Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 20125–20142.
DOI: 10.3934/math.20231025
Received: 28 February 2023
Revised: 11 May 2023
Accepted: 24 May 2023
Published: 19 June 2023

Research article

On the boundedness of the solution set for the ψ-Caputo fractional pantograph equation with a measure of non-compactness via simulation analysis

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Abstract: A large number of physical phenomena can be described and modeled by differential equations. One of these famous models is related to the pantograph, which has been investigated in the history of mathematics and physics with different approaches. Optimizing the parameters involved in the pantograph is very important due to the task of converting the type of electric current in the relevant circuit. For this reason, it is very important to use fractional operators in its modeling. In this work, we will investigate the existence of the solution for the fractional pantograph equation by using a new ψ-Caputo operator. The novelty of this work, in addition to the ψ-Caputo fractional operator, is the use of topological degree theory and numerical results from simulations. Techniques in fixed point theory and the use of inequalities will also help to prove the main results. Finally, we provide two examples with some graphical and numerical simulations to make our results more objective. Our data indicate that the boundedness of the solution set for the desired problem depends on the choice of the $\psi(\kappa)$ function.

Keywords: pantograph equation; fixed-point theory; topological degree theory; ψ-Caputo derivative; measure of non-compactness
Mathematics Subject Classification: 34A08, 34A12

1. Introduction

As we know today, the role of fractional operators or, rather, fractional calculus, in the study and investigation of natural phenomena is undeniable, if not irreplaceable. Certainly, the most important reason for the stunning growth of fractional calculus in the last decade can be seen in its ability and application in modeling biological [1–4] and physical [5–9] phenomena. As one of the most prominent features of fractional operators, we can mention their non-locality. Based on the available results and evidence, modeling by ordinary calculus is not capable of describing the real behavior of phenomena and is often associated with the error of estimating the phenomenon [10]. Researchers in the fields of science and engineering have different approaches to the non-local character of fractional calculus. Physicists' approach to this issue has led to interesting modeling schemes for physical phenomena such as heat flow, hereditary polarization in dielectrics, viscoelasticity and so on [11]. Such phenomena have been modeled with equations which are influenced by the past values of one or more variables, and they are called equations with memory in the literature. Mathematicians have also provided the basis for extending the existing models for different fields by generalizing the fractional operators. The most basic fractional operator is the derivative given by Riemann and Liouville (namely, the Riemann-Liouville derivative). With the publication of a book in 1999, Podlubny may have contributed the most to the systematic presentation of the theory of fractional operators [12]. Taking ideas from his works, we have witnessed the introduction of various operators, such as the Caputo, Hilfer, Atangana-Baleanu, Hadamard, fractal fractional, Caputo-Fabrizio, fractional q-derivative and (p, q)-derivative operators, etc., in the last two decades. For get more information about these contributions, one can refer to [13–21]. There is a certain type of kernel dependency included in all of those definitions. That is, we can consider a general operator from which fractional integral and derivative operators can be extracted by selecting specific kernels [22–24].

One of the most recent generalizations of fractional operators is related to the work of the Portuguese mathematician Ricardo Almeida. In 2017, he presented a new definition of the Caputo derivative, namely, ψ-Caputo, with respect to another non-decreasing function, such as ψ [25]. In his new model, the Riemann-Liouville and Hadamard fractional operators are obtained by choosing $\psi(\kappa) = \kappa$ and $\psi(\kappa) = \ln \kappa$. A year later, he and colleagues investigated the existence and uniqueness of the solution for an initial value problem with his new fractional operator, and, using it, he presented a model for the growth of the world population and the gross domestic product growth rate in the USA [26]. In 2019, Abdo et al. studied the existence and uniqueness of the solution for initial and boundary value problems with the ψ-Caputo derivative [27, 28]. In 2020, Wahash et al. investigated fractional differential equations with singularities by applying the ψ-Caputo operator using the Picard iteration method [29]. Voyiadjis and Sumelka used this new type of fractional operator to provide an important model of brain damage in the framework of anisotropic hyperelasticity [30]. Also, Ahmed et al, presented a model for thermostats by using this fractional operator [31]. For more contributions that include this new fractional operator, the reader can see [32–35]. Here, we are going to present a model for the pantograph equation using the ψ-Caputo fractional operator; however, we will continue this section with a discussion about the pantograph.

Considering the issues raised and the potential in fractional operators, it is not far from expected that we also want to present a model for one of the most important and widely used equations in applied sciences, that is, the pantograph equation. Usually, the pantograph reminds us of a device that is installed on the top of the roof of electric buses. The pantograph problem was first raised by Mr. E. A. Cardwell of the British Railways Technical Center, at the Conference on Applications of Differential Equations in 1969. Ockendon and Tayler, in 1971, presented a mathematical model for Pantograph motion [36]. Today, the pantograph in electric trains is a tool that converts the electric current from direct to alternating current [37]. The pantograph differential equation is widely used in various fields, including number theory, quantum mechanics, statistics and electrodynamics [38, 39]. Many researchers have investigated the pantograph equation from different aspects. For example, numerical solutions via Chebyshev polynomials are presented in [40], the existence of mild solutions to pantograph equations are investigated in [41], the stability of solutions was studied in [42] and a coupled system of pantograph problems using sequential fractional derivatives was examined in [43]. To access more information, see [44–48].

The standard mode of the pantograph equation is formulated as follows:

$$\begin{cases} w'(\kappa) = c_1 w(\kappa) + c_2 w(\varepsilon\kappa), & 0 \le \kappa \le K, \\ w(0) = w_0, \end{cases}$$

such that $0 < \varepsilon < 1$ [49]. The fractional case of the pantograph equation investigated by Balachandran et al. involving the Caputo operator is as follows:

$$\begin{cases} {}^{C}\mathcal{D}^{\eta} w(\kappa) = h(\kappa, w(\kappa), w(\varepsilon\kappa)), & 0 \le \kappa \le K, \\ w(0) = w_0, \end{cases}$$

where $0 < \jmath$ and $\varepsilon < 1$ [49].

To the best of our knowledge, topological degree theory for condensing maps has not been applied to nonlinear pantograph differential equations with ψ-Caputo fractional derivatives under nonlocal boundary conditions. Therefore, inspired by the history mentioned above and previous works, in this paper, we investigate the existence of solutions for the following nonlinear fractional pantograph differential equation:

$$\begin{cases} {}^{C}\mathcal{D}^{\eta,\psi} w(\kappa) = h(\kappa, w(\kappa), w(\varepsilon\kappa)), & \kappa \in \mathcal{K} = [0, K], \\ w'(0) = 0, \quad w(0) + \chi(w) = w_0, \end{cases} \tag{1.1}$$

where ${}^{C}\mathcal{D}^{\eta,\psi}$ is the ψ-Caputo fractional derivative of order $\eta \in (1, 2)$, $\varepsilon \in (0, 1)$, $K > 0$, $h \in C(\mathcal{K} \times \mathbb{R}^2, \mathbb{R})$, $w_0 \in \mathbb{R}$ and χ is the nonlocal term that satisfies some given conditions. The importance of the nonlocal condition, which is better than the classical initial condition, is explained in [50]. Furthermore, in recent research in 2021, Sabatier and Farges showed, by designing some problems and numerical analysis, that the use of fractional derivatives is problematic in some fractional models [51]. According to the above topics, in this work, we also considered the boundary conditions as nonlocal, although the authors, as mentioned have other works with fractional initial conditions applied in this matter [43].

The rest of the paper is structured as follows. In Section 2, we state what we need from fractional calculus and topological degree theory as preliminaries to prove our main results. In Section 3, we first introduce three hypotheses, prove four auxiliary lemmas and prove the existence of solutions for the pantograph equation given by (1.1). In Section 4, we present examples with numerical and graphical simulations to validate our results. The last section concludes this paper.

2. Preliminaries

This section deals with some preliminaries and notations which are used throughout this paper. For more details, we refer the reader to [25].

Definition 2.1. *[26] Assume that $\boldsymbol{w}$ is an integrable function on $\mathcal{K} = [0, K]$, and that $\psi \in C^n(\mathcal{K}, \mathbb{R})$, where $\forall \kappa \in \mathcal{K}$, $\psi'(\kappa) > 0$. Then, the ψ-Riemann-Liouville (ψ-RL) integral and derivative of $\boldsymbol{w}$ of fractional order $\jmath$ is expressed as follows:*

$$\mathcal{I}^{\jmath,\psi}\boldsymbol{w}(\kappa) = \frac{1}{\Gamma(\jmath)} \int_0^\kappa \psi'(p)(\psi(\kappa) - \psi(p))^{\jmath-1}\boldsymbol{w}(p)dp,$$

and

$$\begin{aligned}\mathcal{D}^{\jmath,\psi}\boldsymbol{w}(\kappa) &= \frac{1}{\Gamma(\jmath - n)}\Big(\frac{1}{\psi'(\kappa)}\frac{d}{d\kappa}\Big)^n \int_0^\kappa \psi'(p)(\psi(\kappa) - \psi(p))^{n-\jmath-1}\boldsymbol{w}(p)dp \\ &= \Big(\frac{1}{\psi'(\kappa)}\frac{d}{d\kappa}\Big)^n \mathcal{I}^{n-\jmath,\psi}\boldsymbol{w}(\kappa),\end{aligned}$$

where $n = [\jmath] + 1$.

Definition 2.2. *[26] Suppose that $\boldsymbol{w} \in C^{n-1}(\mathcal{K}, \mathbb{R})$ and $\psi \in C^n(\mathcal{K}, \mathbb{R})$ such that $\forall \kappa \in \mathcal{K}$, $\psi'(\kappa) > 0$. Then, the Ψ-Caputo operator of fractional order $\jmath$ is formulated as follows:*

$${}^C\mathcal{D}^{\jmath,\psi}\boldsymbol{w}(\kappa) = \frac{1}{\Gamma(n - \jmath)} \int_0^\kappa \psi'(p)(\psi(\kappa) - \psi(p))^{n-\jmath-1}\boldsymbol{w}_\psi^{[n]}(p)dp, \tag{2.1}$$

where

$$\boldsymbol{w}_\psi^{[n]}(p) = \left(\frac{1}{\psi'(p)}\frac{d}{dp}\right)^n \boldsymbol{w}(p), \qquad n = [\jmath] + 1.$$

Remark 2.1. *It is obvious that, with correct placement, namely, $\psi(\kappa) = \kappa$ and $\psi(\kappa) = \ln(\kappa)$, in (2.1), the Caputo and Caputo-Hadamard derivatives can be reached.*

Remark 2.2. *If $0 < \jmath < 1$, then we have*

$${}^C\mathcal{D}^{\jmath,\psi}\boldsymbol{w}(\kappa) = \frac{1}{\Gamma(1 - \jmath)}\left(\frac{1}{\psi'(p)}\frac{d}{dp}\right)^1 \int_0^\kappa (\psi(\kappa) - \Psi(p))^{-\jmath}\boldsymbol{w}(p)dp.$$

Theorem 2.1. *[26] If $\jmath > 0$ and $\boldsymbol{w} \in C^{n-1}(\mathcal{K}, \mathbb{R})$, then the following assertions hold:*

1) ${}^C\mathcal{D}^{\jmath,\psi}\mathcal{I}^{\jmath,\psi}\boldsymbol{w}(\kappa) = \boldsymbol{w}(\kappa)$.
2) $\mathcal{I}^{\jmath,\psi}\,{}^C\mathcal{D}^{\jmath,\psi}\boldsymbol{w}(\kappa) = \boldsymbol{w}(\kappa) - \sum_{i=0}^{n-1} \frac{\boldsymbol{w}_\psi^{[i]}(0)}{i!}(\psi(\kappa) - \psi(0))^i$.

Theorem 2.2. *[26] Let $\mu > \nu > 0$ and $\kappa \in \mathcal{K}$; then, we have the following:*

1) $\mathcal{I}^{\mu,\psi}(\psi(\kappa) - \psi(0))^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\mu + \nu)}(\psi(t) - \psi(0))^{\mu+\nu-1}$.

2) $\mathcal{D}^{\mu,\psi}(\psi(t)-\psi(0))^{\nu-1} = \dfrac{\Gamma(\nu)}{\Gamma(\nu-\mu)}(\psi(\kappa)-\psi(0))^{\nu-\mu-1}$.

3) $\mathcal{D}^{\mu,\psi}(\psi(\kappa)-\psi(0))^{p} = 0, \quad \forall p < n \in \mathbb{N}$.

Definition 2.3. *[52] Suppose that $\mathcal{X}$ is a Banach space and $\mathcal{B}_X = \{\mathcal{Y} \subset \mathcal{X} : \mathcal{Y} \neq \emptyset,\ \mathcal{Y}$ is bounded$\}$. The function $\rho : \mathfrak{B}_X \to [0,+\infty)$ is called the Kuratowski measure of non-compactness and defined as follows:*

$$\rho(\mathcal{Y}) = \inf\{r > 0 : \mathcal{Y} \quad \text{admits a finite cover by sets of diameter} \leq r\}.$$

Theorem 2.3. *[52] The measure ρ defined above, namely, Definition 2.3, applies to the following properties.*

(1) $\rho(\mathcal{Y}) = 0$ iff $\mathcal{Y}$ is relativity compact.
(2) $\rho(a\mathcal{Y}) = |a|\rho(\mathcal{Y}), \quad a \in \mathbb{R}$.
(3) $\rho(\mathcal{Y}_1 + \mathcal{Y}_2) \leq \rho(\mathcal{Y}_1) + \rho(\mathcal{Y}_2)$.
(4) If $\mathcal{Y}_1 \subset \mathcal{Y}_2$, then $\rho(\mathcal{Y}_1) \leq \rho(\mathcal{Y}_2)$.
(5) $\rho(\mathcal{Y}_1 \cup \mathcal{Y}_2) = \max\{\rho(\mathcal{Y}_1),\ \rho(\mathcal{Y}_2)\}$.
(6) $\rho(\mathcal{Y}) = \rho(\overline{\mathcal{Y}}) = \rho(conv\mathcal{Y})$, where $\overline{\mathcal{Y}}$ and $conv\mathcal{Y}$ denote the closure and convex hull of $\mathcal{Y}$, respectively.

Definition 2.4. *[52] Assume that the function $\Theta : \mathcal{Y} \subset \mathcal{X} \to \mathcal{X}$ is a continuous and bounded map. The function Θ is called ρ-Lipschitz if $\exists \ell \geq 0$, given that*

$$\rho(\Theta(\mathcal{Y}_*)) \leq \ell\rho(\mathcal{Y}_*), \qquad \mathcal{Y}_* \subset \mathcal{Y}.$$

Definition 2.5. *[52] The function Θ, which is defined in Definition 2.4 is called ρ-condensing if, for every bounded subset $\mathcal{Y}_*$ of $\mathcal{Y}$, the following inequality holds:*

$$\rho(\Theta(\mathcal{Y}_*)) < \rho(\mathcal{Y}_*),$$

such that $\rho(\mathcal{Y}_*) > 0$. Indeed,

$$\rho(\Theta(\mathcal{Y}_*)) \geq \rho(\mathcal{Y}_*) \Rightarrow \rho(\mathcal{Y}_*) = 0.$$

Moreover, we denote the class of all ρ-condensing maps $\Theta : \mathcal{Y} \to \mathcal{X}$ by $\mathbf{C}_\rho(\mathcal{Y})$.

Definition 2.6. *[52] The function $\boldsymbol{w} : \mathcal{Y} \to \mathcal{X}$ is called Lipschitz if $\exists \ell > 0$, given that*

$$\parallel \boldsymbol{w}(y_1) - \boldsymbol{w}(y_2) \parallel \leq \ell \parallel y_1 - y_2 \parallel, \quad \forall y_1, y_2 \in \mathcal{Y}.$$

Lemma 2.1. *[52] Suppose that $\mathbf{w}$ is a Lipschitz function with a constant ℓ; then, $\mathbf{w}$ is $\rho-$Lipschitz with the same constant.*

Lemma 2.2. *[52] Consider the ρ-Lipschitz functions $\Theta, \Delta : \mathcal{Y} \to \mathcal{X}$ with constants ℓ_1, ℓ_2,respectively. Then, the following statements are true:*

- *$\Theta + \Delta : \mathcal{Y} \to \mathcal{X}$, is ρ-Lipschitz because of the $\ell_1 + \ell_2$ constant.*
- *If Θ is compact, then $\ell_1 = 0$.*

Theorem 2.4. *[53] Let $\Theta : \mathcal{Y} \to \mathcal{X}$ such that $\mathcal{Y} \subset \mathcal{X}$ is open and bounded; also suppose that*

$$\mathbf{T} = \left\{ (I - \Theta, \mathcal{Y}, x) : \Theta \in \mathbf{C}_\rho(\bar{\mathcal{Y}}),\ x \in \mathcal{X} \setminus (I - \Theta)(\partial \mathcal{Y}) \right\}$$

is a family of the admissible triplets. Then, there exists one degree function, deg : $\mathbf{T} \to \mathbb{Z}$, such that the following properties are satisfied:

- *$deg(I, \mathcal{Y}, y) = 1$ for every $y \in \mathcal{Y}$.*
- *For every disjoint, open set $\mathcal{Y}_1, \mathcal{Y}_2 \subset \mathcal{Y}$, and every $x \notin (I - \Theta)(\bar{\mathcal{Y}} \setminus (\mathcal{Y}_1 \cup \mathcal{Y}_2))$, we have*

$$deg(I - \Theta, \mathcal{Y}, x) = deg(I - \Theta, \mathcal{Y}_1, x) + deg(I - \Theta, \mathcal{Y}_2, x).$$

- *$deg(I - \mathbf{f}(t, .), \mathcal{Y}, x(t))$ is independent of $t \in [0, 1]$ for every continuous, bounded map $\mathbf{f} : [0, 1] \times \bar{\mathcal{Y}} \to \mathcal{X}$, which satisfies*

$$\rho(\mathbf{f}([0, 1] \times \Omega)) < \rho(\Omega), \quad \forall \Omega \subset \bar{\mathcal{Y}}, \quad with \quad \rho(\Omega) > 0,$$

  and every continuous function $x : [0, 1] \to \mathcal{X}$, which satisfies

$$x(t) \neq z - \mathbf{f}(t, z), \quad \forall t \in [0, 1], \quad \forall z \in \partial \mathcal{Y}.$$

- *$deg(I - \Theta, \mathcal{Y}, x) \neq 0$ implies that $x \in (I - \Theta)(\mathcal{Y})$.*
- *$deg(I - \Theta, \mathcal{Y}, x) = deg(I - \Theta, \mathcal{Y}_1, x)$ for every open set $\mathcal{Y}_1 \subset \mathcal{Y}$, and every $x \notin (I - \Theta)(\bar{\mathcal{Y}} \setminus \mathcal{Y}_1)$.*

Theorem 2.5. *[53] Assume that the map $\Theta : \mathcal{X} \to \mathcal{X}$ is ρ-condensing, $\tau \in [0, 1]$ and $\mathcal{E}_\tau \subset \mathcal{X}$ such that*

$$\mathcal{E}_\tau = \{x \in \mathcal{X} : x = \tau \Theta x \quad for\ some \quad \tau\}.$$

Now, if $\mathcal{E}_\tau$ is a bounded subset of $\mathcal{X}$, then $\exists q > 0$, given that $\mathcal{E}_\tau \subset \mathcal{B}_q(0)$, and we have

$$deg(I - \delta\Theta, \mathcal{B}_q(0), 0) = 1, \quad \forall \delta \in [0, 1].$$

As a result, Θ has at least one fixed point and the set of the fixed points of Θ lies in $\mathcal{B}_q$.

3. Main results

Here, to continue the work, we first introduce the necessary notations and three hypotheses which play a fundamental role in providing a suitable space for using the results of fixed-point theory and its contractions in the sequel. It is worth noting that, as a reminder, we are referring to a closed ball centered at 0 with radius $q > 0$ by using $\mathcal{B}_q$. Also, our Banach space $C := C(\mathcal{K}, \mathbb{R})$ is equipped with the supreme norm, namely, $\| \boldsymbol{w} \| = \sup_{\kappa \in \mathcal{K}} | \boldsymbol{w}(\kappa) |$.

(H_1) $\exists L_\chi > 0$, where

$$|\chi(\boldsymbol{w}) - \chi(\boldsymbol{s})| \leq L_\chi \mid \boldsymbol{w} - \boldsymbol{s} \mid \quad \text{for each } \boldsymbol{w}, \boldsymbol{s} \in C.$$

(H_2) $\exists N_\chi > 0$, $M_\chi \geq 0$ and $0 \leq \alpha \leq 1$, where

$$|\chi(\boldsymbol{w})| \leq N_\chi \mid \boldsymbol{w} \mid^\alpha + M_\chi \quad \text{for each } \boldsymbol{w} \in C.$$

(H_3) $\exists N_h, M_h > 0$ and $0 \leq \beta \leq 1$, where

$$|h(\kappa, \boldsymbol{w}(\kappa), \boldsymbol{w}(\varepsilon\kappa))| \leq N_h \mid \boldsymbol{w} \mid^{\beta} + M_h \quad \text{for each } \boldsymbol{w} \in C.$$

Lemma 3.1. *The solution to problem* (1.1) *is equivalent to the following integral equation:*

$$\boldsymbol{w}(\kappa) = \boldsymbol{w}_0 - \chi(\boldsymbol{w}) + \frac{1}{\Gamma(\eta)} \int_0^{\kappa} \psi'(p)(\psi(\kappa) - \Psi(p))^{\eta-1} h(t, \boldsymbol{w}(p), \boldsymbol{w}(\varepsilon p)) dp. \tag{3.1}$$

Proof. Suppose that $\boldsymbol{w}$ is a solution of (1.1); then, by applying operator $\mathcal{I}^{\eta,\psi}$ on (1.1), we obtain

$$\mathcal{I}^{\eta,\psi} \, {}^{C}\mathcal{D}^{\eta,\psi} \boldsymbol{w}(\kappa) = \mathcal{I}^{\eta,\psi} h(\kappa, \boldsymbol{w}(\kappa), \boldsymbol{w}(\varepsilon\kappa)),$$

and by employing Proposition 2.1, we get

$$\boldsymbol{w}(\kappa) = c_0 + (\psi(\kappa) - \psi(0))c_1 + \mathcal{I}^{\eta,\psi} h(\kappa, \boldsymbol{w}(\kappa), \boldsymbol{w}(\varepsilon\kappa)),$$

where $c_0, c_1 \in \mathbb{R}$. Hence,

$$\boldsymbol{w}'(\kappa) = c_1\psi'(\kappa) + \frac{1}{\Gamma(\eta)} \int_0^{\kappa} \left(\psi'(p)(\psi(\kappa) - \psi(p))^{\eta-1} h(p, \boldsymbol{w}(p), \boldsymbol{w}(\varepsilon p))\right)' dp;$$

since $\boldsymbol{w}(0) + \chi(\boldsymbol{w}) = \boldsymbol{w}_0$ and $\boldsymbol{w}'(0) = 0$, then $c_0 = \boldsymbol{w}_0 - \chi(\boldsymbol{w})$ and $c_1 = 0$. Hence, (3.1) holds.

To show that (3.1) has at least one solution $\boldsymbol{w} \in C$, we define two operators $\mathcal{A}, \mathcal{T} : C \to C$ as follows:

$$\mathcal{A}\boldsymbol{w}(\kappa) = \boldsymbol{w}_0 - \chi(\boldsymbol{w}), \quad \kappa \in \mathcal{K}, \tag{3.2}$$

and

$$\mathcal{T}\boldsymbol{w}(\kappa) = \frac{1}{\Gamma(\eta)} \int_0^{\kappa} \psi'(p)(\psi(\kappa) - \Psi(p))^{\eta-1} h(p, \boldsymbol{w}(p), \boldsymbol{w}(\varepsilon p)) dp, \quad \kappa \in \mathcal{K}. \tag{3.3}$$

Thus, (3.1) can be formulated as follows:

$$\mathcal{F}\boldsymbol{w}(\kappa) = \mathcal{A}\boldsymbol{w}(\kappa) + \mathcal{T}\boldsymbol{w}(\kappa), \quad \kappa \in \mathcal{K}. \tag{3.4}$$

Lemma 3.2. *The operator $\mathcal{A}$ is ρ- Lipschitz with the constant L_χ. Moreover, $\mathcal{A}$ satisfies the following inequality:*

$$\|\mathcal{A}\boldsymbol{w}\|_C \leq |\boldsymbol{w}_0| + N_\chi \|\boldsymbol{w}\|^{\alpha} + M_\chi \;\; for \;\; every \;\; \boldsymbol{w} \in C. \tag{3.5}$$

Proof. At first, we shall show that the operator $\mathcal{A}$ is Lipschitz with the constant $L\chi$. To do this, let $\boldsymbol{w}, \boldsymbol{s} \in C$; then, we have

$$|\mathcal{A}\boldsymbol{w}(\kappa) - \mathcal{A}\boldsymbol{s}(\kappa)| \leq |\chi(\boldsymbol{w}) - \chi(\boldsymbol{s})|;$$

the hypothesis (H_1) yields that

$$|\mathcal{A}\boldsymbol{w}(\kappa) - \mathcal{A}\boldsymbol{s}(\kappa)| \leq L_\chi \|\boldsymbol{w} - \boldsymbol{s}\|,$$

and taking the supremum over κ implies that

$$\|\mathcal{A}\boldsymbol{w} - \mathcal{A}\boldsymbol{s}\| \leq L_\chi \|\boldsymbol{w} - \boldsymbol{s}\|;$$

hence, $\mathcal{A}$ is Lipschitz with L_χ. In view of Lemma 2.1, it follows that $\mathcal{A}$ is ρ-Lipschitz with the same constant L_χ. Now, to prove (3.5), let $w \in C$; then, we have

$$|\mathcal{A}w(\kappa)| = |w_0 - \chi(w)| \leq |w_0| + |\chi(w)|;$$

by using the assumption (H_2), we get

$$\|\mathcal{A}w\| \leq |w_0| + N_\chi \|w\|^\alpha + M_\chi.$$

Lemma 3.3. *The operator $\mathcal{T}$, which is formulated in* (3.3)*, is continuous and satisfies the following inequality:*

$$\|\mathcal{T}w\| \leq \frac{1}{\Gamma(\eta+1)}(N_\chi \|w\|^\beta + M_\chi)(\psi(K) - \psi(0))^\eta, \quad \forall w \in C. \tag{3.6}$$

Proof. For $\mathcal{T}$ to be continuous, assume that $w_n \to w$ in C; hence, $\exists \delta > 0$, given that $\|w_n\| \leq \delta$ and $\|w\| \leq \delta$. Now, let $\kappa \in \mathcal{K}$; we can write

$$\begin{aligned}&|\mathcal{T}w_n(\kappa) - \mathcal{T}w(\kappa)| \\ &\leq \frac{1}{\Gamma(\eta)} \int_0^\kappa \psi'(p)(\psi(\kappa) - \psi(p))^{\eta-1} \, |h(p, w_n(p), w_n(\varepsilon p)) - h(p, w(p), w(\varepsilon p))| \, dp;\end{aligned}$$

since h is continuous, then

$$\lim_{n\to\infty} h(p, w_n(p), w_n(\varepsilon p)) = h(p, w(p), w(\varepsilon p)).$$

On the other hand, by using (H_3), we obtain

$$\begin{aligned}&\frac{1}{\Gamma(\eta)}(\psi'(p)(\psi(\kappa) - \psi(p))^{\eta-1} \, \|h(p, w_n(p), w_n(\varepsilon p)) - h(p, w(p), w(\varepsilon p))\| \\ &\leq \left(N_\chi \delta^\beta + M_\chi\right) \times \frac{1}{\Gamma(\eta)}(\psi'(p)(\psi(\kappa) - \psi(p))^{\eta-1};\end{aligned}$$

since $p \mapsto \frac{1}{\Gamma(\eta)}(\psi'(p)(\psi(\kappa) - \psi(p))^{\eta-1}$ is an integrable function on $[0, \kappa]$, then Lebesgue's dominated convergence theorem implies that

$$\lim_{n\mapsto+\infty} \frac{1}{\Gamma(\eta)}(\psi'(p)(\psi(\kappa) - \psi(p))^{\eta-1} \, \|h(p, w_n(p), w_n(\varepsilon p)) - h(p, w(p), w(\varepsilon p))\| \, dp = 0,$$

which yields that

$$\lim_{n\mapsto+\infty} \| \mathcal{T}w_n - \mathcal{T}w \| = 0;$$

hence, $\mathcal{T}$ is continuous. To show (3.6), let $w(\kappa) \in C$; then, we have

$$|\mathcal{T}w(\kappa)| \leq \frac{1}{\Gamma(\eta)} \int_0^t \psi'(p)(\psi(\kappa) - \psi(p))^{\eta-1} \, |h(p, w(p), w(\varepsilon p))| \, dp;$$

from (H_3), we obtain

$$|\mathcal{T}\boldsymbol{w}(\kappa)| \leq \frac{(N_\chi\|\boldsymbol{w}\|^\beta + M_\chi)}{\Gamma(\eta)} \int_0^\kappa \psi'(p)(\psi(\kappa)-\psi(p))^{\eta-1}dp.$$

Finally, we obtain

$$\| \mathcal{T}\boldsymbol{w} \| \leq \frac{(N_\chi\|\boldsymbol{w}\|^\beta + M_\chi)(\psi(K)-\psi(0))^\eta}{\Gamma(\eta+1)}.$$

Lemma 3.4. *The operator* $\mathcal{T} : C \to C$ *is compact.*

Proof. We shall show that $\mathcal{T}\mathcal{B}_q$ is relatively compact in C. To do this, let $\boldsymbol{w} \in \mathcal{B}_q$; then, from (3.6), we get

$$\| \mathcal{T}\boldsymbol{w} \| \leq \frac{(N_\chi q^\beta + M_\chi)(\psi(T)-\psi(0))^\eta}{\Gamma(\eta+1)} := \xi.$$

It follows that $\mathcal{T}\mathcal{B}_q \subset \mathcal{B}_\xi$. Hence, $\mathcal{T}\mathcal{B}_q$ is bounded. To prove that $\mathcal{T}\mathcal{B}_q$ is equicontinuous, let $\boldsymbol{w} \in \mathcal{T}\mathcal{B}_q$ and $\kappa_1, \kappa_2 \in \mathcal{K}$ such that $\kappa_1 < \kappa_2$; then, we have

$$|\mathcal{T}\boldsymbol{w}(\kappa_2) - \mathcal{T}\boldsymbol{w}(\kappa_1)| \leq \frac{N_\chi|\boldsymbol{w}|^p + M_\chi}{\Gamma(\eta)} \int_{\kappa_1}^{\kappa_2} \psi'(p)(\psi(\kappa_2)-\psi(p))^{\eta-1}dp,$$

$$|\mathcal{T}\boldsymbol{w}(\kappa_2) - \mathcal{T}\boldsymbol{w}(\kappa_1)| \leq \frac{N_\chi q^\beta + M_\chi}{\Gamma(\eta)} \int_{\kappa_1}^{\kappa_2} \psi'(p)(\psi(\kappa_2)-\psi(p))^{\eta-1}dp,$$

$$|\mathcal{T}\boldsymbol{w}(\kappa_2) - \mathcal{T}\boldsymbol{w}(\kappa_1)| \leq \frac{N_\chi q^\beta + M_\chi}{\Gamma(\eta+1)}(\psi(\kappa_2)-\psi(\kappa_1))^\eta.$$

Since Ψ is a continuous function, then we obatin

$$\lim_{\kappa_1 \to \kappa_2} |\mathcal{T}\boldsymbol{w}(\kappa_1) - \mathcal{T}\boldsymbol{w}(\kappa_2)| = 0,$$

which shows that $\mathcal{T}\mathcal{B}_q$ is equicontinuous. Hence, $\mathcal{T}\mathcal{B}_q$ is uniformly bounded and equicontinuous. The Arzelà-Ascoli theorem [54] permits us to conclude that $\mathcal{T}\mathcal{B}_q$ is relatively compact; thus, $\mathcal{T}$ is compact.

Corollary 3.1. $\mathcal{T} : C \to C$ *is* ρ*-Lipschitz with a zero constant.*

Proof. From the compactness of the operator $\mathcal{T}$, and Lemma 2.2, it follows that $\mathcal{T}$ is ρ-Lipschitz with a zero constant.

Now, we have all of the tools to establish our main result.

Theorem 3.1. *Suppose that the hypotheses* (H_1)–(H_3) *are true; then, the fractional pantograph differential equation mentioned in* (1.1) *has at least one solution:* $\boldsymbol{w} \in C$. *Moreover, the set of all solutions for* (1.1) *is bounded in* $C(\mathcal{K}, \mathbb{R})$.

Proof. Let $\mathcal{A},\mathcal{T},\mathcal{F}\colon C\to C$ be the operators formulated in (3.2)–(3.4), respectively. $\mathcal{A},\mathcal{T},\mathcal{F}$ are continuous and bounded. Furthermore, in view of Lemma 3.2 and Corollary 3.1, the operator $\mathcal{A}$ is ρ-Lipschitz given $L_\chi\in[0,1)$, and ρ-Lipschitz with a zero constant. By using Lemma 2.2, we deduce that $\mathcal{F}$ is a strict ρ-contraction with a constant L_χ. Now, for some $\tau\in[0,1]$, we set

$$\mathcal{E}_\tau=\{w\in C: w=\tau\mathcal{F}w\}.$$

We claim that $\mathcal{E}_\tau$ is bounded in C. To prove this claim, suppose that $w\in\mathcal{E}_\tau$; then,

$$w=\tau\mathcal{F}w=\tau(\mathcal{A}w+\mathcal{T}w),$$

which yields that

$$\|w\|=\tau\|\mathcal{F}w\|\le\tau(\|\mathcal{A}w\|+\|\mathcal{T}w\|);$$

by using Lemmas 3.2 and 3.3, we get

$$\|w\|\le\left(|w_0|+N_\chi\|w\|^\alpha+M_\chi+\frac{(N_h\|w\|^\beta+M_h)(\psi(K)-\psi(0))^\eta}{\Gamma(\eta+1)}\right).\tag{3.7}$$

The above inequality, namely, (3.7), yields that $\mathcal{E}_\tau$ is bounded in C given $\alpha<1$ and $\beta<1$.

Suppose that our claim is not true; in this case, let $\xi:=\|w\|\longrightarrow\infty$. Dividing both sides of (3.7) by ξ, and taking $\xi\to\infty$, then we obtain

$$1\le\lim_{\xi\to\infty}\frac{\left(|w_0|+N_\chi\xi^\alpha+M_\chi+\dfrac{(N_h\xi^\beta+M_h)(\psi(T)-\psi(0))^\eta}{\Gamma(\eta+1)}\right)}{\xi}=0,$$

which is a contradiction. By using Theorem 2.5, we conclude that $\mathcal{F}$ has at least one fixed point which is the solution of (1.1) and the set of the fixed points of $\mathcal{F}$ is bounded in C.

Remark 3.1. *If we set $\alpha=\beta=1$ in hypotheses (H_2) and (H_3), then the result of Theorem 3.1 will be as follows:*

$$N_\chi+\frac{N_h(\psi(K)-\psi(0))^\eta}{\Gamma(\eta+1)}<1.$$

4. Examples

In this section, we give two examples to illustrate the usefulness of our main result.

Example 4.1. *Consider the following problem:*

$$\begin{cases}{}^{C}\mathcal{D}^{\frac{3}{2},e^\kappa}w(\kappa)=\dfrac{\kappa^2}{\sqrt{77}}(w(\kappa)+sin^2(w(\kappa))+\dfrac{1}{7\sqrt{\pi}}cos(w(\dfrac{\kappa}{\sqrt{2}})),\quad \kappa\in\mathcal{K}=[0,1]\\ w'(0)=0,\quad w(0)=\sum_{j=1}^{20}\theta_j|w(\kappa_j)|,\quad \theta_j>0,\quad 0<\kappa_j<1,\quad j=1,2,..,20.\end{cases}\tag{4.1}$$

Here, $\varepsilon=\frac{1}{\sqrt{2}}$, $\eta=\frac{3}{2}$, $K=1$ and $\psi(\kappa)=e^\kappa$, and, in this case, we let $\chi(w)=\sum_{j=1}^{20}\theta_j|w(\kappa_j)|$ with $\sum_{j=1}^{20}\theta_j<1$.

Clearly, (H_1) and (H_2) hold with $N_\chi=L_\chi=\sum_{i=j}^{20}\theta_j$, $M_\chi=0$ and $q=1$.

Indeed, we can write

$$|\chi(\boldsymbol{w}(\kappa))| = \left|\sum_{j=1}^{20} \theta_j |\boldsymbol{w}(\kappa_j)\right|;$$

hence,

$$|\chi(\boldsymbol{w})| \leq \sum_{j=1}^{20} \theta_j \, \|\boldsymbol{w}\|;$$

thus, $N_\chi = \sum_{j=1}^{20} \theta_j$, $M_\chi = 0$ *and* $\alpha = 1$. *Alternatively, we have*

$$|\chi(\boldsymbol{w}(\kappa)) - \chi(\boldsymbol{s}(\kappa))| = \left|\sum_{j=1}^{20} \theta_j |\boldsymbol{w}(\kappa_j) - \sum_{j=1}^{20} \theta_j |\boldsymbol{s}(\kappa_j)\right|;$$

hence,

$$|\omega(\boldsymbol{w}) - \omega(\boldsymbol{s})| \leq \sum_{j=1}^{20} \theta_j \, |\boldsymbol{w} - \boldsymbol{s}|;$$

thus, $L_\chi = \sum_{j=1}^{20} \theta_j$.

To check the fulfillment of (H_3)*, let* $\kappa \in \mathcal{K}$ *and* $\boldsymbol{w} \in \mathbb{R}$*; then, we have*

$$|h(\kappa, \boldsymbol{w}(\kappa), \boldsymbol{w}(\varepsilon\kappa))| = \left|\frac{\kappa^2}{\sqrt{77}}(\boldsymbol{w}(\kappa) + sin^2(\boldsymbol{w}(\kappa)) + \frac{1}{7\sqrt{\pi}} cos(\boldsymbol{w}(\frac{\kappa}{\sqrt{2}}))\right|,$$

which implies that

$$|h(\kappa, \boldsymbol{w}(\kappa), \boldsymbol{w}(\varepsilon\kappa))| \leq \frac{1}{\sqrt{77}}|\boldsymbol{w}| + 0.1946.$$

Thus, (H_3) *holds with* $N_h = \dfrac{1}{\sqrt{77}}$, $M_h = 0.1946$ *and* $\beta = 1$. *Consequently, Theorem 3.1 implies that problem (4.1) has at least one solution. Moreover, from the inequality* (3.7)*, we get*

$$\|\boldsymbol{w}\| \leq \xi_* : \frac{0.1946(e-1)^{(\eta)}}{\Gamma(\eta+1) - \frac{1}{\sqrt{77}}(e-1)^{(\eta)}} = \frac{0.1946(e-1)^{(3/2)}}{\Gamma(5/2) - \frac{1}{\sqrt{77}}(e-1)^{(3/2)}} = 0.4086. \tag{4.2}$$

Thus, the set of solutions for (4.1) *is bounded. To better understand this example, graphs of some functions are provided in Figures 1 and 2. The data from Table 1 indicate that the boundedness of the solution set for* (4.1) *depends on the choice of* $\psi(\kappa)$*.*

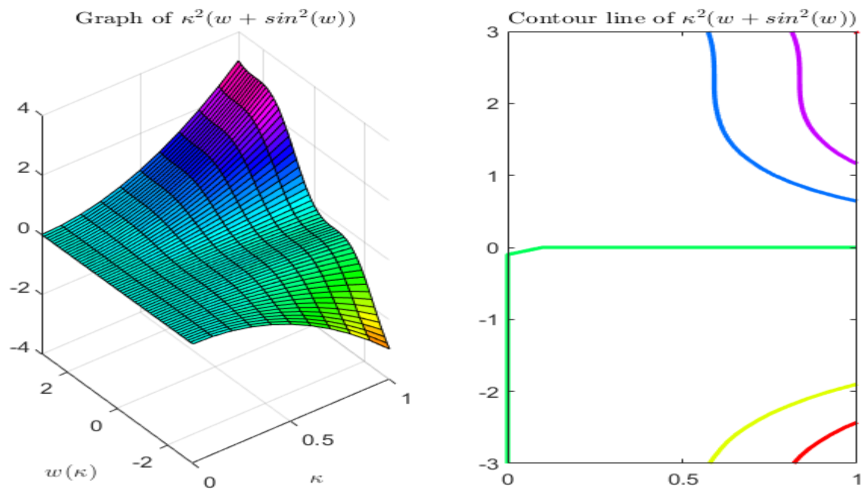

Figure 1. The graph of $h(\kappa, \boldsymbol{w}(\kappa))$ for Example 4.1.

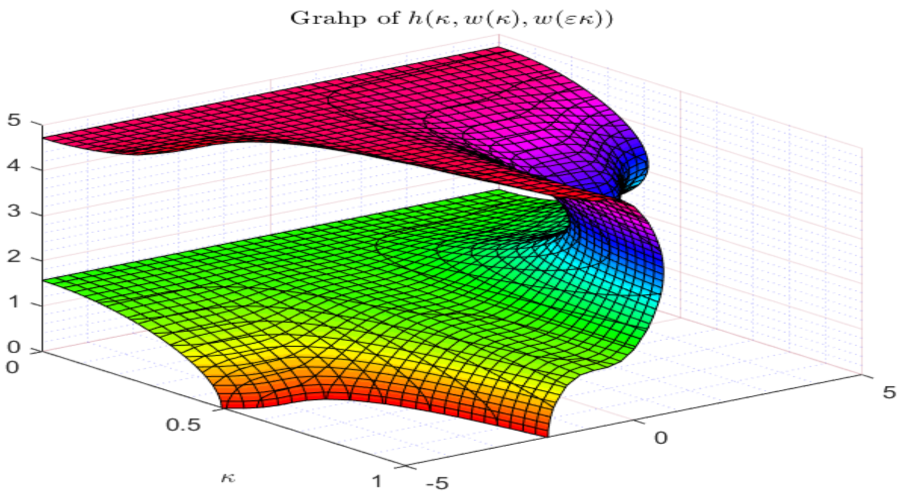

Figure 2. The graph of $h(\kappa, \boldsymbol{w}(\kappa), \boldsymbol{w}(\varepsilon\kappa))$ for Example 4.1.

Table 1. Numerical results for ξ_* based on $\psi(\kappa)$ selection in Example 4.1.

$\psi(\kappa)$	ξ_*
κ	$0.1601 < 1$
e^{κ}	$0.4086 < 1$
2^{κ}	$0.1601 < 1$
3^{κ}	$0.5467 < 1$
4^{κ}	$1.3721 > 1$
5^{κ}	$3.7306 > 1$

Example 4.2. *Consider the following problem:*

$$\begin{cases} {}^{C}\mathcal{D}^{\frac{9}{5},\kappa}\boldsymbol{w}(\kappa) = \dfrac{e^{-\kappa}}{\sqrt{11+\kappa^2}}(\boldsymbol{w}(\kappa) + sin(\kappa) + \dfrac{cos(\boldsymbol{w}(\frac{\kappa}{\sqrt{2}}))}{\sqrt{\pi+\kappa^2}}), \quad \kappa \in \mathcal{K} = [0,1] \\ \boldsymbol{w}'(0) = 0, \quad \boldsymbol{w}(0) = \sum_{j=1}^{20} \theta_j |\boldsymbol{w}(\kappa_j)|, \quad \theta_j > 0, \quad 0 < \kappa_j < 1, \quad j = 1, 2, .., 20. \end{cases} \tag{4.3}$$

In this case, $\varepsilon = \frac{1}{\sqrt{2}}$, $\eta = \frac{9}{5}$, $K = 1$, $\psi(\kappa) = \kappa$ *and* $\chi(\boldsymbol{w}) = \sum_{j=1}^{20} \theta_j |\boldsymbol{w}(\kappa_j)|$ *with* $\sum_{j=1}^{20} \theta_j < 1$. *Similar to the previous example, hypotheses* (H_1) *and* (H_2) *are valid with* $N_\chi = L_\chi = \sum_{i=j}^{20} \theta_j$, $M_\chi = 0$, $q = 1$ *and* $\alpha = 1$. *To check the fulfillment of* (H_3), *we can write*

$$|h(\kappa, \boldsymbol{w}(\kappa), \boldsymbol{w}(\varepsilon\kappa))| = \left| \frac{e^{-\kappa}}{\sqrt{11+\kappa^2}}(\boldsymbol{w}(\kappa) + sin(\kappa) + \frac{cos(\boldsymbol{w}(\frac{\kappa}{\sqrt{2}}))}{\sqrt{\pi+\kappa^2}}) \right|,$$

which implies that

$$|h(\kappa, \boldsymbol{w}(\kappa), \boldsymbol{w}(\varepsilon\kappa))| \le \frac{1}{\sqrt{11}}|\boldsymbol{w}| + 0.4716.$$

Hence, (H_3) *holds with* $N_h = \dfrac{1}{\sqrt{11}}$, $M_h = 0.4716$ *and* $\beta = 1$. *Consequently, Theorem 3.1 implies that problem* (4.3) *has at least one solution. Moreover, from the inequality* (3.7), *we get*

$$\|\boldsymbol{w}\| \le \xi_* : \frac{0.4716}{\Gamma(\eta+1) - \frac{1}{\sqrt{11}}} = \frac{0.4716}{\Gamma(14/5) - \frac{1}{\sqrt{11}}} = 0.3430. \tag{4.4}$$

Thus the set of solutions for (4.3) *is bounded. To better understand this example, graphs of some functions are provided in Figures 3 and 4. The data from Table 2 indicate that the boundedness of the solution set for* (4.3) *depends on the choice of* $\psi(\kappa)$.

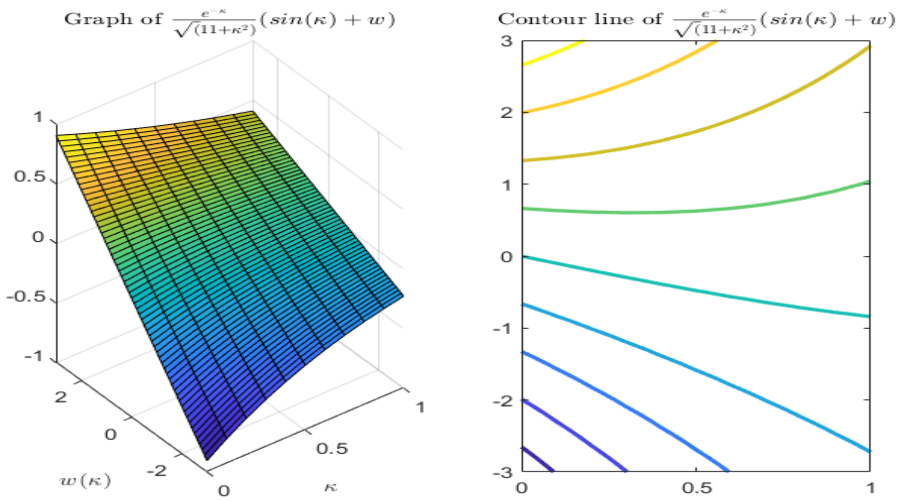

Figure 3. The graph of $h(\kappa, \boldsymbol{w}(\kappa))$ for Example 4.2.

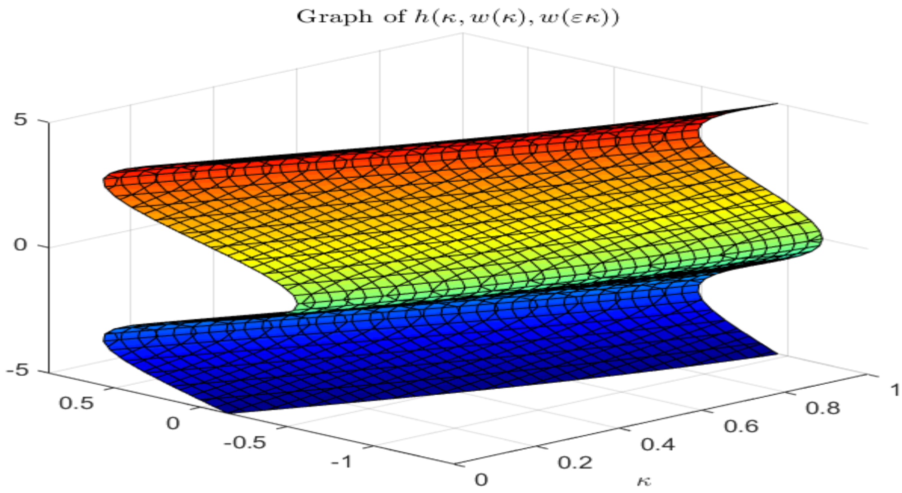

Figure 4. The graph of $h(\kappa, \boldsymbol{w}(\kappa), \boldsymbol{w}(\varepsilon\kappa))$ for Example 4.2.

Table 2. Numerical results for ξ_* based on $\psi(\kappa)$ selection in Example 4.2.

$\psi(\kappa)$	ξ_*
κ	$0.3430 < 1$
e^κ	$1.4238 > 1$
2^κ	$0.3430 < 1$
3^κ	$2.6209 > 1$
4^κ	$-6.7896 < 0$
5^κ	$-2.8888 < 0$

5. Conclusions

Today, we see the presence of fractional calculus in the mathematical modeling of natural phenomena. The non-locality of fractional derivatives gives it the special ability to be used to model and describe physical phenomena. Using this capability, we presented a comprehensive analysis of pantograph modeling by using the fractional derivative of the ψ-Caputo type. We guaranteed the existence of the solution with the help of topological degree theory and the Arzela-Ascoli theorem. Finally, we presented numerical and graphical simulations to validate our results. Our results show that the boundedness of the solution set depends on the type of the $\psi(\kappa)$ function.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

Acknowledgments

The authors extend their appreciation to the Deputyship for Research & Innovation, Ministry of Education in Saudi Arabia for funding this research work through the project number (IF2/PSAU/2022/01/22923).

Conflict of interest

The authors declare no conflict of interest.

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