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Research article

Remarks on the end-topology of some discrete groups

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Abstract: In this note we consider the notion of rate of vanishing of the simple connectivity at infinity, a (growth) function that estimates metrically the topology at infinity of metric spaces. In particular we provide a different (geometric) proof of the linearity of the sci-growth for hyperbolic groups.

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1. Introduction

One of the main problems in low-dimensional topology is the celebrated Poincaré conjecture. A closely related less-known problem is the so-called Universal Covering Conjecture, which states that universal covering spaces of three-manifolds are simply connected at infinity (for more on this see e.g. [12, 13]). Another way to put this is that all fundamental groups of 3-manifolds are simply connected at infinity. This is a topological condition at infinity for non-compact spaces which says that the infinity of the space is simply connected. More precisely, a non-compact, connected topological space X is simply connected at infinity (abbr. sci) if for any compact of X there exists a larger compact such that any loop outside it bounds a disc which sits outside the small compact (for details see [7]).

In terms of Gromov's theory of random groups [8], there are actually very few finitely presented groups which are fundamental groups of three-manifolds. Moreover, Davis [5] showed that the simple connectivity at infinity is indeed a very rare property outside of those three-manifolds groups.

But then, Perelman [11] did not only prove the Poincaré conjecture, he also proved the Thurston conjecture, namely the geometrization of three-manifolds. And one of the crowning items there, is the universal covering conjecture, namely the simple connectivity at infinity for fundamental groups of three-manifolds. This is then really a highly non-trivial fact; actually, even more power of the Ricci flow is necessary for the geometrization than for the Poincaré conjecture itself.

This is just to explain that the issue of the simple connectivity at infinity is important and difficult.

But, in this paper, we will consider this topological notion from another viewpoint. Since given an infinite finitely presented group G, one can construct a compact space X_G with G as fundamental group and a non-compact space, $\tilde{X}_G$, on which it acts in a good way (i.e. the universal covering space of X), one can wonder whether the topological properties of $\tilde{X}_G$ may be considered as group-theoretical notions. This is the case for the simple connectivity at infinity, in the sense that if, given G, $\tilde{X}_G$ is sci, then for any other compact space X' with $\pi_1 X' = G$, then $\tilde{X}'$ will also be sci. Furthermore, being sci is a quasi-isometry invariant of finitely presented groups [7], following Gromov's point of view of discrete groups as geometric objects [8].

In his revolutionary production, Gromov abstracted the classical notion of hyperbolicity and introduced the celebrated class of hyperbolic groups (see the next section for a definition), as the class of finitely presented groups which somehow generalizes the class of fundamental groups of compact Riemannian manifolds with strictly negative sectional curvature [4, 8].

In the present paper we will focus on the behaviour at infinity of sci hyperbolic groups.

Definition 1.1. *Let X be a connected, simply connected, non-compact metric space. If X is also simply connected at infinity (sci), one defines the rate of vanishing of the sci, called also the sci-growth and denoted by $V_X(r)$, as the infimal $N(r)$ with the property that any loop which sits outside the ball $B(N(r))$ of radius $N(r)$ bounds a 2-disk outside $B(r)$.*

It is proved in [7] that the growth of the function $V_X(r)$ (i.e. the equivalence class with respect to the standard equivalence relation for real functions) is a quasi-isometry invariant. In particular, if G is a finitely presented sci group, then $V_G = V_{\tilde{X}_G}$ is a quasi-isometry invariant of G, where $\tilde{X}_G$ is the universal covering space of a compact simplicial complex X_G, with $\pi_1(X_G) = G$. If V_G is defined and linear we say that G has linear sci-growth.

In the next sections we will provide a new, direct and more geometric proof of the following result (proved in [6] with different techniques):

Theorem 1.1. *If G is a sci (simply connected at infinity) hyperbolic group then V_G is linear.*

2. Preliminary lemmas

Let (X, d) be a geodesic metric space, which in our case will be the Cayley graph of a finitely generated group G (see [4, 8] and here below). A geodesic triangle is said to be δ-*slim* if every side of it is contained in the δ-neighbourhood of the union of its other sides (with $\delta \in \mathbb{R}_+$).

Given a geodesic triangle Δ with vertices x, y, z in X, let Δ' be a Euclidean comparison triangle with vertices x', y', z' and sides of the same lengths as those of Δ, and let $f : \Delta \to \Delta'$ be an identification map. Suppose that the maximal inscribed circle C in Δ' meets its sides at $c'_z \in [x', y']$, $c'_x \in [y', z']$ and $c'_y \in [x', z']$. There is a unique isometry f_Δ of Δ' into a metric tripod T (i.e. a tree with one vertex w of degree 3 and three vertices x'', y'', z'' of degree one), such that:

$$d(w, x'') = d(x', c'_z) = d(x', c'_y),$$

$$d(w, y'') = d(y', c'_z) = d(y', c'_x),$$

$$d(w, z'') = d(z', c'_x) = d(z', c'_y).$$

Denote $F = f_\Delta \circ f : \Delta \to T$. The triangle Δ is δ-*thin* if for all $p \in T$, the diameter $diam(F^{-1}(p)) \leqslant \delta$.

The group G is *δ-hyperbolic*, for some real or natural number $\delta \geqslant 0$, if all geodesic triangles in X are δ-thin (and then, all geodesic triangles are also δ-slim). The group G is hyperbolic if it is δ-hyperbolic for some $\delta \geqslant 0$. For proofs and details see [4, 8].

Remark 2.1. *The notion of hyperbolicity for a group is independent of the choice of the presentation. It is also clear that if G is δ-hyperbolic, for some $\delta \geqslant 0$, then it is also δ'-hyperbolic, for any $\delta' > \delta$.*

We recall that a (discrete) group Γ is finitely generated if there exists a finite set S of generators for it (so that every element of Γ can be written as a product of powers of some of these generators $s_i \in S$), while it is finitely presented if, in addition, it has a finite number of relations $r_j \in \mathcal{R}$ (words in terms of the generators, such that, $\forall w \in \Gamma$, $w = 1$ if and only if w can be written as a product of conjugates of these words and their inverses). In such a case one says that Γ has the presentation $\langle S \mid \mathcal{R} \rangle$.

To any finitely generated group Γ with generating set $\langle S \rangle$ one can associate a 'natural' metric on it, called the word metric, by defining the distance (with respect to S) of two elements a, b of Γ as $d_S(a, b) = l_S(a^{-1}b)$, where the length $l_S(g)$ of any element g of Γ is the smallest integer n such that there exists a sequence $(s_1, s_2, \cdots, s_n)$ of generators in S for which $g = s_1 s_2 \ldots s_n$.

With this metric, we may also associate to Γ a graph, called the Cayley graph, whose vertex set is identified with the elements of the group, and such that for any $g \in \Gamma$ and $s \in S$, the vertices corresponding to the elements g and gs are joined by an edge labeled by s.

Whenever the group Γ is finitely presented, one can additionally associate to Γ a 2-dimensional space too. Let $\mathcal{P} = \langle S \mid \mathcal{R} \rangle$ be a finite presentation for the group Γ. The Cayley 2-complex of the presentation is a 2-dimensional complex obtained by gluing a disk on all paths of the Cayley graph labeled by a relator $r \in R$.

Suppose from now on that G is a δ-hyperbolic finitely presented group and X its Cayley graph associated with a finite presentation $\mathcal{P} = \langle S \mid R \rangle$ of G.

Let $\gamma : [0, \infty) \to X$ be a geodesic path, either finite or infinite. For any $x, y \in \gamma$, we denote by $[x, y]_\gamma$ the subpath of γ that connects x to y.

Bestvina and Mess [1] proved the following:

Proposition 2.1. *[1] Let G be a hyperbolic one-ended group. There is a constant $c \geqslant 0$ so that for all $x \in X$ there exists an infinite geodesic ray starting at the identity of G which passes within c of x.*

We say that two geodesic rays are *asymptotic* if their images in X are at finite Hausdorff distance. This defines an equivalence relation on the collection of geodesic rays in X. The boundary ∂X of X is the collection of equivalence classes, under this relation, of geodesic rays in X.

Lemma 2.1. *([4], III.H, Lemma 3.3) Let γ_1, $\gamma_2 : [0, \infty) \to X$ be two asymptotic, unit speed geodesic rays. Then:*

(1) If $\gamma_1(0) = \gamma_2(0)$, then $d(\gamma_1(t), \gamma_2(t)) \leqslant 2\delta$ for all $t > 0$.
(2) In general, there exist t_1, $t_2 \in (0, \infty)$ such that $d(\gamma_1(t_1 + t), \gamma_2(t_2 + t)) \leqslant 5\delta$ for all $t \geqslant 0$.

Given a class $\gamma(\infty) \in \partial X$ of a geodesic ray γ, there is a unit speed geodesic ray starting from the identity of G which is asymptotic to γ. Thus, we identify ∂X with the collection of asymptotic classes of unit speed geodesic rays starting at 1 (see [4, 10]). We say that a geodesic ray $\gamma : [0, \infty) \to X$ connects the point $\gamma(0) \in X$ to a point $x \in \partial X$ if x is the equivalence class of γ, i.e. $x = \gamma(\infty)$. Let $\gamma : (-\infty, \infty) \to X$ be a bi-infinite geodesic in X. We denote by γ^- and γ^+ the geodesic rays whose image is equal to γ restricted to $(-\infty, 0]$ and $[0, \infty)$ respectively. Moreover, we say that γ connects two points x, $y \in \partial X$ if $\{x, y\} = \{\gamma^-(-\infty), \gamma^+(\infty)\}$. Notice also that any two distinct points in ∂X are connected by a bi-infinite geodesic.

All geodesics considered will be assumed to be unit speed geodesics. Let γ_1, $\gamma_2 : [0, \infty) \to X$ be two geodesic rays starting from the identity. We say that γ_1 and γ_2 diverge if they are not asymptotic, i.e. they correspond to different points on ∂X. Moreover, if t_0 is the infimal $t > 0$ such that the distance from $\gamma_1(t)$ to $\gamma_2(t)$ is greater than δ, then we say that γ_1 and γ_2 *diverge at* t_0. The continuous function $d(\gamma_1(t), \gamma_2(t))$ goes from 0 to ∞, therefore $d(\gamma_1(t_0), \gamma_2(t_0)) = \delta$.

Lemma 2.2. *([4], III.H, Lemma 3.2) Let $\gamma_1, \gamma_2 : [0, \infty) \to X$ be two divergent geodesic rays in X, issued from the identity that correspond to points x, $y \in \partial X$. There is a bi-infinite geodesic γ that joins x to y and is contained in the δ-neighborhood of $\gamma_1 \cup \gamma_2$.*

There is a natural topology on $X \cup \partial X$ making it a compact metrizable space. Let $\alpha > 1$ and $x \in X$. We say that a metric d_α on ∂X is a *visual metric* with base point x and visual parameter α if there is $c > 0$, the constant of the visual metric, so that:

(1) The metric d_α induces the natural boundary topology on ∂X.
(2) For any distinct points x, $y \in \partial X$ and any bi-infinite geodesic γ connecting them, we have

$$\frac{1}{c} \cdot \alpha^{-d(\gamma,x)} \leqslant d_\alpha(x, y) \leqslant c \cdot \alpha^{-d(\gamma,x)}.$$

Since (X, d) is a proper δ-hyperbolic space, there is $\alpha_0 > 1$, called the global visual parameter of X, such that for any base point x_0 and any $\alpha \in (1, \alpha_0)$, the boundary ∂X admits a visual metric d_α with respect to x_0 (see [4, 9]).

For the purpose of this paper we will consider a visual metric, d_{2^α}, on ∂X with base point the identity of G and visual parameter $2^\alpha \in (1, \alpha_0)$ for some appropriate $\alpha \in \mathbb{R}$. If c is the constant of this visual metric, let $c_1 \in \mathbb{R}$ be minimal such that $c \leqslant 2^{c_1}$. Then, for all $x, y \in \partial X$ and any bi-infinite geodesic γ that connects x and y, we have

$$2^{-c_1 - \alpha \cdot d(1,\gamma)} \leqslant d_{2^\alpha}(x, y) \leqslant 2^{c_1 - \alpha \cdot d(1,\gamma)}.$$

We say that α and c_1 are the 2-visual parameters of the visual metric d_{2^α}. For sake of simplicity we will use from now on $d_{\partial X}$ for the aforementioned visual metric d_{2^α} on ∂X.

Let $x, y \in \partial X$ and $t > 0$. A *t-chain* from x to y is a sequence of points $l_1 = x, l_2, \ldots, l_k = y$ in ∂X, for some $k > 1$, such that, for all $i \in \{1, 2, \ldots, k-1\}$, $d_{\partial X}(l_i, l_{i+1}) \leqslant t$. The length of a t-chain is the number of points it consists of.

The crucial point in the proof of Theorem 1.1, is the following result due to Bonk and Kleiner [2]:

Proposition 2.2. *[2] Let G be a one-ended hyperbolic group and $d_{\partial X}$ a visual metric on ∂X. There are constants $c, K > 0$ so that for all $x, y \in \partial X$, $t \in \mathbb{Z}_+$ there is a $\frac{1}{2^t} d(x, y)$-chain of length at most c^t that connects x to y and whose diameter is at most $K d_{\partial X}(x, y)$.*

Remark 2.2. *Proposition 2.2 actually states that ∂X is linearly connected and derives from a result of Bowditch, Svenson and Swarup [3, 14, 15] which states that ∂X has no global cut points.*

When X is the Cayley complex of a group G associated with a finite presentation $\mathcal{P} = \langle S \mid R\rangle$, we will only consider geodesics within the Cayley graph, namely the 1-skeleton $X^{(1)}$ of X. Notice that while the Cayley complex may change when adding words equal to the identity to the relators in $\mathcal{P}$, the Cayley graph remains unchanged.

The following lemmas and propositions will also be used in the proof of Theorem 1.1:

Lemma 2.3. *Let G be a δ-hyperbolic group, X its Cayley complex associated with a presentation of G that contains as relators all words of length less than 8δ which are equal to the identity in G. Suppose that $n > 0$ and Δ is a geodesic triangle in X outside the ball $B(n + 1.5\delta)$. If one side of Δ has length less than δ, then Δ can be filled outside $B(n)$ in X.*

Proof. Let α, β, γ be the sides of the geodesic triangle Δ and $x = \alpha \cap \gamma$, $y = \beta \cap \gamma$, $z = \alpha \cap \beta$ its vertices. We may assume that the lengths of its sides satisfy $\ell(\gamma) \leqslant \ell(\alpha) \leqslant \ell(\beta)$. Also, we remark that, since the triangle Δ is δ-slim and $\ell(\gamma) < \delta$, for any $x' \in \alpha$, $y' \in \beta$ with $d(x', z) = d(y', z)$ we have that $d(x', y') \leqslant 3\delta$. For any $i = 1, \dots, \ell(\alpha)$ we consider a geodesic segment w_i that connects $\alpha(i)$ to $\beta(i)$ and define the polygon R_i to be one with sides $[\alpha(i-1), \alpha(i)]_\alpha$, w_i, $[\beta(i-1), \beta(i)]_\beta$ and w_{i-1}, where w_0 is trivially the point z. Each R_i is outside $B(n)$ and corresponds to a word in G of length less than 8δ which is equal to the identity. Thus, R_i can be filled by a disc D_i in X outside the ball $B(n)$. If $\ell(\alpha) = \ell(\beta)$, let $R_{\ell(\alpha)+1} = \emptyset$, otherwise let $R_{\ell(\alpha)+1}$ be the remaining triangle with sides $w_{\ell(\alpha)}$, γ, $[\beta(\ell(\alpha)), y]_\beta$. Then, $R_{\ell(\alpha)+1}$ corresponds to a word in G of length less than 8δ which is equal to the identity and thus can be filled by a disc $D_{\ell(\alpha)+1}$ outside the ball $B(n)$. Therefore, the triangle Δ can be filled outside the ball $B(n)$ by the simplicial disc $D = \bigcup_{i=1}^{\ell(\alpha)+1} D_i$. □

Lemma 2.4. *Let G be a finitely presented group, $X^{(1)}$ its Cayley graph, $x \in X^{(1)}$ a vertex and β a geodesic ray in $X^{(1)}$ starting from the identity. Let $z \in \beta$ with $d(z, x) = d(\beta, x)$. For any $y \in \beta$ with $d(y, 1) \geq d(z, 1)$, if η is a geodesic from x to y, then,*

$$d(1, \eta) \geqslant d(x, 1) - d(x, \beta).$$

Proof. We have $d(y, z) = d(y, 1) - d(z, 1)$, and by the triangle inequality $\ell(\eta) \leqslant d(x, z) + d(y, 1) - d(z, 1)$. Let v be a point on η which is closest to 1. The triangle inequalities applied in the triangles of vertices $1, x, v$ and $1, y, v$ give $\ell(\eta) \geq d(x, 1) + d(y, 1) - 2d(1, v)$. Therefore, $2d(\eta, 1) \geq d(x, 1) - d(x, z) + d(z, 1) \geq 2d(x, 1) - 2d(x, z)$. □

Proposition 2.3. *If γ_1 and γ_2 are two geodesic rays issued from 1 which diverge at t_0 then there is a bi-infinite geodesic γ, that connects $\gamma_1(\infty)$ to $\gamma_2(\infty)$ and*

$$t_0 - 2.5\delta \leq d(1, \gamma) \leq t_0 + \delta.$$

Proof. From Lemma 2.2, we have that there is a bi-infinite geodesic, γ, that joins $\gamma_1(\infty)$ to $\gamma_2(\infty)$ and is contained in the δ-neighborhood of γ_1, γ_2. This implies that the ideal triangle, Δ, of vertices $1, \gamma_1(\infty), \gamma_2(\infty)$ is δ-slim. Suppose that $w \in \gamma$ with $d(w, 1) = d(\gamma, 1)$. For any $t < d(w, 1) - \delta$, we obviously have that $d(\gamma_1(t), \gamma)$, $d(\gamma_2(t), \gamma) > \delta$. The ideal triangle Δ being δ-thin, it follows that $d(\gamma_1(t), \gamma_2(t)) \leq \delta$. This yields that $t_0 \geq d(1, w) - \delta$, which establishes the right hand side inequality.

Now, suppose that $d(1, w) < t_0 - \delta$. We set $\gamma^- = [w, \gamma_1(\infty)]_\gamma$ and $\gamma^+ = [w, \gamma_2(\infty)]_\gamma$. There are $w_1 \in \gamma^-$, $w_2 \in \gamma^+$ so that $d(w_1, 1)$, $d(w_2, 1) = t_0$. It follows that $w \in [w_1, w_2]_\gamma$, and therefore,

$$d(1,w) \geq t_0 - \frac{d(w_1,w_2)}{2}. \tag{2.1}$$

Since $d(1,w) < t_0 - \delta$, it follows from (2.1) that $d(w_1,w_2) > \delta$. The fact that Δ is δ-slim further implies that there are $u,z \in \gamma_1 \cup \gamma_2$ such that $d(w_1,u)$, $d(w_2,z) \leq \delta$. By the triangle inequality in the triangles of vertices $1,u,w_1$ and $1,z,w_2$ we derive that $t_0-\delta \leq d(1,u) \leq t_0+\delta$ and $t_0-\delta \leq d(1,z) \leq t_0+\delta$. We distinguish two cases for u,z: either they belong to the same geodesic ray or to different ones.

In the first case, without loss of generality we assume that $u,z \in \gamma_1$, so there are t_1,t_2 so that $u = \gamma_1(t_1)$ and $z = \gamma_1(t_2)$. Hence, $|t_1 - t_2| \leq 2\delta$ and the triangle inequality shows that $d(w_1,w_2) \leq d(w_1,u) + d(u,z) + d(z,w_2) \leq 4\delta$.

In the second case, without loss of generality we assume that $u \in \gamma_1$, $z \in \gamma_2$, so there are t_1,t_2 so that $u = \gamma_1(t_1)$ and $z = \gamma_2(t_2)$. Hence $|t_0 - t_1|$, $|t_0 - t_2| \leq \delta$ and the triangle inequality shows that $d(w_1,w_2) \leq d(w_1,u) + d(u,\gamma_1(t_0)) + d(\gamma_1(t_0),\gamma_2(t_0)) + d(\gamma_2(t_0),z) + d(z,w_2) \leq 5\delta$.

Hence, in any case, $d(w_1,w_2) \leq 5\delta$. Equation (2.1) then yields that $d(1,w) \geq t_0 - 2.5\delta$, so our left hand side inequality follows. □

Proposition 2.4. *Let γ_1 and γ_2 be two geodesic rays issued from* 1 *which diverge at t_0. If $t \geq 0$ such that $d(\gamma_1(t),\gamma_2(t)) < \delta$, then,*

$$t \leq (t_0 + 3.5\delta).$$

Proof. Assume that $d(\gamma_1(t),\gamma_2(t)) < \delta$ for some $t > t_0 + \delta$, else there is nothing to prove. From Proposition 2.3, we have that there is a bi-infinite geodesic γ, contained in the δ-neighborhood of $\gamma_1 \cup \gamma_2$ and such that $d(1,\gamma) \leq t_0 + \delta$, so $d(1,\gamma) < t$. As in the previous proof, if $w \in \gamma$ with $d(1,w) = d(1,\gamma)$, then there are $w_1,w_2 \in \gamma$ such that $d(w_1,1)$, $d(w_2,1) = t$, $w \in [w_1,w_2]_\gamma$, and therefore,

$$d(1,w) \geq t - \frac{d(w_1,w_2)}{2}. \tag{2.2}$$

Again, the fact that the ideal triangle Δ of vertices $1,\gamma_1(\infty),\gamma_2(\infty)$ is δ-slim, further implies that there are $u,z \in \gamma_1 \cup \gamma_2$ such that $d(w_1,u)$, $d(w_2,z) \leq \delta$. By the triangle inequality in the triangles of vertices $1,u,w_1$ and $1,z,w_2$ we derive that $t-\delta \leq d(1,u) \leq t+\delta$ and $t-\delta \leq d(1,z) \leq t+\delta$. We distinguish two cases for u,z, either they belong to the same geodesic ray or to different ones.

In the first case, without loss of generality we assume that $u,z \in \gamma_1$, so there are t_1,t_2 so that $u = \gamma_1(t_1)$ and $z = \gamma_1(t_2)$. Hence, $|t_1 - t_2| \leq 2\delta$ and the triangle inequality shows that $d(w_1,w_2) \leq d(w_1,u) + d(u,z) + d(z,w_2) \leq 4\delta$.

In the second case, without loss of generality we assume that $u \in \gamma_1$, $z \in \gamma_2$, so there are t_1,t_2 so that $u = \gamma_1(t_1)$ and $z = \gamma_2(t_2)$. Hence, $|t - t_1|$, $|t - t_2| \leq \delta$ and the triangle inequality shows that $d(w_1,w_2) \leq d(w_1,u) + d(u,\gamma_1(t_0)) + d(\gamma_1(t_0),\gamma_2(t_0)) + d(\gamma_2(t_0),z) + d(z,w_2) \leq 5\delta$.

In any case, $d(w_1,w_2) \leq 5\delta$. Equation (2.2) then yields that $d(1,w) \geq t - 2.5\delta$, so our inequality follows. □

Corollary 2.1. *Let γ_1 and γ_2 be two geodesic rays issued from* 1 *which diverge at t_0. If for $p \in \gamma_1$, $q \in \gamma_2$ we have that $d(p,q) < \frac{\delta}{2}$, then $p,q \in B(t_0 + 3.5\delta)$.*

Proof. Let $t_1,t_2 \geq 0$ with $p = \gamma_1(t_1)$ and $q = \gamma_2(t_2)$. Since $d(p,q) < \frac{\delta}{2}$, we get that $|t_1 - t_2| < \frac{\delta}{2}$ and the triangular inequality in the triangle of vertices $p,q,\gamma_2(t_1)$ gives

$$d(\gamma_1(t_1), \gamma_2(t_1)) \leq d(p, q) + d(q, \gamma_2(t_1)) \leq \delta.$$

Therefore, Proposition 2.4 gives that $t_1 \leq t_0 + 3.5\delta$. Similarly, we derive that the same holds for t_2. □

3. Proof of Theorem 1.1

Let us consider first the case when G is a one-ended, sci hyperbolic group. If c_1 is the constant obtained in Proposition 2.1 we can assume that X is δ-hyperbolic and $\delta \in \mathbb{N}$ with $\delta > 4c_1 + 2$.

We consider a visual metric on ∂X, denoted again by $d_{\partial X}$, with base point the identity of G and 2-visual parameters α, c, for appropriate $\alpha, c \in \mathbb{R}$. Suppose that c_2, K are the constants of Proposition 2.2, we can further assume that $\delta > \frac{2c+\log K}{\alpha}$.

Without loss of generality we can also assume that the Cayley complex X is associated with a presentation of G that contains as relators all words of length less that 8δ which are equal to the identity in G.

Let $n \in \mathbb{N}$, with $n > 13\delta$. We will show that every loop f outside $B(n + 13\delta)$ is null homotopic outside $B(n)$. Since G is sci, there is $M > 0$ so that every loop outside $B(M)$ is null homotopic outside $B(n)$. Thus, it is enough to consider the case when f is inside $B(M)$ and consequently that $M > n+13\delta$.

Let p, q be two vertices on f, with $d(p, q) = 1$. There are unit speed geodesic rays, γ_1, γ_2 issued from the identity which pass within c_1 of p, q, respectively. Denote by x, y the corresponding points on ∂X. Also, we denote by p' a closest point on γ_1 to p and q' a closest point on γ_2 to q.

Case 1. Suppose that $x \neq y$, and so γ_1, γ_2 diverge.

Lemma 3.1. *There exist a bi-infinite geodesic γ that connects x to y and*

$$d(1, \gamma) > (n + 6.75\delta).$$

Proof. Suppose that γ_1 and γ_2 diverge at t_0. As $d(p, p') \leq c_1$ and $d(q, q') \leq c_1$ we have $d(p', q') \leq 2c_1 + 1 < \frac{\delta}{2}$. Then, according to Corollary 2.1 we should have $d(p', 1) \leq t_0 + 3.5\delta$. But, $d(1, p') \geq d(1, p) - c_1 > n + 12.75\delta$, so $t_0 > n + 9.25\delta$. Proposition 2.3 then gives us that there is a bi-infinite geodesic γ joining x and y that verifies the desired inequality. □

Therefore, we have

$$d_{\partial X}(x, y) \leqslant 2^{c-\alpha\cdot d(1,\gamma)} < 2^{c-\alpha\cdot(n+6.75\delta)}.$$

Lemma 3.2. *There are $k > 0$ and a sequence of points $(w_1, \ldots, w_k)$ which are interpolated by the path $W(p, q)$ with the following properties:*

(1) $W(p, q) \subset X - B(M + \delta)$;
(2) $w_1 \in \gamma_1$, $w_k \in \gamma_2$;
(3) For all $i \in \{1, \ldots k\}$, $d(w_i, w_{i+1}) < \delta$;
(4) For all $i \in \{1, \ldots k\}$, if η_i is a geodesic path from p to w_i, then $d(1, \eta_i) > n + 1.75\delta$.

Proof. Let

$$T = \min\{t \in \mathbb{Z}_+; t > (M - 3.75\delta - n)\alpha + 2c\}.$$

We recall that c_2, K are the constants of Proposition 2.2, and so there is a $\frac{1}{2^T}d_{\partial X}(x, y)$-chain $L = \{l_1 = x, \ldots, l_k = y\}$ in ∂X of length $k = {c_2}^T$ that joins x to y and $diam(L) \leq K d_{\partial X}(x, y)$. This means that for all $i = 1, \ldots, k - 1$, we have

$$0 < d_{\partial X}(l_i, l_{i+1}) \leq \frac{1}{2^T} d_{\partial X}(x, y) < 2^{c-\alpha\cdot(n+6.75\delta)-T}.$$

Moreover, if L_i is a bi-infinite geodesic in X that connects the points l_i and l_{i+1}, then,

$$d_{\partial X}(l_i, l_{i+1}) \geq 2^{-c-\alpha\cdot d(1,L_i)}.$$

The last two inequalities and the choice of T imply that

$$d(1, L_i) > \frac{\alpha(n + 6.75\delta) + T - 2c}{\alpha} > M + 3\delta. \tag{3.1}$$

Now, for all $i = 1, \ldots, k$ let β_i be a geodesic ray joining 1 to l_i and let w_i be a point of β_i at distance $M + 2\delta$ from 1. Without loss of generality we can assume that $\beta_1 = \gamma_1$ and $\beta_k = \gamma_2$, respectively.

We claim that relation (3.1) implies that β_i and β_{i+1} diverge outside $B(M + 2\delta)$. In fact, if β_i and β_{i+1} diverged at $t_{0,i} \leq M + 2\delta$, then Proposition 2.3 would provide us a geodesic L_i connecting l_i and l_{i+1} such that $d(1, L_i) \leq M + 3\delta$, therefore contradicting (3.1). This claim implies that $d(w_i, w_{i+1}) \leq \delta$ and thus we can join w_i and w_{i+1} with a geodesic path $w(i, i + 1)$ lying outside $B(M + \delta)$. We then set $W(p, q)$ to be the union of these paths: $W(p, q) = \bigcup_{i=1}^{k-1} w(i, i + 1)$.

From Proposition 2.2, we have that $d_{\partial X}(x, l_i) \leq K d_{\partial X}(x, y)$. Hence, if E_i is a bi-infinite geodesic joining x and l_i, as before, we get

$$2^{-c-\alpha d(1,E_i)} \leq d_{\partial X}(x, l_i) \leq 2^{\log K+c-\alpha\cdot(n+6.75\delta)},$$

so that, since $\delta > \frac{2c+\log K}{\alpha}$,

$$d(1, E_i) \geq (n + 5.75\delta).$$

We conclude as before, using Proposition 2.3, that γ_1 and β_i diverge outside the ball $B(n + 4.75\delta)$. Let then $p'' \in \gamma_1$ and $z_i \in \beta_i$ be points at distance $n + 4.75\delta$ from 1. The divergence condition implies that $d(p'', z_i) \leq \delta$. On the other hand the triangle inequality:

$$d(p, z_i) \leq d(p, p') + d(p', p'') + d(p'', z_i)$$

implies that

$$d(p, \beta_i) \leq d(p, z_i) \leq d(p', 1) - n - 2.75\delta \leq d(1, p) - (n + 1.75\delta).$$

Suppose that η_i is a geodesic path from p to w_i. Then Lemma 2.4 yields us

$$d(1, \eta_i) > d(p, 1) - d(p, \beta_i) \geq n + 1.75\delta,$$

and this ends the proof of our lemma. □

Let P, Q be geodesic paths that join p to p' and q to q' respectively. We set $\Phi(p, q)$ to be the following closed loop:

$$\Phi(p, q) = P \cup [p', w_1]_{\gamma_1} \cup W(p, q) \cup [w_k, q']_{\gamma_2} \cup Q^{-1} \cup [p, q]_f^{-1}.$$

Lemma 3.3. *The loop $\Phi(p, q)$ is null homotopic outside $B(n)$.*

Proof. Let $\Sigma = \{s_1 = q', \ldots, s_{k'} = w_k\}$ be a set of points on $[w_k, q']^{-1}_{\gamma_2}$, such that for all $i = 1, \ldots, k'-1$ we have $d(s_i, s_{i+1}) < \delta$. For any $i = 1, \ldots, k'-1$, let Δ_i be a geodesic triangle of vertices p, s_i, s_{i+1} and such that one of its sides is $[s_i, s_{i+1}]_{\gamma_2}$. Also, let Δ_0 be a geodesic triangle of vertices p, q and q' and such that two of its sides are $[p, q]_f$ and Q. From Lemma 2.4 it follows that, if S_i is a geodesic path from p to s_i, then,

$$d(S_i, 1) \geqslant d(p, 1) - d(p, \gamma_2) > n + 12\delta.$$

Therefore, $d(\Delta_i, 1) > n + 12\delta$, and so, from Lemma 2.3 it follows that it can be filled by a disc outside $B(n + 10.5\delta)$.

We proceed similarly to join p to the points w_i on $W(p, q)$. Specifically, from the properties of the path $W(p, q)$ we get that the corresponding triangles are outside the ball $B(n + 1.75\delta)$ and from Lemma 2.3 we get that these triangles can be filled outside $B(n + 0.25\delta)$.

The union, $D(p, q)$, of all the fillings (Van Kampen diagrams) of the triangles we have considered, fills $\Phi(p, q)$ outside $B(n)$, as wanted. □

Case 2. Suppose that $x = y$, and so γ_1 and γ_2 are asymptotic.

By Lemma 2.1 we have that there are t_1, $t_2 > 0$ so that $\gamma_1([t_1, \infty])$ and $\gamma_2([t_2, \infty])$ travel within 5δ of each other. For $i = 1, 2$, there are $w_i \in \gamma_i([t_i, \infty])$ with $d(w_1, 1)$, $d(w_2, 1) > M + 10\delta$ and $d(w_1, w_2) \leq 5\delta$. Let $W(p, q)$ to be a geodesic path that joins them, so that its length is at most 5δ and $W(p, q) \subset X \setminus B(M + 7.5\delta)$. We can now proceed as for Case 1 and show that the corresponding loop $\Phi(p, q)$ can be filled outside $B(n)$.

To sum up, we have considered two cases: $x = y$ and $x \neq y$. But the strategy for both cases has been the same. We started with two points p, q on f at distance $d(p, q) = 1$ and created a closed loop $\Phi(p, q)$ one part of which is $[p, q]_f$ and another is a path, $W(p, q)$, that is outside $B(M + \delta)$. We proved that the closed loop $\Phi(p, q)$ can be filled by a disk $D(p, q)$ outside $B(n)$. Moreover, the paths $W(p, q)$ can be chosen in a way such that their union, over all points p, q of distance 1 on f, creates a closed loop f_1 outside $B(M + \delta)$:

$$f_1 = \bigcup_{\substack{p,q \in f \\ d(p,q)=1}} W(p, q).$$

Since G is simply connected at infinity, the closed loop f_1 can be filled by a disk A_1 outside $B(n)$. On the other hand, we can also fill the ring between f and f_1 with the following simplicial disc A_2 outside $B(n)$:

$$A_2 = \bigcup_{\substack{p,q \in f \\ d(p,q)=1}} D(p, q).$$

Thus, the loop f is filled by $A_1 \cup A_2$ outside $B(n)$ too.

In conclusion, for all $n > 12\delta$, any loop outside $B(n + 13\delta)$ is null homotopic outside $B(n)$, and therefore the group G has linear sci-growth.

If G is not one-ended, we can do the same work for each connected component of $X \setminus B(n + 13\delta)$.

4. Conclusions

In this paper we continued the exploration of the metric measurement of the notion of simple connectivity at infinity. This belongs to a very general research field, namely the study of topological invariants of discrete groups, which is not as developed and exploited as that of invariants of a geometric nature, and there are still many deep open questions to be studied. First of all, even if randomly any finitely presented group is hyperbolic (and hence, if sci, with a linear sci-growth), it is probable that there exists at least one sci group with a super-linear sci-growth, and this would be an interesting strange example of a discrete group with an exotic behaviour at infinity. On the other hand, if one could prove that all finitely presented sci groups have a linear sci-growth, this would be a very fascinating result, because it would provide a typical example of rigidity due to the presence of a group action, since one can easily construct examples of open sci manifolds with a non-linear sci growth.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest in this paper.

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