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Research article

One-parameter Lorentzian spatial kinematics and Disteli's formulae

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Abstract: In this paper, based on the E. Study map, clear terms are offered for the differential equations of one-parameter Lorentzian spatial kinematics that are coordinate system-independent. This cancels the request of demanding coordinate transformations that are typically required in the determination of the canonical systems. With the suggested technique, new proofs of the Disteli formulae of a spacelike line trajectory are instantly gained and their spatial equivalents are studied in detail. As a consequence, we address the kinematic geometry of a point trajectory for the one-parameter Lorentzian spherical and planar movements.

Keywords: spacelike axodes; planar movement; inflection circle
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1. Introduction

Line geometry has a connection to kinematics and thus found applications in mechanism design and robot kinematics. In the area of spatial kinematics, scholars have focused on investigating the intrinsic hallmark of the line trajectory via ruled surfaces in differential geometry [1–3]. It is known in spatial kinematics that, the instantaneous screw axis ($\mathbb{ISA}$) of a movable body creates a pair of ruled surfaces, named the movable and fixed axodes, with $\mathbb{ISA}$ as their common ruling in the movable space and in the stationary space, respectively. Through the movement the axodes slide and roll relative to each other in a specific way such that the tangential contact between the axodes is constantly maintained over the entire length of the two matting rulings (one being in each axode) which together locate the $\mathbb{ISA}$ at any instant. It is considerable that not only does a specific movement give rise to a unique family of axodes but the converse applies as well. This indicates that, should the axodes of any movement be known, the specific movement can be reconstructed without knowing the physical parts of the mechanism, their configuration, the specific dimensions, or the manner by which they are connected [4–6]. The merit of axodes in the procedure of synthesis becomes evident in terms of the realization that the axodes are intermediary in the middle of the physical mechanism and the actual movement of its parts (see Refs. [7–11]).

One of the most proper ways to research the movement of a line space appears to be defining a connection through this space and dual numbers. Dual numbers were first introduced by Clifford after him E. Study utilized it as an apparatus for his developments in differential line geometry and kinematics. He gave a specified view of the parametrization of directed lines by dual unit vectors and specified the mapping that is known by his name (E. Study map): the set of all directed lines in Euclidean 3-space $\mathbb{E}^3$ is represented by a set of points on the dual unit sphere in the dual 3-space $\mathbb{D}^3$. Supplementary details on the E. Study map and screw calculus can be found in [4–10]. If we occupy $\mathbb{E}^3_1$ (the Minkowski 3-space $\mathbb{E}^3_1$) instead of $\mathbb{E}^3$ the situation is much more interesting than the Euclidean case. In $\mathbb{E}^3_1$, the metric function $<,>$ can be positive, negative or zero, whereas the metric function in the Euclidean space can only be positive. Then, we have to disconnect directed lines on the basis of whether the metric function is positive, negative, or zero. Directed lines with $<,>< 0$ ($<,>> 0$) are named timelike (spacelike) directed lines and directed lines with $<,>= 0$ are named null lines. Many works have been published which deal with the spatial kinematics of line trajectories in both Euclidean space and Minkowski space [6–10,12–19].

In this paper, the invariants of the axodes of one-parameter Lorentzian spatial kinematics are examined. New proofs for Euler-Savary and Disteli formulae are specified which demonstrate the stylishness and logicality of the E. Study map in Lorentzian spatial kinematics. As a result, theoretical expressions of point trajectories with private values of velocity and acceleration, which can be addressed as a Lorentzian form of the Euler-Savary formula, are acquired for spherical and planar movements.

2. Basic concepts

In this section, we give a short outline of the theory of dual numbers and dual Lorentzian vectors [10–19]. If a, and a^* are two real numbers, the equation: $\widehat{a} = a + \varepsilon a^*$ is named a dual number, such that ε is a dual unit subject to $\varepsilon \neq 0$, and $\varepsilon^2 = 0$. This is in fact very comparable to the idea of a complex number, with the main distinction being that, given a complex number $\varepsilon^2 = -1$. Then the set

$$\mathbb{D}^3 = \{\widehat{\mathbf{a}} := \mathbf{a} + \varepsilon \mathbf{a}^* = (\widehat{a}_1, \widehat{a}_2, \widehat{a}_3)\},$$

together with the Lorentzian scalar product

$$< \widehat{\mathbf{a}}, \widehat{\mathbf{a}} > = \widehat{a}_1^2 - \widehat{a}_2^2 + \widehat{a}_3^2,$$

defines the named dual Lorentzian 3-space $\mathbb{D}^3_1$. Therefore, a point $\widehat{\mathbf{a}} = (\widehat{a}_1, \widehat{a}_2, \widehat{a}_3)^t$ has dual coordinates $\widehat{a}_i = (a_i + \varepsilon a_i^*) \in \mathbb{D}$. The norm $\left\|\widehat{\mathbf{a}}\right\|$ of $\widehat{\mathbf{a}}$ is defined by

$$\begin{aligned}
\left\|\widehat{\mathbf{a}}\right\| &= \sqrt{\left|< \widehat{\mathbf{a}}, \widehat{\mathbf{a}} >\right|} = \sqrt{|< \mathbf{a}, \mathbf{a} >|} + \varepsilon \frac{1}{2\sqrt{|< \mathbf{a}, \mathbf{a} >|}} \frac{< \mathbf{a}, \mathbf{a} >}{|< \mathbf{a}, \mathbf{a} >|} .2 < \mathbf{a}, \mathbf{a}^* > \\
&= \|\mathbf{a}\| + \varepsilon \frac{1}{\|\mathbf{a}\|} \frac{< \mathbf{a}, \mathbf{a} >}{|< \mathbf{a}, \mathbf{a} >|} < \mathbf{a}, \mathbf{a}^* > .
\end{aligned}$$

If a is timelike, we have

$$\left\|\widehat{\mathbf{a}}\right\| = \|\mathbf{a}\| - \varepsilon \frac{1}{\|\mathbf{a}\|} < \mathbf{a}, \mathbf{a}^* >= \|\mathbf{a}\| \left(1 - \varepsilon \frac{1}{\|\mathbf{a}\|^2} < \mathbf{a}, \mathbf{a}^* >\right).$$

If a is spacelike, we have

$$\left\|\widehat{\mathbf{a}}\right\| = \|\mathbf{a}\| + \varepsilon \frac{1}{\|\mathbf{a}\|} < \mathbf{a}, \mathbf{a}^* >= \|\mathbf{a}\| \left(1 + \varepsilon \frac{1}{\|\mathbf{a}\|^2} < \mathbf{a}, \mathbf{a}^* >\right).$$

The Lorentzian and hyperbolic Lorentzian dual unit spheres, respectively, are

$$\mathbb{S}_1^2 = \left\{\widehat{\mathbf{a}} \in \mathbb{D}_1^3 \mid \left\|\widehat{\mathbf{a}}\right\|^2 = \widehat{a}_1^2 - \widehat{a}_2^2 + \widehat{a}_3^2 = 1\right\},$$

and

$$\mathbb{H}_+^2 = \left\{\widehat{\mathbf{a}} \in \mathbb{D}_1^3 \mid \left\|\widehat{\mathbf{a}}\right\|^2 = \widehat{a}_1^2 - \widehat{a}_2^2 + \widehat{a}_3^2 = -1, \widehat{a}_2 \geq 1\right\}.$$

Theorem 2.1. *(E. Study map). There is a one-to-one exemplification between spacelike (timelike) directed lines in Minkowski 3-space $\mathbb{E}_1^3$ and ordered pairs of vectors $(\mathbf{a}, \mathbf{a}^*) \in \mathbb{E}_1^3 \times \mathbb{E}_1^3$ such that*

$$\left\|\widehat{\mathbf{a}}\right\|^2 = \pm 1 \Longleftrightarrow \|\mathbf{a}\|^2 = \pm 1, \ < \mathbf{a}, \mathbf{a}^* >= 0, \tag{2.1}$$

where a_i, $a_i^*(i = 1, 2, 3)$ of a, and a^* are the normalized Plücker coordinates of the non-lightlike line.

The E. Study map yields the following: the ring shaped hyperboloid compatibility with the set of spacelike lines, the mutual asymptotic cone compatibility with the set of null lines and the oval shaped hyperboloid compatibility with the set of timelike lines (see Figure 1) Therefore, a smooth curve on $\mathbb{H}_+^2$ acts as a timelike ruled surface in $\mathbb{E}_1^3$. Also a regular curve on $\mathbb{S}_1^2$ acts as a spacelike or timelike ruled surface in $\mathbb{E}_1^3$.

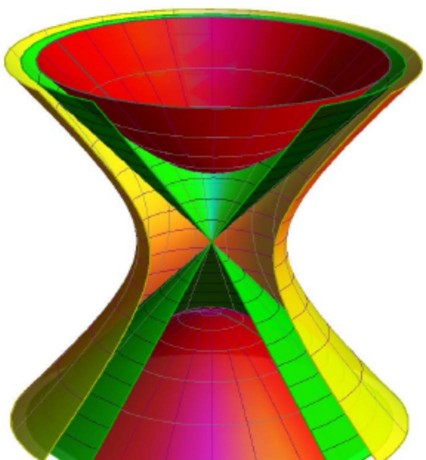

Figure 1. The dual hyperbolic and dual Lorentzian unit spheres.

Definition 2.1. *For any two (non-null) dual vectors $\widehat{\zeta}$ and $\widehat{\xi}$ in $\mathbb{D}_1^3$, we have the following [10–17]:*
a) If $\widehat{\zeta}$ and $\widehat{\xi}$ are two dual spacelike vectors:

- *If they span a dual spacelike plane, there is a single dual number $\widehat{\theta} = \theta + \varepsilon\theta^*$, $0 \leq \theta \leq \pi$, and $\theta^* \in \mathbb{R}$ such that $< \widehat{\zeta}, \widehat{\xi} >= \left\|\widehat{\zeta}\right\| \left\|\widehat{\xi}\right\| \cos\widehat{\theta}$. This number is named the spacelike dual angle between $\widehat{\zeta}$ and $\widehat{\xi}$.*

• *If they span a dual timelike plane, there is a single dual number* $\widehat{\theta} = \theta + \varepsilon\theta^* \geq 0$ *such that* $<\widehat{\zeta},\widehat{\xi}>= \epsilon\left\|\widehat{\zeta}\right\|\left\|\widehat{\xi}\right\|\cosh\widehat{\theta}$*, where* $\epsilon = +1$ *or* $\epsilon = -1$ *given* $sign(\widehat{\zeta}_2) = sign(\widehat{y}_2)$ *or* $sign(\widehat{\zeta}_2) \neq sign(\widehat{\xi}_2)$*, respectively. This number is named the central dual angle between* $\widehat{\zeta}$ *and* $\widehat{\xi}$*.*

b) If $\widehat{\zeta}$ *and* $\widehat{\xi}$ *are two dual timelike vectors, then there is a single dual number* $\widehat{\theta} = \theta + \varepsilon\theta^* \geq 0$ *such that* $<\widehat{\zeta},\widehat{\xi}>= \epsilon\left\|\widehat{\zeta}\right\|\left\|\widehat{\xi}\right\|\cosh\widehat{\theta}$*, where* $\epsilon = -1$ *or* $\epsilon = +1$ *given* $\widehat{\zeta}$ *and* $\widehat{\xi}$ *has the same time-direction or different time directions, respectively. This dual number is named the Lorentzian timelike dual angle between* $\widehat{\zeta}$ *and* $\widehat{\xi}$*.*

c) If $\widehat{\zeta}$ *is dual spacelike and* $\widehat{\xi}$ *is dual timelike, then there is a single dual number* $\widehat{\theta} = \theta+\varepsilon\theta^* \geq 0$ *such that* $<\widehat{\zeta},\widehat{\xi}>= \epsilon\left\|\widehat{\zeta}\right\|\left\|\widehat{\xi}\right\|\sinh\widehat{\theta}$*, where* $\epsilon = +1$ *or* $\epsilon = -1$ *given* $sign(\widehat{\zeta}_2) = sign(\widehat{\xi}_1)$ *or* $sign(\widehat{\zeta}_2) \neq sign(\widehat{\xi}_1)$*, respectively. This number is named the Lorentzian timelike dual angle between* $\widehat{\zeta}$ *and* $\widehat{\xi}$*.*

3. One-parameter Lorentzian spatial movements

Let a Lorentzian movable space $\mathbb{L}_m$ perform a one-parameter spatial movement against the Lorentzian stationary space $\mathbb{L}_f$. We assume that the dual coordinate frames $\{\mathbf{p};\ \widehat{\mathbf{e}}_1,\ \widehat{\mathbf{e}}_2(\text{timelike}), \widehat{\mathbf{e}}_3\}$ and $\{\mathbf{0}_f;\ \widehat{\mathbf{f}}_1,\ \widehat{\mathbf{f}}_2(\text{timelike}), \widehat{\mathbf{f}}_3\}$ are rigidly linked to the Lorentzian spaces $\mathbb{L}_m$ and $\mathbb{L}_f$, respectively. The directed lines $\widehat{\mathbf{e}}_i$ and $\widehat{\mathbf{f}}_i$ are specified by

$$\widehat{\mathbf{e}}_i = \mathbf{e}_i + \varepsilon\mathbf{e}_i^*, \text{ and } \widehat{\mathbf{f}}_i = \mathbf{f}_i + \varepsilon\mathbf{f}_i^*,\ (i = 1,\ 2,\ 3), \tag{3.1}$$

where $\mathbf{e}_i^* = \mathbf{p}\times\mathbf{e}_i$, and $\mathbf{f}_i^* = \mathbf{0}\mathbf{0}_f\times\mathbf{f}_i$, in which $\mathbf{0}$ is a stationary point as the origin of $\mathbb{E}_1^3$. This movement is named a one-parameter Lorentzian spatial movement and will be denoted by $\mathbb{L}_m/\mathbb{L}_f$. Through the movement $\mathbb{L}_m/\mathbb{L}_f$, let $\{\mathbf{S};\ \widehat{\mathbf{r}}_1,\ \widehat{\mathbf{r}}_2(\text{timelike}),\ \widehat{\mathbf{r}}_3\}$ be a further right-handed movable relative Blaschke frame which is specified as $\widehat{\mathbf{r}}_1(t) = \mathbf{r}_1(t) + \varepsilon\mathbf{r}_1^*(t)$, i.e., the $\mathbb{ISA}$, and $\widehat{\mathbf{r}}_2(t) := \mathbf{r}_2(t) + \varepsilon\mathbf{r}_2^*(t) = \widehat{\mathbf{r}}_1' \left\|\widehat{\mathbf{r}}_1'\right\|^{-1}$ is the mutual normal of $\widehat{\mathbf{r}}_1(t)$ and $\widehat{\mathbf{r}}_1(t + dt)$. A third dual unit vector is specified by $\widehat{\mathbf{r}}_3(t) = \widehat{\mathbf{r}}_1 \times \widehat{\mathbf{r}}_2$. In this case, the $\mathbb{ISA}$ will create two spacelike ruled surfaces named the stationary axode $\pi_f \subseteq \mathbb{L}_f$ and the movable axode $\pi_m \subseteq \mathbb{L}_m$. The directed lines $\widehat{\mathbf{r}}_i$ intersect at the mutual striction point $\mathbf{S}$ of the axodes π_j $(j = m, f)$. The dual unit vectors $\widehat{\mathbf{r}}_2$ and $\widehat{\mathbf{r}}_3$ are known as the central normal and the central tangent of the spacelike axodes, respectively. The origin $\mathbf{S}$ is the mutual central principal point of the movable and stationary spacelike axodes created by the $\mathbb{ISA}$s. Then,

$$\left.\begin{array}{c} <\widehat{\mathbf{r}}_1,\widehat{\mathbf{r}}_1> = -<\widehat{\mathbf{r}}_2,\widehat{\mathbf{r}}_2>=<\widehat{\mathbf{r}}_3,\widehat{\mathbf{r}}_3>=1, \\ \widehat{\mathbf{r}}_3 = \widehat{\mathbf{r}}_1\times\widehat{\mathbf{r}}_2, \widehat{\mathbf{r}}_1 = \widehat{\mathbf{r}}_2\times\widehat{\mathbf{r}}_3, \widehat{\mathbf{r}}_2 = \widehat{\mathbf{r}}_1\times\widehat{\mathbf{r}}_3. \end{array}\right\}$$

Here, and in what follows, the derivative with respect to t is indicated by a dash over the function symbol. Hence, for the Blaschke formulae with respect to $\mathbb{L}_j$, we find the following:

$$\begin{pmatrix} \widehat{\mathbf{r}}_1' \\ \widehat{\mathbf{r}}_2' \\ \widehat{\mathbf{r}}_3' \end{pmatrix} = \begin{pmatrix} 0 & \widehat{p} & 0 \\ \widehat{p} & 0 & \widehat{q}_j \\ 0 & \widehat{q}_j & 0 \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{r}}_1 \\ \widehat{\mathbf{r}}_2 \\ \widehat{\mathbf{r}}_3 \end{pmatrix} = \widehat{\omega}_j\times \begin{pmatrix} \widehat{\mathbf{r}}_1 \\ \widehat{\mathbf{r}}_2 \\ \widehat{\mathbf{r}}_3 \end{pmatrix}, \tag{3.2}$$

where $\widehat{\omega}_j(t) := \omega_j(t)+\varepsilon\omega_j^*(t)=\widehat{q}_j\widehat{\mathbf{r}}_1-\widehat{p}\widehat{\mathbf{r}}_3$ is the Darboux vector. The dual functions $\widehat{p}(t) = p(t)+\varepsilon p^*(t) = \left\|\widehat{\mathbf{r}}_1'\right\|$ and $\widehat{q}_j(t) = q_j(t) + \varepsilon q_j^*(t) = \left\langle\widehat{\mathbf{r}}_1'', \widehat{\mathbf{r}}_1'\times\widehat{\mathbf{r}}_1\right\rangle\left\|\widehat{\mathbf{r}}_1'\right\|^{-2}$ are named Blaschke invariants of the axodes. The tangent of the striction curve $\mathbf{S}(t)$ is

$$\mathbf{S}^{'}(t) = \cos\sigma_j \mathbf{r}_1 + \sin\sigma_j \mathbf{r}_3,$$

where $\sigma_j(t)$ is the striction angle of the axode π_j. Then, the Blaschke invariants become $\widehat{p}(t) = p + \varepsilon\sin\sigma_j$ and $\widehat{q}_i(t) = q_j + \varepsilon\cos\sigma_j$. The mutual distribution parameter of the spacelike axodes can be calculated as

$$\mu(t) = \frac{\det(\mathbf{S}^{'}(t), \mathbf{r}_1(t), \mathbf{r}_1^{'}(t))}{\left\|\mathbf{r}_1^{'}(t)\right\|^2} = \frac{\sin\sigma_j}{p}. \tag{3.3}$$

Corollary 3.1. *Throughout the movement $\mathbb{L}_m/\mathbb{L}_f$, the spacelike axodes have a mutual spacelike tangent plane on the $\mathbb{ISA}$ and slide on each other; that is, the movable spacelike axode makes contact with the stationary spacelike axode on the $\mathbb{ISA}$ in the 1st order at any instant t.*

Comparable with the spatial three-axis theorem, we obtain

$$\widehat{\omega}(t) := \omega(t) + \varepsilon\omega^*(t) = \widehat{\omega}_f(t) - \widehat{\omega}_m(t) = \widehat{\omega}(t)\widehat{\mathbf{r}}_1, \tag{3.4}$$

where $\widehat{\omega}(t) = \omega(t) + \varepsilon\omega^*(t) = \widehat{q}_f(t) - \widehat{q}_m(t)$ is the relative dual speed of the spacelike axodes. We shall set that $\omega^* \neq 0$ to leave out the pure translational movements. Also, we disregard zero divisors $\omega = 0$, that is, we shall exmaine only non-torsional movements so that the spacelike axodes are non-developable ruled surfaces ($\mu \neq 0$).

Corollary 3.2. *Throughout the movement L_m/L_f, at any instant $t \in \mathbb{R}$, the pitch may be obtained by*

$$h(t) = \frac{<\omega, \omega^*>}{\|\omega\|^2} = \mu\frac{\cot\sigma_f - \cot\sigma_m}{\cot\varphi_f - \cot\varphi_m};$$

$\varphi_j(t)$ is the apex angle of the osculating cone of the spacelike axodes π_j.

3.0.1. Disteli formulae for the spacelike axodes

In planar kinematics, at each point of a smooth curve, there exists only one osculating circle, which is frequently named the curvature circle of the curve. The radius and center of this circle can be specified by the famous Euler-Savary formulae if the location of the point is given in the movable plane [1–3]. In spite of the fact that the famous Euler-Savary formulae of a line trajectory had been proved for various types of geometry [1–3,12–19], some notations should be filtered.

The spacelike Disteli-axis of the axode π_j can be specified as

$$\widehat{\mathbf{b}}_j(t) = \frac{\widehat{\mathbf{r}}_1^{'} \times \widehat{\mathbf{r}}_1^{''}}{\left\|\widehat{\mathbf{r}}_1^{'} \times \widehat{\mathbf{r}}_1^{''}\right\|} = \frac{\widehat{q}_j\widehat{\mathbf{r}}_1 - \widehat{p}\widehat{\mathbf{r}}_3}{\sqrt{\widehat{q}_j^2 + \widehat{p}^2}} = \frac{\widehat{\omega}_j(t)}{\left\|\widehat{\omega}_j(t)\right\|}. \tag{3.5}$$

Let $\widehat{\varphi}_j = \varphi_j + \varepsilon\varphi_j^*$ be the apex dual angle (radius of curvature between $\widehat{\mathbf{r}}_1$ and $\widehat{\mathbf{b}}_j$). Then, for $\widehat{p} \neq 0$, the spacelike Disteli-axis becomes

$$\widehat{\mathbf{b}}_j(t) = \cos\widehat{\varphi}_j\widehat{\mathbf{r}}_1 - \sin\widehat{\varphi}_j\widehat{\mathbf{r}}_3, \text{ with } \widehat{\varphi}_j = \cot^{-1}\left(\widehat{q}_j/\widehat{p}\right). \tag{3.6}$$

The dual geodesic curvature of the spacelike axodes π_j is

$$\widehat{\gamma}_j(t) := \gamma_j(t) + \varepsilon\gamma_j^* = \frac{\widehat{q}_j}{\widehat{p}} = \cot\widehat{\varphi}_j. \tag{3.7}$$

From Eq (3.7), it follows that

$$\widehat{\gamma}_f - \widehat{\gamma}_m = \cot\widehat{\varphi}_f - \cot\widehat{\varphi}_m = \frac{\widehat{\omega}}{\widehat{p}}. \tag{3.8}$$

This equation is a new Lorentzian dual Euler-Savary formula of the axodes of L_m/L_f. If we separate the real and dual parts of Eq (3.8), respectively, we have

$$\cot\varphi_f - \cot\varphi_m = \frac{\omega}{p}, \tag{3.9}$$

and

$$\frac{\varphi_m^*}{\sin^2\varphi_m} - \frac{\varphi_f^*}{\sin^2\varphi_f} = \frac{\omega}{p}(h-\mu). \tag{3.10}$$

Equations (3.9) and (3.10) are new Lorentzian Disteli formulae for the spacelike axodes. At the same time, Eq (3.9) is a Lorentzian version of the Euler-Savary formula for the polodes of real spherical movement [1–3]. Note that the scalars ω, ω^* and h are invariants of the choice of the reference point.

Velocity and acceleration For $\mathbb{L}_m/\mathbb{L}_f$, each stationary spacelike line of the moveable space $\mathbb{L}_m$ generally generates a timelike or spacelike ruled surface in the fixed space $\mathbb{L}_f$ that will be indicated by $(\widehat{x})$, and its generator in $\mathbb{L}_m$ by $\widehat{\mathbf{x}}$. We assume a spacelike ruled surface in our study. Then, we may write

$$\widehat{\mathbf{x}}(\widehat{s})=\widehat{x}^t\widehat{\mathbf{r}},\ \ \widehat{x} = \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \end{pmatrix} = \begin{pmatrix} x_1+\varepsilon x_1^* \\ x_2+\varepsilon x_2^* \\ x_3+\varepsilon x_3^* \end{pmatrix},\ \widehat{\mathbf{r}} = \begin{pmatrix} \widehat{\mathbf{r}}_1 \\ \widehat{\mathbf{r}}_2 \\ \widehat{\mathbf{r}}_3 \end{pmatrix}, \tag{3.11}$$

where

$$x_1^2 - x_2^2 + x_3^2 = 1,$$
$$x_1x_1^* - x_2x_2^* + x_3x_3^* = 0.$$

By the instantaneous screw $\widehat{\omega}(t)=\widehat{\omega}(t)\widehat{\mathbf{r}}_1$, the velocity $\widehat{\mathbf{x}}'$ and the acceleration $\widehat{\mathbf{x}}''$ of $\widehat{\mathbf{x}}$ with respect to $\mathbb{L}_f$, respectively, we have

$$\widehat{\mathbf{x}}'(t) = \widehat{\omega}\times\widehat{\mathbf{x}}, \tag{3.12}$$

and

$$\widehat{\mathbf{x}}'' = \widehat{x}_3\widehat{p}\widehat{\omega}\widehat{\mathbf{r}}_1 + (\widehat{x}_2^2\widehat{\omega}^2 + \widehat{x}_3\widehat{\omega}')\widehat{\mathbf{r}}_2 + (\widehat{x}_2\widehat{\omega}' - \widehat{x}_1\widehat{p}\widehat{\omega} + \widehat{x}_3\widehat{\omega}^2)\widehat{\mathbf{r}}_3. \tag{3.13}$$

Then,

$$\widehat{\mathbf{x}}' \times \widehat{\mathbf{x}}'' = \widehat{\omega}^2[(1-\widehat{x}_1^2)\widehat{\omega}\widehat{\mathbf{r}}_1 - \widehat{p}\widehat{x}_3\widehat{\mathbf{x}}]. \tag{3.14}$$

3.1. Disteli formulae for a spacelike line trajectory

Analogous with the Euclidean Disteli formulae in [1–9], we will give the Lorentzian Disteli formulae in view of dual angle approximations. This means that we consider the spacelike line $\widehat{\mathbf{x}}$ in the moveable space $\mathbb{L}_m$, which occurs at a constant spacelike dual angle from a given spacelike line $\widehat{\mathbf{y}}$ in the stationary space $\mathbb{L}_f$. Then, we consider the spacelike dual angle

$$\widehat{\rho}(t) = \cos^{-1}(<\widehat{\mathbf{y}}, \widehat{\mathbf{x}}>)$$

such that $\widehat{\mathbf{y}}$ and $\widehat{\rho}$ remain stationary up to the 2nd order at $t = t_0$, that is,

$$\widehat{\rho}'(t) \mid_{t=t_0} = 0, \quad \widehat{\mathbf{x}}'(t) \mid_{t=t_0} = \mathbf{0},$$

and

$$\widehat{\rho}''(t) \mid_{t=t_0} = 0, \; \widehat{\mathbf{x}}''(t) \mid_{t=t_0} = \mathbf{0}.$$

Then, for the 1st-order, we have

$$<\widehat{\mathbf{x}}', \widehat{\mathbf{y}}> \mid_{t=t_0} = 0,$$

and, for the 2nd-order properties,

$$<\widehat{\mathbf{x}}'', \widehat{\mathbf{y}}> \mid_{t=t_0} = 0.$$

From the above two equations, we find that

$$\widehat{\mathbf{y}} = \frac{\widehat{\mathbf{x}}' \times \widehat{\mathbf{x}}''}{\left\| \widehat{\mathbf{x}}' \times \widehat{\mathbf{x}}'' \right\|}. \tag{3.15}$$

Hence, $\widehat{\rho}$ will be invariant in the 2nd approximation iff $\widehat{\mathbf{y}}$ is the spacelike Disteli-axis $\widehat{\mathbf{b}}$ of (X), that is,

$$\widehat{\rho}' = \widehat{\rho}'' = 0 \Leftrightarrow \widehat{\mathbf{y}} = \frac{\widehat{\mathbf{x}}' \times \widehat{\mathbf{x}}''}{\left\| \widehat{\mathbf{x}}' \times \widehat{\mathbf{x}}'' \right\|} = \pm \widehat{\mathbf{b}}. \tag{3.16}$$

Substituting Eq (3.14) into the Eq (3.16), we have

$$\pm \widehat{\mathbf{b}}(t) = \frac{\widehat{\omega}^2[(1 - \widehat{x}_1^2)\widehat{\omega}\widehat{\mathbf{r}}_1 - \widehat{p}\widehat{x}_3\widehat{\mathbf{x}}]}{\left\| \widehat{\mathbf{x}}' \times \widehat{\mathbf{x}}'' \right\|}. \tag{3.17}$$

Since (X) is a spacelike ruled surface, then Eq (3.12) shows that $\widehat{\mathbf{x}}'$ is a timelike dual vector orthogonal to both $\widehat{\omega}$ and $\widehat{\mathbf{x}}$. Therefore, we define the spacelike dual angles $\widehat{\vartheta} = \vartheta + \varepsilon\vartheta^*$, $0 \leq \vartheta \leq 2\pi$, $\vartheta^* \in \mathbb{R}$ and $\widehat{\varphi} = \varphi + \varepsilon\varphi^*$ ($\varphi^* \in \mathbb{R}$, and $\varphi \geq 0$) to identify its directions, that is,

$$\widehat{\mathbf{x}} = \cos\widehat{\vartheta}\widehat{\mathbf{r}}_1 + \sin\widehat{\vartheta}\widehat{\mathbf{m}}, \text{ with } \widehat{\mathbf{m}} = \sinh\widehat{\varphi}\widehat{\mathbf{r}}_2 + \cosh\widehat{\varphi}\widehat{\mathbf{r}}_3. \tag{3.18}$$

Similarly, a set of coordinates may be utilized to match the spacelike Disteli-axis by using the equation

$$\widehat{\mathbf{b}} = \cos\widehat{\alpha}\widehat{\mathbf{r}}_1 + \sin\widehat{\alpha}\widehat{\mathbf{m}}, \tag{3.19}$$

where $\widehat{\alpha} = \alpha + \varepsilon\alpha^*$. Thus, from Eqs (3.17) and (3.19), one finds that

$$\frac{(1 - \widehat{x}_1^2)\widehat{\omega} + \widehat{p}\widehat{x}_3\widehat{x}_1}{\cos\widehat{\alpha}} = \frac{\widehat{p}\widehat{x}_2\widehat{x}_3,}{\sin\widehat{\alpha}\sinh\widehat{\varphi}} = \frac{\widehat{p}\widehat{x}_3^2}{\sin\widehat{\alpha}\cosh\widehat{\varphi}}. \tag{3.20}$$

From Eqs (3.11), (3.18) and (3.20), we have

$$\left(\cot\widehat{\alpha}-\cot\widehat{\vartheta}\right)\cosh\widehat{\varphi}=\frac{\widehat{\omega}}{\widehat{p}}. \tag{3.21}$$

Equation (3.21) is the dual Lorentzian Euler-Savary formula for point trajectories in both planar and spherical kinematics (compared with [1–3]). If we separate Eq (3.21) into real and dual parts, we get

$$(\cot\alpha-\cot\vartheta)\cosh\varphi=\frac{\omega}{p}, \tag{3.22}$$

and

$$\varphi^{*}(\cot\alpha-\cot\vartheta)\sinh\varphi-(\frac{\alpha^{*}}{\sin^{2}\alpha}-\frac{\vartheta^{*}}{\sin^{2}\vartheta})\cosh\varphi=\frac{\omega}{p}\,(h-\mu)\,. \tag{3.23}$$

Equations (3.22) and (3.23) are new Lorentzian Disteli formulae for the spacelike line trajectory of a one-parameter Lorentzian spatial movement. In view of Figure 2, the sign of α^{*} (+ or -) defines that the Disteli-axis is situated on the negative or positive orientation of the timelike central normal vector $\widehat{\mathbf{t}}$=$\widehat{\mathbf{x}}' \left\|\widehat{\mathbf{x}}'\right\|^{-1}$ of the spacelike trajectory $\widehat{\mathbf{x}}$ at the central point $\mathbf{c}$, while the location of $\widehat{\mathbf{t}}$ is acquired by $\widehat{\mathbf{t}}(t)=\cosh\widehat{\varphi}\widehat{\mathbf{r}}_{2}+\sinh\widehat{\varphi}\widehat{\mathbf{r}}_{3}$. Using the fact that the central points $\mathbf{c}$ of (X) are existing on the spacelike plane Sp{$\widehat{\mathbf{x}}$, $\widehat{\mathbf{x}}\times\widehat{\mathbf{t}}$}, indicated by π, Eq (3.23) shows that, if the spacelike rulings $\widehat{x}$ are realized by the dual pairs $(\widehat{\vartheta},\widehat{\varphi})$ with respect to the fixed axode π_{f}, we can obtain the Disteli formula in the spacelike plane π. Therefore, any arbitrary point $\mathbf{c}(\varphi^{*},\vartheta^{*})$ on π is inspected as the central point of the spacelike ruled surface (X) whose ruling is a spacelike directed line $\widehat{\mathbf{x}}$, and the radius ϑ^{*} would be calculated in Eq (3.23); Figure 2 shows its length at the point $\mathbf{p}$ to $\mathbf{c}$ on π. The timelike vector from $\mathbf{p}$ to $\mathbf{c}$ will have the same orientation as $\widehat{\mathbf{t}}$ if $\vartheta^{*}>0$, and the opposite orientation of $\widehat{\mathbf{t}}$ if $\vartheta^{*}<0$. The central point $\mathbf{c}(\varphi^{*},\vartheta^{*})$ of (X) can be on the $\mathbb{ISA}$ if $\vartheta^{*}=0$, and on the Disteli-axis if $\vartheta^{*}=\alpha^{*}$. Then, the central points of (X) can be acquired by applying $\alpha^{*}=0$ in Eq (3.23), which is simplified as a linear equation:

$$L:\left(\frac{\omega}{p}\tanh\varphi\right)\varphi^{*}+\left(\frac{\cosh\varphi}{\sin^{2}\vartheta}\right)\vartheta^{*}-\frac{\omega}{p}\,(h-\mu)=0. \tag{3.24}$$

Therefore, at any instant $t\in\mathbb{R}$, the spacelike lines are in a given direction, which is fixed in L_{m} and lie on π. The spacelike line L changes its location if ϑ is given as a variable value while φ is constant. However, the spacelike lines would create a Lorentzian curve on π. Moreover, π varies in position if φ varies in value while ϑ is constant. Hence, the spacelike lines given be Eq (3.24) denotes a spacelike line congruence for all values of $(\varphi^{*},\vartheta^{*})$.

Once again, we can find a second Lorentzian dual version of the Euler-Savary formula, as follows: Let $\widehat{\sigma}=\sigma+\varepsilon\sigma^{*}$ be the spacelike dual angle among the spacelike directed lines $\widehat{\mathbf{x}}$ and $\widehat{\mathbf{b}}$. Then, the following relation holds:

$$\cot\widehat{\sigma}=\frac{\left\langle\widehat{\mathbf{x}}'',\widehat{\mathbf{x}}'\times\widehat{\mathbf{x}}\right\rangle}{\left\|\widehat{\mathbf{x}}'\right\|^{3}}=\frac{\widehat{\omega}\widehat{x}_{1}(\widehat{x}_{1}^{2}-1)+\widehat{p}\widehat{x}_{3}}{\widehat{\omega}(1-\widehat{x}_{1}^{2})^{\frac{3}{2}}}. \tag{3.25}$$

From Eqs (3.18) and (3.25), we have

$$\cot\widehat{\sigma}-\cot\widehat{\vartheta}=\frac{\widehat{p}\sinh\widehat{\sigma}}{\widehat{\omega}\sin^{2}\widehat{\vartheta}}. \tag{3.26}$$

Equation (3.26) gives the relationships between $\widehat{\mathbf{x}}(t)$, which parametrize the spacelike ruled surface (X), and its spacelike Disteli-axis $\widehat{\mathbf{b}}$ at any instant. From the real and dual parts, we have

$$\left.\begin{array}{l}\cot\sigma-\cot\vartheta=\frac{p\sinh\varphi}{\omega\sin^{2}\vartheta},\\ \sigma^{*}=\frac{\sin^{2}\varphi}{\sin^{2}\vartheta}\left[\left(1-2\frac{p}{\omega}\sinh\varphi\cot\vartheta\right)\vartheta^{*}-\frac{p}{\omega}(\mu+\varphi^{*}\coth\varphi)\sinh\varphi\right].\end{array}\right\}\tag{3.27}$$

Equation (3.27) shows the new Lorentzian Disteli formulae for the movement $\mathbb{L}_m/\mathbb{L}_f$; the first equation reveals the correlation among the locations of $\widehat{\mathbf{x}}$ and its spacelike Disteli-axis $\widehat{\mathbf{b}}$ in $\mathbb{L}_m$. Whereas, the second one describes the Lorentzian distance from $\widehat{\mathbf{x}}$ to the spacelike Disteli-axis $\widehat{\mathbf{b}}$.

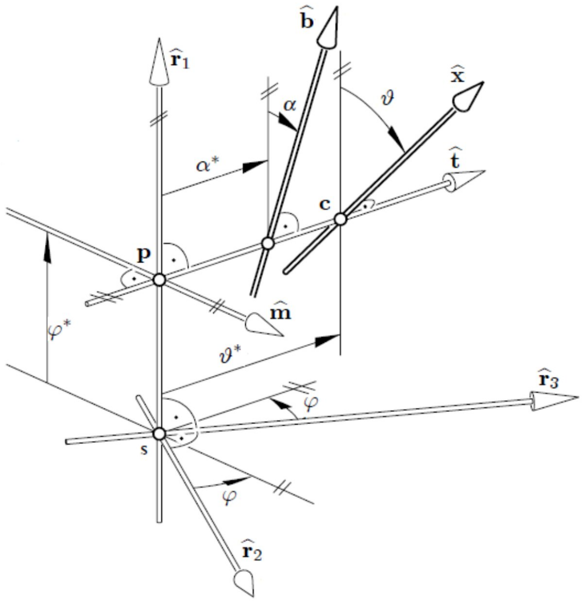

Figure 2. The line $\widehat{\mathbf{x}}$ and its Disteli-axis $\widehat{\mathbf{b}}$.

3.2. *Kinematic geometry of a point trajectory*

For $\mathbb{L}_m/\mathbb{L}_f$, at any instant $t\in\mathbb{R}$, a generic stationary point in $\mathbb{L}_m$, generally, will create a curve in $\mathbb{L}_f$. In kinematics, this curve is referred to as the point trajectory. Point trajectories with private values of velocity and acceleration have some certain characteristics in kinematics.

Via Eq (3.4), the velocity of a stationary point $\mathbf{q}(x,y,z)\in L_m$ can be defined as follows:

$$\mathbf{v}(t):=\mathbf{q}^{'}=\omega^{*}+\omega\times\mathbf{q}=(h\omega,\omega z,\omega y).\tag{3.28}$$

From Eq (3.28), the acceleration is

$$\mathbf{J}(t)=:\mathbf{q}^{''}=\begin{pmatrix}J_1\\J_2\\J_3\end{pmatrix}=\begin{pmatrix}\omega^{*'}\\p\omega^{*}+p^{*}\omega\\0\end{pmatrix}+\begin{pmatrix}0&0&p\omega\\0&\omega^{2}&\omega^{'}\\-p\omega&\omega^{'}&\omega^{2}\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}.\tag{3.29}$$

Thus, we may write the curvature of $\mathbf{q}(t)$ as follows:

$$\kappa(t):=\frac{\left\|\mathbf{q}^{'}\times\mathbf{q}^{''}\right\|}{\|\mathbf{q}^{'}\|^{3}}=\frac{\omega\sqrt{\left|(zJ_3-yJ_2)^{2}-(hJ_3-yJ_1)^{2}+(hJ_2-zJ_1)^{2}\right|}}{\sqrt{\left|h^{2}+\omega^{2}\left(-y^{2}+z^{2}\right)\right|^{3}}}.\tag{3.30}$$

Equation (3.30) is named the Lorentzian curvature equation, shows the curvature of the trajectory of point $\mathbf{q}$ as a function of the coordinates $(x, y,\ z)$ and the instantaneous invariants up to 2nd order of the spacelike axodes. We may state that the trajectory of points, which are relatively stationary with respect to the moveable spacelike axode, which has trajectories with the same assigned value of curvature, at any instant, lie on a spacelike or timelike surface which may be named the curvature surface given by Eq (3.30). However, various significant private cases for $\mathbb{L}_m/\mathbb{L}_f$ created by the specific spacelike axodes exist as described below.

3.2.1. Real Lorentzian spherical movements

In spherical movements, there is one stationary point, i.e., the central point of the $\mathbb{ISA}$, and all points in the movement generate their paths upon Lorentzian spheres that are concentric around this stationary point. The spacelike axodes are special types of developable ruled surfaces (spacelike cones) whose rulings all have a mutual point at the Lorentzian sphere's center. The $\mathbb{ISA}$ governing the movement changes their orientation, but the central point is connected to all of them, that is, $h = \mu = 0$. Then, from Eq (3.30), we have the following equation:

$$\kappa(t)\sqrt{\left|-y^2+z^2\right|^3} = \sqrt{\left|(zJ_3 - yJ_2)^2 + (-y^2+z^2)\,J_1^2\right|}. \tag{3.31}$$

By using Lorentzian spherical coordinates, we have

$$x = r\cos\vartheta,\ \ y = r\sin\vartheta\sinh\varphi,\ \ z = r\sin\vartheta\cosh\varphi. \tag{3.32}$$

Equation (3.31) reduces to

$$\kappa\omega r\sin\vartheta = \sqrt{(\omega - p\cot\vartheta\cosh\varphi)^2 + (p\cosh\varphi)^2}. \tag{3.33}$$

Also, for the Lorentzian spherical curve $\mathbf{q}(t)$, the curvature is

$$\kappa(t) = \frac{1}{r\sin\vartheta}. \tag{3.34}$$

Substituting Eq (3.34) into Eq (3.33) and simplifying it, we have

$$\frac{\omega}{p} - \frac{\cosh\varphi}{\sin 2\vartheta} = 0,$$

or, in view of Eq (3.9), we have

$$\cot\varphi_f - \cot\varphi_m = \frac{\cosh\varphi}{\sin 2\vartheta}. \tag{3.35}$$

Equation (3.35) is a new Euler-Savary formula for the polodes of real Lorentzian spherical movement.

3.2.2. Lorentzian planar movements

The Lorentzian planar movement occurs with the pitch along the $\mathbb{ISA}$ equal to zero ($h = 0$); points in timelike planes orthogonal to the $\mathbb{ISA}$ will existing within their respective timelike planes as the movement occurs ($p^* = x = 0$). The axodes are spacelike cylindrical surfaces; all of their rulings are parallel to the $\mathbb{ISA}$. The striction curves become indeterminate, and the spacelike axodes project onto a plane to give spacelike pole curves. They are coincident with each other at the pole point **p**, and the relative frame is $\{\mathbf{p}; \mathbf{r}_2, \mathbf{r}_3\}$; see Figure 3. This means that the moveable spacelike pole curve π_m and stationary spacelike pole curve π_f roll on each other without sliding. In this case, then we have

$$\left.\begin{array}{c} \mathbf{v}(t) = \omega z\mathbf{r}_2 + \omega z\mathbf{r}_3, \\ \mathbf{J}(t)=\left(\omega^2 y + \omega' z\right)\mathbf{r}_2 + \left(-p\omega + \omega^2 y + \omega' z\right)\mathbf{r}_3. \end{array}\right\} \tag{3.36}$$

Then, Eqs (3.30) and (3.36) lead to

$$\kappa(t) = \frac{z^2 - y^2 - \eta z}{\sqrt{\left|y^2 - z^2\right|^3}}, \text{ with } \eta = \frac{p}{\omega}. \tag{3.37}$$

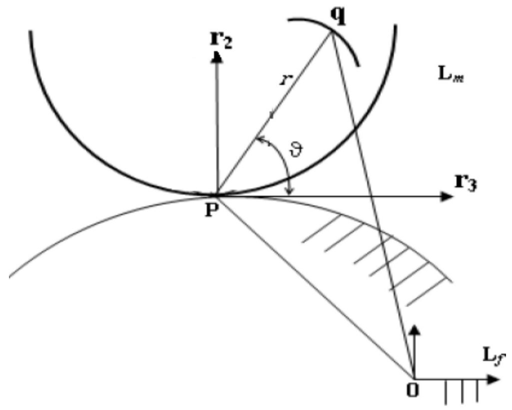

Figure 3. π_m / π_f pole curves.

Lorentzian inflection circle We now aim to have a closer look at inflection points on the moveable timelike plane $\mathbb{L}_m$. Those are points at which their curve changes from being convex (concave downward) to concave (concave upward), or vice versa, so their radius of curvature is instantaneously infinite. Such a point has acceleration **J** that is directed tangential to the curve, as is its **v** velocity. Then, the state of collinearity among those two implies that

$$\left\|\mathbf{q}' \times \mathbf{q}''\right\| = 0 \Leftrightarrow \kappa = 0 \Leftrightarrow z^2 - y^2 - \eta z = 0. \tag{3.38}$$

Thus, we may state that all points on a moveable timelike plane $\mathbb{L}_m$, which are inflection points of their path, are located on a circle, particularly, the inflection circle in $\mathbb{L}_f$.

The Euler-Savary formula If we introduce polar coordinates (r, ϑ) with oz as the polar axis, we have that $z = r\cosh\vartheta$ and $y = r\sinh\vartheta$, $\vartheta \geq 0$, $-\infty < r < \infty$. Then, we obtain the following from Eq (3.37):

$$\kappa = \frac{r - \eta\cosh\vartheta}{r^2}, \; r \neq 0. \tag{3.39}$$

If the curvature radius ρ is replaced by the curvature κ, another form of the above equation is

$$\rho = \frac{r^2}{r - \eta\cosh\vartheta}. \tag{3.40}$$

Equation (3.40) is comparable to the quadratic form of the Euler-Savary formula for planar Euclidean movement [1–3]. When the denominator in Eq (3.40) vanishes, the curvature radius $\rho = \pm\infty$. So, the equation $r - \eta\cosh\vartheta = 0$ is a purely geometric, necessary condition for any point $\mathbf{q}$ located on the Lorentzian inflection circle. The attitude of points in the moveable timelike plane $\mathbb{L}_m$ with trajectories having a given curvature radius is a function of the diameter of the inflection circle depends only on η. Let $\mathbf{c}$ be the curvature center of the point $\mathbf{q}$. These points and the instantaneous rotation of the spacelike pole $\mathbf{p}$ stay on a spacelike line, that is, on an instantaneous normal path that is connected to $\mathbf{q}$ at $t \in \mathbb{R}$. In general, a curvature center with respect to a point of a planar curve stays on the normal plane of the curve with respect to that point. From Eq (3.40), through the points $\mathbf{p}$ and $\mathbf{c}(\overline{r}, \vartheta)$ with $\overline{r} = r - \rho$, we then have

$$\left(\frac{1}{r} - \frac{1}{\overline{r}}\right)\cosh\vartheta = \frac{1}{\eta}. \tag{3.41}$$

Then, Eq (3.41) is the Minkowski version of the Euler-Savary formula, whose geometrical meaning is shown by rotation of the spacelike pole $\mathbf{p}$ and points $\mathbf{q}$ and $\mathbf{c}$ in Figure 4. Herein, $\overline{r}$ and r have to be interpreted as directed quantities. Equation (3.41) assumes them to be unidirectional. If $\mathbf{q}$ and $\mathbf{c}$ have opposite directions, the minus in the parentheses has to be changed to a plus. From Eq (3.41), we also see that $r = \pm\infty$ if and only if $\overline{r} + \eta\cosh\vartheta = 0$, that is, $\overline{r} = -\eta\cosh\vartheta$, which is a reflection of the Lorentzian inflection circle into the y axis. Also, Eq (3.41) can be rewritten as

$$\eta\cosh\vartheta = \frac{\overline{r}r}{\overline{r} - r}. \tag{3.42}$$

Once the angle ϑ is known ($\cosh\vartheta \neq 0$), Eq (3.42) gives the correspondence between r and $\overline{r}$ in terms of the second order invariant η, that is, every point $\mathbf{c}$ belongs to one point $\mathbf{q}$.

Once again, we start with one further implementation of the main formula, i.e., Eq (3.40), as follows: All points of the moveable plane $\mathbb{L}_m$, possessing the same curvature radius of their paths, are situated on a spacelike curve named the ρ-curve. We get it by resolving Eq (3.40) for r:

$$r_{1,2} = \frac{\rho}{2}\left(1 \pm \sqrt{1 - \frac{4\eta\cosh\vartheta}{\rho}}\right). \tag{3.43}$$

The spacelike curve of corresponding curvature centers results from substituting $\overline{r} = r - \rho$ in Eq (3.43) and resolving for $\overline{r}$. Here, we yield

$$\overline{r}_{1,2} = -\frac{\rho}{2}\left(1 \mp \sqrt{1 - \frac{4\eta\cosh\vartheta}{\rho}}\right). \tag{3.44}$$

When the given curvature radius goes to infinity, the ρ-curve approaches the inflection circle, that is, $r = \eta\cosh\vartheta$.

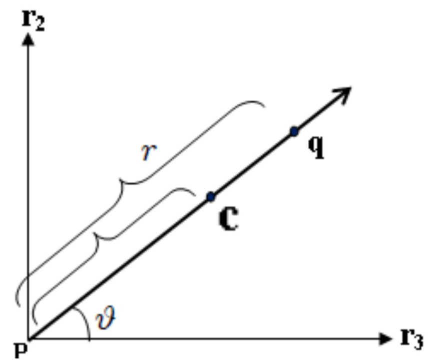

Figure 4. ρ-curve and inflection circle.

4. Conclusions

The main aim of this work was to exmaine one-parameter Lorentzian spatial movements by means of the E. Study map. With the suggested technique, new proofs for the Euler-Savary and Disteli formulae were acquired. In addition, new metric properties have been defined for the Disteli-axis of a spacelike trajectory ruled surface under one-parameter Lorentzian spatial movement of a body in Minkowski 3-space.

We hope that our work may contribute to the application of dual Lorentzian spherical motions, four-bar mechanisms, the theory of mechanism synthesis for higher order approximations, gear theory and spatial mechanisms in engineering design.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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