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*Research article*

# Two-step inertial method for solving split common null point problem with multiple output sets in Hilbert spaces

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**Abstract:** In this paper, an algorithm with two-step inertial extrapolation and self-adaptive step sizes is proposed to solve the split common null point problem with multiple output sets in Hilbert spaces. Weak convergence analysis are obtained under some easy to verify conditions on the iterative parameters in Hilbert spaces. Preliminary numerical tests are performed to support the theoretical analysis of our proposed algorithm.

**Keywords:** Hilbert space; metric projection; self-adaptive step size; two-step inertial; split common null point problem
**Mathematics Subject Classification:** 47H09, 47H10, 49J53, 90C25

## 1. Introduction

Throughout this paper, $\mathcal{H}$ denotes a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the induced $\|\cdot\|$, $I$ the identity operator on $\mathcal{H}$, $\mathbb{N}$ the set of all natural numbers and $\mathbb{R}$ the set of all real numbers. For a self-operator $T$ on $\mathcal{H}$, $F(T)$ denotes the set of all fixed points of $T$.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real Hilbert spaces and let $\mathcal{T} : \mathcal{H}_1 \to \mathcal{H}_2$ be bounded linear operator. Let $\{U_j\}_{j=1}^t : \mathcal{H}_1 \to \mathcal{H}_1$ and $\{T_i\}_{i=1}^r : \mathcal{H}_2 \to \mathcal{H}_2$ be two finite families of operators, where $t, r \in \mathbb{N}$. The split common fixed point problem (SCFPP) is formulated as finding a point $x^* \in \mathcal{H}_1$ such that

$$x^* \in \cap_{j=1}^t F(U_j) \quad \text{such that} \quad \mathcal{T} x^* \in \cap_{i=1}^r F(T_i). \tag{1.1}$$

In particular, if $t = r = 1$, the SCFPP (1.1) reduces to finding a point $x^* \in \mathcal{H}_1$ such that

$$x^* \in F(U) \text{ such that } \mathcal{T} x^* \in F(T). \tag{1.2}$$

The above problem is usually called the two-set SCFPP.

In recent years, the SCFPP (1.1) and the two-set SCFPP (1.2) have been studied and extended by many authors, see for instance [15, 20, 23, 27, 36–40, 47–49]. It is known that the SCFPP includes the multiple-set split feasibility problem and split feasibility problem as a special case. In fact, let $\{C_j\}_{j=1}^t$ and $\{Q_i\}_{i=1}^r$ be two finite families of nonempty closed convex subsets in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $U_j = P_{C_j}$ and $T_i = P_{Q_i}$; then SCFPP (1.1) becomes the multiple-set split feasibility problem (MSSFP) as follows:

$$\text{find } x^* \in \cap_{j=1}^t C_j \text{ such that } \mathcal{T} x^* \in \cap_{i=1}^r Q_i. \tag{1.3}$$

When $t = r = 1$ the MSSFP (1.3) is reduced to the split feasibility problem (SFP) which is described as finding a point $x^* \in \mathcal{H}_1$ satisfying the following property

$$x^* \in C \text{ such that } \mathcal{T} x^* \in Q. \tag{1.4}$$

The SFP was first introduced by Censor and Elfving [22] with the aim of modeling certain inverse problems. It has turned out to also play an important role in, for example, medical image reconstruction and signal processing (see [2, 4, 15, 17, 21]). Since then, several iterative algorithms for solving (1.4) have been presented and analyzed. See, for instance [1, 5, 14–16, 18, 19, 23, 24, 27] and references therein.

The $CQ$ algorithm has been extended by several authors to solve the multiple-set split convex feasibility problem. See, for instance, the papers by Censor and Segal [25], Elfving, Kopf and Bortfeld [23], Masad and Reich [35], and by Xu [53, 54].

In 2020, Reich and Tuyen [45] proposed and analyzied the following split feasibility problem with multiple output sets in Hilbert spaces: let $\mathcal{H}, \mathcal{H}_i, i = 1, 2, \ldots, m$ be real Hilbert spaces. Let $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}_i,\ i = 1, 2, \ldots, m$, be bounded linear operators. Furthermore, let $C$ and $Q_i$ be nonempty, closed and convex subsets of $\mathcal{H}$ and $\mathcal{H}_i, i = 1, 2, \ldots, m$, respectively. Find an element $u^\dagger$, such that:

$$u^\dagger \in \Omega^{SFP} = C \cap \left(\cap_{i=1}^m \mathcal{T}_i^{-1}(Q_i)\right) \neq \emptyset; \tag{1.5}$$

that is, $u^\dagger \in C$ and $\mathcal{T} u^\dagger \in Q_i$, for all $i = 1, 2, \ldots, m$.

To solve problem (1.5), Reich et al. [46] proposed the following iterative methods: for any $u_0, v_0 \in C$, let $\{u_n\}$ and $\{v_n\}$ be two sequence generated by:

$$u_{n+1} = P_C\left[u_n - \gamma \sum_{i=1}^m \mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i u_n\right], \tag{1.6}$$

$$v_{n+1} = \alpha_n f(v_n) + (1 - \alpha_n) P_C\left[v_n - \gamma_n \sum_{i=1}^m \mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i v_n\right], \tag{1.7}$$

where $f : C \to C$ is a strict $k$-contraction with $k \in [0, 1)$, $\{\gamma_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$. They established the weak and strong convergence of iterative methods (1.6) and (1.7), respectively.

In 2021, Reich and Tuyen [44] considered the following split common null point problem with multiple output sets in Hilbert spaces: let $\mathcal{H}, \mathcal{H}_i, i = 1, 2, \ldots, N$, be real Hilbert spaces and let $\mathcal{T}_i : \mathcal{H} \to \mathcal{H}_i, i = 1, 2, \ldots, N$, be bounded linear operators. Let $\mathcal{B} : \mathcal{H} \to 2^{\mathcal{H}_i}, \ i = 1, 2, \ldots, N$ be maximal monotone operators. Given $\mathcal{H}, \mathcal{H}_i$ and $\mathcal{T}_i$ as defined above, the split common null point problem with multiple output sets is to find a point $u^\dagger$ such that

$$u^\dagger \in \Omega := \mathcal{B}^{-1}(0) \cap \left(\cap_{i=1}^N \mathcal{T}_i^{-1}(\mathcal{B}_i^{-1}(0))\right) \neq \emptyset. \tag{1.8}$$

To solve problem (1.8), Reich and Tuyen [44] proposed the following iterative method:

**Algorithm 1.1.** *For any $u_0 \in \mathcal{H}$, Let $\mathcal{H}_0 = \mathcal{H}, \mathcal{T}_0 = I^{\mathcal{H}}, \mathcal{B}_0 = \mathcal{B}$, and let $\{u_n\}$ be the sequence generated by:*

$$\begin{aligned} v_n &= \sum_{i=0}^N \beta_{i,n}\left[u_n - \tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_i} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i u_n\right] \\ u_{n+1} &= \alpha_n f(u_n) + (1-\alpha_n)v_n, \ n \geq 0, \end{aligned}$$

*where $\{\alpha_n\} \subset (0,1)$, and $\{\beta_{i,n}\}$ and $\{r_{i,n}\}, i = 0, 1, \ldots, N$, are sequences of positive real numbers, such that $\{\beta_{i,n}\} \subset [a,b] \subset (0,1)$ and $\sum_{i=0}^N \beta_{i,n} = 1$ for each $n \geq 0$, and $\tau_{i,n} = \rho_{i,n}\frac{\|(\mathcal{I}^{\mathcal{H}_i} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i u_n\|^2}{\|\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_i} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i u_n\|^2 + \theta_{i,n}}$, where $\{\rho_{i,n}\} \subset [c,d] \subset (0,2)$ and $\{\theta_{i,n}\}$ is a sequence of positive real numbers for each $i = 0, 1, \ldots, N$, and $f : \mathcal{H} \to \mathcal{H}$ is a strict contraction mapping $\mathcal{H}$ into itself with the contraction coefficient $k \in [0,1)$.*

They established the strong convergence of the sequence $\{u_n\}$ generated by Algorithm 1.1 which is a solution of the Problem (1.8)

Alvarez and Attouch [7] applied the following inertial technique to develop an inertial proximal method for finding the zero of a monotone operator, i.e.,

$$\text{find } x \in H \ \text{ such } \ \text{that } 0 \in G(x). \tag{1.9}$$

where $G : H \to 2^H$ is a set-valued monotone operator. Given $x_{n-1}, \ x_n \in H$ and two parameters $\theta_n \in [0,1), \lambda_n > 0$, find $x_{n+1} \in H$ such that

$$0 \in \lambda_n G(x_{n+1}) + x_{n+1} - x_n - \theta_n(x_n - x_{n-1}). \tag{1.10}$$

Here, the inertia is induced by the term $\theta_n(x_n - x_{n-1})$. The equation (1.10) may be thought as coming from the implicit discretization of the second-other differential system

$$0 \in \frac{d^2x}{dt^2}(t) + \rho\frac{dx}{dt}(t) + G(x(t)) \ a.e. \ t > 0, \tag{1.11}$$

where $\rho > 0$ is a damping or a friction parameter. This point of view inspired various numerical methods related to the inertial terminology which has a nice convergence property [6–8, 28, 29, 33] by incorporating second order information and helps in speeding up the convergence speed of an algorithm (see, e.g., [3, 7, 9–13, 51, 52] and the references therein).

Recently, Thong and Hieu [50] introduced an inertial algorithm to solve split common fixed point problem (1.1). The algorithm is of the form

$$\begin{cases} x_0, x_1 \in \mathcal{H}_1, \\ y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n \sum_{j=1}^{r} w_j U_j \left(I + \sum_{i=1}^{t} \eta_i \gamma \mathcal{T}^*(T_i - \mathcal{I})\mathcal{T}\right) y_n. \end{cases} \tag{1.12}$$

Under approximate conditions, they show that the sequence $\{x_n\}$ generated by (1.12) converges weakly to some solution of SCFPP (1.1).

It was shown in [43, Section 4] by example that one-step inertial extrapolation $w_n = x_n + \theta(x_n - x_{n-1})$, $\theta \in [0,1)$ may fail to provide acceleration. It was remarked in [32, Chapter 4] that the use of inertial of more than two points $x_n, x_{n-1}$ could provide acceleration. For example, the following two-step inertial extrapolation

$$y_n = x_n + \theta(x_n - x_{n-1}) + \delta(x_{n-1} - x_{n-2}) \tag{1.13}$$

with $\theta > 0$ and $\delta < 0$ can provide acceleration. The failure of one-step inertial acceleration of ADMM was also discussed in [42, Section 3] and adaptive acceleration for ADMM was proposed instead. Polyak [41] also discussed that the multi-step inertial methods can boost the speed of optimization methods even though neither the convergence nor the rate result of such multi-step inertial methods was established in [41]. Some results on multi-step inertial methods have recently be studied in [26].

**Our Contributions.** Motivated by [44, 50], in this paper, we consider the following split common null point problem with multiple output sets in Hilbert spaces: Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real Hilbert spaces. Let $\{U_j\}_{j=1}^r : \mathcal{H}_1 \to \mathcal{H}_1$ be a finite family of quasi-nonexpansive operators and $\mathcal{B}_i : \mathcal{H}_2 \to 2^{\mathcal{H}_2}, i = 1, 2, \dots, t.$ be maximal monotone operators and $\{\mathcal{T}_i\}_{i=1}^t : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. The split common null point problem with multiple output set is to find a point $x^* \in \mathcal{H}_1$ such that

$$x^* \in \cap_{j=1}^r F(U_j) \cap \left(\cap_{i=1}^t \mathcal{T}_i^{-1}(\mathcal{B}_i^{-1}0)\right) \neq \emptyset. \tag{1.14}$$

Let $\Upsilon$ be the solution set of (1.14). We propose a two-step inertial extrapolation algorithm with self-adaptive step sizes for solving problem (1.14) and give the weak convergence result of our problem in real Hilbert spaces. We give numerical computations to show the efficiency of our proposed method.

## 2. Preliminaries

Let $C$ be a nonempty, closed, and convex subset of a real Hilbert spaces $\mathcal{H}$. We know that for each point $u^* \in \mathcal{H}$, there is a unique element $P_C u^* \in C$, such that:

$$\|u^* - P_C u^*\| = \inf_{v \in C} \|u^* - v\|. \tag{2.1}$$

We recall that the mapping $P_C : \mathcal{H} \to C$ defined by (2.1) is said to be metric projection of $\mathcal{H}$ onto $C$. Moreover, we have (see, for instance, Section 3 in [31]):

$$\langle u^* - P_C u^*, v - P_C u^* \rangle \leq 0, \;\; \forall\, u^* \in \mathcal{H},\; v \in C. \tag{2.2}$$

**Definition 2.1.** Let $T : \mathcal{H} \to \mathcal{H}$ be an operator with $F(T) \neq \emptyset$. Then

- $T : \mathcal{H} \to \mathcal{H}$ is called nonexpansive if

$$\|Tu - Tv\| \leq \|u - v\|, \ \ \forall\, u, v \in \mathcal{H}, \tag{2.3}$$

- $T : \mathcal{H} \to \mathcal{H}$ is quasi-nonexpansive if

$$\|Tu - v\| \leq \|u - v\|, \ \ \forall\, v \in F(T),\ u \in \mathcal{H}. \tag{2.4}$$

We denote by $F(T)$ the set of fixed points of mapping $T$; that is, $F(T) = \{u^* \in C : Tu^* = u^*\}$. Given an operator $\mathcal{E} : \mathcal{H} \to 2^{\mathcal{H}}$, its domain, range, and graph are defined as follows:

$$\begin{aligned} \mathcal{D}(\mathcal{E}) &:= \{u^* \in \mathcal{H} : \mathcal{E}(u^*) \neq \emptyset\}, \\ \mathcal{R}(\mathcal{E}) &:= \cup\{\mathcal{E}(u^*) : u^* \in \mathcal{D}(\mathcal{E})\} \end{aligned}$$

and

$$\mathcal{G}(\mathcal{E}) := \{(u^*, v^*) \in \mathcal{H} \times \mathcal{H} : u^* \in \mathcal{D}(\mathcal{E}),\ v^* \in \mathcal{E}(u^*)\}.$$

The inverse operator $\mathcal{E}^{-1}$ of $\mathcal{E}$ is defined by:

$$u^* \in \mathcal{E}^{-1}(v^*) \text{ if and only if } v^* \in \mathcal{E}(u^*).$$

Recall that the operator $\mathcal{E}$ is said to be monotone if, for each $u^*, v^* \in \mathcal{D}(\mathcal{E})$, we have $\langle f - g, u^* - v^*\rangle \geq 0$ for all $f \in \mathcal{E}(u^*)$ and $g \in \mathcal{E}(v^*)$. We denote by $\mathcal{I}^{\mathcal{H}}$ the identity mapping on $\mathcal{H}$. A monotone operator $\mathcal{E}$ is said to be maximal monotone if there is no proper monotone extension of $\mathcal{E}$ or, equivalently, by Minty's theorem, if $\mathcal{R}(\mathcal{I}^{\mathcal{H}} + \lambda\mathcal{E}) = \mathcal{H}$, for all $\lambda > 0$. If $\mathcal{E}$ is maximal monotone, then we can define, for each $\lambda > 0$, a nonexpansive single-valued operator $J_\lambda^{\mathcal{E}} : \mathcal{R}(\mathcal{I}^{\mathcal{H}} + \lambda\mathcal{E}) \to \mathcal{D}(\mathcal{E})$ by

$$J_\lambda^{\mathcal{E}} = (\mathcal{I}^{\mathcal{H}} + \lambda\mathcal{E})^{-1}.$$

This operator is called the resolvent of $\mathcal{E}$. It is easy to see that $\mathcal{E}^{-1}(0) = F(J_\lambda^{\mathcal{E}})$, for all $\lambda > 0$.

**Lemma 2.2.** *[45] Suppose that $\mathcal{E} : \mathcal{D}(\mathcal{E}) \subset \mathcal{H} \to 2^{\mathcal{H}}$ is a monotone operator. Then, we have the following statements:*

*(i) For $r \geq s > 0$, we have:*

$$\|u - J_s^{\mathcal{E}} u\| \leq 2\|u - J_r^{\mathcal{E}} u\|,$$

*for all elements $u \in \mathcal{R}(\mathcal{I}^{\mathcal{H}} + r\mathcal{E}) \cap \mathcal{R}(\mathcal{I}^{\mathcal{H}} + s\mathcal{E})$.*

*(ii) For all numbers $r > 0$ and for all points $u, v \in \mathcal{R}(\mathcal{I}^{\mathcal{H}} + r\mathcal{E})$, we have:*

$$\langle u - v, J_r^{\mathcal{E}} u - J_r^{\mathcal{E}} v\rangle \geq \|J_r^{\mathcal{E}} u - J_r^{\mathcal{E}} v\|^2.$$

*(iii) For all numbers $r > 0$ and for all points $u, v \in \mathcal{R}(\mathcal{I}^{\mathcal{H}} + r\mathcal{E})$, we have:*

$$\langle (\mathcal{I}^{\mathcal{H}} - J_r^{\mathcal{E}})u - (\mathcal{I}^{\mathcal{H}} - J_r^{\mathcal{E}})v, u - v\rangle \geq \|(\mathcal{I}^{\mathcal{H}} - J_r^{\mathcal{E}})u - (\mathcal{I}^{\mathcal{H}} - J_r^{\mathcal{E}})v\|^2.$$

*(iv) If $S = \mathcal{E}^{-1}(0) \neq \emptyset$, then for all elements $u^* \in S$ and $u \in \mathcal{R}(\mathcal{I}^{\mathcal{H}} + r\mathcal{E})$, we have:*

$$\|J_r^{\mathcal{E}} u - u^*\|^2 \leq \|u - u^*\|^2 - \|u - J_r^{\mathcal{E}} u\|^2.$$

**Lemma 2.3.** *[30] Suppose that $\mathcal{T}$ is a nonexpansive mapping from a closed and convex subset of a Hilbert space $\mathcal{H}$ into $\mathcal{H}$. Then, the mapping $\mathcal{I}^{\mathcal{H}} - \mathcal{T}$ is demiclosed on $C$; that is, for any $\{u_n\} \subset C$, such that $u_n \rightharpoonup u \in C$ and the sequence $(\mathcal{I}^{\mathcal{H}} - \mathcal{T})(u_n) \to v$, we have $(\mathcal{I}^{\mathcal{H}} - \mathcal{T})(u) = v$.*

**Lemma 2.4.** *[34] Given an integer $N \geq 1$. Assume that for each $i = 1, \dots, N$, $T_i : \mathcal{H} \to \mathcal{H}$ is a $k_i$-demicontractive operator such that $\cap_{i=1}^N F(T_i) \neq \emptyset$. Assume that $\{w_i\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N w_i = 1$. Setting $U = \sum_{i=1}^N w_i T_i$, then the following results hold:*

*(i)* $F(U) = \cap_{i=1}^N F(T_i)$.
*(ii)* *$U$ is $\lambda$-demicontractive operator, where $\lambda = \max\{k_i | i = 1, \dots, N\}$.*
*(iii)* $\langle x - Ux, x - z\rangle \geq \frac{1-\lambda}{2} \sum_{i=1}^N w_i \|x - T_i x\|^2$ *for all $x \in \mathcal{H}$ and $z \in \cap_{i=1}^N F(T_i)$.*

**Lemma 2.5.** *Let $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{R}$. Then*

$$\begin{aligned} \|(1+\alpha)x - (\alpha-\beta)y - \beta z\|^2 &= (1+\alpha)\|x\|^2 - (\alpha-\beta)\|y\|^2 - \beta\|z\|^2 + (1+\alpha)(\alpha-\beta)\|x-y\|^2 \\ &\quad + \beta(1+\alpha)\|x-z\|^2 - \beta(\alpha-\beta)\|y-z\|^2. \end{aligned}$$

## 3. Main result

We give the following assumptions in order to obtain our convergence analysis.

**Assumptions 3.1.** *We assume that the inertial parameters $\theta \in [0, \frac{1}{2})$, $\rho \in (0,1)$ and $\delta \in (-\infty, 0]$ satisfies the following conditions.*

*(a)*

$$0 \leq \theta < \frac{1-\rho}{1+\rho};$$

*(b)*

$$\max\left\{\frac{2\theta\rho}{1-\rho} - (1-\theta), \frac{\theta-1}{\theta+1}[\rho(2\theta+1) - (\theta-1)]\right\} < \delta \leq 0;$$

*(c)*

$$(2\rho-1)(\theta^2+\delta^2) + (2-\rho)(\theta-\delta) + \rho - 2\delta\theta - 1 < 0.$$

Now, we present our proposed method and our convergence analysis as follows:

**Algorithm 3.2.** *Two-Step Inertial for Split Common Null Point Problem*

***Step 1.*** *Choose $\delta \in (-\infty, 0]$ and $\theta \in [0, 1/2)$ such that Assumption 3.1 is fulfilled. Pick $x_{-1}, x_0, x_1 \in \mathcal{H}_1$ and set $n = 1$.*

***Step 2.*** *Given $x_{n-2}, x_{n-1}$ and $x_n$, compute $x_{n+1}$ as follows*

$$\begin{cases} y_n = x_n + \theta(x_n - x_{n-1}) + \delta(x_{n-1} - x_{n-2}), \\ x_{n+1} = (1-\rho)y_n + \rho\sum_{j=1}^r w_j U_j\left(\mathcal{I} - \sum_{i=1}^t \delta_{i,n}\tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i\right)y_n, \end{cases} \tag{3.1}$$

*where $\{\delta_{i,n}\}$ and $\{r_{i,n}\}$, $i = 1, 2, \dots t$, are sequences of positive real numbers, such $\{\delta_{i,n}\} \subset [a,b] \subset (0,1)$ and $\sum_{i=1}^t \delta_{i,n} = 1$, for each $n \geq 1$ and*

$$\tau_{i,n} = \rho_{i,n}\frac{\|(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i y_n\|^2}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i y_n\|^2 + \theta_{i,n}}, \tag{3.2}$$

*where $\{\rho_{i,n}\} \subset [c,d] \subset (0,2)$ and $\{\theta_{i,n}\}$ are sequences of positive real numbers for each $i = 1,2,\ldots,t$, and $\{U_j\}_{j=1}^{r}$ is a finite family of quasi-nonexpansive operators.*

***Step 3.*** *Set $n \leftarrow n+1$ and go to **Step 2.***

**Lemma 3.3.** *For $t \in \mathbb{N}$, let $\{\mathcal{B}_i\}_{i=1}^{t} : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be a finite family maximal monotone operators. Let $\{T_i\}_{i=1}^{t} : \mathcal{H}_1 \to \mathcal{H}_2$, be a finite family of bounded linear operators. Define the operator $V : \mathcal{H}_1 \to \mathcal{H}_1$ by*

$$V := \mathcal{I} - \sum_{i=1}^{t} \delta_{i,n}\tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i, \tag{3.3}$$

*where $\tau_{i,n}$ is as defined in* (3.2), *$\{\delta_{i,n}\}_{i=1}^{t} \subset (0,1)$ and $\sum_{i=1}^{t}\delta_{i,n} = 1$. Then we have the following results:*

*(1)*

$$\|Vx - z\|^2 \le \|x - z\|^2 - \sum_{i=1}^{t} \delta_{i,n}\rho_{i,n}(2-\rho_{i,n})\frac{\|(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 + \theta_{i,n}}.$$

*(2) $x \in F(V)$ if and only if $T_i x \in \cap_{i=1}^{t} F(J_{r_{i,n}}^{\mathcal{B}_i})$, for $i = 1,2,\ldots,t$.*

*Proof.* (1) Given a point $z \in \Upsilon$, it follows from the convexity of the function $\|\cdot\|^2$ that:

$$\begin{aligned}
\|Vx - z\|^2 &= \left\| x - \sum_{i=1}^{t} \delta_{i,n}\tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x - z \right\|^2 \\
&= \left\| \sum_{i=1}^{t} \delta_{i,n}(x - \tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x - z) \right\|^2 \\
&\le \sum_{i=1}^{t} \delta_{i,n}\|x - \tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x - z\|^2.
\end{aligned} \tag{3.4}$$

Using $J_{r_{i,n}}^{\mathcal{B}_i}(\mathcal{T}_i z) = \mathcal{T}_i z$ and Lemma 2.2(iii), for each $i = 1,2,\ldots,t$, we see that

$$\begin{aligned}
&\|x - \tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x - z\|^2 \\
= \ &\|x - z\|^2 - 2\tau_{i,n}\langle \mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x, x - z\rangle + \tau_{i,n}^2\|\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 \\
= \ &\|x - z\|^2 - 2\tau_{i,n}\langle (\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x, \mathcal{T}_i x - \mathcal{T}_i z\rangle + \tau_{i,n}^2\|\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 \\
= \ &\|x - z\|^2 - 2\tau_{i,n}\langle (\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x - (\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i z, \mathcal{T}_i x - \mathcal{T}_i z\rangle \\
&+\tau_{i,n}^2\|\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 \\
\le \ &\|x - z\|^2 - 2\tau_{i,n}\|(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 + \tau_{i,n}^2(\|\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 + \theta_{i,n}) \\
= \ &\|x - z\|^2 - \rho_{i,n}(2-\rho_{i,n})\frac{\|(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^4}{\|\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 + \theta_{i,n}}.
\end{aligned} \tag{3.5}$$

Hence, from (3.4) and (3.5), we get

$$\|Vx - z\|^2 \le \|x - z\|^2 - \sum_{i=1}^{t} \delta_{i,n}\rho_{i,n}(2-\rho_{i,n})\frac{\|(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 + \theta_{i,n}}.$$

(2) It is obvious that if $\mathcal{T}_i x \in \cap_{i=1}^t F(J_{r_{i,n}}^{\mathcal{B}_i})$ then $x \in F(V)$. We show the converse, let $x \in F(V)$ and $z \in \mathcal{T}_i^{-1}\left(F(J_{r_{i,n}}^{\mathcal{B}_i})\right)$ we have

$$\|x - z\|^2 = \|Vx - z\|^2 \leq \|x - z\|^2 - \sum_{i=1}^{t} \delta_{i,n}\rho_{i,n}(2 - \rho_{i,n}) \frac{\|(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 + \theta_{i,n}}. \tag{3.6}$$

Since $\rho_{i,n} \subset (0, 2)$, we obtain

$$(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x = 0, \quad \forall\, i = 1, 2, \ldots, t.$$

That is, $\mathcal{T}_i x \in \cap_{i=1}^t F(J_{r_{i,n}}^{\mathcal{B}_i})$.

□

**Lemma 3.4.** *For $t, r \in \mathbb{N}$, let $\{\mathcal{B}_i\}_{i=1}^t : \mathcal{H}_2 \to \mathcal{H}_2$ be maximal monotone operators such that $\cap_{i=1}^t F(J_{r_{i,n}}^{\mathcal{B}_i}) \neq \emptyset$ and $\{U_j\}_{j=1}^r : \mathcal{H}_1 \to \mathcal{H}_1$ be a finite family of quasi-nonexpansive operators such that $\cap_{j=1}^r F(U_j) \neq \emptyset$. Assume that $\{\mathcal{I} - U_j\}_{j=1}^r$ and $\{\mathcal{I} - J_{r_{i,n}}^{\mathcal{B}_i}\}_{i=1}^t$ are demiclosed at zero. Let $\{\mathcal{T}_i\}_{i=1}^t : \mathcal{H}_1 \to \mathcal{H}_2$, be bounded linear operators suppose $\Upsilon \neq \emptyset$. Let $S : \mathcal{H}_1 \to \mathcal{H}_1$ be defined by*

$$Sx = \sum_{j=1}^{r} w_j U_j \left( \mathcal{I} - \sum_{i=1}^{t} \delta_{i,n}\tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i \right) x,$$

*where $\{\tau_{i,n}\}$, is as defined in (3.2), $\{w_j\}_{j=1}^r$ and $\{\delta_{i,n}\}_{i=1}^t$ are in $(0, 1)$ with $\sum_{j=1}^r w_j = 1$ and $\sum_{i=1}^t \delta_{i,n} = 1$. Assume that the following conditions are satisfied*

*(A1)* $\min_{i=1,2,\ldots,t}\{\inf_n\{r_{i,n}\}\} = r > 0$;
*(A2)* $\max_{i=1,2,\ldots,t}\{\sup_n\{\theta_{i,n}\}\} = K < \infty$.

*Then the following hold:*

*(a) The operator $S$ is quasi-nonexpansive.*
*(b) $F(S) = \Upsilon$.*
*(c) $\mathcal{I} - S$ is demiclosed at zero.*

*Proof.* From the definition of $V$ we can rewrite the operator $S$ as

$$Sx = \sum_{j=1}^{r} w_j U_j V x.$$

We show the following

(i) $\{U_j V\}_{j=1}^r$ is a finite family of quasi-nonexpansive operator,
(ii) $\cap_{j=1}^r F(U_j V) = \Upsilon$,
(iii) for each $j = 1, 2, \ldots, r$ then $\mathcal{I} - U_j V$ is demiclosed at zero.

By Lemma 3.3, $V$ is quasi-nonexpansive. Therefore, for each $j = 1, 2, \ldots, r$ the operator $U_j V$ is quasi-nonexpansive. Next, we show that for each $j = 1, 2, \ldots, r$, then

$$F(U_j V) = F(U_j) \cap F(V).$$

Indeed, it suffices to show that for each $j = 1, 2, \ldots, r$ $F(U_jV) \subset F(U_j) \cap F(V)$. Let $p \in F(U_jV)$. It is enough to show that $p \in F(V)$. Now, taking $z \in F(U_j) \cap F(V)$; we have

$$\begin{aligned} \|p - z\|^2 &= \|U_jVp - z\|^2 \le \|Vp - z\|^2 \\ &\le \|p - z\|^2 - \sum_{i=1}^{t} \delta_{i,n}\rho_{i,n}(2 - \rho_{i,n}) \frac{\|(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 + \theta_{i,n}}. \end{aligned}$$

This implies that

$$\sum_{i=1}^{t} \delta_{i,n}\rho_{i,n}(2 - \rho_{i,n}) \frac{\|(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 + \theta_{i,n}} = 0.$$

That is $J_{r_{i,n}}^{\mathcal{B}_i}(\mathcal{T}_i p) = \mathcal{T}_i p, \quad \forall \ i = 1, 2, \ldots, t$. This implies that $\mathcal{T}_i p \in \cap_{i=1}^{t} F(J_{r_{i,n}}^{\mathcal{B}_i})$. Thus, $p \in F(V)$. Therefore, $F(U_j) \cap F(V) = F(U_jV), \ \ \forall j = 1, \ldots, r$. We now show that

$$\begin{aligned} \Upsilon &= \{p \in \cap_{j=1}^{r} F(U_j) \quad \text{such that} \ \ \mathcal{T}_i p \in \cap_{i=1}^{t} F(J_{r_{i,n}}^{\mathcal{B}_i})\} \\ &= \cap_{j=1}^{t} F(U_jV). \end{aligned}$$

By Lemma 3.3, we have

$$\begin{aligned} \Upsilon &= \{x \in \cap_{j=1}^{r} F(U_j) | \mathcal{T}_i x \in \cap_{i=1}^{r} F(J_{r_{i,n}}^{\mathcal{B}_i})\} \\ &= \{x \in \cap_{i=1}^{r} F(U_j) | x \in F(V)\} \\ &= \cap_{j=1}^{r} F(U_j) \cap F(V) \\ &= \cap_{j=1}^{r} F(U_jV). \end{aligned}$$

Finally, we show that for each $j = 1, ..r$, $\mathcal{I} - U_jV$ is demiclosed at zero. Let $\{x_n\} \subset \mathcal{H}_1$ be a sequence such that $x_n \rightharpoonup z \in \mathcal{H}_1$ and $U_jVx_n - x_n \to 0$ we have

$$0 \le \|x_n - z\| - \|U_jVx_n - z\| \le \|x_n - U_jVx_n\| \to 0.$$

This implies that

$$\|x_n - z\|^2 - \|U_jVx_n - z\|^2 \to 0.$$

By Lemma 3.3, we have

$$\begin{aligned} \|U_jVx_n - z\|^2 &\le \|Vx_n - z\|^2 \\ &\le \|x_n - z\|^2 - \sum_{i=0}^{t} \delta_{i,n}\rho_{i,n}(2 - \rho_{i,n}) \frac{\|(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|^2 + \theta_{i,n}}. \end{aligned} \tag{3.7}$$

This implies that

$$\sum_{i=0}^{t} \delta_{i,n}\rho_{i,n}(2 - \rho_{i,n}) \frac{\|(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x\|^2 + \theta_{i,n}} \le \|x_n - z\|^2 - \|U_jVx_n - z\|^2.$$

Since $\{\delta_{i,n}\} \subset [a, b] \subset (0, 1)$, $\{\rho_{i,n}\} \subset [c, d] \subset (0, 2)$, and (3.7) implies

$$\frac{\|(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|^2 + \theta_{i,n}} \to 0, \quad \text{as} \quad n \to \infty, \tag{3.8}$$

$\forall \quad i \quad = \quad 0,1,2\ldots t$. It follows from the boundedness of the sequence $\{x_n\}$ that $L := max_{i=0,1,\ldots,N}\{\sup\{\|\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|^2\}\} < \infty$. Thus from Condition (A2), It follows that

$$\frac{\|(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|^4}{\|\mathcal{T}_i^*(I^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|^2 + \theta_{i,n}} \geq \frac{\|(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|^4}{L+K}.$$

Combining this with (3.8), we deduce that

$$\|(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\| \to 0 \tag{3.9}$$

$\forall\ i = 0,1,\ldots,N$, Lemma 2.2(i) and Condition (A1) now imply that

$$\|(\mathcal{I}^{\mathcal{H}_2} - J_r^{\mathcal{B}_i})\mathcal{T}_i x_n\| \leq 2\|(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|, \tag{3.10}$$

$\forall\ i = 0,1,\ldots,N$. Thus using (3.9) and (3.10), we are able to deduce that

$$\|(\mathcal{I}^{\mathcal{H}_2} - J_r^{\mathcal{B}_i})\mathcal{T}_i x_n\| \to 0, \tag{3.11}$$

$\forall i = 0,1,\ldots,N$.

From $\|Vx_n - x_n\| = \|\sum_{i=0}^t \delta_{i,n}\tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i x_n\|$, the assumptions on $\{\delta_{i,n}\}$ and $\{\tau_{i,n}\}$ and (3.10), it follows that

$$\|Vx_n - x_n\| \to 0.$$

On the other hand

$$\|U_j V x_n - V x_n\| \leq \|U_j V x_n - x_n\| + \|V x_n - x_n\| \to 0. \tag{3.12}$$

Since $x_n \rightharpoonup z$, we have $Vx_n \rightharpoonup z$ and by the demiclosedness of $U_j$ we have $z \in F(U_j)$. Since, for each $i = 1,2,\ldots,N$, $\mathcal{T}_i$ is a bounded linear operator, it follows that $\mathcal{T}_i x_{n_k} \rightharpoonup \mathcal{T}_i z$. Thus by Lemma 2.3 and (3.11) implies that $\mathcal{T}_i z \in F(J_r^{\mathcal{B}_i})\ \ \forall i = 1,\ldots,t$ that is $\mathcal{T}_i z \in \cap_{i=1}^t F(J_r^{\mathcal{B}_i})$. By Lemma 3.3 we get $z \in F(V)$. Therefore, $z \in F(U_j) \cap F(V) = F(U_j V)$.

By Claim (i) and Lemma 2.4, we obtain $Sx = \sum_{j=1}^r w_j U_j V x$ is quasi-nonexpansive and $F(S) = \cap_{j=1}^t F(U_j V) = \Upsilon$.

Finally, we show that $\mathcal{I} - S$ is demiclosed at zero. Indeed, Let $\{x_n\} \subset \mathcal{H}_1$ be a sequence such that $x_n \rightharpoonup z \in \mathcal{H}_1$ and $\|x_n - Sx_n\| \to 0$. Let $p \in F(S)$ by Lemma 2.4, we have

$$\langle x_n - Sx_n, x_n - p\rangle \geq \frac{1}{2}\sum_{j=1}^t \|x_n - U_j V x_n\|^2.$$

This imples that, for each $j = 1,\ldots,t$ we have

$$\|x_n - U_j V x_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

By the demiclosedness of $\mathcal{I} - U_j V$ we have $z \in F(U_j V)$. Therefore $z \in \cap_{j=1}^t F(U_j V) = F(S)$. □

**Theorem 3.5.** *For $t, r \in \mathbb{N}$. Let $\{\mathcal{B}_i\}_{i=1}^t : \mathcal{H}_2 \to \mathcal{H}_2$ be a finite family of maximal monotone operators such that $\cap_{i=1}^t F(J_r^{\mathcal{B}_i}) \neq \emptyset$ and $\{U_j\}_{j=1}^r : \mathcal{H}_1 \to \mathcal{H}_1$ be a finite family of quasi-nonexpansive operators such that $\cap_{j=1}^r F(U_j) \neq \emptyset$. Assume that $\{\mathcal{I} - U_j\}_{j=1}^r$ and $\{\mathcal{I} - J_r^{\mathcal{B}_i}\}_{i=1}^t$ are demiclosed at zero. Let $\mathcal{T}_i : \mathcal{H}_1 \to \mathcal{H}_2$, $i = 1, 2, \ldots, N$ be bounded linear operators. Suppose $\Upsilon \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by Algorithm 3.2. and suppose that Assumptions* (3.1) *(a)-(c) are fulfilled. Then $\{x_n\}$ converges weakly to an element of $\Upsilon$.*

*Proof.* Let $S = \sum_{j=1}^r w_j U_j \left(\mathcal{I} - \sum_{i=1}^t \delta_{i,n}\tau_{i,n}\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J_{r_{i,n}}^{\mathcal{B}_i})\mathcal{T}_i\right)$, then the sequence $\{x_{n+1}\}$ can be rewritten as follows

$$x_{n+1} = (1-\rho)y_n + \rho S y_n. \tag{3.13}$$

By Lemma 3.3, we have that $S$ is quasi-nonexpansive. Let $z \in \Upsilon$, from (3.13), we get

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|(1-\rho)(y_n - z) + \rho(S y_n - z)\|^2 \\ &= (1-\rho)\|y_n - z\|^2 + \rho\|S y_n - z\|^2 - \rho(1-\rho)\|y_n - S y_n\|^2 \\ &\leq \|y_n - z\|^2 - \rho(1-\rho)\|y_n - S y_n\|^2. \end{aligned} \tag{3.14}$$

Observe that

$$\begin{aligned} y_n - z &= x_n + \theta(x_n - x_{n-1}) + \delta(x_{n-1} - x_{n-2}) - z \\ &= (1+\theta)(x_n - z) - (\theta - \delta)(x_{n-1} - z) - \delta(x_{n-2} - z). \end{aligned}$$

Hence by Lemma 2.5, we have

$$\begin{aligned} \|y_n - z\|^2 &= \|(1+\theta)(x_n - z) - (\theta - \delta)(x_{n-1} - z) - \delta(x_{n-2} - z)\|^2 \\ &= (1+\theta)\|x_n - z\|^2 - (\theta - \delta)\|x_{n-1} - z\|^2 - \delta\|x_{n-2} - z\|^2 \\ &\quad +(1+\theta)(\theta - \delta)\|x_n - x_{n-1}\|^2 + \delta(1+\theta)\|x_n - x_{n-2}\|^2 \\ &\quad -\delta(\theta - \delta)\|x_{n-1} - x_{n-2}\|^2. \end{aligned} \tag{3.15}$$

Note also that

$$\begin{aligned} 2\theta\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle &= 2\langle\theta(x_{n+1} - x_n), x_n - x_{n-1}\rangle \\ &\leq 2|\theta|\|x_{n+1} - x_n\|\|x_n - x_{n-1}\| \\ &= 2\theta\|x_{n+1} - x_n\|\|x_n - x_{n-1}\|, \end{aligned}$$

and so,

$$-2\theta\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \geq -2\theta\|x_{n+1} - x_n\|\|x_n - x_{n-1}\|. \tag{3.16}$$

Also,

$$\begin{aligned} 2\delta\langle x_{n+1} - x_n, x_{n-1} - x_{n-2}\rangle &= 2\langle\delta(x_{n+1} - x_n), x_{n-1} - x_{n-2}\rangle \\ &\leq 2|\delta|\|x_{n+1} - x_n\|\|x_{n-1} - x_{n-2}\|, \end{aligned}$$

which implies that

$$-2\delta\langle x_{n+1}-x_n, x_{n-1}-x_{n-2}\rangle \geq -2|\delta|\|x_{n+1}-x_n\|\|x_{n-1}-x_{n-2}\|. \tag{3.17}$$

Similarly, we have

$$2\delta\theta\langle x_{n-1}-x_n, x_{n-1}-x_{n-2}\rangle \leq 2|\delta|\theta\|x_n-x_{n-1}\|\|x_{n-1}-x_{n-2}\|,$$

and thus

$$2\delta\theta\langle x_n-x_{n-1}, x_{n-1}-x_{n-2}\rangle \geq -2|\delta|\theta\|x_n-x_{n-1}\|\|x_{n-1}-x_{n-2}\|. \tag{3.18}$$

By (3.16)–(3.18) and Cauchy-Schwartz inequality one has

$$\begin{aligned}
\|x_{n+1}-y_n\|^2 &= \|x_{n+1}-(x_n+\theta(x_n-x_{n-1})+\delta(x_{n-1}-x_{n-2}))\|^2 \\
&= \|x_{n+1}-x_n-\theta(x_n-x_{n-1})-\delta(x_{n-1}-x_{n-2})\|^2 \\
&= \|x_{n+1}-x_n\|^2-2\theta\langle x_{n+1}-x_n, x_n-x_{n-1}\rangle-2\delta\langle x_{n+1}-x_n, x_{n-1}-x_{n-2}\rangle \\
&\quad +\theta^2\|x_n-x_{n-1}\|^2+2\delta\theta\langle x_n-x_{n-1}, x_{n-1}-x_{n-2}\rangle+\delta^2\|x_{n-1}-x_{n-2}\|^2 \\
&\geq \|x_{n+1}-x_n\|^2-2\theta\|x_{n+1}-x_n\|\|x_n-x_{n-1}\|-2|\delta|\|x_{n+1}-x_n\|\|x_{n-1}-x_{n-2}\| \\
&\quad +\theta^2\|x_n-x_{n-1}\|^2-2|\delta|\theta\|x_n-x_{n-1}\|\|x_{n-1}-x_{n-2}\|+\delta^2\|x_{n-1}-x_{n-2}\|^2 \\
&\geq \|x_{n+1}-x_n\|^2-\theta\|x_{n+1}-x_n\|^2-\theta\|x_n-x_{n-1}\|^2-|\delta|\|x_{n+1}-x_n\|^2 \\
&\quad -|\delta|\|x_{n-1}-x_{n-2}\|^2+\theta^2\|x_n-x_{n-1}\|^2-|\delta|\theta\|x_n-x_{n-1}\|^2 \\
&\quad -|\delta|\theta\|x_{n-1}-x_{n-2}\|^2+\delta^2\|x_{n-1}-x_{n-2}\|^2 \\
&= (1-|\delta|-\theta)\|x_{n+1}-x_n\|^2+(\theta^2-\theta-|\delta|\theta)\|x_n-x_{n-1}\|^2 \\
&\quad +(\delta^2-|\delta|-|\delta|\theta)\|x_{n-1}-x_{n-2}\|^2.
\end{aligned} \tag{3.19}$$

Observe that

$$\|Sy_n-y_n\|^2=\frac{1}{\rho^2}\|x_{n+1}-y_n\|^2. \tag{3.20}$$

Putting (3.20) in (3.14), we get

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\leq \|y_n-z\|^2-\rho(1-\rho)\|Sy_n-y_n\|^2 \\
&= \|y_n-z\|^2-\frac{1-\rho}{\rho}\|x_{n+1}-y_n\|^2.
\end{aligned} \tag{3.21}$$

Combining (3.15) and (3.19) in (3.21) with noting that $\delta \leq 0$ we obtain

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= (1+\theta)\|x_n-z\|^2-(\theta-\delta)\|x_{n-1}-z\|^2-\delta\|x_{n-2}-z\|^2 \\
&\quad +(1+\theta)(\theta-\delta)\|x_n-x_{n-1}\|^2+\delta(1+\theta)\|x_n-x_{n-2}\|^2-\delta(\theta-\delta)\|x_{n-1}-x_{n-2}\|^2 \\
&\quad -\frac{(1-\rho)}{\rho}(1-|\delta|-\theta)\|x_{n+1}-x_n\|^2-\frac{(1-\rho)}{\rho}(\theta^2-\theta-|\delta|\theta)\|x_n-x_{n-1}\|^2 \\
&\quad -\frac{(1-\rho)}{\rho}(\delta^2-|\delta|-|\delta|\theta)\|x_{n-1}-x_{n-2}\|^2
\end{aligned}$$

$$
\begin{aligned}
&= (1+\theta)\|x_n - z\|^2 - (\theta-\delta)\|x_{n-1} - z\|^2 - \delta\|x_{n-2} - z\|^2 + \delta(1+\theta)\|x_n - x_{n-2}\|^2 \\
&\quad + \left[(1+\theta)(\theta-\delta) - \frac{(1-\rho)}{\rho}(\theta^2 - \theta - |\delta|\theta)\right]\|x_n - x_{n-1}\|^2 \\
&\quad - \left[\delta(\theta-\delta) + \frac{(1-\rho)}{\rho}(\delta^2 - |\delta| - |\delta|\theta)\right]\|x_{n-1} - x_{n-2}\|^2 \\
&\quad - \frac{(1-\rho)}{\rho}(1 - |\delta| - \theta)\|x_{n+1} - x_n\|^2 \\
&\le (1+\theta)\|x_n - z\|^2 - (\theta-\delta)\|x_{n-1} - z\|^2 - \delta\|x_{n-2} - z\|^2 \\
&\quad + \left[(1+\theta)(\theta-\delta) - \frac{(1-\rho)}{\rho}(\theta^2 - \theta + \delta\theta)\right]\|x_n - x_{n-1}\|^2 \\
&\quad - \left[\delta(\theta-\delta) + \frac{(1-\rho)}{\rho}(\delta^2 + \delta + \delta\theta)\right]\|x_{n-1} - x_{n-2}\|^2 \\
&\quad - \frac{(1-\rho)}{\rho}(1 + \delta - \theta)\|x_{n+1} - x_n\|^2. \qquad (3.22)
\end{aligned}
$$

By rearranging we get

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 - \theta\|x_n - z\|^2 - \delta\|x_{n-1} - z\|^2 + \frac{(1-\rho)}{\rho}(1+\delta-\theta)\|x_{n+} - x_n\|^2 \\
&\le \|x_n - z\|^2 - \theta\|x_{n-1} - z\|^2 - \delta\|x_{n-2} - z\|^2 + \frac{(1-\rho)}{\rho}(1+\delta-\theta)\|x_n - x_{n-1}\|^2 \\
&+ \left[(1+\theta)(\theta-\delta) - \frac{(1-\rho)}{\rho}(\theta^2 - 2\theta + \delta\theta + \delta + 1)\right] \\
&- \left[\delta(\theta-\delta) + \frac{(1-\rho)}{\rho}(\delta^2 + \delta + \delta\theta)\right]\|x_{n-1} - x_{n-2}\|^2. \qquad (3.23)
\end{aligned}
$$

Define

$$
\Upsilon_n := \|x_n - z\|^2 - \theta\|x_{n-1} - z\|^2 - \delta\|x_{n-2} - z\| + \frac{(1-\rho)}{\rho}(1+\delta-\theta)\|x_n - x_{n-1}\|^2.
$$

Let us show that $\Upsilon_n \ge 0, \forall n \ge 1$. Now

$$
\begin{aligned}
\Upsilon_n &= \|x_n - z\|^2 - \theta\|x_{n-1} - z\|^2 - \delta\|x_{n-2} - z\|^2 + \frac{(1-\rho)}{\rho}(1+\delta-\theta)\|x_n - x_{n-1}\|^2 \\
&\ge \|x_n - z\|^2 - 2\theta\|x_n - x_{n-1}\|^2 - 2\theta\|x_n - z\|^2 - \delta\|x_{n-2} - z\|^2 \\
&\quad + \frac{(1-\rho)}{\rho}(1+\delta-\theta)\|x_n - x_{n-1}\|^2 \\
&= (1-2\theta)\|x_n - z\|^2 + \left[\frac{(1-\rho)}{\rho}(1+\delta-\theta) - 2\theta\right]\|x_n - x_{n-1}\|^2 - \delta\|x_{n-2} - z\|^2. \qquad (3.24)
\end{aligned}
$$

Since $\theta < 1/2$, $\delta \le 0$, $\frac{2\theta\rho}{1-\rho} - (1-\theta) < \delta$ and $0 \le \theta < \frac{1-\rho}{1+\rho}$, it follows from (3.24) that $\Upsilon_n \ge 0$. Furthermore, we drive from (3.23)

$$
\Upsilon_{n+1} - \Upsilon_n \le \left[(1+\theta)(\theta-\delta) - \frac{(1-\rho)}{\rho}(\theta^2 - 2\theta + \delta\theta + \delta + 1)\right]\|x_n - x_{n-1}\|^2
$$

$$
\begin{aligned}
& -\left[\delta(\theta-\delta)+\frac{(1-\rho)}{\rho}(\delta^2+\delta+\delta\theta)\right]\|x_{n-1}-x_{n-2}\|^2 \\
= & -\left[(1+\theta)(\theta-\delta)-\frac{(1-\rho)}{\rho}(\theta^2-2\theta+\delta\theta+\delta+1)\right] \\
& \times\left(\|x_{n-1}-x_{n-2}\|-\|x_n-x_{n-1}\|^2\right) \\
& +[(1+\theta)(\theta-\delta)-\frac{(1-\rho)}{\rho}(\theta^2-2\theta+\delta\theta+\delta+1) \\
& -\delta(\theta-\delta)-\frac{(1-\rho)}{\rho}(\delta^2+\delta+\delta\theta)]\|x_{n-1}-x_{n-2}\|^2 \\
= & \; q_1[\|x_{n-1}-x_{n-2}\|^2-\|x_n-x_{n-1}\|^2]-q_2\|x_{n-1}-x_{n-2}\|^2, \qquad (3.25)
\end{aligned}
$$

where

$$
q_1 := -\left[(1+\theta)(\theta-\delta)-\frac{(1-\rho)}{\rho}(\theta^2-2\theta+\delta\theta+\delta+1)\right]
$$

and

$$
\begin{aligned}
q_2 : \; = & -[(1+\theta)(\theta-\delta)-\frac{(1-\rho)}{\rho}(\theta^2-2\theta+\delta\theta+\delta+1) \\
& -\delta(\theta-\delta)-\frac{(1-\rho)}{\rho}(\delta^2+\delta+\delta\theta)]\|x_{n-1}-x_{n-2}\|^2. \qquad (3.26)
\end{aligned}
$$

By our assumption, it holds that

$$
\frac{\theta-1}{\theta+1}[\rho(2\theta+1)-(\theta-1)]<\delta. \tag{3.27}
$$

As a result $q_1>0$. Also $q_2>0$ by Assumption 3.1 (c). Then by (3.25) we have

$$
\Upsilon_{n+1}+q_1\|x_n-x_{n-1}\|^2 \le \Upsilon_n+q_1\|x_{n-1}-x_{n-2}\|^2-q_2\|x_{n-1}-x_{n-2}\|^2. \tag{3.28}
$$

Letting $\bar{\Upsilon}_n := \Upsilon_n+q_1\|x_{n-1}-x_{n-2}\|^2$. Then $\bar{\Upsilon}_n \ge 0,\ \forall n\ge 1$. Also, it follows from (3.28) that

$$
\bar{\Upsilon}_{n+1} \le \bar{\Upsilon}_n. \tag{3.29}
$$

These facts imply that the sequence $\{\bar{\Upsilon}_n\}$ is decreasing and bounded from below and thus $\lim\limits_{n\to\infty}\bar{\Upsilon}_n$ exists. Consequently, we get from (3.28) and the squeeze theorem that

$$
\lim_{n\to\infty} q_1\|x_{n-1}-x_{n-2}\|^2=0. \tag{3.30}
$$

Hence

$$
\lim_{n\to\infty} \|x_{n-1}-x_{n-2}\|^2=0. \tag{3.31}
$$

As a result

$$
\|x_{n+1}-y_n\| \;=\; \|x_{n+1}-x_n-\theta(x_n-x_{n-1})-\delta(x_{n-1})\|
$$

$$\leq \quad \|x_{n+1} - x_n\| + \theta\|x_n - x_{n-1}\| + |\delta|\|x_{n-1} - x_{n-2}\| \to 0$$

as $n \to \infty$. By $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, one has

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0, \text{ as } n \to \infty.$$

By (3.31) and the existence of $\lim_{n\to\infty}\bar{\Upsilon}_n$, we have that $\lim_{n\to\infty}\Upsilon_n$ exists and hence $\{\Upsilon_n\}$ is bounded. Now, since $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we have from the definition of $\Upsilon_n$ that

$$\lim_{n\to\infty}[\|x_n - z\|^2 - \theta\|x_{n-1} - z\|^2 - \delta\|x_{n-2} - z\|^2] \tag{3.32}$$

exists. Using the boundedness of $\{\Upsilon_n\}$, we obtain from (3.24) that $\{x_n\}$ is bounded. Consequently $\{y_n\}$ is bounded. From (3.8), we obtain

$$\rho(1-\rho)\|Sy_n - y_n\| \leq \|y_n - z\|^2 - \|x_{n+1} - z\|^2.$$

This implies that

$$\lim_{n\to\infty}\|Sy_n - y_n\| = 0. \tag{3.33}$$

Finally, we show that the sequence $\{x_n\}$ converges weakly to $x^* \in \Upsilon$. Indeed, since $\{x_n\}$ is bounded we assume that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^* \in H$. Since $\|x_n - y_n\| \to 0$, we also have $y_{n_j} \rightharpoonup x^*$. Then by the demiclosedness of $I - S$, we obtain $x^* \in F(S) = \Upsilon$.

Now, we show that $\{x_n\}$ has unique weak limit point in $\Upsilon$. Suppose that $\{x_{m_j}\}$ is another subsequence of $\{x_n\}$ such that $x_{m_j} \rightharpoonup v^*$ as $j \to \infty$. Observe that

$$2\langle x_n, x^* - v^*\rangle = \|x_n - v^*\|^2 - \|x_n - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2 \tag{3.34}$$

$$2\langle x_{n-1}, x^* - v^*\rangle = \|x_{n-1} - v^*\|^2 - \|x_{n-1} - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2$$

and

$$2\langle x_{n-2}, x^* - v^*\rangle = \|x_{n-2} - v^*\|^2 - \|x_{n-2} - x^*\|^2 - \|v^*\|^2 + \|x^*\|^2.$$

Therefore

$$2\langle -\theta x_{n-1}, x^* - v^*\rangle = -\theta\|x_{n-1} - v^*\|^2 + \theta\|x_{n-1} - x^*\|^2 + \theta\|v^*\|^2 - \theta\|x^*\|^2. \tag{3.35}$$

and

$$2\langle -\delta x_{n-2}, x^* - v^*\rangle \quad = \quad -\delta\|x_{n-2} - v^*\|^2 + \delta\|x_{n-2} - x^*\|^2 + \delta\|v^*\|^2 - \delta\|x^*\|^2. \tag{3.36}$$

Addition of (3.34)–(3.36) gives

$$\begin{aligned} & 2\langle x_n - \theta x_{n-1} - \delta x_{n-2}, x^* - v^*\rangle \qquad (3.37) \\ = \quad & \left(\|x_n - v^*\|^2 - \theta\|x_{n-1} - v^*\|^2 - \delta\|x_{n-2} - v^*\|^2\right) \end{aligned}$$

$$-\left(\|x_n - x^*\|^2 - \theta\|x_{n-1} - x^*\|^2 - \delta\|x_{n-2} - x^*\|^2\right) + (1-\theta-\delta)(\|x^*\| - \|v^*\|^2). \tag{3.38}$$

According to (3.32), we get

$$\lim_{n\to\infty}[\|x_n - x^*\|^2 - \theta\|x_{n-1} - x^*\|^2 - \delta\|x_{n-2} - x^*\|^2] \tag{3.39}$$

exists and

$$\lim_{n\to\infty}[\|x_n - v^*\|^2 - \theta\|x_{n-1} - v^*\|^2 - \delta\|x_{n-2} - v^*\|^2] \tag{3.40}$$

exists. This implies from (3.37) that $\lim_{n\to\infty}\langle x_n - \theta x_{n-1} - \delta x_{n-2}, x^* - v^*\rangle$ exists. Consequently,

$$\begin{aligned}
\langle v^* - \theta v^* - \delta v^*, x^* - v^*\rangle &= \lim_{j\to\infty}\langle x_{n_j} - \theta x_{n_{j-1}} - \beta x_{n_{j-2}}, x^* - v^*\rangle \\
&= \lim_{n\to\infty}\langle x_{n-1} - \theta x_{n-1} - \delta x_{n-2}, x^* - v^*\rangle \\
&= \lim_{j\to\infty}\langle x_{m_j} - \theta x_{m_{j-1}} - \beta x_{m_{j-2}}, x^* - v^*\rangle \\
&= \langle x^* - \theta x^* - \beta x^*, x^* - v^*\rangle.
\end{aligned}$$

Hence

$$(1-\theta-\delta)\|x^* - v^*\|^2 = 0.$$

Since $\delta \le 0 < 1-\theta$, we obtain that $x^* = v^*$. Therefore, the sequence $\{x_n\}$ converges weakly to $x^* \in \Upsilon$. This completes the proof. □

## 4. Numerical examples

In this section, we give a numerical description to illustrate how our proposed algorithm can be implemented in the setting of the real Hilbert space $\mathbb{R}$. Furthermore, we shall show the effect of the double inertia in the fast convergence of the sequence generated by our proposed Algorithm 3.2. First, we give the set of parameters that satisfy the conditions given in assumption 3.1. To this end, fix $\rho = \frac{1}{2}$ and take

$$\theta = \frac{1}{4},\ \delta = -\frac{1}{5} \text{ and } -\frac{1}{1000};\quad \theta = \frac{1}{5},\ \delta = -\frac{1}{10} \text{ and } -\frac{1}{1000};\quad \theta = \frac{1}{6},\ \delta = -\frac{1}{100} \text{ and } -\frac{1}{1000}.$$

Clearly, these parameters satisfy the conditions given in assumption 3.1. Next, we define the operators to be used in the implementation of Algorithm 3.2. In Algorithm 3.2, fix $t = N = r = 3$. Set $H_1 = H_2 = H_3 = \mathbb{R}$. Let $\delta_{i,n} = \frac{1}{3}$, $\rho_{i,n} = \theta_{i,n} = \frac{2}{3}$, $r_{i,n} = \frac{1}{2}$ and $w_j = \frac{1}{3}$, where $i, j \in \{1,2,3\}$ and $n \ge 1$. Let $B_i,\ T_i,\ U_j : \mathbb{R} \to \mathbb{R}$ be defined by

$$B_i x = 2ix, \text{ then } J^{B_i}_{r_{i,n}} x = \frac{x}{1+2ir_{i,n}}, \quad T_i x = ix, \quad U_j x = jx.$$

Then,

$$\mathcal{T}_i^*(\mathcal{I}^{\mathcal{H}_2} - J^{\mathcal{B}_i}_{r_{i,n}})\mathcal{T}_i x = \frac{2i^3 r_{i,n}}{1+2ir_{i,n}}, \quad \text{and} \quad \tau_{i,n} = \frac{2}{3}\frac{(2i^2 r_{i,n} y_n)^2}{(2i^3 r_{i,n})^2 + 2(1+2ir_{i,n})^2}.$$

With this, we are ready to implement our proposed Algorithm 3.2 on MATLAB. Choosing $x_0 = 1$, $x_1 = -2$ and $x_2 = 0.5$, and setting maximum number of iterations to 150 or $10^{-16}$, as our stopping criteria, we varied the double inertial parameters as given above. We obtained the following successive approximations:

**Table 1.** Results of the numerical simulations.

| No. Iter | Inertia Para. | $\lvert x_{n+1} - x_n \rvert$ |
|---|---|---|
| 120 | $\theta = \frac{1}{4}$ $\delta$ =-$\frac{1}{5}$ | 1.11E-16 |
| 80 | $\theta = \frac{1}{4}$ $\delta$ =-$\frac{1}{1000}$ | 1.11E-16 |
| 107 | $\theta = \frac{1}{5}$ $\delta$ =-$\frac{1}{10}$ | 1.11E-16 |
| 88 | $\theta = \frac{1}{5}$ $\delta$ =-$\frac{1}{1000}$ | 1.11E-16 |
| 95 | $\theta = \frac{1}{6}$ $\delta$ =-$\frac{1}{100}$ | 1.11E-16 |
| 116 | $\theta = \frac{1}{6}$ $\delta$ =-$\frac{1}{1000}$ | 1.11E-16 |

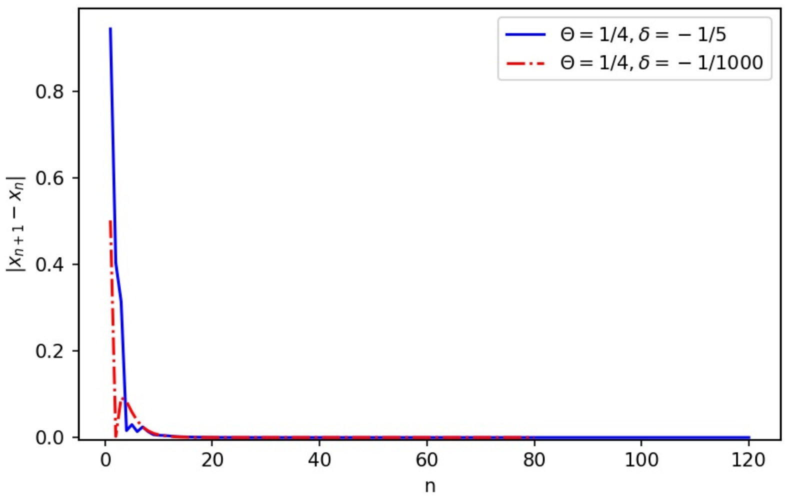


**Figure 1.** Graph of the iterates of Algorithm 3.2 when $\theta = \frac{1}{4}$, $\delta$ =-$\frac{1}{5}$ and $\delta$ =-$\frac{1}{1000}$.

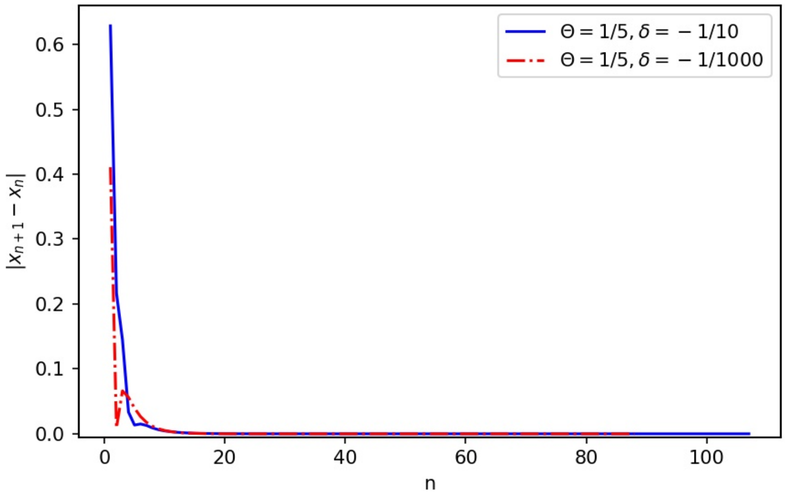


**Figure 2.** Graph of the iterates of Algorithm 3.2 when $\theta = \frac{1}{5}$, $\delta$ =-$\frac{1}{10}$ and $\delta$ =-$\frac{1}{1000}$.

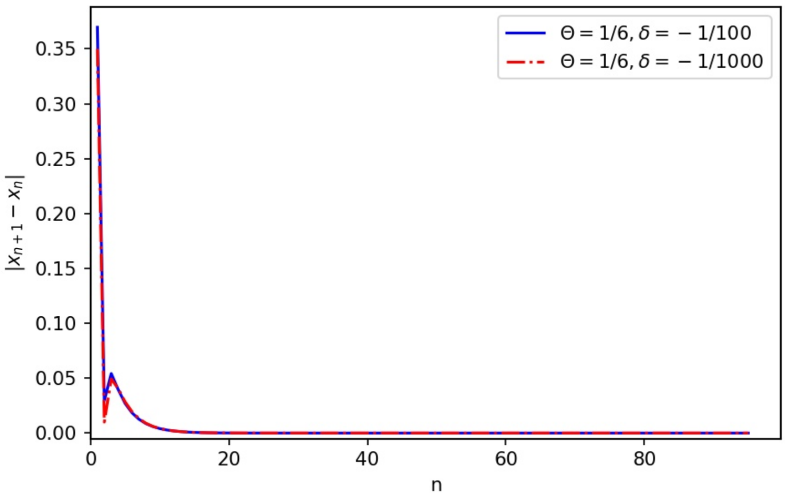


**Figure 3.** Graph of the iterates of Algorithm 3.2 when $\theta = \frac{1}{6}$, $\delta$ =-$\frac{1}{100}$ and $\delta$ =-$\frac{1}{1000}$.

## 5. Discussion

From the numerical simulations presented in Table 1 and Figures 1–3, we saw that in this example, the best choice for the double inertial parameters is $\theta = \frac{1}{4}$ and $\delta = -\frac{1}{1000}$. Furthermore, we observed that as $\theta$ decreases and $\delta$ approaches 0, the number of iterations required to satisfy the stopping criteria increases.

### Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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### Conflict of interest

All authors declare no conflicts of interest in this paper.

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