Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 20154–20168.
DOI: 10.3934/math.20231027
Received: 10 March 2023
Revised: 03 June 2023
Accepted: 06 June 2023
Published: 19 June 2023

*Research article*

# Common fixed point theorems for multi-valued mappings in bicomplex valued metric spaces with application

**Afrah Ahmad Noman Abdou***

Department of Mathematics, University of Jeddah, P.O. Box 80327, Jeddah 21589, Saudi Arabia

* **Correspondence:** Email: aabdou@uj.edu.sa.

**Abstract:** The aim of this article is to introduce a generalized Hausdorff distance function in the setting of a bicomplex valued metric space. Using this, we obtain common fixed point theorems for generalized contractions. Our outcomes extend and generalize some conventional fixed point results in the literature. We also furnish a significant example to express the genuineness of the presented results. As an application, we derive some common fixed point results for self mappings, including the leading results of [*Demonstr. Math.*, 54 (2021), 474–487] and [*Int. J. Nonlinear Anal. Appl.*, 12 (2021), 717–727].

**Keywords:** common fixed point; multi-valued mappings; bicomplex valued metric space; generalized Hausdorff distance function
**Mathematics Subject Classification:** 46S40, 47H10, 54H25

## 1. Introduction

Gauss gave the theory of complex numbers in $17^{th}$ century, but his research work was not on record. Later on, Cauchy initiated an in-depth review of complex numbers in the year 1840, and he is familiar as a successful originator of complex analysis. The study of complex numbers had its beginning because the solution of the quadratic equation $ax^2 + bx + c = 0$ did not exist for $b^2 - 4ac < 0$ in the set of real numbers. Under this background, Euler was the first mathematician who presented the symbol $i$ for $\sqrt{-1}$, with the property, $i^2 = -1$.

On the other hand, the beginning of bicomplex numbers was set up by Segre [1], and they provide a commutative replacement to the skew field of quaternions. These numbers extrapolated complex numbers more firmly and precisely to quaternions. For a better extensive study of investigation in bicomplex numbers, we refer the readers to [2]. In 2007, Huang et al. [3] presented the notion of a cone metric space (CMS) as an expansion of a traditional metric space (MS) and determined fixed point results for contractive mappings. Later on, Azam et al. [4] introduced the concept of a complex valued metric space (CVMS) as a particular case of a CMS. Mebawondu et al. [5] investigated the existence of solutions of differential equations by fixed point results in complex valued $b$-metric spaces. Vairaperumal et al. [6] established some common fixed point results for rational contractions in complex valued extended $b$-metric spaces. Okeke et al. [7] introduced the notion of complex valued convex metric spaces and proved certain fixed point results. In 2017, Choi et al. [8] introduced the notion of bicomplex valued metric spaces (bi-CVMS) by combining bicomplex numbers and CVMS. They proved some common fixed point theorems for weakly compatible mappings. Subsequently, Jebril et al. [9] used the idea of this novel space and presented theorems for two self mappings in the framework of bi-CVMS. In 2021, Beg et al. [10] reinforced the conception of bi-CVMS and proved extrapolated fixed point results. Afterward, Gnanaprakasam et al. [11] presented results for a contractive type condition in the framework of bi-CVMSs and explored the solution of linear equations. Later on, Tassaddiq et al. [12] involved control functions in the contractive inequality and established common fixed point results. Recently, Albargi et al. [13] obtained common fixed points of six self mapping in the setting of bi-CVMS. Mlaiki et al. [14] introduced locally contractive mappings in bi-CVMS and proved common fixed point theorems. For more details on CVMS and bi-CVMS, we refer the readers to [15–28].

In this research work, we introduce a generalized Hausdorff distance function in the framework of bi-CVMS and obtain common fixed point theorems for generalized contractions. We also furnish a significant example to illustrate the originality of the obtained results.

## 2. Preliminaries

We represent by $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$ the set of real numbers, the set of complex numbers and the set of bicomplex numbers, respectively. Segre [1] defined the idea of a bicomplex number as follows:

$$\ell = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2,$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$, and the independent units $i_1, i_2$ are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. We define the set of bicomplex numbers $\mathbb{C}_2$ as

$$\mathbb{C}_2 = \{\ell : \ell = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 : a_1, a_2, a_3, a_4 \in \mathbb{C}_0\},$$

that is,

$$\mathbb{C}_2 = \{\ell : \ell = z_1 + i_2 z_2 : z_1, z_2 \in \mathbb{C}_1\},$$

where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$. If $\ell = z_1 + i_2 z_2, \rho = \omega_1 + i_2 \omega_2 \in \mathbb{C}_2$, then the sum is

$$\ell \pm \rho = (z_1 + i_2 z_2) \pm (\omega_1 + i_2 \omega_2) = (z_1 \pm \omega_1) + i_2 ((z_2 \pm \omega_2)),$$

and the product is

$$\ell \cdot \rho = (z_1 + i_2 z_2) \cdot (\omega_1 + i_2 \omega_2) = (z_1 \omega_1 - z_2 \omega_2) + i_2 (z_1 \omega_2 + z_2 \omega_1).$$

There are four idempotent elements in $\mathbb{C}_2$, which are, $0, 1, e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$, out of which $e_1$ and $e_2$ are nontrivial such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Every bicomplex number $z_1 + i_2 z_2$ can uniquely be expressed as a combination of $e_1$and $e_2$, namely,

$$\ell = z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2.$$

This description of $\ell$ is familiar as the idempotent representation of $\ell$, and the complex coefficients $\ell_1 = (z_1 - i_1 z_2)$ and $\ell_2 = (z_1 + i_1 z_2)$ are known as idempotent components of the bicomplex number $\ell$.

An element $\ell = z_1 + i_2 z_2 \in \mathbb{C}_2$ is invertible if there exists $\rho \in \mathbb{C}_2$ such that $\ell\rho = 1$. In this way, the element $\rho$ is the multiplicative inverse of $\ell$. As a consequence, $\ell$ is the multiplicative inverse of $\rho$.

An element $\ell = z_1 + i_2 z_2 \in \mathbb{C}_2$ is non-singular if and only if $\left|z_1^2 + z_2^2\right| \neq 0$ and singular if and only if $\left|z_1^2 + z_2^2\right| = 0$. The inverse of $\ell$ is defined as

$$\ell^{-1} = \rho = \frac{z_1 - i_2 z_2}{z_1^2 + z_2^2}.$$

Zero is the only member in $\mathbb{C}_0$ that does not possess a multiplicative inverse, and in $\mathbb{C}_1$, $0 = 0 + i0$ is the only member that does not possess a multiplicative inverse. We represent the sets of singular members of $\mathbb{C}_0$ and $\mathbb{C}_1$ by $\mathcal{W}_0$ and $\mathcal{W}_1$, respectively. There are many members in $\mathbb{C}_2$ that do not have multiplicative inverse. We represent this set by $\mathcal{W}_2$, and evidently $\mathcal{W}_0 = \mathcal{W}_1 \subset \mathcal{W}_2$.

A bicomplex number $\ell = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 \in \mathbb{C}_2$ is said to be degenerated if the matrix

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_{2\times 2}$$

is degenerated. In that case, $\ell^{-1}$ exists, and it is also degenerated.

The norm $\|\cdot\|$ of $\mathbb{C}_2$ is a positive real valued function, and $\|\cdot\| : \mathbb{C}_2 \to \mathbb{C}_0^+$ is defined by

$$\begin{aligned} \|\ell\| &= \|z_1 + i_2 z_2\| = \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} \\ &= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2}\right]^{\frac{1}{2}} \\ &= \left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right)^{\frac{1}{2}}, \end{aligned}$$

where $\ell = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

A linear space $\mathbb{C}_2$ with regard to norm $\|\cdot\|$ is a normed linear space, and since $\mathbb{C}_2$ is complete, thus $\mathbb{C}_2$ is the Banach space. If $\ell, \rho \in \mathbb{C}_2$, then

$$\|\ell\rho\| \leq \sqrt{2}\, \|\ell\|\, \|\rho\|$$

holds instead of

$$\|\ell\rho\| \leq \|\ell\|\, \|\rho\|.$$

Therefore $\mathbb{C}_2$ is not the Banach algebra. The partial order relation $\preceq_{i_2}$ on $\mathbb{C}_2$ is defined as follows:

Let $\mathbb{C}_2$ be the set of bicomplex numbers and $\ell = z_1 + i_2 z_2$, $\rho = \omega_1 + i_2 \omega_2 \in \mathbb{C}_2$. Then

$$\ell \preceq_{i_2} \rho \Leftrightarrow \operatorname{Re}(z_1) \leq \operatorname{Re}(\omega_1) \text{ and } \operatorname{Im}(z_2) \leq \operatorname{Im}(\omega_2).$$

It follows that

$$\ell \preceq_{i_2} \rho$$

if one of these assertions is satisfied:

$$
\begin{aligned}
(a)\ z_1 &= \omega_1,\ z_2 \prec \omega_2,\\
(b)\ z_1 &\prec \omega_1,\ z_2 = \omega_2,\\
(c)\ z_1 &\prec \omega_1,\ z_2 \prec \omega_2,\\
(d)\ z_1 &= \omega_1,\ z_2 = \omega_2.
\end{aligned}
$$

In particular, we can write $\ell \precsim_{i_2} \rho$ if $\ell \preceq_{i_2} \rho$ and $\ell \neq \rho$, that is, one of (a), (b) and (c) is satisfied, and we will write $\ell = \rho$ if only (d) is satisfied. For any two bicomplex numbers $\ell, \rho \in \mathbb{C}_2$, we can verify the followings:

(i) $\ell \preceq_{i_2} \rho \Longrightarrow \|\ell\| \leq \|\rho\|$,
(ii) $\|\ell + \rho\| \leq \|\ell\| + \|\rho\|$,
(iii) $\|a\ell\| \leq a\|\rho\|$, where $a$ is a non-negative real number,
(iv) $\|\ell\rho\| \leq \sqrt{2}\|\ell\|\|\rho\|$,
(v) $\left\|\ell^{-1}\right\| = \|\ell\|^{-1}$,
(vi) $\left\|\frac{\ell}{\rho}\right\| = \frac{\|\ell\|}{\|\rho\|}$.

Choi et al. [8] defined the notion of a bi-CVMS as follows.

**Definition 2.1.** [8] Let $O \neq \emptyset$ and $\kappa : O \times O \to \mathbb{C}_2$ be a function satisfying

(i) $0 \preceq_{i_2} \kappa(\sigma, \varrho)$ and $\kappa(\sigma, \varrho) = 0 \Leftrightarrow \sigma = \varrho$,
(ii) $\kappa(\sigma, \varrho) = \kappa(\rho, \varrho)$,
(iii) $\kappa(\sigma, \varrho) \preceq_{i_2} \kappa(\sigma, \nu) + \kappa(\nu, \varrho)$,

for all $\sigma, \varrho, \nu \in O$. Then $(O, \kappa)$ is a bi-CVMS.

**Example 2.1.** [10] Let $O = \mathbb{C}_2$ and $\sigma, \varrho \in O$. Define $\kappa : O \times O \to \mathbb{C}_2$ by

$$\kappa(\sigma, \varrho) = |z_1 - \omega_1| + i_2 |z_2 - \omega_2|$$

where $\sigma = z_1 + i_2 z_2$, $\varrho = \omega_1 + i_2\omega_2 \in \mathbb{C}_2$. Then, $(O, \kappa)$ is a bi-CVMS.

**Lemma 2.1.** *[10] Let $\{\sigma_n\} \subseteq (O, \kappa)$. Then, $\{\sigma_n\}$ converges to $\ell$ if and only if $\|\kappa(\sigma_n, \sigma)\| \to 0$ as $n \to \infty$.*

**Lemma 2.2.** *[10] Let $\{\sigma_n\} \subseteq (O, \kappa)$. Then, $\{\sigma_n\}$ is a Cauchy sequence if and only if $\|\kappa(\sigma_n, \sigma_{n+m})\| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

Let $(O, \kappa)$ be a bi-CVMS. We denote by $N(O)$ ( resp. $CB(O)$) the collection of nonempty (resp. the collection of nonempty, closed and bounded ) subsets of $(O, \kappa)$. Now, we denote generalized Hausdorff distance function as $\wp$ and define

$$\wp(\ell) = \{\rho \in \mathbb{C}_2 : \ \ell \preceq_{i_2} \rho\}$$

for $\ell \in \mathbb{C}_2$, and

$$\wp(\sigma, B) = \bigcup_{\varrho \in B} \wp(\kappa(\sigma, \varrho)) = \bigcup_{\varrho \in B} \{\ell \in \mathbb{C}_2 : \ \kappa(\sigma, \varrho) \preceq_{i_2} \ell\}$$

for $\sigma \in O$ and $B \in CB(O)$. For $A, B \in CB(O)$, we denote

$$\wp(A, B) = \left(\bigcap_{\sigma \in A} \wp(\sigma, B)\right) \cap \left(\bigcap_{\varrho \in B} \wp(\sigma, A)\right).$$

*Remark* 2.1. Let $(\mathcal{O}, \kappa)$ be a bi-CVMS. If we take $a_2 = a_3 = a_4$, then $(\mathcal{O}, \kappa)$ is a metric space. Furthermore, for $A, B \in CB(\mathcal{O})$,

$$H(A, B) = \inf \wp(A, B)$$

is the Hausdorff distance induced by $\kappa$.

Let $\beth : \mathcal{O} \to CB(\mathcal{O})$ be a multi-valued mapping. For $\sigma \in \mathcal{O}$, and $A \in CB(\mathcal{O})$, define

$$W_\sigma(A) = \{\kappa(\sigma, \varrho) : \varrho \in A\}.$$

Thus, for $\sigma, \varrho \in \mathcal{O}$

$$W_\sigma(\beth\varrho) = \{\kappa(\sigma, \varrho) : \varrho \in \beth\varrho\}.$$

**Definition 2.2.** Let $(\mathcal{O}, \kappa)$ be a bi-CVMS. A nonempty subset $A$ of $\mathcal{O}$ is called bounded from below if there exists $\ell \in \mathbb{C}_2$, such that $\ell \preceq_{i_2} \rho$, for all $\rho \in A$.

**Definition 2.3.** Let $(\mathcal{O}, \kappa)$ be a bi-CVMS. A mapping $\mathcal{L} : \mathcal{O} \to 2^{\mathbb{C}_2}$ is said to be bounded from below if for $\sigma \in \mathcal{O}$, if there exists $\ell_\sigma \in \mathbb{C}$ such that

$$\ell_\sigma \preceq_{i_2} \rho, \quad \text{for all}\, \rho \in \mathcal{L}_\sigma.$$

**Definition 2.4.** Let $(\mathcal{O}, \kappa)$ be a bi-CVMS. A mapping $\beth : \mathcal{O} \to CB(\mathcal{O})$ is said to have lower bound (l.b) property on $(\mathcal{O}, \kappa)$, if for any $\sigma \in \mathcal{O}$, the multi-valued mapping $\mathcal{L}_\sigma : \mathcal{O} \to 2^{\mathbb{C}_2}$ defined by,

$$\mathcal{L}_\sigma(\beth\varrho) = W_\sigma(\beth\varrho)$$

is bounded from below, that is, for $\sigma, \varrho \in \mathcal{O}$, there exists $l_\sigma(\beth\varrho) \in \mathbb{C}$ such that;

$$l_\sigma(\beth\varrho) \preceq_{i_2} \rho$$

for all $\rho \in W_\sigma(\beth\varrho)$, where $l_\sigma(\beth\varrho)$ is called the lower bound of $\beth$ associated with $(\sigma, \varrho)$.

**Definition 2.5.** Let $(\mathcal{O}, \kappa)$ be a bi-CVMS. A mapping $\beth : \mathcal{O} \to CB(\mathcal{O})$ is said to satisfy greatest lower bound (g.l.b) property on $(\mathcal{O}, \kappa)$ if a greatest lower bound of $W_\sigma(\beth\varrho)$ exists in $\mathbb{C}_2$, for all $\sigma, \varrho \in \mathcal{O}$. We represent by $\kappa(\sigma, \beth\varrho)$ the g.l.b. of $W_\sigma(\beth\varrho)$, that is,

$$\kappa(\sigma, \beth\varrho) = \inf\{\kappa(\sigma, \mu) : \mu \in \beth\varrho\}.$$

## 3. Main result

Throughout this section, we consider $(\mathcal{O}, \kappa)$ as a complete bi-CVMS, and the mappings $\beth_1, \beth_2 : \mathcal{O} \to CB(\mathcal{O})$ satisfy the g.l.b. property.

**Theorem 3.1.** *Let $(\mathcal{O}, \kappa)$ be a complete bi-CVMS, and let $\beth_1, \beth_2 : (\mathcal{O}, \kappa) \to CB(\mathcal{O})$ be such that*

$$\aleph_1 \kappa(\sigma, \varrho) + \aleph_2 \frac{\kappa(\sigma, \beth_1\sigma)\, \kappa(\varrho, \beth_2\varrho)}{1 + \kappa(\sigma, \varrho)} + \aleph_3 \frac{\kappa(\varrho, \beth_1\sigma)\, \kappa(\sigma, \beth_2\varrho)}{1 + \kappa(\sigma, \varrho)} \in \wp(\beth_1\sigma, \beth_2\varrho) \tag{3.1}$$

*for all $\sigma, \varrho \in \mathcal{O}$ and $\aleph_1, \aleph_2, \aleph_3 \in [0, 1)$ with $\aleph_1 + \sqrt{2}\aleph_2 + \sqrt{2}\aleph_3 < 1$. Then, there exists $\varpi \in \mathcal{O}$ such that $\varpi \in \beth_1\varpi \cap \beth_2\varpi$.*

*Proof.* Let $\sigma_0 \in O$ be an arbitrary point and $\sigma_1 \in \beth_1\sigma_0$. From (3.1), we have

$$\aleph_1\kappa(\sigma_0,\sigma_1)+\aleph_2\frac{\kappa(\sigma_0,\beth_1\sigma_0)\,\kappa(\sigma_1,\beth_2\sigma_1)}{1+\kappa(\sigma_0,\sigma_1)}+\aleph_3\frac{\kappa(\sigma_1,\beth_1\sigma_0)\,\kappa(\sigma_0,\beth_2\sigma_1)}{1+\kappa(\sigma_0,\sigma_1)}\in\wp(\beth_1\sigma_0,\beth_2\sigma_1).$$

This implies that

$$\aleph_1\kappa(\sigma_0,\sigma_1)+\aleph_2\frac{\kappa(\sigma_0,\beth_1\sigma_0)\,\kappa(\sigma_1,\beth_2\sigma_1)}{1+\kappa(\sigma_0,\sigma_1)}+\aleph_3\frac{\kappa(\sigma_1,\beth_1\sigma_0)\,\kappa(\sigma_0,\beth_2\sigma_1)}{1+\kappa(\sigma_0,\sigma_1)}\in\bigcap_{\sigma\in\beth_1\sigma_0}\wp(\sigma,\beth_2\sigma_1)$$

$$\aleph_1\kappa(\sigma_0,\sigma_1)+\aleph_2\frac{\kappa(\sigma_0,\beth_1\sigma_0)\,\kappa(\sigma_1,\beth_2\sigma_1)}{1+\kappa(\sigma_0,\sigma_1)}+\aleph_3\frac{\kappa(\sigma_1,\beth_1\sigma_0)\,\kappa(\sigma_0,\beth_2\sigma_1)}{1+\kappa(\sigma_0,\sigma_1)}\in\wp(\sigma,\beth_2\sigma_1)$$

for $\sigma \in \beth_1\sigma_0$. Since $\sigma_1 \in \beth_1\sigma_0$, we have

$$\aleph_1\kappa(\sigma_0,\sigma_1)+\aleph_2\frac{\kappa(\sigma_0,\beth_1\sigma_0)\,\kappa(\sigma_1,\beth_2\sigma_1)}{1+\kappa(\sigma_0,\sigma_1)}+\aleph_3\frac{\kappa(\sigma_1,\beth_1\sigma_0)\,\kappa(\sigma_0,\beth_2\sigma_1)}{1+\kappa(\sigma_0,\sigma_1)}\in\wp(\sigma_1,\beth_2\sigma_1).$$

This implies that

$$\aleph_1\kappa(\sigma_0,\sigma_1)+\aleph_2\frac{\kappa(\sigma_0,\beth_1\sigma_0)\,\kappa(\sigma_1,\beth_2\sigma_1)}{1+\kappa(\sigma_0,\sigma_1)}+\aleph_3\frac{\kappa(\sigma_1,\beth_1\sigma_0)\,\kappa(\sigma_0,\beth_2\sigma_1)}{1+\kappa(\sigma_0,\sigma_1)}\in\bigcup_{\sigma\in\beth_2\sigma_1}\wp(\sigma_1,\sigma).$$

So, there exists $\sigma_2 \in \beth_2\sigma_1$ and we have

$$\aleph_1\kappa(\sigma_0,\sigma_1)+\aleph_2\frac{\kappa(\sigma_0,\beth_1\sigma_0)\,\kappa(\sigma_1,\beth_2\sigma_1)}{1+\kappa(\sigma_0,\sigma_1)}+\aleph_3\frac{\kappa(\sigma_1,\beth_1\sigma_0)\,\kappa(\sigma_0,\beth_2\sigma_1)}{1+\kappa(\sigma_0,\sigma_1)}\in\wp(\kappa(\sigma_1,\sigma_2)).$$

Therefore,

$$\kappa(\sigma_1,\sigma_2)\preceq_{i_2}\aleph_1\kappa(\sigma_0,\sigma_1)+\aleph_2\frac{\kappa(\sigma_0,\beth_1\sigma_0)\,\kappa(\sigma_1,\beth_2\sigma_1)}{1+\kappa(\sigma_0,\sigma_1)}+\aleph_3\frac{\kappa(\sigma_1,\beth_1\sigma_0)\,\kappa(\sigma_0,\beth_2\sigma_1)}{1+\kappa(\sigma_0,\sigma_1)}.$$

Since the pair $(\beth_1, \beth_2)$ satisfies g.l.b. property, we get

$$\begin{aligned}\kappa(\sigma_1,\sigma_2)&\preceq_{i_2}\aleph_1\kappa(\sigma_0,\sigma_1)+\aleph_2\frac{\kappa(\sigma_0,\sigma_1)\,\kappa(\sigma_1,\sigma_2)}{1+\kappa(\sigma_0,\sigma_1)}+\aleph_3\frac{\kappa(\sigma_1,\sigma_1)\,\kappa(\sigma_0,\sigma_2)}{1+\kappa(\sigma_0,\sigma_1)}\\&=\aleph_1\kappa(\sigma_0,\sigma_1)+\aleph_2\frac{\kappa(\sigma_0,\sigma_1)\,\kappa(\sigma_1,\sigma_2)}{1+\kappa(\sigma_0,\sigma_1)}.\end{aligned}$$

This implies

$$\begin{aligned}\|\kappa(\sigma_1,\sigma_2)\|&\leq\aleph_1\|\kappa(\sigma_0,\sigma_1)\|+\aleph_2\left\|\frac{\kappa(\sigma_0,\sigma_1)\,\kappa(\sigma_1,\sigma_2)}{1+\kappa(\sigma_0,\sigma_1)}\right\|\\&\leq\aleph_1\|\kappa(\sigma_0,\sigma_1)\|+\sqrt{2}\aleph_2\frac{\|\kappa(\sigma_0,\sigma_1)\|}{\|1+\kappa(\sigma_0,\sigma_1)\|}\|\kappa(\sigma_1,\sigma_2)\|\\&\leq\aleph_1\|\kappa(\sigma_0,\sigma_1)\|+\sqrt{2}\aleph_2\|\kappa(\sigma_1,\sigma_2)\|\end{aligned}$$

because $\frac{\|\kappa(\sigma_0,\sigma_1)\|}{\|1+\kappa(\sigma_0,\sigma_1)\|} < 1$. This yields

$$\|\kappa(\sigma_1,\sigma_2)\|\leq\frac{\aleph_1}{1-\sqrt{2}\aleph_2}\|\kappa(\sigma_0,\sigma_1)\|. \tag{3.2}$$

Similarly, for $\sigma_2 \in \beth_2\sigma_1$ and from (3.1), we have

$$\aleph_1\kappa(\sigma_1,\sigma_2)+\aleph_2\frac{\kappa(\sigma_2,\beth_1\sigma_2)\,\kappa(\sigma_1,\beth_2\sigma_1)}{1+\kappa(\sigma_2,\sigma_1)}$$

$$+\aleph_3\frac{\kappa(\sigma_1,\beth_1\sigma_2)\,\kappa(\sigma_2,\beth_2\sigma_1)}{1+\kappa(\sigma_2,\sigma_1)}\in\wp(\beth_1\sigma_2,\beth_2\sigma_1)=\wp(\beth_2\sigma_1,\beth_1\sigma_2).$$

This implies that

$$\aleph_1\kappa(\sigma_1,\sigma_2)+\aleph_2\frac{\kappa(\sigma_2,\beth_1\sigma_2)\,\kappa(\sigma_1,\beth_2\sigma_1)}{1+\kappa(\sigma_2,\sigma_1)}+\aleph_3\frac{\kappa(\sigma_1,\beth_1\sigma_2)\,\kappa(\sigma_2,\beth_2\sigma_1)}{1+\kappa(\sigma_2,\sigma_1)}\in\bigcap_{\sigma\in\beth_2\sigma_1}\wp(\sigma,\beth_1\sigma_2)$$

$$\aleph_1\kappa(\sigma_1,\sigma_2)+\aleph_2\frac{\kappa(\sigma_2,\beth_1\sigma_2)\,\kappa(\sigma_1,\beth_2\sigma_1)}{1+\kappa(\sigma_2,\sigma_1)}+\aleph_3\frac{\kappa(\sigma_1,\beth_1\sigma_2)\,\kappa(\sigma_2,\beth_2\sigma_1)}{1+\kappa(\sigma_2,\sigma_1)}\in\wp(\sigma,\beth_1\sigma_2)$$

for $\sigma \in \beth_2\sigma_1$. Since $\sigma_2 \in \beth_2\sigma_1$, we have

$$\aleph_1\kappa(\sigma_1,\sigma_2)+\aleph_2\frac{\kappa(\sigma_2,\beth_1\sigma_2)\,\kappa(\sigma_1,\beth_2\sigma_1)}{1+\kappa(\sigma_2,\sigma_1)}+\aleph_3\frac{\kappa(\sigma_1,\beth_1\sigma_2)\,\kappa(\sigma_2,\beth_2\sigma_1)}{1+\kappa(\sigma_2,\sigma_1)}\in\wp(\sigma_2,\beth_1\sigma_2).$$

This implies that

$$\aleph_1\kappa(\sigma_1,\sigma_2)+\aleph_2\frac{\kappa(\sigma_2,\beth_1\sigma_2)\,\kappa(\sigma_1,\beth_2\sigma_1)}{1+\kappa(\sigma_2,\sigma_1)}+\aleph_3\frac{\kappa(\sigma_1,\beth_1\sigma_2)\,\kappa(\sigma_2,\beth_2\sigma_1)}{1+\kappa(\sigma_2,\sigma_1)}\in\bigcup_{\sigma\in\beth_1\sigma_2}\wp(\sigma_2,\sigma).$$

So, there exists $\sigma_3 \in \beth_1\sigma_2$, and we have

$$\aleph_1\kappa(\sigma_1,\sigma_2)+\aleph_2\frac{\kappa(\sigma_2,\beth_1\sigma_2)\,\kappa(\sigma_1,\beth_2\sigma_1)}{1+\kappa(\sigma_2,\sigma_1)}+\aleph_3\frac{\kappa(\sigma_1,\beth_1\sigma_2)\,\kappa(\sigma_2,\beth_2\sigma_1)}{1+\kappa(\sigma_2,\sigma_1)}\in\wp(\kappa(\sigma_2,\sigma_3)).$$

Therefore,

$$\kappa(\sigma_2,\sigma_3)\preceq_{i_2}\aleph_1\kappa(\sigma_1,\sigma_2)+\aleph_2\frac{\kappa(\sigma_2,\beth_1\sigma_2)\,\kappa(\sigma_1,\beth_2\sigma_1)}{1+\kappa(\sigma_2,\sigma_1)}+\aleph_3\frac{\kappa(\sigma_1,\beth_1\sigma_2)\,\kappa(\sigma_2,\beth_2\sigma_1)}{1+\kappa(\sigma_2,\sigma_1)}.$$

Since the pair $(\beth_1,\beth_2)$ satisfies g.l.b. property, we get

$$\kappa(\sigma_2,\sigma_3)\preceq_{i_2}\aleph_1\kappa(\sigma_1,\sigma_2)+\aleph_2\frac{\kappa(\sigma_2,\sigma_3)\,\kappa(\sigma_1,\sigma_2)}{1+\kappa(\sigma_2,\sigma_1)}+\aleph_3\frac{\kappa(\sigma_1,\sigma_3)\,\kappa(\sigma_2,\sigma_2)}{1+\kappa(\sigma_2,\sigma_1)}$$

$$=\aleph_1\kappa(\sigma_1,\sigma_2)+\aleph_2\frac{\kappa(\sigma_2,\sigma_3)\,\kappa(\sigma_1,\sigma_2)}{1+\kappa(\sigma_1,\sigma_2)}.$$

This implies

$$\begin{aligned}\|\kappa(\sigma_2,\sigma_3)\| &\leq \aleph_1\|\kappa(\sigma_1,\sigma_2)\|+\aleph_2\left\|\frac{\kappa(\sigma_2,\sigma_3)\,\kappa(\sigma_1,\sigma_2)}{1+\kappa(\sigma_1,\sigma_2)}\right\|\\ &\leq \aleph_1\|\kappa(\sigma_1,\sigma_2)\|+\sqrt{2}\aleph_2\frac{\|\kappa(\sigma_1,\sigma_2)\|}{\|1+\kappa(\sigma_1,\sigma_2)\|}\|\kappa(\sigma_2,\sigma_3)\|\\ &\leq \aleph_1\|\kappa(\sigma_1,\sigma_2)\|+\sqrt{2}\aleph_2\|\kappa(\sigma_2,\sigma_3)\|\end{aligned}$$

since $\frac{\|\kappa(\sigma_1,\sigma_2)\|}{\|1+\kappa(\sigma_1,\sigma_2)\|} < 1$. This yields

$$\|\kappa(\sigma_2, \sigma_3)\| \leq \frac{\aleph_1}{1 - \sqrt{2}\aleph_2} \|\kappa(\sigma_1, \sigma_2)\| . \tag{3.3}$$

Let $\frac{\aleph_1}{1-\sqrt{2}\aleph_2} = \aleph < 1$. Then, from (3.2) and (3.3), we have

$$\|\kappa(\sigma_2, \sigma_3)\| \leq \aleph \|\kappa(\sigma_1, \sigma_2)\| \leq \aleph^2 \|\kappa(\sigma_1, \sigma_2)\| .$$

Thus, we can generate a sequence $\{\sigma_n\}$ in $O$ such that

$$\sigma_{2n+1} \in \beth_1\sigma_{2n} \text{ and } \sigma_{2n+2} \in \beth_2\sigma_{2n+1},$$

and

$$\|\kappa(\sigma_n, \sigma_{n+1})\| \leq \aleph \|\kappa(\sigma_{n-1}, \sigma_n)\| \leq ... \leq \aleph^n \|\kappa(\sigma_0, \sigma_1)\| .$$

for $n \in \mathbb{N}$. Now, for $m > n$ and by the triangle inequality, we have

$$\begin{aligned} \|\kappa(\sigma_n, \sigma_m)\| &\leq \aleph^n \|\kappa(\sigma_0, \sigma_1)\| \\ &\quad +\aleph^{n+1} \|\kappa(\sigma_0, \sigma_1)\| \\ &\quad + \cdots +\aleph^{m-1} \|\kappa(\sigma_0, \sigma_1)\| \\ &\leq \left[\aleph^n + \aleph^{n+1} + \cdots + \aleph^{m-1}\right] \|\kappa(\sigma_0, \sigma_1)\| . \end{aligned}$$

Now, by taking $n \to \infty$, we get

$$\|\kappa(\sigma_n, \sigma_m)\| \to 0.$$

Thus, $\{\sigma_n\}$ is a Cauchy sequence in $O$ by Lemma 2.2. Therefore there exists $\varpi \in O$ such that $\lim_{n\to\infty} \sigma_n = \varpi$. Then, also, $\lim_{n\to\infty} \sigma_{2n} = \varpi$, and $\lim_{n\to\infty} \sigma_{2n+1} = \varpi$. Now, we show that $\varpi \in \beth_1\varpi$ and $\varpi \in \beth_2\varpi$. From (3.1), we have

$$\aleph_1\kappa(\sigma_{2n}, \varpi) + \aleph_2\frac{\kappa(\sigma_{2n}, \beth_1\sigma_{2n})\,\kappa(\varpi, \beth_2\varpi)}{1 + \kappa(\sigma_{2n}, \varpi)} + \aleph_3\frac{\kappa(\varpi, \beth_1\sigma_{2n})\,\kappa(\sigma_{2n}, \beth_2\varpi)}{1 + \kappa(\sigma_{2n}, \varpi)} \in \wp(\beth_1\sigma_{2n}, \beth_2\varpi),$$

which implies that

$$\aleph_1\kappa(\sigma_{2n}, \varpi) + \aleph_2\frac{\kappa(\sigma_{2n}, \beth_1\sigma_{2n})\,\kappa(\varpi, \beth_2\varpi)}{1 + \kappa(\sigma_{2n}, \varpi)} + \aleph_3\frac{\kappa(\varpi, \beth_1\sigma_{2n})\,\kappa(\sigma_{2n}, \beth_2\varpi)}{1 + \kappa(\sigma_{2n}, \varpi)} \in \bigcap_{\sigma\in\beth_1\sigma_{2n}} \wp(\sigma, \beth_2\varpi)$$

$$\aleph_1\kappa(\sigma_{2n}, \varpi) + \aleph_2\frac{\kappa(\sigma_{2n}, \beth_1\sigma_{2n})\,\kappa(\varpi, \beth_2\varpi)}{1 + \kappa(\sigma_{2n}, \varpi)} + \aleph_3\frac{\kappa(\varpi, \beth_1\sigma_{2n})\,\kappa(\sigma_{2n}, \beth_2\varpi)}{1 + \kappa(\sigma_{2n}, \varpi)} \in \wp(\sigma, \beth_2\varpi)$$

for $\sigma \in \beth_1\sigma_{2n}$. Since $\sigma_{2n+1} \in \beth_1\sigma_{2n}$, we have

$$\aleph_1\kappa(\sigma_{2n}, \varpi) + \aleph_2\frac{\kappa(\sigma_{2n}, \beth_1\sigma_{2n})\,\kappa(\varpi, \beth_2\varpi)}{1 + \kappa(\sigma_{2n}, \varpi)} + \aleph_3\frac{\kappa(\varpi, \beth_1\sigma_{2n})\,\kappa(\sigma_{2n}, \beth_2\varpi)}{1 + \kappa(\sigma_{2n}, \varpi)} \in \wp(\sigma_{2n+1}, \beth_2\varpi) .$$

By definition

$$\aleph_1\kappa(\sigma_{2n},\varpi)+\aleph_2\frac{\kappa(\sigma_{2n},\beth_1\sigma_{2n})\,\kappa(\varpi,\beth_2\varpi)}{1+\kappa(\sigma_{2n},\varpi)}$$
$$+\aleph_3\frac{\kappa(\varpi,\beth_1\sigma_{2n})\,\kappa(\sigma_{2n},\beth_2\varpi)}{1+\kappa(\sigma_{2n},\varpi)}\in\wp(\sigma_{2n+1},\beth_2\varpi)=\bigcup_{\sigma^/\in\beth_2\varpi}\wp\left(\kappa(\sigma_{2n+1},\sigma^/)\right).$$

There exists $\sigma_n\in\beth_2\varpi$ such that

$$\aleph_1\kappa(\sigma_{2n},\varpi)+\aleph_2\frac{\kappa(\sigma_{2n},\beth_1\sigma_{2n})\,\kappa(\varpi,\beth_2\varpi)}{1+\kappa(\sigma_{2n},\varpi)}+\aleph_3\frac{\kappa(\varpi,\beth_1\sigma_{2n})\,\kappa(\sigma_{2n},\beth_2\varpi)}{1+\kappa(\sigma_{2n},\varpi)}\in\ \wp(\kappa(\sigma_{2n+1},\sigma_n)).$$

By definition

$$\kappa(\sigma_{2n+1},\varpi_n)\preceq_{i_2}\aleph_1\kappa(\sigma_{2n},\varpi)+\aleph_2\frac{\kappa(\sigma_{2n},\beth_1\sigma_{2n})\,\kappa(\varpi,\beth_2\varpi)}{1+\kappa(\sigma_{2n},\varpi)}+\aleph_3\frac{\kappa(\varpi,\beth_1\sigma_{2n})\,\kappa(\sigma_{2n},\beth_2\varpi)}{1+\kappa(\sigma_{2n},\varpi)}.$$

Since the pair $(\beth_1,\beth_2)$ satisfies g.l.b. property, we get

$$\kappa(\sigma_{2n+1},\varpi_n)\preceq_{i_2}\aleph_1\kappa(\sigma_{2n},\varpi)+\aleph_2\frac{\kappa(\sigma_{2n},\sigma_{2n+1})\,\kappa(\varpi,\varpi_n)}{1+\kappa(\sigma_{2n},\varpi)}+\aleph_3\frac{\kappa(\varpi,\sigma_{2n+1})\,\kappa(\sigma_{2n},\varpi_n)}{1+\kappa(\sigma_{2n},\varpi)}. \tag{3.4}$$

By the triangle inequality, we have

$$\kappa(\varpi,\varpi_n)\preceq_{i_2}\kappa(\varpi,\sigma_{2n+1})+\kappa(\sigma_{2n+1},\varpi_n).$$

Now, using (3.4), we have

$$\begin{aligned}\kappa(\varpi,\varpi_n)&\preceq_{i_2}\kappa(\varpi,\sigma_{2n+1})+\kappa(\sigma_{2n+1},\varpi_n)\\&\preceq_{i_2}\kappa(\varpi,\sigma_{2n+1})+\aleph_1\kappa(\sigma_{2n},\varpi)\\&\quad+\aleph_2\frac{\kappa(\sigma_{2n},\sigma_{2n+1})\,\kappa(\varpi,\varpi_n)}{1+\kappa(\sigma_{2n},\varpi)}+\aleph_3\frac{\kappa(\varpi,\sigma_{2n+1})\,\kappa(\sigma_{2n},\varpi_n)}{1+\kappa(\sigma_{2n},\varpi)}\end{aligned}$$

which implies

$$\begin{aligned}\|\kappa(\varpi,\varpi_n)\|&\leq\|\kappa(\varpi,\sigma_{2n+1})\|+\aleph_1\|\kappa(\sigma_{2n},\varpi)\|\\&\quad+\aleph_2\left\|\frac{\kappa(\sigma_{2n},\sigma_{2n+1})\,\kappa(\varpi,\varpi_n)}{1+\kappa(\sigma_{2n},\varpi)}\right\|+\aleph_3\left\|\frac{\kappa(\varpi,\sigma_{2n+1})\,\kappa(\sigma_{2n},\varpi_n)}{1+\kappa(\sigma_{2n},\varpi)}\right\|\\&\leq\|\kappa(\varpi,\sigma_{2n+1})\|+\aleph_1\|\kappa(\sigma_{2n},\varpi)\|+\sqrt{2}\aleph_2\left\|\frac{\kappa(\sigma_{2n},\sigma_{2n+1})}{1+\kappa(\sigma_{2n},\varpi)}\right\|\|\kappa(\varpi,\varpi_n)\|\\&\quad+\sqrt{2}\aleph_3\left\|\frac{\kappa(\varpi,\sigma_{2n+1})}{1+\kappa(\sigma_{2n},\varpi)}\right\|\|\kappa(\sigma_{2n},\varpi_n)\|.\end{aligned}$$

Taking the limit as $n\to\infty$, we have $\|\kappa(\varpi,\varpi_n)\|\to 0$ as $n\to\infty$. Thus, $\varpi_n\to\varpi$ as $n\to\infty$. Since $\beth_2\varpi$ is closed, we have $\varpi\in\beth_2\varpi$. Similarly, we can prove that $\varpi\in\beth_1\varpi$. Therefore, $\varpi$ is a common fixed point of $\beth_1$ and $\beth_2$. □

**Corollary 3.1.** *Let $(O,\kappa)$ be a complete bi-CVMS and let $\beth:(O,\kappa)\to CB(O)$ be a multi-valued mapping with g.l.b. property such that*

$$\aleph_1\kappa(\sigma,\varrho)+\aleph_2\frac{\kappa(\sigma,\beth\sigma)\,\kappa(\varrho,\beth\varrho)}{1+\kappa(\sigma,\varrho)}+\aleph_3\frac{\kappa(\varrho,\beth\sigma)\,\kappa(\sigma,\beth\varrho)}{1+\kappa(\sigma,\varrho)}\in\wp(\beth\sigma,\beth\varrho) \tag{3.5}$$

*for all $\sigma,\varrho\in O$ and $\aleph_1,\aleph_2,\aleph_3\in[0,1)$ with $\aleph_1+\sqrt{2}\aleph_2+\sqrt{2}\aleph_3<1$. Then, there exists $\varpi\in O$ such that $\varpi\in\beth\varpi$.*

*Proof.* Take $\beth_1 = \beth_2 = \beth$ in Theorem 3.1. □

**Corollary 3.2.** *Let $(O, \kappa)$ be a complete bi-CVMS and let $\beth_1, \beth_2 : (O, \kappa) \to CB(O)$ be such that*

$$\aleph_1 \kappa(\sigma, \varrho) + \aleph_2 \frac{\kappa(\sigma, \beth_1\sigma)\,\kappa(\varrho, \beth_2\varrho)}{1 + \kappa(\sigma, \varrho)} \in \wp(\beth_1\sigma, \beth_2\varrho) \tag{3.6}$$

*for all $\sigma, \varrho \in O$ and $\aleph_1, \aleph_2 \in [0, 1)$ with $\aleph_1 + \sqrt{2}\aleph_2 < 1$. Then, there exists $\varpi \in O$ such that $\varpi \in \beth_1\varpi \cap \beth_2\varpi$.*

*Proof.* Take $\aleph_3 = 0$ in Theorem 3.1. □

**Corollary 3.3.** *Let $(O, \kappa)$ be a complete bi-CVMS and $\beth : (O, \kappa) \to CB(O)$ be such that*

$$\aleph_1 \kappa(\sigma, \varrho) + \aleph_2 \frac{\kappa(\sigma, \beth\sigma)\,\kappa(\varrho, \beth\varrho)}{1 + \kappa(\sigma, \varrho)} \in \wp(\beth\sigma, \beth\varrho) \tag{3.7}$$

*for all $\sigma, \varrho \in O$ and $\aleph_1, \aleph_2 \in [0, 1)$ with $\aleph_1 + \sqrt{2}\aleph_2 < 1$. Then, there exists $\varpi \in O$ such that $\varpi \in \beth\varpi$.*

*Proof.* Take $\beth_1 = \beth_2 = \beth$ in Corollary 3.2. □

**Corollary 3.4.** *Let $(O, \kappa)$ be a complete bi-CVMS and $\beth_1, \beth_2 : O \to CB(O)$ be such that*

$$\aleph_1 \kappa(\sigma, \varrho) + \aleph_3 \frac{\kappa(\varrho, \beth_1\sigma)\,\kappa(\sigma, \beth_2\varrho)}{1 + \kappa(\sigma, \varrho)} \in \wp(\beth_1\sigma, \beth_2\varrho) \tag{3.8}$$

*for all $\sigma, \varrho \in O$ and $\aleph_1, \aleph_3 \in [0, 1)$ with $\aleph_1 + \sqrt{2}\aleph_3 < 1$. Then, there exists $\varpi \in O$ such that $\varpi \in \beth_1\varpi \cap \beth_2\varpi$.*

*Proof.* Take $\aleph_2 = 0$ in Theorem 3.1. □

**Corollary 3.5.** *Let $(O, \kappa)$ be a complete bi-CVMS and let $\beth : (O, \kappa) \to CB(O)$ be such that*

$$\aleph_1 \kappa(\sigma, \varrho) + \aleph_3 \frac{\kappa(\varrho, \beth\sigma)\,\kappa(\sigma, \beth\varrho)}{1 + \kappa(\sigma, \varrho)} \in \wp(\beth\sigma, \beth\varrho) \tag{3.9}$$

*for all $\sigma, \varrho \in O$ and $\aleph_1, \aleph_3 \in [0, 1)$ with $\aleph_1 + \sqrt{2}\aleph_3 < 1$. Then, there exists $\varpi \in O$ such that $\varpi \in \beth\varpi$.*

*Proof.* Set $\beth_1 = \beth_2 = \beth$ in Corollary 3.4. □

**Corollary 3.6.** *Let $(O, \kappa)$ be a complete bi-CVMS and let $\beth_1, \beth_2 : (O, \kappa) \to CB(O)$ be such that*

$$\aleph_1 \kappa(\sigma, \varrho) \in \wp(\beth_1\sigma, \beth_2\varrho) \tag{3.10}$$

*for all $\sigma, \varrho \in O$ and $\aleph_1 \in [0, 1)$ with $\aleph_1 < 1$. Then, there exists $\varpi \in O$ such that $\varpi \in \beth_1\varpi \cap \beth_2\varpi$.*

*Proof.* Choose $\aleph_2 = \aleph_3 = 0$ in Theorem 3.1. □

**Corollary 3.7.** *Let $(O, \kappa)$ be a complete bi-CVMS and let $\beth : (O, \kappa) \to CB(O)$ be such that*

$$\aleph_1 \kappa(\sigma, \varrho) \in \wp(\beth\sigma, \beth\varrho) \tag{3.11}$$

*for all $\sigma, \varrho \in O$ and $\aleph_1 \in [0, 1)$. Then, there exists $\varpi \in O$ such that $\varpi \in \beth\varpi$.*

*Proof.* Take $\beth_1 = \beth_2 = \beth$ in Corollary 3.6. □

**Example 3.1.** Let $O = [0, 1]$. Define $\kappa : O \times O \to \mathbb{C}_2$ by

$$\kappa(\sigma, \varrho) = (1 + i_2) |\sigma - \varrho|.$$

Then, $(O, \kappa)$ is a complete bi-CVMS. Consider the mappings $\beth_1, \beth_2 : O \to CB(O)$ defined by

$$\beth_1\sigma = [0, \frac{1}{5}\sigma] \text{ and } \beth_2\sigma = [0, \frac{1}{10}\sigma].$$

If $\sigma = \varrho = 0$, then obviously the contractive condition is satisfied. Now, we assume that $\sigma < \varrho$ Then, we have

$$\begin{aligned}
\kappa(\sigma, \varrho) &= (1 + i_2) |\varrho - \sigma|, \\
\kappa(\sigma, \beth_1\sigma) &= (1 + i_2) \left|\sigma - \frac{\sigma}{6}\right|, \\
\kappa(\varrho, \beth_2\varrho) &= (1 + i_2) \left|\varrho - \frac{\varrho}{12}\right|, \\
\kappa(\varrho, \beth_1\sigma) &= (1 + i_2) \left|\varrho - \frac{\sigma}{6}\right|, \\
\kappa(\sigma, \beth_2\varrho) &= (1 + i_2) \left|\sigma - \frac{\varrho}{12}\right|,
\end{aligned}$$

and

$$\wp(\beth_1\sigma, \beth_2\varrho) = \wp\left((1 + i_2) \left|\frac{\sigma}{6} - \frac{\varrho}{12}\right|\right).$$

Consider,

$$\begin{aligned}
& \aleph_1\kappa(\sigma, \varrho) + \aleph_2 \frac{\kappa(\sigma, \beth_1\sigma)\, \kappa(\varrho, \beth_2\varrho)}{1 + \kappa(\sigma, \varrho)} + \aleph_3 \frac{\kappa(\varrho, \beth_1\sigma)\, \kappa(\sigma, \beth_2\varrho)}{1 + \kappa(\sigma, \varrho)} \\
= \; & \aleph_1 |\varrho - \sigma| + \aleph_2 \frac{\left|\sigma - \frac{\sigma}{6}\right| \left|\varrho - \frac{\varrho}{12}\right|}{1 + |\varrho - \sigma|} + \aleph_3 \frac{\left|\varrho - \frac{\sigma}{6}\right| \left|\sigma - \frac{\varrho}{12}\right|}{1 + |\varrho - \sigma|}.
\end{aligned}$$

Then, for any values of $\aleph_2$ and $\aleph_3$ and $\aleph_1 = \frac{1}{6}$, we have

$$\left|\frac{\sigma}{6} - \frac{\varrho}{12}\right| \leq \frac{1}{6} |\varrho - \sigma| + \aleph_2 \frac{\left|\sigma - \frac{\sigma}{6}\right| \left|\varrho - \frac{\varrho}{12}\right|}{1 + |\varrho - \sigma|} + \aleph_3 \frac{\left|\varrho - \frac{\sigma}{6}\right| \left|\sigma - \frac{\varrho}{12}\right|}{1 + |\varrho - \sigma|}.$$

Hence,

$$\aleph_1\kappa(\sigma, \varrho) + \aleph_2 \frac{\kappa(\sigma, \beth_1\sigma)\, \kappa(\varrho, \beth_2\varrho)}{1 + \kappa(\sigma, \varrho)} + \aleph_3 \frac{\kappa(\varrho, \beth_1\sigma)\, \kappa(\sigma, \beth_2\varrho)}{1 + \kappa(\sigma, \varrho)} \in \wp(\beth_1\sigma, \beth_2\varrho).$$

Thus, all the axioms of Theorem 3.1 hold, and the pair $(\beth_1, \beth_2)$ has a fixed point 0.

Now, if we consider $\beth_1(\sigma) = \{\sigma\}$ and $\beth_2(\varrho) = \{\varrho\}$ in Theorem 3.1, then we can derive the key result of Gnanaprakasam et al. [11] in this manner.

**Corollary 3.8.** *[11] Let $(\mathcal{O},\kappa)$ be a complete bi-CVMS and let $\beth_1,\beth_2:(\mathcal{O},\kappa)\to(\mathcal{O},\kappa)$ be such that*

$$\kappa(\beth_1\sigma,\beth_2\varrho)\preceq_{i_2}\aleph_1\kappa(\sigma,\varrho)+\aleph_2\frac{\kappa(\sigma,\beth_1\sigma)\,\kappa(\varrho,\beth_2\varrho)}{1+\kappa(\sigma,\varrho)}+\aleph_3\frac{\kappa(\varrho,\beth_1\sigma)\,\kappa(\sigma,\beth_2\varrho)}{1+\kappa(\sigma,\varrho)}$$

*for all $\sigma,\varrho\in\mathcal{O}$ and $\aleph_1,\aleph_2,\aleph_3\in[0,1)$ with $\aleph_1+\sqrt{2}\aleph_2+\sqrt{2}\aleph_3<1$. Then, there exists $\varpi\in\mathcal{O}$ such that $\varpi=\beth_1\varpi=\beth_2\varpi$.*

**Corollary 3.9.** *Let $(\mathcal{O},\kappa)$ be a complete bi-CVMS and let $\beth:(\mathcal{O},\kappa)\to(\mathcal{O},\kappa)$ be such that*

$$\kappa(\beth\sigma,\beth\varrho)\preceq_{i_2}\aleph_1\kappa(\sigma,\varrho)+\aleph_2\frac{\kappa(\sigma,\beth\sigma)\,\kappa(\varrho,\beth\varrho)}{1+\kappa(\sigma,\varrho)}+\aleph_3\frac{\kappa(\varrho,\beth\sigma)\,\kappa(\sigma,\beth\varrho)}{1+\kappa(\sigma,\varrho)}$$

*for all $\sigma,\varrho\in\mathcal{O}$ and $\aleph_1,\aleph_2,\aleph_3\in[0,1)$ with $\aleph_1+\sqrt{2}\aleph_2+\sqrt{2}\aleph_3<1$. Then, there exists $\varpi\in\mathcal{O}$ such that $\varpi=\beth\varpi$.*

*Proof.* Take $\beth_1=\beth_2=\beth$ in Corollary 3.8. □

**Corollary 3.10.** *Let $(\mathcal{O},\kappa)$ be a complete bi-CVMS and let $\beth_1,\beth_2:(\mathcal{O},\kappa)\to(\mathcal{O},\kappa)$ be such that*

$$\kappa(\beth_1\sigma,\beth_2\varrho)\preceq_{i_2}\aleph_1\kappa(\sigma,\varrho)+\aleph_2\frac{\kappa(\sigma,\beth_1\sigma)\,\kappa(\varrho,\beth_2\varrho)}{1+\kappa(\sigma,\varrho)}$$

*for all $\sigma,\varrho\in\mathcal{O}$ and $\aleph_1,\aleph_2\in[0,1)$ with $\aleph_1+\sqrt{2}\aleph_2<1$. Then, there exists $\varpi\in\mathcal{O}$ such that $\varpi=\beth_1\varpi=\beth_2\varpi$.*

*Proof.* Take $\aleph_3=0$ in Corollary 3.8. □

**Corollary 3.11.** *[10] Let $(\mathcal{O},\kappa)$ be a complete bi-CVMS and let $\beth:(\mathcal{O},\kappa)\to(\mathcal{O},\kappa)$ be such that*

$$\kappa(\beth\sigma,\beth\varrho)\preceq_{i_2}\aleph_1\kappa(\sigma,\varrho)+\aleph_2\frac{\kappa(\sigma,\beth\sigma)\,\kappa(\varrho,\beth\varrho)}{1+\kappa(\sigma,\varrho)}$$

*for all $\sigma,\varrho\in\mathcal{O}$ and $\aleph_1,\aleph_2\in[0,1)$ with $\aleph_1+\sqrt{2}\aleph_2<1$. Then, there exists $\varpi\in\mathcal{O}$ such that $\varpi=\beth\varpi$.*

*Proof.* Take $\beth_1=\beth_2=\beth$ in the above Corollary. □

## 4. Conclusions

In this paper, we have introduced a generalized Hausdorff distance function in the setting of bi-CVMS and obtained common fixed point results for rational contractions. We hope that the established theorems in this paper will form contemporary associations for researchers who are working in bi-CVMS. As an application of our main results, we have derived some results for self mappings in the context of bi-CVMS, including the leading results of [*Demonstr. Math.*, 54 (2021), 474–487] and [*Int. J. Nonlinear Anal. Appl.*, 12 (2021), 717–727].

## Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

This work was funded by the University of Jeddah, Jeddah, Saudi Arabia, under grant No. (UJ-21-DR-68). The author, therefore, thanks the University of Jeddah for its technical and financial support.


## Conflict of interest

The author declares that she has no conflict of interest.

## References


1. C. Segre, Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici, *Math. Ann.*, **40** (1892), 413–467. https://doi.org/10.1007/BF01443559
2. G. B. Price, *An introduction to multicomplex spaces and functions*, Boca Raton: CRC Press, 1991. https://doi.org/10.1201/9781315137278
3. L. G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332** (2007), 1468–1476. https://doi.org/10.1016/j.jmaa.2005.03.087
4. A. Azam, B. Fisher, M. Khan, Common fixed point theorems in complex valued metric spaces, *Numer. Funct. Anal. Optim.*, **32** (2011), 243–253. https://doi.org/10.1080/01630563.2011.533046
5. A. A. Mebawondu, H. A. Abass, M. O. Aibinu, O. K. Narain, Existence of solution of differential equation via fixed point in complex valued $b$-metric spaces, *Nonlinear Funct. Anal. Appl.*, **26** (2021), 303–322. https://doi.org/10.22771/nfaa.2021.26.02.05
6. V. Vairaperumal, J. Carmel Pushpa Raj, J. Maria Joseph, M. Marudai, Common fixed point theorems under rational contractions in complex valued extended $b$-metric spaces, *Nonlinear Funct. Anal. Appl.*, **26** (2021), 685–700. https://doi.org/10.22771/nfaa.2021.26.04.03
7. G. A. Okeke, S. H. Khan, J. K. Kim, Fixed point theorems in complex valued convex metric spaces, *Nonlinear Funct. Anal. Appl.*, **26** (2021), 117–135. https://doi.org/10.22771/nfaa.2021.26.01.09
8. J. Choi, S. K. Datta, T. Biswas, M. N. Islam, Some fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces, *Honam Math. J.*, **39** (2017), 115–126. https://doi.org/10.5831/HMJ.2017.39.1.115
9. I. H. Jebril, S. K. Datta, R. Sarkar, N. Biswas, Common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces, *J. Interdiscip. Math.*, **22** (2019), 1071–1082. https://doi.org/10.1080/09720502.2019.1709318
10. I. Beg, S. K. Datta, D. Pal, Fixed point in bicomplex valued metric spaces, *Int. J. Nonlinear Anal. Appl.*, **12** (2021), 717–727. https://doi.org/10.22075/ijnaa.2019.19003.2049
11. A. J. Gnanaprakasam, S. M. Boulaaras, G. Mani, B. Cherif, S. A. Idris, Solving system of linear equations via bicomplex valued metric space, *Demonstr. Math.*, **54** (2021), 474–487. https://doi.org/10.1515/dema-2021-0046

12. A. Tassaddiq, J. Ahmad, A. E. Al-Mazrooei, D. Lateef, F. Lakhani, On common fixed point results in bicomplex valued metric spaces with application, *AIMS Mathematics*, **8** (2023), 5522–5539. https://doi.org/10.3934/math.2023278

13. A. H. Albargi, A. E. Shammaky, J. Ahmad, Common fixed point results in bicomplex valued metric spaces with application, *Mathematics*, **11** (2023), 1207. https://doi.org/10.3390/math11051207

14. N. Mlaiki, J. Ahmad, A. E. Al-Mazrooei, D. Santina, Common fixed points of locally contractive mappings in bicomplex valued metric spaces with application to Urysohn integral equation, *AIMS Mathematics*, **8** (2023), 3897–3912. https://doi.org/10.3934/math.2023194

15. Z. Mitrovic, G. Mani, A. J. Gnanaprakasam, R. George, The existence of a solution of a nonlinear Fredholm integral equations over bicomplex $b$-metric spaces, *Gulf J. Math.*, **14** (2022), 69–83. https://doi.org/10.56947/gjom.v14i1.984

16. G. Mani, A. J. Gnanaprakasam, O. Ege, N. Fatima, N. Mlaiki, Solution of Fredholm integral equation via common fixed point theorem on bicomplex valued $b$-metric space, *Symmetry*, **15** (2023), 297. https://doi.org/10.3390/sym15020297

17. Z. H. Gu, G. Mani, A. J. Gnanaprakasam, Y. J. Li, Solving a system of nonlinear integral equations via common fixed point theorems on bicomplex partial metric space, *Mathematics*, **9** (2021), 1584. https://doi.org/10.3390/math9141584

18. C. Klin-eam, C. Suanoom, Some common fixed point theorems for generalized contractive type mappings on complex valued metric spaces, *Abstr. Appl. Anal.*, **2013** (2013), 604215. https://doi.org/10.1155/2013/604215

19. F. Rouzkard, M. Imdad, Some common fixed point theorems on complex valued metric spaces, *Comput. Math. Appl.*, **64** (2012), 1866–1874. https://doi.org/10.1016/j.camwa.2012.02.063

20. W. Sintunavarat, P. Kumam, Generalized common fixed point theorems in complex valued metric spaces and applications, *J. Inequal. Appl.*, **2012** (2012), 84. https://doi.org/10.1186/1029-242X-2012-84

21. K. Sitthikul, S. Saejung, Some fixed point theorems in complex valued metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 189. https://doi.org/10.1186/1687-1812-2012-189

22. J. Ahmad, C. Klin-eam, A. Azam, Common fixed points for multivalued mappings in complex valued metric spaces with applications, *Abstr. Appl. Anal.*, **2013** (2013), 854965. https://doi.org/10.1155/2013/854965

23. M. S. Abdullahi, A. Azam, Multivalued fixed points results via rational type contractive conditions in complex valued metric spaces, *J. Int. Math. Virtual Inst.*, **7** (2017), 119–146. https://doi.org/10.7251/JIMVI1701119A

24. A. Azam, J. Ahmad, P. Kumam, Common fixed point theorems for multi-valued mappings in complex-valued metric spaces, *J. Inequal. Appl.*, **2013** (2013), 578. https://doi.org/10.1186/1029-242X-2013-578

25. M. A. Kutbi, J. Ahmad, A. Azam, N. Hussain, On fuzzy fixed points for fuzzy maps with generalized weak property, *J. Appl. Math.*, **2014** (2014), 549504. https://doi.org/10.1155/2014/549504

26. M. Humaira, G. N. V. Kishore, Fuzzy fixed point results for $\varphi$ contractive mapping with applications, *Complexity*, **2018** (2018), 5303815. https://doi.org/10.1155/2018/5303815

27. J. Carmel Pushpa Raj, A. Arul Xavier, J. Maria Joseph, M. Marudai, Common fixed point theorems under rational contractions in complex valued extended $b$-metric spaces, *Int. J. Nonlinear Anal. Appl.*, **13** (2022), 3479–3490. https://doi.org/10.22771/nfaa.2021.26.04.03

28. L. C. Ceng, N. J. Huang, C. F. Wen, On generalized global fractional-order composite dynamical systems with set-valued perturbations, *J. Nonlinear Var. Anal.*, **6** (2022), 149–163. https://doi.org/10.23952/jnva.6.2022.1.09