Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 20169–20186.
DOI: 10.3934/math.20231028
Received: 03 May 2023
Revised: 24 May 2023
Accepted: 29 May 2023
Published: 19 June 2023

Research article

The lower bound on the measure of sets consisting of Julia limiting directions of solutions to some complex equations associated with Petrenko's deviation

Guowei Zhang*

School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan 455000, China

* **Correspondence:** E-mail: herrzgw@foxmail.com.

Abstract: In the value distribution theory of complex analysis, Petrenko's deviation is to describe more precisely the quantitative relationship between $T(r, f)$ and $\log M(r, f)$ when the modulus of variable $|z| = r$ is sufficiently large. In this paper we introduce Petrenko's deviations to the coefficients of three types of complex equations, which include difference equations, differential equations and differential-difference equations. Under different assumptions we study the lower bound of limiting directions of Julia sets of solutions of these equations, where Julia set is an important concept in complex dynamical systems. The results of this article show that the lower bound of limiting directions mentioned above is closely related to Petrenko's deviation, and our conclusions improve some known results.

Keywords: Petrenko's deviation; Julia set; entire function; complex equation
Mathematics Subject Classification: 30D05, 37F10, 37F50

1. Introduction

The content of this paper is related to the value distribution theory and complex dynamic theory in complex analysis, the relevant literatures are available at [9, 10, 25, 29]. On the complex plane, let g be an entire function, its order and lower order denote by $\sigma(g)$ and $\mu(g)$ respectively, see reference [25] for details.

In Nevanlinna theory, for an entire function g, $T(r, g)$ represents its characteristic function and $M(r, g)$ represents its maximum modulus in the domain $|z| < r$. For these two quantities, they have a certain magnitude relationship. It's well known that, if $0 < r < R < +\infty$, then $M(r, g)$ and $T(r, g)$ satisfy the inequalities (see [10])

$$\frac{R+r}{R-r}T(R, g) \geq \log M(r, g) \geq T(r, g). \tag{1.1}$$

If one takes R as a multiple of r and r goes to infinity, these two quantities are very similar. In order to study the exact relationship between these two quantities, Petrenko [18] introduced the deviation of entire function g at ∞ and defined it as follows.

$$\beta^-(\infty, g) = \liminf_{r\to\infty} \frac{\log M(r, g)}{T(r, g)} \text{ and } \beta^+(\infty, g) = \limsup_{r\to\infty} \frac{\log M(r, g)}{T(r, g)}. \tag{1.2}$$

Later they were called Petrenko's deviation. Particularly, $\beta^-(\infty, g) < \beta^+(\infty, g)$ is likely to hold, as stated in [18, §4]. As described in [6], for any increasing function $\phi(r)$ which is convex for $\log r$, there exists an entire function g that satisfies the following similarity relation

$$T(r, g) \sim \phi(r) \sim \log M(r, g). \tag{1.3}$$

From the above formula (1.3), we know that there should be many entire functions satisfying $\beta^+(\infty, g) = \beta^-(\infty, g)$.

In the following we will give some specific examples of functions with Petrenko's deviations. For any entire function g with finite lower order μ, let's define a variable $\mathfrak{B}(\mu)$ with respect to μ,

$$\mathfrak{B}(\mu) := \begin{cases} \dfrac{\pi\mu}{\sin(\pi\mu)}, & \text{if } 0 \le \mu < 1/2, \\ \pi\mu, & \text{if } \mu \ge 1/2. \end{cases} \tag{1.4}$$

It is shown in [18, Theorem 1] that Petrenko's deviation of g satisfies

$$\mathfrak{B}(\mu) \ge \beta^-(\infty, g) \ge 1. \tag{1.5}$$

Both inequalities in (1.5) can hold strictly, see details in [6, 18]. For example, the well known Airy integral function $Ai(z)$, which is a solution of $g''(z) - zg(z) = 0$, has lower order $3/2$. It's Petrenko's deviation satisfies

$$\frac{3\pi}{2} > \lim_{r\to\infty} \frac{\log M(r, Ai)}{T(r, Ai)} = \frac{3\pi}{4} > 1.$$

If the upper deviation $\beta^+(\infty, g)$ and the lower deviation $\beta^-(\infty, g)$ are equal and finite, then there exists a constant $\nu \in (0, 1]$ such that $T(r, f)$ and $\log M(r, f)$ satisfy

$$T(r, g) \sim \nu \log M(r, g) \tag{1.6}$$

as $r(\notin E) \to \infty$, where $E \subset (0, +\infty)$ is an exceptional set. For the function $g(z) = e^z$ that we're most familiar with, $T(r, e^z) = \frac{r}{\pi}$ and $\log M(r, e^z) = r$ satisfies (1.6) with $\nu = 1/\pi$. Generally, the exponential polynomials can also satisfy (1.6) for suitable ν as $r(\notin E) \to \infty$, where $E \subset (0, +\infty)$ is a set of zero density [17].

Another example is entire function with Fabry gaps. For an entire function represented by a power series, $g(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$, if it satisfies the gaps condition $\frac{\lambda_n}{n} \to \infty$ as $n \to \infty$, we call it having Fabry gaps, see [3]. The maximum modulus of such a function is similar to its minimum modulus, that is,

$$\log M(r, g) \sim \log L(r, g), \tag{1.7}$$

as $r(\notin E) \to \infty$, where $E \subset (0, +\infty)$ is a set of zero density and $L(r, g) = \min_{|z|=r} |g(z)|$ is the minimum modulus, see [8]. The orders of this kinds of functions are positive [11, p.651]. Consequently, entire functions with Fabry gaps satisfy (1.6) with $\nu = 1$ as $r(\notin E) \to \infty$.

In 2021, Heittokangas and Zemirni [12] investigated the zeros of solutions of second order complex differential equation

$$f'' + A(z)f = 0, \tag{1.8}$$

in which the coefficient $A(z)$ satisfies (1.6). In fact, they found the lower bound of the zero convergence exponent of solution to this equation is closely related to Petrenko's deviation of $A(z)$. In the same article, they also proved the growth of every nontrivial solution to

$$f'' + A(z)f' + B(z)f = 0, \tag{1.9}$$

is infinite under the hypothesise that $A(z)$ is associated with Petrenko's deviation and $B(z)$ is non-transcendental growth in some angular domains.

For the sake of the following statement, we give the condition that an entire function $g(z)$ has an angular domain in which it is non-transcendental growth. Set

$$\Xi(g) := \left\{\theta \in [0, 2\pi) : \limsup_{r\to\infty} \frac{\log^+ |g(re^{i\theta})|}{\log r} < \infty\right\} \tag{1.10}$$

and

$$\xi(g) := \frac{1}{2\pi} \cdot meas\,(\Xi(g))\,.$$

More specifically, the authors of paper [12] was under the condition that

$$\beta^-(\infty, B) < \frac{1}{1 - \xi(A)} \tag{1.11}$$

to prove the infinite growth of solutions of (1.9).

The following content is some introduction about Julia set and the results of its limiting directions. The Julia set is one of two important subjects of the research of complex dynamics of transcendental meromorphic functions, see more details in [4, 28]. Below we briefly recall the known results on the limiting direction of Julia set of meromorphic functions. To prevent confusion, we use $f^{\circ n}(n \in \mathbb{N})$ to denote the n-th iteration of meromorphic function f. The sequence of functions $\{f^{\circ n}\}(n \in \mathbb{N})$ forms a normal family in some domains of the complex plane, and such domains are called the Fatou set, denoted as $F(f)$. Its complement on the complex plane is Julia set, denoted as $J(f)$. The focus of our research is $J(f)$, it's well known that $J(f)$ a nonempty closed set. For transcendental entire functions, the distribution of $J(f)$ in the complex plane is so complicated that it cannot be constrained in a finite number of straight lines, see Baker [1]. However, this is not true for transcendental meromorphic functions, such as the Julia set of $\tan z$ being the whole real axis.

Therefore, the research direction began to focus on the distribution of $J(f)$ on rays, and Qiao introduced this concept in [19]. To explain this concept, let's first give a notation for an angular domain,

$$\Omega(\alpha, \beta) = \{z \in \mathbb{C} : \arg z \in (\alpha, \beta) \subset [0, 2\pi)\}.$$

So, for any $\theta \in [0, 2\pi)$ and any small $\varepsilon > 0$, if the set $\Omega(\theta - \varepsilon, \theta + \varepsilon) \cap J(f)$ is unbounded, then the ray $\arg z = \theta$ is called a limiting direction of $J(f)$. To see how many such rays there are, the researchers introduced a set called $\Delta(f)$, which contains all the limiting direction of $J(f)$. $\Delta(f)$ has some properties, such as it is closed and measurable, so one can find a lower bound on the measure of $\Delta(f)$. The research on this aspect shows that when f has finite lower order $\mu(f)$, the lower bound of meas$\Delta(f)$ is closely related to $\mu(f)$, see [19, 30]. So what happens if $\mu(f)$ is infinite? Later, Huang et al. [13, 14] studied the limiting directions of $J(f)$ of nontrivial solutions to some complex differential equations, these solutions have infinite order or lower order, more results have followed, such as [21–23, 26]. Below we list a few results that are useful for this article.

In [13] the authors obtained the lower bound of limiting directions of $J(f)$ of nontrivial solutions to linear complex differential equations

$$f^{(n)} + A(z)f = 0. \tag{1.12}$$

Their result showed that the lower bound is related to the order of $A(z)$. In more depth, the authors of paper [21] investigated the common limiting directions of $J(f)$ of nontrivial solutions of high order linear complex differential equation

$$f^{(n)} + A_{n-1}f^{(n-1)} + \cdots + A_0 f = 0, \tag{1.13}$$

where A_0 is the dominated coefficient function comparing to other coefficients functions. The common limiting direction of $J(f)$ which just mentioned is defined as follows:

$$L(f) := \bigcap_{n\in\mathbb{Z}} \Delta(f^{(n)}).$$

In the above definition, when $n \geq 0$, $f^{(n)}$ denotes n times of the ordinary derivative of f, when $n < 0$, $f^{(n)}$ means $|n|$ times integral operation.

Combining the concept of Petrenko's deviation with the results of limiting directions of Julia set of solutions to complex differential equations, the authors of article [27] studied the lower bound of common limiting directions of Julia set of solutions to some complex differential equations with coefficient function related to Petrenko's deviation, the two conclusions are stated below.

Theorem 1.1. *[27] Suppose that f is a nontrivial solution to the complex differential equation* (1.9)*, $A(z)$ is an entire function and has non-transcendental growth angular domain such that $\xi(A) > 0$ and $B(z)$ is a transcendental entire function and has a Petrenko's deviation with $\beta^-(\infty, B) < \frac{1}{1-\xi(A)}$, then the common limiting direction of Julia set of f satisfies*

$$meas(L(f)) \geq \min\left\{2\pi, 2\pi\left(\frac{1}{\beta^-(\infty, B)} + \xi(A) - 1\right)\right\}.$$

Theorem 1.2. *[27] Suppose that the coefficient $A(z)$ of equation* (1.12) *is a transcendental entire function and satisfies* (1.6) *as $r \to \infty$ outside a set G with* $\log dens(G) < 1$*, then the common limiting direction of Julia set of every nontrivial solution f to* (1.12) *satisfies*

$$meas(L(f)) \geq 2\pi\nu.$$

Moreover, let $f_1, f_2, \cdots, f_n$ be a solution base of equation (1.12) *and their product be $E := f_1 \cdot f_2 \cdots f_n$, then the common limiting direction of Julia set of E satisfies*

$$meas(L(E)) \geq 2\pi\nu.$$

While studying the limiting directions of Julia set of solutions to complex differential equations, the limiting direction of Julia set for the solution of difference equation is also concerned, for examples [5, 7, 16, 24]. In order to state the following Theorem 1.3 which is proved in [5], we give some denotations at first. We set

$$\overline{P(z,f)} := P(f(z+c_1), f(z+c_2), \cdots, f(z+c_m)),$$

where $f(z+c_i)$ are some different shift forms of the complex function $f(z)$ with $c_i(i=1,\cdots,m)$ being distinct complex numbers, these shifts are closely related to the difference of f. To be specific, $\overline{P(z,f)}$ is a polynomial in m variables with degree less than d,

$$\overline{P(z,f)} := \sum_{\lambda=(k_1,\cdots,k_m)\in\Lambda} a_\lambda \prod_{i=1}^{m} (f(z+c_i))^{k_i}$$

where a_λ are constant coefficients and are nonzeros, every element of Λ is a multi-indices and has the form $\lambda = (k_1,\cdots,k_m), k_i \in \mathbb{N}$, and the degree of $\overline{P(z,f)}$ is $\max_{\lambda\in\Lambda}\{\sum_{i=1}^{m} k_i\} < d$. In addition, the common limiting directions of Julia sets of distinct shifts of f is defined by

$$\overline{L(f)} := \bigcap_{i=1}^{n} \Delta(f(z+\eta_i)),$$

where n is a finite positive integer, and $\eta_i(i=1,2\cdots,n)$ are distinct complex numbers. Chen et al. [5] obtained the following result about the limiting directions of Julia set of solutions to some complex difference equations.

Theorem 1.3. *[5] Suppose that $\overline{P_i(z,f)}$ $(i=0,1,\cdots,n)$ are distinct polynomials of degree less than d, and the coefficients $A_i(z)$ are entire functions. Assume that A_0 is transcendental with finite lower order, and it's dominant compared to the other coefficients, that is, $T(r,A_j) = o(T(r,A_0))$ as $r\to\infty$, $j=1,2,\cdots,n$. For any nontrivial entire solution f of*

$$A_n(z)\overline{P_n(z,f)} + \cdots + A_1(z)\overline{P_1(z,f)} = A_0(z), \tag{1.14}$$

we have meas$(\overline{L(f)}) \geq \min\{2\pi, \pi/\mu(A_0)\}$. Furthermore, if $\sigma(A_0) > \max\{\sigma(A_1),\cdots,\sigma(A_n)\}$, then $\overline{L(f)}$ contains a closed interval $[v_1,v_2]$ with $v_2 - v_1 \geq \min\{2\pi, \pi/\sigma(A_0)\}$.

Motivated by Theorems 1.2 and 1.3, we study the limiting directions of Julia sets of solutions to complex difference equations with a coefficient related to Petrenko's deviation and obtain the following theorem as our first conclusion of this article.

Theorem 1.4. *Suppose that $\overline{P_j(z,f)}(j=1,2,\cdots,n)$ are distinct polynomials with degree less than d, and A_0 is a transcendental entire function with finite lower order $\mu(A_0)$ and satisfies (1.6) as $r(\notin G)\to\infty$ with $\log dens(G) < 1$. The other coefficients $A_i(z)(i=1,2,\cdots,n)$ are distinct entire functions satisfying*

$$\lambda = \max\{\sigma(A_1),\cdots,\sigma(A_n)\} < \mu(A_0).$$

Then the common limiting direction of Julia set of every nontrivial entire solution f to (1.14) satisfies meas$(\overline{L(f)}) \geq 2\pi\nu$.

In the paper [16], the authors studied a nonlinear differential-difference equation, it belongs to the general form of Tumura-Clunie equation. This equation is shown as below.

$$f^n(z) + A(z)\widetilde{P(z,f)} = h(z), \tag{1.15}$$

where the coefficient functions $A(z), h(z)$ are entire, and $\widetilde{P(z,f)}$ denotes a polynomial in the derivatives of the shifts of f,

$$\widetilde{P(z,f)} := \sum_{j=1}^{s} a_j(z) \prod_{i=0}^{l} (f^{(i)}(z+c_i))^{n_{ij}}, (i, j \in \mathbb{N})$$

where c_i are distinct complex numbers and $a_j(z)$ are polynomials. $\deg\widetilde{P(z,f)} = \max_{1\le j\le s} \sum_{i=0}^{l} n_{ij}$ denotes the degree of $\widetilde{P(z,f)}$, where n_{ij} are positive integers. For more details, see [16, Theorems 1.4 and 1.6]. This inspired us to consider a more general nonlinear complex difference equation than (1.15), combined with the coefficient having Petrenko's deviation. In fact, we obtain the following theorem as our second conclusion of this article.

Theorem 1.5. *Suppose that $A_0(z)$ is a transcendental entire function and has Petrenko's deviation with* (1.6)*. $A_1(z)$ is an entire function and non-transcendental growth in some angulars such that $\xi(A_1) > 0$. Moreover, $\beta^-(\infty, A_0) < \frac{1}{1-\xi(A_1)}$. Then the measure of common limiting direction of Julia set of every nontrivial solution f of equation*

$$\widetilde{P_2(z,f)} + A_1(z)\widetilde{P_1(z,f)} = A_0(z) \tag{1.16}$$

satisfies

$$meas(\widetilde{L(f)}) \ge \min\left\{2\pi, 2\pi\left(\frac{1}{\beta^-(\infty, A_0)} + \xi(A_1) - 1\right)\right\},$$

where $\widetilde{L(f)} := \bigcap_{n\in\mathbb{Z}} \Delta(f^{(n)}(z+\eta))$ and η is any complex number.

The third conclusion of this article is about the dynamic properties of solutions to nonlinear complex differential equations. Let's first recall the definition of a differential polynomial for an entire function f. This differential polynomial $P(z, f)$ consists of adding a finite number of differential monomials together, that is

$$P(z,f) := \sum_{j=1}^{l} a_j(z) f^{n_{0j}} (f')^{n_{1j}} \cdots (f^{(k)})^{n_{kj}}, \tag{1.17}$$

where $a_j(z)$ are coefficient functions and all of them are meromorphic, and $n_{0j}, n_{1j}, \ldots, n_{kj}$ are non-negative integer powers. The minimum degree of $P(z, f)$ is denoted by

$$\gamma_P := \min_{1\le j\le l} \left(\sum_{i=0}^{k} n_{ij} \right).$$

Wang et al. [23] observed the limiting directions of Julia set of solutions to the following Eq (1.18), they found when F grows fast in a certain direction, that direction is the limiting direction of Julia set of solution f.

Theorem 1.6. *[23] Let n, s be integers. Suppose that the coefficient $F(z)$ is a transcendental entire function with finite lower order, $P(z, f)$ is a differential polynomial with respect to f, its minimum degree satisfies $\gamma_P \geq s$ and its coefficients $a_j (j = 1, 2, \cdots, l)$ are entire functions with $\sigma(a_j) < \mu(F)$. Then, for every nontrivial transcendental entire solution f of the differential equation*

$$P(z, f) + F(z) f^s = 0, \tag{1.18}$$

we have $meas(L(f)) \geq \min\{2\pi, \pi/\mu(F)\}$.

We improve the above Theorem 1.6 by introducing Petrenko's deviation for one coefficient of the equation and assuming the existence of angular domain of non-transcendental growth for the other coefficient. We state our result as follow.

Theorem 1.7. *Let $P_1(z, f), P_2(z, f)$ be two differential polynomials with respect to f with $\min\{\gamma_{P_1}, \gamma_{P_2}\} \geq s$, where the coefficients $a_{1j}, a_{2j} (j = 1, 2, \cdots, l)$ are entire functions such that $\max\{\sigma(a_{1j}), \sigma(a_{2j})\} < \mu(A_0)$. Suppose that the coefficient $A_0(z)$ is a transcendental entire function with finite lower order and satisfies (1.6), $A_1(z)$ is an entire function such that $\xi(A_1) > 0$ and $\beta^-(\infty, A_0) < \frac{1}{1-\xi(A_1)}$. s is an integer. Let f be a nontrivial transcendental entire solution of differential equation*

$$P_2(z, f) + A_1(z) P_1(z, f) + A_0(z) f^s = 0. \tag{1.19}$$

Then the measure of common limiting direction of Julia set of f satisfying

$$meas(L(f)) \geq \min\left\{2\pi, 2\pi\left(\frac{1}{\beta^-(\infty, A_0)} + \xi(A_1) - 1\right)\right\}.$$

2. Preliminary lemmas

The following lemma is crucial, which describes that when f maps an angular domain to a hyperbolic domain with some special property, f grows by no more than a polynomial in a slightly smaller angular domain. In order to better express this lemma, let us first give some explanations and notations. $\widehat{\mathbb{C}}$ denotes the extended complex plane and U is a set contained in $\widehat{\mathbb{C}}$. If $\widehat{\mathbb{C}} \backslash U$ contains at least three points, then we say U is a hyperbolic domain. For any $a \in \mathbb{C} \backslash U$, we define the quantity $C_U(a) := \inf\{\lambda_U(z)|z - a| : \forall z \in U\}$, where $\lambda_U(z)$ is the hyperbolic density on U. In particular, $C_U(a) \geq 1/2$ when every component of U is simply connected, see [30].

Lemma 2.1. *([30, Lemma 2.2]) Suppose that $f(z)$ is an analytic mapping and $f : \Omega(r_0, \theta_1, \theta_2) \to U$, where $\Omega(r_0, \theta_1, \theta_2) := \{re^{i\theta} : \theta \in (\theta_1, \theta_2), r \geq r_0\}$, and U is a hyperbolic domain. If there exists a point $a \in \partial U \backslash \{\infty\}$ such that $C_U(a) > 0$, then there exists a constant $d > 0$ such that, for sufficiently small $\varepsilon > 0$, we have*

$$|f(z)| = O(|z|^d), \ z \in \Omega(r_0, \theta_1 + \varepsilon, \theta_2 - \varepsilon), \ |z| \to \infty.$$

Because in the process of proving our theorems, we need to use the Nevanlinna theory in the angular domain, so here we list the notations related to it, which are given in accordance with the literatures [9, 29]. Suppose that in the closed angular domain $\overline{\Omega}(\alpha, \beta)$, $g(z)$ is meromorphic. Set

$w = \pi/(\beta - \alpha)$, and $b_n = |b_n|e^{i\beta_n}$ are poles of $g(z)$ on $\overline{\Omega}(\alpha, \beta)$ considering their multiplicities. The three main basic notations are

$$\begin{aligned} A_{\alpha,\beta}(r, g) &= \frac{w}{\pi}\int_1^r \left(\frac{1}{t^w} - \frac{t^w}{r^{2w}}\right)\{\log^+|g(te^{i\alpha})| + \log^+|g(te^{i\beta})|\}\frac{dt}{t}; \\ B_{\alpha,\beta}(r, g) &= \frac{2w}{\pi r^w}\int_\alpha^\beta \log^+|g(re^{i\theta})|\sin w(\theta - \alpha)d\theta; \\ C_{\alpha,\beta}(r, g) &= 2\sum_{1<|b_n|<r}\left(\frac{1}{|b_n|^w} - \frac{|b_n|^w}{r^{2w}}\right)\sin w(\beta_n - \alpha). \end{aligned}$$

The sum of these three basic notations is the characteristic function in $\Omega(\alpha, \beta)$, that is,

$$S_{\alpha,\beta}(r, g) = A_{\alpha,\beta}(r, g) + B_{\alpha,\beta}(r, g) + C_{\alpha,\beta}(r, g).$$

Moreover, its growth is defined as

$$\sigma_{\alpha,\beta}(g) := \limsup_{r\to\infty}\frac{\log S_{\alpha,\beta}(r, g)}{\log r}. \tag{2.1}$$

The following result is a lemma for the estimate of the first two basic notations above of the logarithmic derivative of a meromorphic function f in an angular domain, which is related to the logarithm of the characteristic function of f in a larger angular domain.

Lemma 2.2. *([29, Theorem 2.5.1]) Let $f(z)$ be a meromorphic function on $\Omega(\alpha - \varepsilon, \beta + \varepsilon)$ for $\varepsilon > 0$ and $0 < \alpha < \beta \le 2\pi$. Then*

$$A_{\alpha,\beta}\left(r, \frac{f'}{f}\right) + B_{\alpha,\beta}\left(r, \frac{f'}{f}\right) \le K(\log^+ S_{\alpha-\varepsilon,\beta+\varepsilon}(r, f) + \log r + 1)$$

for $r > 1$ possibly except a set with finite linear measure.

Further, the estimate of the modulus of the logarithmic derivative of the analytic function in the angular domain is obtained from the following lemma. Let's start with a definition of R set, see [15]. $B(z_n, r_n)$ are a list of disks, that is, $B(z_n, r_n) = \{z : |z - z_n| < r_n\}$. If $\sum_{n=1}^{\infty} r_n < \infty$ and $z_n \to \infty$, then $\cup_{n=1}^{\infty} B(z_n, r_n)$ is called an R-set. Obviously, the linear measure of set $\{|z| : z \in \cup_{n=1}^{\infty} B(z_n, r_n)\}$ is finite.

Lemma 2.3. *([13, Lemma 7]) Let $g(z)$ be analytic in $\Omega(r_0, \alpha, \beta)$ with $\sigma_{\alpha,\beta}(g) < \infty$ for some $r_0 > 0$, and $z = re^{i\psi}, r > r_0 + 1$ and $\alpha \le \psi \le \beta$. Suppose that $n(\ge 2)$ is an integer, and that choose $\alpha < \alpha_1 < \beta_1 < \beta$. Then, for every $\varepsilon_j \in (0, (\beta_j - \alpha_j)/2)(j = 1, 2, \dots, n-1)$ outside a set of linear measure zero with*

$$\alpha_j = \alpha + \sum_{s=1}^{j-1}\varepsilon_s, \quad \beta_j = \beta - \sum_{s=1}^{j-1}\varepsilon_s, \quad j = 2, 3, \dots, n-1.$$

There exist $K > 0$ and $M > 0$ only depending on g, $\varepsilon_1, \dots, \varepsilon_{n-1}$ and $\Omega(\alpha_{n-1}, \beta_{n-1})$, not depending on z, such that

$$\left|\frac{g'(z)}{g(z)}\right| \le Kr^M(\sin k(\psi - \alpha))^{-2}$$

and

$$\left|\frac{g^{(n)}(z)}{g(z)}\right| \le Kr^M\left(\sin k(\psi - \alpha)\prod_{j=1}^{n-1}\sin k_{\varepsilon_j}(\psi - \alpha_j)\right)^{-2}$$

for all $z \in \Omega(\alpha_{n-1}, \beta_{n-1})$ outside an R-set, where $k = \pi/(\beta - \alpha)$ and $k_{\varepsilon_j} = \pi/(\beta_j - \alpha_j)(j = 1, 2, \dots, n-1)$.

3. Proof of Theorem 1.4

By the condition $\lambda = \max\{\sigma(A_1), \cdots, \sigma(A_n)\} < \mu := \mu(A_0)$, we define a constant $\kappa := \frac{\lambda+\mu}{2}$, a set

$$D := \{z \in \mathbb{C} : |A_0(z)| > e^{r^{\kappa}}, |z| = r\}$$

and a set of directions of rays

$$H(r) := \{\theta \in [0, 2\pi) : z = re^{i\theta} \in D\}.$$

Noting that A_0 is an entire function and recalling the definition of Nevanlinna characteristic function in the complex plane, for some $r_1 > 0$, if $r > r_1$, we obtain

$$\begin{aligned} 2\pi T(r, A_0) &= \int_{H(r)} \log^+ |A_0(re^{i\theta})| d\theta + \int_{[0,2\pi)\setminus H(r)} \log^+ |A_0(re^{i\theta})| d\theta \\ &\leq meas(H(r)) \log M(r, A_0) + r^{\kappa}(2\pi - meas(H(r))). \end{aligned} \tag{3.1}$$

Consequently,

$$2\pi \leq meas(H(r)) \frac{\log M(r, A_0)}{T(r, A_0)} + \frac{r^{\kappa}}{T(r, A_0)}(2\pi - meas(H(r))). \tag{3.2}$$

Recall that the definition of lower order, we have

$$\mu(A_0) = \liminf_{r\to\infty} \frac{\log^+ T(r, A_0)}{\log r},$$

and since A_0 has Petrenko's deviation with (1.6) outside G, we deduce that, for $r \notin G$,

$$\liminf_{r\to\infty} meas(H(r)) \geq 2\pi\nu. \tag{3.3}$$

Then there exists a sequence $\{r_n\}(\subset (r_1, +\infty) \setminus G)$ with $\{r_n\} \to \infty$ as $n \to \infty$ such that

$$\liminf_{n\to\infty} meas(H(r_n)) \geq 2\pi\nu. \tag{3.4}$$

We define

$$B_n := \bigcup_{j=n}^{\infty} H(r_j),$$

where $n = 1, 2, \cdots$. Obviously, the set B_n is measurable with $meas(B_n) \leq 2\pi$, and monotone decreasing as $n \to \infty$. Their intersection is

$$\widetilde{H} := \bigcap_{n=1}^{\infty} B_n,$$

so $\widetilde{H}$ is independent of the value r. In view of (3.4) and the monotone convergence theorem [20, Theorem 1.19], it's easy to get

$$meas(\widetilde{H}) = \lim_{n\to\infty} meas(B_n) = \lim_{n\to\infty} meas(\bigcup_{n=j}^{\infty} H(r_j)) \geq 2\pi\nu. \tag{3.5}$$

Let's use proof by contradiction to prove our conclusion and suppose that $meas(\overline{L(f)}) < 2\pi\nu$. Compared with (3.5), there exists an interval $(\alpha,\beta) \subset [0, 2\pi)$ such that

$$(\alpha,\beta) \subset \widetilde{H},\ (\alpha,\beta) \cap \overline{L(f)} = \emptyset.$$

Therefor, for any $\theta \in (\alpha,\beta)$, the ray $\{z : \arg z = \theta\}$ is not a limiting direction of Julia set of $f(z+\eta_\theta)$ for some complex number $\eta_\theta \in \{\eta_i : i = 1, 2, \cdots, n\}$ depending on θ. Immediately, by the definition of limiting direction of Julia set, for a constant ξ_θ depending on θ, we have an angular domain $\Omega(\theta - \xi_\theta, \theta + \xi_\theta)$ such that

$$(\theta - \xi_\theta, \theta + \xi_\theta) \subset (\alpha,\beta),\ \ \Omega(r, \theta - \xi_\theta, \theta + \xi_\theta) \cap J(f(z + \eta_\theta) = \emptyset \tag{3.6}$$

as $r(\notin G)$ sufficiently large. Then, for a larger r_θ, the angular domain $\Omega(r_\theta, \theta-\xi_\theta, \theta+\xi_\theta)$ is contained in an unbounded Fatou component, say U_θ, of $F(f(z+\eta_\theta))$ (see [2]). Suppose that $\Gamma \subset \partial U_\theta$ is unbounded and connected, then the mapping

$$f(z + \eta_\theta) : \Omega(r_\theta, \theta - \xi_\theta, \theta + \xi_\theta) \to \mathbb{C} \setminus \Gamma$$

is analytic. Obviously, $\mathbb{C} \setminus \Gamma$ is a simply connected hyperbolic domain, then by the explanation before Lemma 2.1, we have $C_{\mathbb{C}\setminus\Gamma} \geq 1/2$ for any $a \in \Gamma \setminus \{\infty\}$. From Lemma 2.1, the mapping $f(z + \eta_\theta)$ in $\Omega(r_\theta, \theta - \xi_\theta, \theta + \xi_\theta)$ satisfies

$$|f(z + \eta_\theta)| = O(|z|^{d_1}) \tag{3.7}$$

for $z \in \Omega(r_\theta, \theta - \xi_\theta + \varepsilon, \theta + \xi_\theta - \varepsilon)$, where ε is a small positive number and d_1 is a positive number. For the sake of simplicity, set $\overline{\alpha} := \theta - \xi_\theta + \varepsilon$ and $\overline{\beta} := \theta + \xi_\theta - \varepsilon$.

Choosing suitable $r^*_\theta(> r_\theta)$, for $z \in \Omega(r^*_\theta, \overline{\alpha}+\varepsilon, \overline{\beta}-\varepsilon)$ we have $z+c_j-\eta_\theta \in \Omega(r_\theta, \overline{\alpha}, \overline{\beta})$ $(j = 1, 2, \cdots, m)$. By (3.7), we obtain

$$|f(z + c_j)| = |f((z + c_j - \eta_\theta) + \eta_\theta)| = O(|z + c_j - \eta_\theta|^{d_1}) = O(|z|^{M_0})$$

for $z \in \Omega(r^*_\theta, \overline{\alpha} + \varepsilon, \overline{\beta} - \varepsilon)$ as $|z| \to \infty$, where M_0 is a positive constant. Repeating the above argument some times, we can obtain

$$|\overline{P_i(z, f)}| \leq K|z|^M, (i = 1, 2, \cdots, n) \tag{3.8}$$

for $z \in \Omega(r^*_\theta, \overline{\alpha} + \varepsilon, \overline{\beta} - \varepsilon)$ as $|z| \to \infty$, where K and M are positive constants. From (1.14), we have

$$e^{r_n^\kappa} < |A_0(r_n e^{i\theta})| \leq \sum_{i=1}^{n} |A_i(r_n e^{i\theta})||\overline{P_i(r_n e^{i\theta}, f)}| \leq K e^{r_n^\lambda} r_n^M, \tag{3.9}$$

where $z_n = r_n e^{i\theta} \in \Omega(r^*_\theta, \alpha^* + \varepsilon, \beta^* - \varepsilon)$ as $r_n(\notin G) \to \infty$, K and M are two large enough constants. This is impossible since $\kappa > \lambda$. Hence, we obtain $meas(\overline{L(f)}) \geq 2\pi\nu$.

4. Proof of Theorem 1.5

By the assumption and the definition of Petrenko's deviation, for $\xi(A_1) > 0$, we have

$$1 \le \beta^{-}(\infty, A_0) < \frac{1}{1-\xi(A_1)}.$$

We choose two constants ε and d satisfying

$$0 < \varepsilon < \frac{1}{\beta^{-}(\infty, A_0)} - (1 - \xi(A_1))$$

and

$$1 > d > \frac{2}{2+\varepsilon}.$$

It's easy to get

$$\frac{2(1-d)}{d} < \varepsilon < \frac{1}{\beta^{-}(\infty, A_0)} - (1 - \xi(A_1)). \tag{4.1}$$

Set

$$I_d(r) := \{\theta \in [0, 2\pi) : \log |A_0(re^{i\theta})| \ge (1-d)\log M(r, A_0)\}. \tag{4.2}$$

Since A_0 is entire function, we have

$$\begin{aligned} 2\pi T(r, A_0) &= \int_{I_d(r)} \log^{+} |A_0(re^{i\theta})| d\theta + \int_{[0,2\pi)\setminus I_d(r)} \log^{+} |A_0(re^{i\theta})| d\theta \\ &\le meas(I_d(r)) \log M(r, A_0) + (2\pi - meas(I_d(r)))(1-d)\log M(r, A_0). \end{aligned} \tag{4.3}$$

In view the definition of Petrenko's deviation, see (1.2), the formula (4.3) can convert into

$$\limsup_{r\to\infty} meas(I_d(r)) \ge 2\pi\left(\frac{1}{d\beta^{-}(\infty, A_0)} - \frac{1-d}{d}\right). \tag{4.4}$$

Noting that ε and d satisfy (4.1), we can choose an infinite sequence $\{r_n\}$ such that

$$\begin{aligned} meas(I_d(r_n)) &\ge \frac{2\pi}{d\beta^{-}(\infty, A_0)} - \frac{2\pi(1-d)}{d} - \pi\varepsilon \\ &\ge \frac{2\pi}{d\beta^{-}(\infty, A_0)} - 2\pi\varepsilon \\ &\ge \frac{2\pi}{d\beta^{-}(\infty, A_0)} - 2\pi\left(\frac{1}{\beta^{-}(\infty, A_0)} - (1-\xi(A_1))\right) \\ &> 2\pi(1-\xi(A_1)). \end{aligned} \tag{4.5}$$

Then we get a lower bound of the measure of $I_d(r_n)$. Set $D_n := \bigcup_{n=j}^{\infty} I_d(r_j)$ and $\widetilde{I_d} := \bigcap_{n=1}^{\infty} D_n$, so $\widetilde{I_d}$ is independent of r. By the same arguments in section 3, we deduce that

$$meas(\widetilde{I_d}) = \lim_{n\to\infty} meas(D_n) = \lim_{n\to\infty} meas(\bigcup_{n=j}^{\infty} I_d(r_j)) > 2\pi(1-\xi(A_1)).$$

Thus, we estimate the difference between $meas(\widetilde{I_d})$ and $2\pi(1-\xi(A_1))$ as follows.

$$\begin{aligned} meas(\widetilde{I_d})-2\pi(1-\xi(A_1)) &> \frac{2\pi}{d\beta^-(\infty,A_0)}-\frac{2\pi(1-d)}{d}-\pi\varepsilon-2\pi(1-\xi(A_1)) \\ &= 2\pi\left[\xi(A_1)-\frac{1}{d}\left(1-\frac{1}{\beta^-(\infty,A_0)}\right)-\frac{\varepsilon}{2}\right] \\ &> 2\pi\left[\xi(A_1)-\frac{2+\varepsilon}{2}\left(1-\frac{1}{\beta^-(\infty,A_0)}\right)-\frac{\varepsilon}{2}\right] \\ &= 2\pi\left(\frac{1}{\beta^-(\infty,A_0)}+\xi(A_1)-1\right)-\pi\varepsilon\left(2-\frac{1}{\beta^-(\infty,A_0)}\right). \end{aligned} \tag{4.6}$$

By the arbitrariness of ε and the condition $\beta^-(\infty,A_0)<\frac{1}{1-\xi(A_1)}$, we have

$$meas(\widetilde{I_d})-2\pi(1-\xi(A_1))\geq 2\pi\left(\frac{1}{\beta^-(\infty,A_0)}+\xi(A_1)-1\right)>0. \tag{4.7}$$

Now let's prove the conclusion of the theorem by contradiction, on the contrary we assume

$$meas(\widetilde{L(f)})<2\pi\left(\frac{1}{\beta^-(\infty,A_0)}+\xi(A_1)-1\right). \tag{4.8}$$

Noting the definition of $\Xi(A_1)$, see (1.10), and (4.7), (4.8), we can find an interval $(\alpha,\beta)\subset(0,2\pi]$ satisfying

$$(\alpha,\beta)\subset\widetilde{I_d}\cap\Xi(A_1),\ \ (\alpha,\beta)\cap\widetilde{L(f)}=\emptyset. \tag{4.9}$$

Therefore, for any $\theta\in(\alpha,\beta)$, the corresponding ray is not a limiting direction of Julia set of $f^{(n_\theta)}(z+\eta)$, where n_θ is an integer and depends on θ. By the definition of limiting direction of Julia set, we can take an angular domain $\Omega(\theta-\xi_\theta,\theta+\xi_\theta)$ with $(\theta-\xi_\theta,\theta+\xi_\theta)\subset(\alpha,\beta)$ satisfying

$$\Omega(r,\theta-\xi_\theta,\theta+\xi_\theta)\cap J(f^{(n_\theta)}(z+\eta))=\emptyset \tag{4.10}$$

when $r(\notin G)$ is large enough, ξ_θ is small enough, and η is a complex constant.

For a corresponding r_θ, we have $\Omega(r_\theta,\theta-\xi_\theta,\theta+\xi_\theta)\subset U_\theta$, where U_θ is an unbounded Fatou component of $F(f^{(n_\theta)}(z+\eta))$ (see [2]). Choose $\Gamma\subset\partial U_\theta$ such that Γ is unbounded and connected set, then $\mathbb{C}\setminus\Gamma$ is a simply connected hyperbolic domain. Applying Lemma 2.1 to the analytic mapping

$$f^{(n_\theta)}(z+\eta):\Omega(r_\theta,\theta-\xi_\theta,\theta+\xi_\theta)\to\mathbb{C}\setminus\Gamma,$$

we deduce that

$$|f^{(n_\theta)}(z+\eta)|=O(|z|^{d_1}) \tag{4.11}$$

for $r_\theta\to\infty$ and $z\in\Omega(r_\theta,\theta-\xi_\theta+\varepsilon,\theta+\xi_\theta-\varepsilon)$, where ε is a sufficiently small positive constant and d_1 is a positive constant. Moreover, we can take a smaller angular domain $\Omega(r'_\theta,\theta-\xi_\theta+2\varepsilon,\theta+\xi_\theta-2\varepsilon)$ where $r'_\theta>r_\theta$, such that

$$z+c_i-\eta\in\Omega(r_\theta,\theta-\xi_\theta+\varepsilon,\theta+\xi_\theta-\varepsilon)$$

for all $z \in \Omega(r'_\theta, \theta - \xi_\theta + 2\varepsilon, \theta + \xi_\theta - 2\varepsilon)$. For simplicity, denote $\overline{\alpha} := \theta - \xi_\theta + 2\varepsilon$ and $\overline{\beta} := \theta + \xi_\theta - 2\varepsilon$. Therefore, it yields that

$$|f^{(n_\theta)}(z + c_i)| = O(|z|^{d_1}) \tag{4.12}$$

for all $z \in \Omega(r'_\theta, \overline{\alpha}, \overline{\beta})$ as $|z| = r \to \infty$, where $c_i (i = 0, 1, \cdots, l)$ are complex numbers. Now, there are two cases based on the sign of n_θ.

If $n_\theta > 0$, by integral we get

$$f^{(n_\theta - 1)}(z + c_i) = \int_0^z f^{(n_\theta)}(\zeta + c_i) d\zeta + c, \tag{4.13}$$

this integral is path independent since $f(z)$ is entire. By (4.12) and (4.13), we obtain

$$|f^{(n_\theta - 1)}(z + c_i)| = O(|z|^{d_1 + 1})$$

for $z \in \Omega(r'_\theta, \overline{\alpha}, \overline{\beta})$. Repeating the above integral we get

$$|f(z + c_i)| = O(|z|^{d_1 + n_\theta}) \tag{4.14}$$

for $z \in \Omega(r'_\theta, \overline{\alpha}, \overline{\beta})$.

If $n_\theta < 0$. From (4.12), we have

$$S_{\overline{\alpha}, \overline{\beta}}(r, f^{(n_\theta)}(z + c_i)) = O(1) \tag{4.15}$$

and $\sigma_{\overline{\alpha}, \overline{\beta}}(r, f^{(n_\theta)}(z + c_i)) = 0$. Noting the fact $n_\theta < 0$ and applying Lemma 2.3 to $f^{(n_\theta)}(z + c_i)$, we obtain

$$\left| \frac{f(z + c_i)}{f^{(n_\theta)}(z + c_i)} \right| \le K r^M \tag{4.16}$$

when $z \in \Omega(r'_\theta, \overline{\alpha} + \varepsilon', \overline{\beta} - \varepsilon')$ outside an R set, where K and M are two positive constants and ε' is a small positive constant. Then by (4.12) and (4.16), it's easy to obtain

$$|f(z + c_i)| = O(|z|^{d_1 + M}) \tag{4.17}$$

for all $z \in \Omega(r'_\theta, \overline{\alpha} + \varepsilon', \overline{\beta} - \varepsilon')$ outside an R-set.

By (4.14) and (4.17), for any $n_\theta \in \mathbb{Z}$ in (4.12), we always can get

$$S_{\alpha^*, \beta^*}(r, f(z + c_i)) = O(1)$$

in the corresponding angular domain. When $n_\theta \ge 0$, $\alpha^* = \overline{\alpha}, \beta^* = \overline{\beta}$; and when $n_\theta < 0$, $\alpha^* = \overline{\alpha} + \varepsilon', \beta^* = \overline{\beta} - \varepsilon'$. This yields that $\sigma_{\alpha^*, \beta^*}(r, f(z + c_i)) = 0$, then by Lemma 2.3, for every $i = 1, 2, \cdots, l$ and all $z \in \Omega(r'_\theta, \alpha^* + \varepsilon, \beta^* - \varepsilon)$ outside an R-set, we can take two constants $K_i > 0$ and $M_i > 0$ such that

$$\left| \frac{f^{(i)}(z + c_i)}{f(z + c_i)} \right| \le K_i r^{M_i}. \tag{4.18}$$

From (4.17) and (4.18), it's easy to get that, for all $z \in \Omega(\alpha^* + \varepsilon, \beta^* - \varepsilon)$ outside an R-set, the modulus of $(f^{(i)}(z + c_i))^{n_{ij}}$ satisfies

$$|(f^{(i)}(z + c_i))^{n_{ij}}| = \left| \left(\frac{f^{(i)}(z + c_i)}{f(z + c_i)} \right)^{n_{ij}} f^{n_{ij}}(z + c_i) \right| = O(|z|^M), \tag{4.19}$$

where M is an appropriate constant. Recall the representations of $\widetilde{P_i(z, f)}$ we can obtain that

$$|\widetilde{P_i(z, f)}| \leq \sum_{j=1}^{s} |a_j(z)| \prod_{i=0}^{l} \left| \left(\frac{f^{(i)}(z + c_i)}{f(z + c_i)} \right)^{n_{ij}} f^{n_{ij}}(z + c_i) \right| = O(|z|^M), (i = 1, 2) \tag{4.20}$$

for all $z \in \Omega(\alpha^* + \varepsilon, \beta^* - \varepsilon)$ outside an R-set, and an appropriate constant M. From (1.16), we deduce that, for a suitable sequence $\{r_n\}$ with $z_n = r_n e^{i\theta} \in \Omega(r'_\theta, \alpha^* + \varepsilon, \beta^* - \varepsilon)$,

$$[M(r_n, A_0)]^{1-d} < |A_0(z_n)| \leq |\widetilde{P_2(z_n, f)}| + |A_1(z)||\widetilde{P_1(z_n, f)}| = O(r_n^M) \tag{4.21}$$

as $r_n \to \infty$. This inequality contradicts the transcendence of A_0. So our conclusion is valid.

5. Proof of Theorem 1.7

Since the entire coefficients $A_0(z), A_1(z)$ of differential equation (1.19) have the same properties as in Theorem 1.5, then the fact in (4.9) also holds. Then, for some integer n_θ which depends on θ, every $\theta \in (\alpha, \beta)$ is not a limiting direction of Julia set of $f^{(n_\theta)}$. Therefore, we can choose a constant ξ_θ which depends on θ, so the interval $(\theta - \xi_\theta, \theta + \xi_\theta) \subset (\alpha, \beta)$, and the corresponding angular domain $\Omega(\theta - \xi_\theta, \theta + \xi_\theta)$ satisfies

$$\Omega(r, \theta - \xi_\theta, \theta + \xi_\theta) \cap J(f^{(n_\theta)}) = \emptyset \tag{5.1}$$

for sufficiently large $r(\notin G)$. Therefore, for a large enough $r_\theta > r$, we have $\Omega(r_\theta, \theta - \xi_\theta, \theta + \xi_\theta) \subset U_\theta$, where U_θ is an unbounded Fatou component of $F(f^{(n_\theta)})$ (see [2]). Taking an unbounded and connected curve $\Gamma \subset \partial U_\theta$ and noting that $\mathbb{C} \setminus \Gamma$ is simply connected hyperbolic domain, then the mapping

$$f^{(n_\theta)} : \Omega(r_\theta, \theta - \xi_\theta, \theta + \xi_\theta) \to \mathbb{C} \setminus \Gamma$$

meets the conditions of Lemma 2.1. Consequently, we have

$$|f^{(n_\theta)}(z)| = O(|z|^{d_1}) \tag{5.2}$$

for all $z \in \Omega(r_\theta, \theta - \xi_\theta + \varepsilon, \theta + \xi_\theta - \varepsilon)$, where ε is a sufficiently small positive constant and d_1 is a positive constant. Denote $\overline{\alpha} := \theta - \xi_\theta + \varepsilon$ and $\overline{\beta} := \theta + \xi_\theta - \varepsilon$ for simplicity.

If $n_\theta > 0$, similar as in section 4, we integrate $f^{(n_\theta)}$ for n_θ times and note the modulus in (5.2), then

$$|f(z)| = O(|z|^{d_1 + n_\theta}) \tag{5.3}$$

for $z \in \Omega(r_\theta, \overline{\alpha}, \overline{\beta})$. Thus, the Nevanlinna angular characteristic of f is

$$S_{\overline{\alpha}, \overline{\beta}}(r, f) = O(1). \tag{5.4}$$

If $n_\theta < 0$, from [29, p.49] and Lemma 2.2, we get

$$\begin{aligned} & S_{\overline{\alpha}+\varepsilon',\overline{\beta}-\varepsilon'}(r, f^{(n_\theta+1)}) \\ \leq\ & S_{\overline{\alpha}+\varepsilon',\overline{\beta}-\varepsilon'}\left(r, \frac{f^{(n_\theta+1)}}{f^{(n_\theta)}}\right) + S_{\overline{\alpha}+\varepsilon',\overline{\beta}-\varepsilon'}(r, f^{(n_\theta)}) \\ \leq\ & O(\log^+ S_{\overline{\alpha},\overline{\beta}}(r, f^{(n_\theta)}) + \log r) + S_{\overline{\alpha}+\varepsilon',\overline{\beta}-\varepsilon'}(r, f^{(n_\theta)}) \end{aligned} \tag{5.5}$$

for $\varepsilon' = \frac{\varepsilon}{|n_\theta|}$. By (5.2) we get $S_{\overline{\alpha},\overline{\beta}}(r, f^{(n_\theta)}) = O(1)$. Combining these together, we deduce

$$S_{\overline{\alpha}+\varepsilon',\overline{\beta}-\varepsilon'}(r, f^{(n_\theta+1)}) = O(\log r).$$

Repeating the above argument $|n_\theta|$ times, we obtain

$$S_{\overline{\alpha}+\varepsilon,\overline{\beta}-\varepsilon}(r, f) = O(\log r). \tag{5.6}$$

From (5.4), (5.6) and combining the two cases above, we deduce that $S_{\alpha^*,\beta^*}(r, f) = O(\log r)$, where

$$\begin{cases} \alpha^* = \overline{\alpha}, \beta^* = \overline{\beta}, \text{for } n_\theta \geq 0; \\ \alpha^* = \overline{\alpha} + \varepsilon, \beta^* = \overline{\beta} - \varepsilon, \text{for } n_\theta < 0. \end{cases} \tag{5.7}$$

By the definition of order in angular domain, see (2.1), we have $\sigma_{\alpha^*,\beta^*}(r, f) = 0$. Applying Lemma 2.3 to f in the angular domain $\Omega(\alpha^*,\beta^*)$, we can find two constants $K > 0$ and $M > 0$ such that

$$\left|\frac{f^{(n)}(z)}{f(z)}\right| \leq K r^M \tag{5.8}$$

for $n \in \mathbb{N}$ and all $z \in \Omega(\alpha^* + \varepsilon, \beta^* - \varepsilon)$ outside an R-set. We rewrite (1.19) as

$$\begin{aligned} -A_0(z) &= \frac{P_2(z, f)}{f^s} + A_1(z)\frac{P_1(z, f)}{f^s} \\ &= \sum_{j=1}^{l_2} a_{2j}(z)\left(\frac{f'}{f}\right)^{n_{21j}} \cdots \left(\frac{f^{(k)}}{f}\right)^{n_{2kj}} f^{n_{20j}+n_{21j}+\cdots n_{2kj}-s} \\ &\quad + A_1(z)\sum_{j=1}^{l_1} a_{1j}(z)\left(\frac{f'}{f}\right)^{n_{11j}} \cdots \left(\frac{f^{(k)}}{f}\right)^{n_{1kj}} f^{n_{10j}+n_{11j}+\cdots n_{1kj}-s}. \end{aligned} \tag{5.9}$$

From (1.10), (4.2), (5.3), (5.8) and (5.9), we deduce that, for a suitable sequence $z_n = r_n e^{i\theta} \in \Omega(r_\theta, \alpha^* + \varepsilon, \beta^* - \varepsilon)$,

$$[M(r_n, A_0)]^{1-d} < |A_0(z_n)| \leq O(r_n^M)\left(\sum_{j=1}^{l_1} |a_{1j}(z_n)| + \sum_{j=1}^{l_2} |a_{2j}(z_n)|\right) \tag{5.10}$$

as $r_n \to \infty$. Noting the assumption that $\max\{\sigma(a_{1j}), \sigma(a_{2j})\} < \mu(A_0)$ and A_0 is transcendental, the above inequality is not valid.

6. Conclusions

By using the Nevanlinna theory in angular domain, three theorems (Theorems 1.4, 1.5 and 1.7) about the lower bounds on the measure of sets consisting of Julia limiting directions of solutions to three corresponding complex equations were proved. The three equations include the differential or difference of entire functions. The feature of this paper is that the coefficients of these equations associated with Petrenko's deviation. The results of this paper show that the lower bounds mentioned above have close relation with Petrenko's deviation. Meanwhile, the results also extend some conclusions in the related literatures referenced by this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

This work was supported by Natural Science Foundation of Henan (No. 232300420129) and the Key Scientific Research Project of Colleges and Universities in Henan Province (No. 22A110004), China. The author would like to thank the referee for a careful reading of the manuscript and valuable comments.

Conflict of interest

The author declares no conflicts of interest.

References

1. I. N. Baker, Sets of non-normality in iteration theory, *J. Lond. Math. Soc.*, **40** (1965), 499–502.

2. I. N. Baker, The domains of normality of an entire function, *Ann. Acad. Sci. Fenn. Ser. AI Math.*, **1** (1975), 277–283.

3. P. D. Barry, Some theorems related to the $\cos\pi\rho$ theorem, *Proc. London Math. Soc.*, **21** (1970), 334–360.

4. W. Bergweiler, Iteration of meromorphic functions, *Bull. Amer. Math. Soc.*, **29** (1993), 151–188.

5. J. C. Chen, Y. Z. Li, C. F. Wu, Radial distribution of Julia sets of entire solutions to complex difference equations, *Mediterr. J. Math.*, **17** (2020), 184. https://doi.org/10.1007/s00009-020-01627-y

6. J. Clunie, On integral functions having prescribed asymptotic growth, *Canadian J. Math.*, **17** (1965), 396–404. https://doi.org/10.4153/CJM-1965-040-8

7. H. X. Dai, Q. Y. Qiao, T. B. Cao, On limiting directions of entire solutions of complex differential-difference equations, *Anal. Math.*, 2023. https://doi.org/10.1007/s10476-023-0213-7

8. W. H. J. Fuchs, Proof of a conjecture of G. Pólya concerning gap series, *Illinois J. Math.*, **7** (1963), 661–667.

9. A. A. Gol'dberg, I. V. Ostrovskii, *Value Distribution of Meromorphic Function*, Washington: American Mathematical Society, 2008.

10. W. Hayman, *Meromorphic Functions*, Oxford: Clarendon Press, 1964.

11. W. K. Hayman, J. Rossi, Characteristic, maximum modulus and value distribution, *Trans. Amer. Math. Soc.*, **284** (1984), 651–664.

12. J. Heittokangas, M. A. Zemirni, On Petrenko's deviations and second order differential equations, *Kodai Math. J.*, **44** (2021), 181–193. https://doi.org/10.2996/kmj44111

13. Z. Huang, J. Wang, On the radial distribution of Julia sets of entire solutions of $f^{(n)} + A(z)f = 0$, *J. Math. Anal. Appl.*, **387** (2012), 1106–1113. https://doi.org/10.1016/j.jmaa.2011.10.016

14. Z. Huang, J. Wang, On limit directions of Julia sets of entire solutions of linear differential equations, *J. Math. Anal. Appl.*, **409** (2014), 478–484. https://doi.org/10.1016/j.jmaa.2013.07.026

15. I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Berlin: Walter de Gruyter, 1993.

16. Y. Z. Li, H. Q. Sun, A note on the Julia sets of entire solutions to delay differential equations, *Acta Math. Sci.*, **43** (2023), 143–155. https://doi.org/10.1007/s10473-023-0109-4

17. J. R. Long, J. Heittokangas, Z. Ye, On the relationship between the lower order of coefficients and the growth of solutions of differential equations, *J. Math. Anal. Appl.*, **444** (2016), 153–166. https://doi.org/10.1016/j.jmaa.2016.06.030

18. V. P. Petrenko, Growth of meromorphic function of finite order, *Math. USSR Izv.*, **33** (1969), 414–454. https://doi.org/10.1070/IM1969v003n02ABEH000786

19. J. Qiao, On limiting directions of Julia set, *Ann. Acad. Sci. Fenn. Math.*, **26** (2001), 391–399.

20. W. Rudin, *Real and Complex Analysis*, New York: McGraw-Hill, 1987.

21. J. Wang, Z. Chen, Limiting directions of Julia sets of entire solutions to complex differential equations, *Acta Math. Sci.*, **37** (2017), 97–107. https://doi.org/10.1016/S0252-9602(16)30118-7

22. J. Wang, X. Yao, On Julia limiting directions of meromorphic functions, *Israel J. Math.*, **238** (2020), 405–430. https://doi.org/10.1007/s11856-020-2037-5

23. J. Wang, X. Yao, C. Zhang, Julia limiting directions of entire solutions of complex differential equations, *Acta Math. Sci.*, **41** (2021), 1275–1286. https://doi.org/10.1007/s10473-021-0415-7

24. Z. Wang, Z. G. Huang, Limiting directions of Julia sets of entire solutions of complex difference equations, *Filomat*, **36** (2022), 3745–3754.

25. C. C. Yang, H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Beijing: Science Press, 2003.

26. G. W. Zhang, J. Ding, L. Z. Yang, Radial dsitribution of Julia sets of derivetives of solutions to complex linear differential equations, in Chinese, *Sci. Sin. Math.*, **44** (2014), 693–700.

27. G. W. Zhang, L. Z. Yang, On Petrenko's deviations and the Julia limiting directions of solutions of complex differential equations, *J. Math. Anal. Appl.*, **519** (2023), 126799. https://doi.org/10.1016/j.jmaa.2022.126799

28. J. H. Zheng, *Dynamics of Meromorphic Functions*, in Chinese, Beijing: Tsinghua University Press, 2006.

29. J. H. Zheng, *Value Distribution of Meromorphic Function*, Beijing: Tsinghua University Press, 2010.

30. J. H. Zheng, S. Wang, Z. Huang, Some properties of Fatou and Julia sets of transcendental meromorphic functions, *Bull. Aust. Math. Soc.*, **66** (2002), 1–8. https://doi.org/10.1017/S000497270002061X