lambda - b = 0$, are distinct. Then $a - \lambda_1 = \lambda_2$ and

$$A^n = (\lambda_1)^n B + (\lambda_2)^n C,\ n \geq 0, \tag{2.15}$$

where $B = \frac{1}{\lambda_2-\lambda_1}\begin{pmatrix} \lambda_2 - a & -b \\ -1 & \lambda_2 \end{pmatrix}$ and $C = \frac{1}{\lambda_2-\lambda_1}\begin{pmatrix} a - \lambda_1 & b \\ 1 & -\lambda_1 \end{pmatrix}$.

Hence,

$$\alpha_n u_0 + \beta_n = \frac{(\lambda_1)^n}{\lambda_2 - \lambda_1}(u_0(\lambda_2 - a) - b) + \frac{(\lambda_2)^n}{\lambda_2 - \lambda_1}(u_0(a - \lambda_1) + b)$$

and consequently,

$$u_n = \frac{(\lambda_1)^n (u_0(\lambda_2 - a) - b) + (\lambda_2)^n (u_0(a - \lambda_1) + b)}{(\lambda_1)^{n-1} (u_0(\lambda_2 - a) - b) + (\lambda_2)^{n-1} (u_0(a - \lambda_1) + b)},\ n \geq 1. \tag{2.16}$$

Next we show that the Eq (2.14) is Ulam stable if and only if $|\lambda_1| \neq 1$ and $|\lambda_2| \neq 1$. Indeed, to prove the necessity, let us suppose without loss of generality that $|\lambda_1| = 1$. Taking $u_0 = \lambda_2$, we get $u_n = \lambda_2$ and $\lim_{n\to\infty} |a_n - u_{n+1}| = |\lambda_1| = 1$. Hence, according to Theorem 2.6, the Eq (2.14) is not Ulam stable and we get a contradiction. Conversely, if $|\lambda_1| \neq 1$ and $|\lambda_2| \neq 1$, it is sufficient to consider $u_0 = \lambda_1$. Hence $u_n = \lambda_1$, $\lim_{n\to\infty} |u_n| \neq 1$ and $\lim_{n\to\infty} |a_n - u_{n+1}| = |\lambda_2| \neq 1$. It follows that the Eq (2.14) is Ulam stable and the proof is complete.

Similarly, one can easily remark that the previous assertion remains also true for $\lambda_1 = \lambda_2$. In this case, $\lambda_1 = \lambda_2 = \frac{a}{2}$, $b = -\frac{a^2}{4}$ and

$$A^n = (\lambda_1)^n (Cn + B),\ n \geq 0, \tag{2.17}$$

where $B = I_2$ and $C = \begin{pmatrix} 1 & -\frac{a}{2} \\ \frac{2}{a} & -1 \end{pmatrix}$, for $a \neq 0$.

Hence,

$$\alpha_n u_0 + \beta_n = \left(\frac{a}{2}\right)^n \left((n+1)u_0 - \frac{a}{2}n\right)$$

and consequently,

$$u_n = \frac{\frac{a}{2}\left((n+1)u_0 - \frac{a}{2}n\right)}{nu_0 - \frac{a}{2}(n-1)},\ n \geq 1.$$

Furthermore, $\lim\limits_{n\to\infty} |u_n| = \lim\limits_{n\to\infty} |a_n - u_{n+1}| = |\frac{a}{2}|$ and the Eq (2.14) is Ulam stable if and only if $|a| \neq 2$. □

Corollary 2.11. *Let $(b_n)_{n\geq 0}$ be a sequence in $K \setminus \{0\}$ and $(c_n)_{n\geq 0}$ a sequence in X. The linear difference equation*

$$x_{n+2} = b_n x_n + c_n,\ n \geq 0 \tag{2.18}$$

is Ulam stable if the following two limits exist and

$$\lim_{n\to\infty} \left| \frac{\prod_{k=1}^{n} b_{2k-1} u_0}{\prod_{k=0}^{n-1} b_{2k}} \right| = \lim_{n\to\infty} \left| \frac{\prod_{k=0}^{n} b_{2k}}{\prod_{k=1}^{n} b_{2k-1} u_0} \right| \neq 1. \tag{2.19}$$

Proof. Indeed, for $a_n = 0$ in (2.1), we get $u_{n+1} = \frac{b_n}{u_n}$, hence $A_n = \begin{pmatrix} 0 & b_n \\ 1 & 0 \end{pmatrix}$, $n \geq 0$. Further, applying Remark 2.3, we get

$$u_{2n} = \frac{\prod_{k=1}^{n} b_{2k-1} u_0}{\prod_{k=0}^{n-1} b_{2k}}, \quad u_{2n+1} = \frac{\prod_{k=0}^{n} b_{2k}}{\prod_{k=1}^{n} b_{2k-1} u_0} \tag{2.20}$$

for all $n \geq 1$. Finally, taking into account Theorem 2.5, we obtain the desired conclusion. □

Corollary 2.12. *The linear difference equation*

$$x_{n+2} = a_n x_{n+1} + (\alpha^2 - \alpha a_n) x_n + c_n, \ n \geq 0 \tag{2.21}$$

is Ulam stable if for some $\delta_0 \in \mathbb{K}$*, the sequence*

$$\delta_n = \alpha^n \delta_0 + (\delta_0 - \alpha) \sum_{k=1}^{n} \alpha^{n-k} \prod_{s=0}^{k-1} (a_s - \alpha), \ n \geq 1 \tag{2.22}$$

satisfies the relations $\lim_{n\to\infty} |\frac{\delta_n}{\delta_{n-1}}| \neq 1$ *and* $\lim_{n\to\infty} |a_n - \frac{\delta_{n+1}}{\delta_n}| \neq 1$.

Proof. By Remark 2.3,

$$u_n = \frac{\alpha_n u_0 + \beta_n}{\alpha_{n-1} u_0 + \beta_{n-1}}, \ n \geq 1, \tag{2.23}$$

where

$$\begin{cases} \alpha_n &= a_{n-1}\alpha_{n-1} + \alpha(\alpha - a_{n-1})\alpha_{n-2} \\ \beta_n &= a_{n-1}\beta_{n-1} + \alpha(\alpha - a_{n-1})\beta_{n-2}, \ n \geq 2 \end{cases}$$

and $\alpha_0 = 1, \beta_0 = 0, \alpha_1 = a_0, \beta_1 = \alpha^2 - \alpha a_0$. Observe now that

$$\alpha_n - \alpha\alpha_{n-1} = (a_{n-1} - \alpha)(\alpha_{n-1} - \alpha\alpha_{n-2}).$$

Denoting further $\gamma_n = \alpha_n - \alpha\alpha_{n-1}$ ($n \geq 1$), where $\gamma_1 = a_0 - \alpha$, one obtains $\gamma_n := (a_{n-1} - \alpha)\gamma_{n-1}$ ($n \geq 2$) and consequently,

$$\gamma_n = \prod_{k=0}^{n-1} (a_k - \alpha).$$

Thus, $\alpha_n = \alpha\alpha_{n-1} + \prod_{k=0}^{n-1}(a_k - \alpha)$, $n \geq 1$. Further, putting $\frac{y_{n-1}}{y_n}$ ($n \geq 1$) with $y_0 = 1$ instead of the first α below, we may deduce that $y_n = \frac{1}{\alpha^n}$, and finally

$$\alpha_n = a_0 \alpha^{n-1} + \sum_{k=2}^{n} \alpha^{n-k} \prod_{s=0}^{k-1} (a_s - \alpha), \ n \geq 2.$$

Similarly, one may show that

$$\beta_n = (\alpha - a_0)\alpha^n - \sum_{k=2}^{n} \alpha^{n-k+1} \prod_{s=0}^{k-1} (a_s - \alpha), \ n \geq 2.$$

Hence, if we take $\delta_0 = u_0$ and we substitute the above expressions in (2.23), one obtains an explicit formula for u_n, more precisely $u_n = \frac{\delta_n}{\delta_{n-1}}$, $n \geq 1$. Finally, applying Theorem 2.5 we finish the proof. □

Example 2.13. The linear recurrence

$$x_{n+2} = 2\frac{2n+3}{n+1}x_{n+1} - 4\frac{n+2}{n+1}x_n,\ n \geq 0 \tag{2.24}$$

is Ulam stable.

Indeed, since the Eq (2.24) is a particular case of (2.21) with $\alpha = 2$, $a_n = 2\frac{2n+3}{n+1}$ and $c_n = 0$, we deduce that

$$\delta_n = (n+1)(n+2)2^{n-1}(\delta_0 - 2) + 2^{n+1},\ n \geq 1$$

and, consequently,

$$\frac{\delta_n}{\delta_{n-1}} = 2\frac{(n+1)(n+2)(\delta_0 - 2) + 4}{n(n+1)(\delta_0 - 2) + 4},\ n \geq 1.$$

Moreover, $\lim\limits_{n\to\infty} |\frac{\delta_n}{\delta_{n-1}}| = \lim\limits_{n\to\infty} |a_n - \frac{\delta_{n+1}}{\delta_n}| = 2$. Hence, using Corollary 2.12, we conclude that the linear difference equation (2.1) is Ulam stable.

3. Third order equations

Consider in the following the third order linear difference equation

$$x_{n+3} = a_n x_{n+2} + b_n x_{n+1} + c_n x_n + d_n,\ n \geq 0, \tag{3.1}$$

where $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, $(c_n)_{n\geq 0}$ are sequences in $\mathbb{K}$ and $(d_n)_{n\geq 0}$ is a sequence in X.

Lemma 3.1. *Suppose that* $(x_n)_{n\geq 0}$ *satisfies* (3.1). *Consider*

$$z_n = x_{n+1} - u_n x_n,\ n \geq 0 \tag{3.2}$$

and

$$z'_n = x_{n+1} - u'_n x_n,\ n \geq 0 \tag{3.3}$$

where $(u_n)_{n\geq 0}$ *and* $(u'_n)_{n\geq 0}$ *are sequences in* $\mathbb{K}$ *defined as follows:*

$$u_{n+2} = a_n + \frac{b_n}{u_{n+1}} + \frac{c_n}{u_n u_{n+1}},\ n \geq 0,\ u_0, u_1 \in \mathbb{K} \tag{3.4}$$

and

$$u'_{n+1} = a_n - u_{n+2} + \frac{(a_n - u_{n+2})u_{n+1} + b_n}{u'_n},\ n \geq 0,\ u'_0 \in \mathbb{K}. \tag{3.5}$$

Then

$$z_{n+2} = (a_n - u_{n+2})z_{n+1} + ((a_n - u_{n+2})u_{n+1} + b_n)\, z_n + d_n,\ n \geq 0 \tag{3.6}$$

and

$$z'_{n+1} = (a_n - u_{n+2} - u'_{n+1})z'_n + d_n,\ n \geq 0. \tag{3.7}$$

Proof. From (3.4) it follows that

$$-\frac{c_n}{u_n} = (a_n - u_{n+2})u_{n+1} + b_n$$

and hence, using (3.1) and (3.6) one gets

$$\begin{aligned}
z_{n+2} &= (a_n - u_{n+2})x_{n+2} + b_n x_{n+1} + c_n x_n + d_n \\
&= (a_n - u_{n+2})(x_{n+2} - u_{n+1}x_{n+1}) + (a_n - u_{n+2})u_{n+1}x_{n+1} \\
&\quad + b_n x_{n+1} + c_n x_n + d_n \\
&= (a_n - u_{n+2})z_{n+1} + [(a_n - u_{n+2})u_{n+1} + b_n](x_{n+1} - u_n x_n) \\
&\quad - \frac{c_n}{u_n}u_n x_n + c_n x_n + d_n \\
&= (a_n - u_{n+2})z_{n+1} + [(a_n - u_{n+2})u_{n+1} + b_n]z_n + d_n.
\end{aligned}$$

Finally, proceeding similarly to the proof of 2.1 one can show that $(z'_n)_{n\geq 0}$ satisfies (3.7). □

Remark 3.2. Observe that the sequence $(u_n)_{n\geq 0}$ defined above can be written in the following form

$$u_{n+2} = \frac{a_n u_n u_{n+1} + b_n u_n + c_n}{u_n u_{n+1}},\ n \geq 0$$

and, consequently

$$u_n = \frac{\alpha_n u_0 u_1 + \beta_n u_0 + \gamma_n}{\alpha_{n-1} u_0 u_1 + \beta_{n-1} u_0 + \gamma_{n-1}},\ n \geq 1$$

where $A_n = \begin{pmatrix} a_n & b_n & c_n \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ with $A_0 = \begin{pmatrix} \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_0 & \beta_0 & \gamma_0 \end{pmatrix}$ and

$$A_{n-2} \cdot \ldots \cdot A_0 = \begin{pmatrix} \alpha_n & \beta_n & \gamma_n \\ \alpha_{n-1} & \beta_{n-1} & \gamma_{n-1} \\ \alpha_{n-2} & \beta_{n-2} & \gamma_{n-2} \end{pmatrix},\ n \geq 2.$$

Lemma 3.3. *If the equation $x_{n+1} - u_n x_n - z_n = 0$ is Ulam stable with the constant L_1 and* (3.6) *is Ulam stable with the constant L_2, then the Eq* (3.1) *is Ulam stable with the constant $L_1 L_2$.*

Proof. Let $\varepsilon > 0$ and let $(x_n)_{n\geq 0}$ be a sequence in X such that

$$\|x_{n+3} - a_n x_{n+2} - b_n x_{n+1} - c_n x_n - d_n\| \leq \varepsilon,\ n \geq 0.$$

Put $z_n = x_{n+1} - u_n x_n$, where $(u_n)_{n\geq 0}$ satisfies relation (3.4). Then

$$\|z_{n+2} - (a_n - u_{n+2})z_{n+1} - [(a_n - u_{n+2})u_{n+1} + b_n]z_n - d_n\| \leq \varepsilon,\ n \geq 0$$

and there exists $(w_n)_{n\geq 0}$,

$$w_{n+2} = (a_n - u_{n+2})w_{n+1} + [(a_n - u_{n+2})u_{n+1} + b_n]w_n + d_n,\ n \geq 0, \tag{3.8}$$

such that

$$\|z_n - w_n\| \le L_2\varepsilon,\ n \ge 0.$$

Taking account of (3.2) we get

$$\|x_{n+1} - u_n x_n - w_n\| \le L_2\varepsilon,\ n \ge 0.$$

Now, since the Eq (3.2) is Ulam stable and its stability does not depend on z_n, it follows that there exists a sequence $(y_n)_{n\ge 0}$,

$$y_{n+1} = u_n y_n + w_n,\ n \ge 0, \tag{3.9}$$

such that

$$\|x_n - y_n\| \le L_1 L_2\varepsilon,\ n \ge 0. \tag{3.10}$$

To complete the proof it remains for us to show that $(y_n)_{n\ge 0}$ satisfies the Eq (3.1). For this we replace $(w_n)_{n\ge 0}$ from (3.9) to (3.8) and we obtain

$$y_{n+3} = a_n y_{n+2} + b_n y_{n+1} - [(a_n - u_{n+2})u_n u_{n+1} + b_n u_n]y_n + d_n.$$

Finally, taking account of relation (3.4), we obtain

$$y_{n+3} = a_n y_{n+2} + b_n y_{n+1} + c_n y_n + d_n.$$

□

The following result holds as a direct consequence of Lemma 2.4 and Lemma 3.3.

Lemma 3.4. *If the equation $x_{n+1} - u_n x_n - z_n = 0$ is Ulam stable with the constant L_1, $z_{n+1} - u'_n z_n - z'_n = 0$ is Ulam stable with the constant L_2 and* (3.7) *is Ulam stable with the constant L_3, then (3.1) is Ulam stable with the constant $L_1 L_2 L_3$.*

As a consequence of Theorem 1.5 and Lemma 3.4, we get the following result on Ulam stability for the Eq (3.1).

Theorem 3.5. *Suppose that for the sequences $(u_n)_{n\ge 0}$ and $(u'_n)_{n\ge 0}$ given by* (3.4) *and* (3.5)*, respectively, $\lim_{n\to\infty} |u_n|$, $\lim_{n\to\infty} |u'_n|$, $\lim_{n\to\infty} |a_n - u_{n+2} - u'_{n+1}|$ exist and:*
1) $\lim_{n\to\infty} |u_n| \ne 1$;
2) $\lim_{n\to\infty} |u'_n| \ne 1$;
3) $\lim_{n\to\infty} |a_n - u_{n+2} - u'_{n+1}| \ne 1$.
Then the Eq (3.1) *is Ulam stable.*

Example 3.6. The following linear recurrence

$$x_{n+3} = \frac{4n+5}{n+1}x_{n+2} - \frac{1+4(n+1)(n+2)^2}{(n+2)(n+1)^2}x_{n+1} + \frac{2+4(n+1)(n+2)}{(n+2)(n+1)^2}x_n$$

is Ulam stable.

Indeed, in this case, if we take $u_0 = u_1 = 2$ and $u'_0 = 1$, one can see that $u_n = 2$ and $u'_n = \frac{1}{n+1}$ for all $n \ge 0$. Thus, all the assumptions of Theorem 3.5 are fulfilled, which means that we have stability.

A nonstability result for the Eq (3.1), similar to the one obtained for the second order linear difference equation (2.1), holds.

Theorem 3.7. *If the Eq (3.6) is not Ulam stable and there exist $u_0, u_1 \in \mathbb{K}$ such that $(u_n)_{n\geq 0}$ is bounded, then the Eq* (3.1) *is not Ulam stable.*

Proof. Let $\varepsilon > 0$. Since (3.6) is not Ulam stable, i.e. there exists a sequence $(\bar{z}_n)_{n\geq 0}$ in X, satisfying the inequality

$$\|\bar{z}_{n+2} - (a_n - u_{n+2})\bar{z}_{n+1} - [(a_n - u_{n+2})u_{n+1} + b_n]\bar{z}_n - d_n\| \leq \varepsilon,\ n \geq 0 \tag{3.11}$$

such that for every sequence $(\bar{y}_n)_{n\geq 0}$ with

$$\bar{y}_{n+2} = (a_n - u_{n+2})\bar{y}_{n+1} + [(a_n - u_{n+2})u_{n+1} + b_n]\bar{y}_n + d_n,\ n \geq 0 \tag{3.12}$$

we have

$$\sup_{n\geq 0} \|\bar{y}_n - \bar{z}_n\| = \infty. \tag{3.13}$$

Let $(x_n)_{n\geq 0}$ be such that

$$x_{n+1} - u_n x_n = \bar{z}_n,\ n \geq 0 \tag{3.14}$$

(it suffices to take $x_0 = 0$ in order to determine $(x_n)_{n\geq 0}$ step by step). Inequality 3.11 implies that the sequence $(x_n)_{n\geq 0}$ satisfies

$$\|x_{n+3} - a_n x_{n+2} - b_n x_{n+1} - c_n x_n - d_n\| \leq \varepsilon,\ n \geq 0. \tag{3.15}$$

Let now $(y_n)_{n\geq 0}$ be an arbitrary sequence defined by

$$y_{n+3} = a_n y_{n+2} + b_n y_{n+1} + c_n y_n + d_n$$

and $(\bar{y}_n)_{n\geq 0}$ be the sequence given by

$$\bar{y}_n = y_{n+1} - u_n y_n,\ n \geq 0. \tag{3.16}$$

Then the relations (3.12) and (3.13) hold.

Finally, we have to prove that $\sup_{n\geq 0} \|\bar{y}_n - \bar{z}_n\| = \infty$. Suppose the contrary. Then there exists $M > 0$ such that

$$\|x_n - y_n\| \leq M,\ n \geq 0.$$

From (3.14) and (3.16) it follows that

$$\begin{aligned}\|\bar{y}_n - \bar{z}_n\| &= \|y_{n+1} - u_n y_n - x_{n+1} + u_n x_n\| \\ &\leq \|y_{n+1} - x_{n+1}\| + |u_n| \cdot \|y_n - x_n\| \\ &\leq (1 + |u_n|) \cdot M,\end{aligned}$$

for every $n \geq 0$, which contradicts relation (3.13), if we take also into account that $(u_n)_{n\geq 0}$ is bounded. □

The next corollary gives sufficient conditions for Eq (3.1) to be not Ulam stable. To prove this, just take into account Lemma 3.1, Theorem 2.6 and Theorem 3.7.

Corollary 3.8. *If there exist $u_0, u_1, u'_0 \in \mathbb{K}$ such that $(u_n)_{n\geq 0}$ and $(u'_n)_{n\geq 0}$ are bounded and $\lim_{n\to\infty} |a_n - u_{n+2} - u'_{n+1}| = 1$, then the Eq* (3.1) *is not Ulam stable.*

Example 3.9. The following linear recurrence

$$x_{n+3} = \frac{1}{n+2}x_{n+2} + \frac{n^2+3n+1}{(n+1)(n+2)}x_{n+1} - \frac{1}{n+1}x_n$$

is not Ulam stable. Indeed, letting $u_0 = u_1 = -1$ and $u'_0 = 1$, we get $u_n = -1$ and $u'_n = \frac{1}{n+1}$ for all $n \geq 0$ and consequently $\lim_{n\to\infty} |a_n - u_{n+2} - u'_{n+1}| = 1$, which means that we do not have stability.

For the case of constant coefficients in the Eq (3.1), we get the result given in [22].

Corollary 3.10. *The third order linear difference equation with constant coefficients*

$$x_{n+3} = ax_{n+2} + bx_{n+1} + cx_n + d,\ n \geq 0, \tag{3.17}$$

is Ulam stable if and only if none of the roots of the characteristic equation $\lambda^3 - a\lambda^2 - b\lambda - c = 0$ lie on the unit circle.

Proof. Indeed, let λ_1, λ_2 and λ_3 be the roots of the characteristic equation and suppose without loss of generality that $|\lambda_1| = 1$ and take $u_0 = u_1 = \lambda_2$, $u'_0 = \lambda_3$. Then, using Corollary 3.8 and Vieta's formulas, i.e. the relations $\lambda_1 + \lambda_2 + \lambda_3 = a$, $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = -b$ and $\lambda_1\lambda_2\lambda_3 = c$, one can easily check that $u_n = \lambda_2, n \geq 2$ and $u'_n = \lambda_3, n \geq 1$. Moreover, $\lim_{n\to\infty} |a - u_{n+2} - u'_{n+1}| = |a - \lambda_2 - \lambda_3| = |\lambda_1|$, which means that (3.17) is not Ulam stable, a contradiction. Conversely, if $|\lambda_1| \neq 1$, $|\lambda_2| \neq 1$ and $|\lambda_3| \neq 1$, it is sufficient to consider $u_0 = u_1 = \lambda_1$ and $u'_0 = \lambda_2$. Indeed, due to the choice of the initial values it follows that $\lim_{n\to\infty} |u_n| = |\lambda_1| \neq 1$, $\lim_{n\to\infty} |u'_n| = |\lambda_2| \neq 1$ and $\lim_{n\to\infty} |a - u_{n+2} - u'_{n+1}| = |a - \lambda_1 - \lambda_2| = |\lambda_3| \neq 1$. Finally, the Eq (3.17) is Ulam stable, by Theorem 3.5. □

Finally, it is worth mentioning here that the result on nonstability of (3.1) contains in particular the case of nonstability for constant coefficients proved in [6, 9, 10, 22, 24].

4. Conclusions

In this paper we give some results on Ulam stability for the second order and for the third order linear difference equation with nonconstant coefficients in a Banach space. As far as we know there are few results on Ulam stability for such equations (see, e.g., [11–13]). The novelty of this approach consists in the fact that we decompose the second order linear difference equation in a Riccati difference equation and a linear difference equation. In this way we can use the results on Ulam stability of the first order linear difference equation.

The importance of these results consists in the fact that they are related to the theory of perturbation of a discrete dynamical system (see [15]). Remark that for difference equations with constant coefficients we get the results obtained in the paper [3].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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