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Research article

Subharmonic solutions for degenerate periodic systems of Lotka-Volterra type with impulsive effects

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Abstract: In this paper, we are concerned with the existence of subharmonic solutions for the degenerate periodic systems of Lotka-Volterra type with impulsive effects. In our degenerate model, the variation of the predator and prey populations may vanish on a time interval, which imitates the (real) possibility that the predation is seasonally absent. Our proof is based on the Poincaré-Birkhoff theorem. By using phase plane analysis, we can find the large gap in the rotation numbers between the "small" solutions and the "large" solutions, which guarantees a suitable twist property. By applying the Poincaré-Birkhoff theorem, we then obtain the existence of subharmonic solutions. Our main theorem extends the associated results by J. López-Gómez et al.

Keywords: periodic systems of Lotka-Volterra type; subharmonic solutions; degenerate systems; impulsive effects; Poincaré-Birkhoff theorem
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1. Introduction

In this paper, we consider the periodic systems of Lotka-Volterra type with impulsive effects

$$\begin{cases} x' = -\alpha(t)f(y), \\ y' = \beta(t)g(x), \\ \triangle x(t_j) = I_j(x(t_j^-), y(t_j^-)), \\ \triangle y(t_j) = J_j(x(t_j^-), y(t_j^-)), \;\; j = \pm 1, \pm 2, \cdots, \end{cases} \tag{1.1}$$

where α, β are nonnegative T-periodic continuous functions, $\triangle x(t_j) = x(t_j^+) - x(t_j^-)$, $\triangle y(t_j) = y(t_j^+) - y(t_j^-)$, the impulses $I_j,\ J_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and area-preserving for $j = \pm 1, \pm 2, \cdots$.

In addition, we assume that the impulsive time is T-periodic and there exists an integer $l > 0$ such that

$$0 \le t_1 < t_2 < \cdots < t_l < T, \quad t_{j+l} = t_j + T \quad \text{and} \quad I_{j+l} = I_j,\ J_{j+l} = J_j,\ j = \pm 1, \pm 2, \cdots. \tag{1.2}$$

Throughout this paper, we write

$$A := \int_0^T \alpha(t)dt \quad \text{and} \quad B := \int_0^T \beta(t)dt.$$

Extensive and interesting research has been conducted on the existence and multiplicity of periodic solutions for second order differential equations without impulsive effects, see [1–4] by using critical point theory, [5–9] by using Poincaré-Birkhoff theorem, and other references. At the same time, many researchers are also interested in periodic solutions for impulsive systems, see [10–13]. For the Lotka-Volterra type systems, one can refer to [14–18].

In above previous researches, there has often been some requirements for the preservation of the sign of equations. For example, in (1.1), one of the two coefficients $\alpha(t)$ and $\beta(t)$ is required to be strictly positive. From the point of view of using the Poincaré-Birkhoff theorem and phase plane analysis, these strictly positive conditions are used to obtain suitable twist properties. Otherwise, if both coefficient functions change sign or vanish on a subinterval, the associated phase plane analysis is considerably more difficult.

Recently, many researchers have been paying attention to the system where coefficient functions are degenerate, that is, $\alpha(t)$ and $\beta(t)$ could both be zero. These degenerate conditions imitate the (real) possibility that the predation is seasonally absent.

In [19], López-Gómez et al. considered a Hamiltonian system

$$\begin{cases} x' = -\lambda\alpha(t)f(y), \\ y' = \lambda\beta(t)g(x), \end{cases} \tag{1.3}$$

where parameter $\lambda > 0$. By the Poincaré-Birkhoff theorem, the authors obtained the existence of subharmonic solutions for sufficiently large λ, under the following assumptions:

(A_1) Let $\alpha \gneq 0$ and $\beta \gneq 0$ be T-periodic continuous functions such that

$$\alpha(t_0)\beta(t_0) > 0 \quad \text{for some } t_0 \in [0, T]. \tag{1.4}$$

(A_2) Let $f, g \in C(\mathbb{R})$ be locally Lipschitz functions such that $f, g \in C^1$ on a neighborhood of the origin and

$$\begin{cases} f(0) = 0, & f(y)y > 0 \text{ for all } y \neq 0, \\ g(0) = 0, & g(x)x > 0 \text{ for all } x \neq 0, \\ f'(0) > 0, & g'(0) > 0. \end{cases}$$

(A_3) Either f, or g, satisfies, at least, one of the following conditions:

(f_-) f is bounded in $\mathbb{R}^-$, (f_+) f is bounded in $\mathbb{R}^+$,
(g_-) g is bounded in $\mathbb{R}^-$, (g_+) g is bounded in $\mathbb{R}^+$.

Meanwhile, a predator-prey model of Lotka-Volterra type has been considered in [20], in which the coefficients satisfy $\alpha\beta = 0$, that is,

$$\begin{cases} u' = \alpha(t)u(1-v), \\ v' = \beta(t)v(-1+u), \end{cases} \tag{1.5}$$

where $\alpha(t)$ and $\beta(t)$ are real continuous T-periodic functions such that

$$\alpha(t) > 0, \quad \forall t \in \left(0, \frac{T}{2}\right), \quad \alpha(t) = 0, \quad \forall t \in \left[\frac{T}{2}, T\right];$$
$$\beta(t) = 0, \quad \forall t \in \left[0, \frac{T}{2}\right], \quad \beta(t) > 0, \quad \forall t \in \left(\frac{T}{2}, T\right).$$

By constructing the iterates of the monodromy operator of the system, it was shown in [20] that the system (1.5) possesses exactly two $2T$-periodic solutions if $AB > 4$.

Further, López-Gómez et al. [21] considered the degenerate model (1.3) with (A_1) and (A_2). Here "degenerate" means that the set

$$Z := \operatorname{supp}(\alpha) \cap \operatorname{supp}(\beta)$$

has Lebesgue measure zero, that is $|Z| = 0$. According to some geometric configurations of α and β, for large $\lambda > 0$, the existence of a large number of subharmonic solutions is obtained in [21].

The strategy used in [19, 21] is to expand the parameter $\lambda > 0$ such that the rotational motions of "large" solutions and "small" solutions produce enough angular gap in mT time in phase plane, which implies a twist property. Then the Poincaré-Birkhoff theorem can be applied. Thus, a natural and interesting problem is whether we can apply Poincaré-Birkhoff theorem to degenerate impulsive systems (1.1) without expanding parameter λ. This is one of the motivations for the research presented in this paper.

Due to the presence of pulses, even the simplest pulse function can potentially give rise to complex dynamical phenomena, posing challenges for subsequent research. There are only a few results on the existence of periodic solutions to impulsive equations of Lotka-Volterra type. In [22], Tang and Chen investigated a classical periodic Lotka-Volterra predator-prey system with impulsive effect. By using the method of coincidence degree, the authors proved the existence of strictly positive periodic solution.

In the following, we consider the impulsive periodic systems of Lotka-Volterra type (1.1). We preserve (A_2)–(A_3) and assume that

(H_1) Let $\alpha \gneq 0$, $\beta \gneq 0$ be T-periodic continuous functions such that $A > 0$ and $B > 0$.

By (A_2), there exists a constant $\eta > 0$ such that

$$\min\{f'(0), g'(0)\} > \eta. \tag{1.6}$$

Then by the limit definition, we find $\varepsilon_0 > 0$ such that,

$$f(\zeta)\zeta \geq \eta\zeta^2, \quad g(\zeta)\zeta \geq \eta\zeta^2, \quad \text{for } |\zeta| \leq \varepsilon_0.$$

Moreover, we give the following assumptions about the impulse functions.

(H_2) For some η satisfying (1.6), there exists a constant δ_1 , such that

$$|I_j(x,y)| < \mu r, \ |J_j(x,y)| < \mu r, \quad \text{for } r < \delta_1,$$

where $r = \sqrt{x^2 + y^2}$ and the constant μ satisfies

$$l \arcsin(\sqrt{2}\mu) < \mu^* := \min\left\{\frac{\pi}{6(l+1)}, \frac{\pi}{12}, \eta A \sin^2\frac{\pi}{12}, \eta B \sin^2\frac{\pi}{12}\right\},$$

where l as defined in (1.2).

(H_3) There exists $M_0 > 0$ such that

$$|I_j(x,y)| < M_0, \quad |J_j(x,y)| < M_0, \quad j = \pm 1, \pm 2, \cdots.$$

The main result of this paper is the following:

Theorem 1.1. *Assume that* (A_2)–(A_3), (H_1)–(H_3) *hold. Then for every positive integer* $k \geq 1$, *there exists a positive integer* $m^*(k)$ *such that, for every integer* $m \geq m^*(k)$, *the system* (1.1) *has at least two* mT*-periodic solutions having* k *as a rotation number.*

Remark 1.1. *Theorem 1.1 generalizes the relevant results [19, 21] of López-Gómez et al.*

(1) In contrast to the non-degenerate condition (A_1), *condition* (H_1) *allows the coefficient functions* α *and* β *to vanish on a subinterval. This implies that the solutions near the origin may enter the origin in the phase plane which leads to a bad evaluation of the rotations. By careful phase plane analysis, we can overcome this problem.*

(2) Our model does not have a parameter λ. *In [19, 21], as the parameter* λ *increases, the gap in the rotation number between the small and the large solutions increases. But this approach is not directly applicable to our case. For our system* (1.1) *without parameter, it is difficult to obtain enough gap between the rotation numbers of small and large solutions. In Section 3, we overcome all difficulties by phase plane analysis.*

The rest of the paper is organized as follows. In Section 2, we consider the degenerate systems without impulses and give some preliminary results. In Section 3, we provide the proof of Theorem 1.1. Firstly, we prove that the solutions of (1.1) near the origin can complete many turns around the origin via phase plane analysis. Next, we show that the solutions of (1.1) far from the origin cannot complete one turn. Finally, we obtain the existence of subharmonic solutions by Poincaré-Birkhoff theorem.

2. Preliminary

For the sake of convenience, we introduce some basic lemmas and tools that will be used in the next section. First, we consider the Hamiltonian system without impulsive terms

$$\begin{cases} x' = -\alpha(t) f(y), \\ y' = \beta(t) g(x), \end{cases} \tag{2.1}$$

where α, β, f and g satisfy (A_2)–(A_3) and (H_1).

Apparently, $(0,0)$ is a solution of (2.1). By the uniqueness of the solution, if the initial value $(x(0), y(0)) \neq (0,0)$, then the solution $(x(t), y(t)) \neq (0,0)$ of (2.1), for $t \in \mathbb{R}$. Introducing polar coordinates

$$x(t) = r(t)\cos\theta(t), \quad y(t) = r(t)\sin\theta(t).$$

Let

$$\mathcal{D}_1 = \{(x, y): \ x > 0, \ y \geq 0\}, \quad \mathcal{D}_2 = \{(x, y): \ x \leq 0, \ y > 0\},$$

$$\mathcal{D}_3 = \{(x, y): \ x < 0, \ y \leq 0\}, \quad \mathcal{D}_4 = \{(x, y): \ x \geq 0, \ y < 0\}.$$

Similar to the discussion in [19, Remark 1], by (A_2) and (H_1), all solutions of (2.1) exists globally. Further, we have the following spiral property.

Lemma 2.1. *Assume that (A_2)–(A_3) and (H_1) hold. For any fixed positive integer m, there are a sufficiently large $R^* > 0$ and two strictly monotonically increasing functions $\xi_m^{\pm} : [R^*, +\infty) \to \mathbb{R}$ such that*

$$\xi_m^{\pm}(s) \to +\infty \Longleftrightarrow s \to +\infty.$$

Moreover, let $z(t) = (x(t), y(t))$ be the solution of (2.1) *with $R_0 = |z(0)| \geq R^*$. Then we have either*

$$\xi_m^{-}(R_0) \leq |z(t)| \leq \xi_m^{+}(R_0), \quad t \in [0, mT],$$

or there exists $t^* \in (0, mT)$ such that

$$\theta(t^*) - \theta(t_0) = 2\pi,$$

and

$$\xi_m^{-}(R_0) \leq |z(t)| \leq \xi_m^{+}(R_0), \quad t \in [0, t_*].$$

Proof. Without loss of generality we assume that $z(t)$ is the solution of (2.1) with $|z(0)| = R_0$, where R_0 is sufficiently large.

Step 1. The estimation of upper bound on the solution $z(t)$.

As a first case, we assume that $z(0) \in \mathcal{D}_2$. If $z(t) \in \mathcal{D}_2$ for all $t \in [0, mT]$. Notice that $x'(t) \leq 0$ and $y'(t) \leq 0$, we have $0 \leq y(t) \leq y(0)$. Taking

$$M = \max\{|f(y)| : 0 \leq y \leq y(0)\},$$

then

$$x(t) = \int_0^t x' dt \geq - \int_0^{mT} \alpha(t) M dt = -mAM := -R_1. \tag{2.2}$$

Otherwise, there exists a $t_1 \in [0, mT]$, such that $y(t_1) = 0$ and $z(t) \in \mathcal{D}_2$ for $t \in [0, t_1)$, then (2.2) still holds.

If $z(t) \in \mathcal{D}_3$ for all $t \in [t_1, mT]$. Then, we have $x'(t) \geq 0$, $y'(t) \leq 0$. Therefore, $-R_1 \leq x(t) \leq 0$. Taking

$$N = \max\{|g(x)| : -R_1 \leq x \leq 0\},$$

then we have

$$y(t) = \int_{t_1}^t y' dt \geq - \int_0^{mT} \beta(t) N dt = -mBN := -R_2. \tag{2.3}$$

Otherwise, there exists a $t_2 \in (t_1, mT]$, such that $x(t_2) = 0$ and $z(t) \in \mathcal{D}_3$ for $t \in [t_1, t_2)$. Thus, for $t \in [t_1, t_2]$, (2.3) still holds.

If $z(t) \in \mathcal{D}_4$ for all $t \in [t_2, mT]$, we have $x'(t) \geq 0$, $y'(t) \geq 0$, which implies that $-R_2 \leq y \leq 0$. Taking

$$M_1 = \max\{|f(y)| : -R_2 \leq y \leq 0\},$$

$$x(t) = \int_{t_2}^{t} x' dt \le \int_0^{mT} \alpha(t) M_1 dt = mAM_1 := R_3. \tag{2.4}$$

Otherwise, there exists a $t_3 \in (t_2, mT]$, such that $y(t_3) = 0$ and $z(t) \in \mathcal{D}_4$ for $t \in [t_2, t_3)$. Then for $t \in [t_2, t_3]$, (2.4) still holds.

If $z(t) \in \mathcal{D}_1$ for all $t \in [t_3, mT]$, we have $x'(t) \le 0$, $y'(t) \ge 0$, which implies that $0 \le x(t) \le R_3$. Taking

$$N_1 = \max\{|g(x)| : 0 \le x \le R_3\}.$$

It follows that

$$y(t) = \int_{t_1}^{t} y' dt \le \int_0^{mT} \beta(t) N_1 dt = mBN_1 := R_4. \tag{2.5}$$

Otherwise, there exists a $t^* \in (t_3, mT]$, such that $x(t^*) = 0$ and $z(t) \in \mathcal{D}_1$ for $t \in [t_3, t^*)$. Then for $t \in [t_3, t^*]$, (2.5) still holds.

Let $\xi_m^+(z(0)) = \sqrt{2} \max\{R_0, R_1, R_2, R_3, R_4\}$. Then

$$\xi_m^+(z(0)) \to +\infty \Longleftrightarrow |z(0)| \to +\infty.$$

For the other case, that is $z(0) \in \mathcal{D}_3$, $z(0) \in \mathcal{D}_4$ and $z(0) \in \mathcal{D}_1$, similar to the discussion above, we can find the an upper bound on each quadrant. For briefness, we still denote by R_i, $i = 0, \cdots, 4$ the associated upper bound on each quadrant. Note that R_i, $i = 0, \cdots, 4$, depends on $z(0) \in \mathbb{R}^2$, and therefore, the choice of ξ_m^+ only depends on $z(0)$.

Step 2. The estimation of lower bound on the solution $z(t)$.

To complete our analysis, we need to look at what happens in the every quadrants. At first, we consider the case of $z(t) \in \mathcal{D}_2$.

For $z(t) \in \mathcal{D}_2$, there exists $t_1' \in [0, t_1]$ such that $y(t_1') = -x(t_1')$. we claim that

$$|x(t_1')| \to +\infty \Longleftrightarrow |z_0| \to +\infty. \tag{2.6}$$

For otherwise, we assume that there exists a constant J_1 which is independent of $|z_0|$, such that

$$0 < -x(t) \le J_1, \quad \text{for } t \in (0, t_1'].$$

On the other hand,

$$y(t_1') = y(0) + \int_0^{t_1'} y' dt \ge |z_0| - \left(\left[\frac{t_1'}{T} \right] + 1 \right) BM_g',$$

where $M_g' = \max\{|g(x)| : 0 < -x \le J_1\}$. Then $y(t_1') \to +\infty$ as $|z_0| \to +\infty$. But this is a contradiction with the fact that $y(t_1') \le J$.

Since $x' \le 0$ and $y' \le 0$, we have

$$|z(t)| \ge y(t) \ge y(t_1') \quad \text{for } t \in [0, t_1'] \tag{2.7}$$

and

$$|z(t)| \ge -x(t) \ge -x(t_1') \quad \text{for } t \in [t_1', t_1]. \tag{2.8}$$

Combine (2.6)–(2.8), there exists $\xi_{m,2}^-(z_0)$ such that

$$\xi_{m,2}^-(z(0)) \to +\infty \Longleftrightarrow |z(0)| \to +\infty \quad \text{and} \quad |z(t)| \ge \xi_{m,2}^-(z(0)) \text{ for } t \in [0, t_1].$$

In particular, $|z(t_1)| \to +\infty \Longleftrightarrow |z_0| \to +\infty$.

Similarly, we can discuss the cases of $z(t) \in \mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{D}_1$. In conclusion, we can find $\xi^-_{m,3}(z(t_1))$, $\xi^-_{m,4}(z(t_2))$ and $\xi^-_{m,1}(z(t_3))$ such that

$$\xi^-_{m,3}(z(t_1)) \to +\infty \Longleftrightarrow |z(t_1)| \to +\infty, \quad |z(t)| \geq \xi^-_{m,3}(z(t_1)) \text{ for } t \in [t_1, t_2],$$

$$\xi^-_{m,4}(z(t_2)) \to +\infty \Longleftrightarrow |z(t_2)| \to +\infty, \quad |z(t)| \geq \xi^-_{m,4}(z(t_2)) \text{ for } t \in [t_2, t_3],$$

$$\xi^-_{m,1}(z(t_3)) \to +\infty \Longleftrightarrow |z(t_3)| \to +\infty, \quad |z(t)| \geq \xi^-_{m,1}(z(t_3)) \text{ for } t \in [t_3, t^*].$$

Set

$$\xi^-_m(z(0)) = \xi^-_{m,1} \circ \xi^-_{m,4} \circ \xi^-_{m,3} \circ \xi^-_{m,2}(z(0)).$$

Then,

$$|z(t)| \geq \xi^-_m(z(0)) \quad \text{for } t \in [0, t^*].$$

Notice that the discussions in Steps 1 and 2 are true for not only for initial time $t_0 = 0$ but also for all initial time $t_0 \in [0, T]$. Therefore, we can rewrite $\xi^\pm_m(z(0))$ as $\xi^\pm_m(R_0)$, which means that the upper and lower bounds of the solution depend only on $R_0 = |z(t_0)|$. The proof of the lemma is complete.

Now let us return to the impulsive Eq (1.1). Notice that the motion of the solution of (1.1) is the same as that of (2.1) until it meets the next impulsive time. We consider the behavior of small solutions. Denote by D_R the disc of radius R centered at zero.

According to [23], (A_2) guarantee the existence and uniqueness of the solution. By boundedness of impulses, similar to the proof of [19, Proposition 1], we have the following result.

Proposition 1. *Assume that* (A_2)–(A_3), (H_1)–(H_3) *hold. For every integer* $m \geq 1$ *and* $\varepsilon > 0$, *there exists* $\delta = \delta(m, \varepsilon) > 0$ *such that if* $(x_0, y_0) \in D_\delta$, *then the unique solution of* (1.1), $(x(t), y(t))$, *with* $(x(0), y(0)) = (x_0, y_0)$, *satisfies* $(x(t), y(t)) \in D_\varepsilon$ *for all* $t \in [0, mT]$.

By Proposition 1, there exists $\delta_0 = \delta_0(m, \delta_1)$ such that if $z(t) = (x(t), y(t))$ is any solution of (1.1) with initial value $z_0 = (x(0), y(0)) \in D_{\delta_0}$, then $z(t) \in D_{\delta_1}$. For the nontrivial solution $z(t) \in D_{\delta_1}$, by (H_2), we have $\sqrt{2}\mu < \dfrac{1}{2}$, then $\Delta r(t_j) = \sqrt{(\triangle x(t_j))^2 + (\triangle y(t_j))^2} < \dfrac{1}{2} r(t_j^-)$. This implies that $z(t)$ never passes the origin, that is, if $z_0 \neq (0, 0)$ then $z(t) \neq (0, 0)$ for $t \in \mathbb{R}$. Then the rotation number of the solution $z(t)$ associated with the initial value z_0 can be defined as

$$\operatorname{rot}(z_0; [0, mT]) := \frac{\theta(mT) - \theta(0)}{2\pi}.$$

The rotation number is an algebraic counter of the counterclockwise turns of the solution $z(t)$ around the origin during the time-interval $[0, mT]$.

Now we introduce a generalized version of the Poincaré-Birkhoff theorem [11, Theorem 2.1] to prove the existence of subharmonic solutions.

Theorem 2.1. *Let* $\mathcal{A}$ *be an annular region bounded by two strictly star-shaped curves around the origin,* Γ_-*and* $\Gamma_+, \Gamma_- \subset \operatorname{int}(\Gamma_+)$, *where* $\operatorname{int}(\Gamma_+)$*denotes the interior domain bounded by* Γ_+. *Suppose that* $F : \overline{\operatorname{int}(\Gamma_+)} \to \mathbb{R}^2$ *is an area-preserving homeomorphism and* $F|_{\mathcal{A}}$ *admits a lifting, with the standard covering projection* $\Pi : (\theta, r) \mapsto z = (r\cos\theta, r\sin\theta)$, *of the form*

$$\widetilde{F}\Big|_{\mathcal{A}} : (\theta, r) \mapsto (\theta + h(\theta, r), w(\theta, r)),$$

where h and w are continuous functions of period T in the first variable. Correspondingly, for $\widetilde{\Gamma}_- = \Pi^{-1}(\Gamma_-)$ and $\widetilde{\Gamma}_+ = \Pi^{-1}(\Gamma_+)$, assume the twist condition

$$h(\theta, r) > 0 \quad on \ \widetilde{\Gamma}_-, \quad h(\theta, r) < 0 \quad on \ \widetilde{\Gamma}_+. \tag{2.9}$$

Then, F has two fixed points z_1, z_2 in the interior of $\mathcal{A}$, such that

$$h\left(\Pi^{-1}(z_1)\right) = h\left(\Pi^{-1}(z_2)\right) = 0.$$

Remark 2.1. *In (2.9), the function $h(\theta, r)$ means $\Delta\theta_F(z)$, the difference between polar angles of z and its image through $F(z)$. The continuity of this function is essential to verify the twist condition. In fact, Poincaré map of impulsive system is an iterative map by the section maps of flows and the jump maps. See [11] for more details.*

3. The proof of Theorem 1.1.

In order to apply the Poincaré-Birkhoff theorem to the impulsive Eq (1.1) where α and β may vanish on a subinterval, we need to characterize a suitable twist condition. Therefore, we should evaluate the rotation number of small and large solutions (see Lemmas 3.1 and 3.3).

Let $z(t; z_0)$ be the solution of (1.1) and $(\theta(t; \theta_0, r_0), r(t; \theta_0, r_0))$ the polar coordinates of $z(t; z_0)$. First of all, we find some small solutions of (1.1) with a given lower bound on the rotation number on the interval $[0, mT]$. More precisely, we have the following lemma.

Lemma 3.1. *Assume that (A_2), (H_1)–(H_3) hold. For any given positive integer $k > 1$, there exists a integer $m^*(k) \geq 1$ such that, for every integer $m \geq m^*(k)$, there exists $r_-(m) > 0$ such that for*

$$\theta(mT; \theta_0, r_0) - \theta_0 > 2k\pi, \quad for \ \theta_0 \in \mathbb{R} \ and \ r_0 = r_-(m).$$

Proof. For simplicity, let $z(t) = z(t; z_0)$ and $(\theta(t), r(t)) = (\theta(t; \theta_0, r_0), r(t; \theta_0, r_0))$. We assume that $(x(t), y(t)) \in D_\varepsilon$. Let's introduce notations

$$\Delta_j^i\theta =: \theta(t_j^-) - \theta(t_{j-1}^+), \quad j = (i-1)l + 1, \cdots, il,$$

$$\Delta_{il+1}^i\theta =: \theta(iT) - \theta(t_{il}^+) \quad \text{and} \quad \Delta_i\Theta =: \sum_{j=(i-1)l+1}^{il} (\theta(t_j^+) - \theta(t_j^-)),$$

where l as defined in (1.2).

We first consider the case where $i = 1$. For $(x(t), y(t)) \in D_{\delta_1}$ $t \in [0, T]$, $t \neq t_j$, $j = 1, \cdots, l$, we have

$$\begin{aligned}\theta'(t) = \frac{y'(t)x(t) - y(t)x'(t)}{x^2(t) + y^2(t)} &= \frac{\alpha(t)f(y(t))y(t) + \beta(t)g(x(t))x(t)}{x^2(t) + y^2(t)} \\ &\geq \eta(\alpha(t)\sin^2\theta + \beta(t)\cos^2\theta).\end{aligned}$$

By (H_2), we get

$$\sin(|\theta(t_j^+) - \theta(t_j^-)|) = \frac{\sqrt{\triangle^2 x(t_j) + \triangle^2 y(t_j)}}{x^2(t_j^-) + y^2(t_j^-)} < \frac{\sqrt{2}\mu r(t_j^-)}{r(t_j^-)} = \sqrt{2}\mu,$$

it follows that $|\theta(t_j^+) - \theta(t_j^-)| < \arcsin(\sqrt{2}\mu)$, thus

$$|\Delta_1\Theta| \le \sum_{j=1}^{l} |\theta(t_j^+) - \theta(t_j^-)| < l \arcsin(\sqrt{2}\mu). \tag{3.1}$$

There are two cases to be considered.

Case 1.1. Let $\theta(0) \in \left[\frac{1}{6}\pi, \frac{2}{3}\pi\right] \cup \left[\frac{7}{6}\pi, \frac{5}{3}\pi\right]$. For $t \in [0, T]$, $t \ne t_j$, $j = 1, \cdots, l$, we have $\theta'(t) \ge 0$. So either

$$\Delta_j^1\theta \ge \frac{\pi}{6(l+1)}, \quad \text{for some } j \in \{1, \cdots, l+1\},$$

or

$$0 \le \Delta_j^1\theta \le \frac{\pi}{6(l+1)}, \quad \text{for all } j \in \{1, \cdots, l+1\}.$$

For the former, by (H_2) and (3.1), we have

$$\theta(T) - \theta(0) = \sum_{j=1}^{l+1} \Delta_j^1\theta + \Delta_1\Theta \ge \frac{\pi}{6(l+1)} - l \arcsin(\sqrt{2}\mu) > 0.$$

For the latter, by (H_2) and (3.1), we obtain $|\Delta_i\Theta| < \frac{\pi}{12}$. Hence,

$$\theta(t) \in \left[\frac{1}{12}\pi, \frac{11}{12}\pi\right] \bigcup \left[\frac{13}{12}\pi, \frac{23}{12}\pi\right], \quad \text{for } t \in [0, T],$$

which implies that

$$\theta'(t) \ge \eta\alpha(t) \sin^2 \frac{1}{12}\pi, \quad \text{for } \ t \ne t_j, \ j = 1, \cdots, l.$$

Integrating both sides of the above inequality from 0 to T, we have

$$\theta(T) - \theta(0) = \sum_{j=1}^{l} \int_{t_{j-1}}^{t_j} \theta'(t)dt + \int_{t_l}^{T} \theta'(t)dt + \Delta_1\Theta \ge \eta A \sin^2 \frac{1}{12}\pi - l \arcsin(\sqrt{2}\mu) > 0.$$

Case 1.2. Let $\theta(0) \in \left[-\frac{1}{3}\pi, \frac{1}{6}\pi\right] \cup \left[\frac{2}{3}\pi, \frac{7}{6}\pi\right]$. Similarly, either

$$\Delta_j^1\theta \ge \frac{\pi}{6(l+1)}, \quad \text{for some } j \in \{1, \cdots, l+1\},$$

or

$$0 \le \Delta_j^1\theta \le \frac{\pi}{6(l+1)}, \quad \text{for all } j \in \{1, \cdots, l+1\}.$$

For the former, we get

$$\theta(T) - \theta(0) = \sum_{j=1}^{l+1} \Delta_j^1\theta + \Delta_1\Theta \ge \frac{\pi}{6(l+1)} - l \arcsin(\sqrt{2}\mu) > 0.$$

For the latter, we see

$$\theta(t) \in \left[-\frac{5}{12}\pi, \frac{5}{12}\pi\right] \bigcup \left[\frac{7}{12}\pi, \frac{17}{12}\pi\right], \quad \text{for } t \in [0, T].$$

Therefore,

$$\theta'(t) \geq \eta\beta(t)\cos^2\frac{5}{12}\pi = \eta\beta(t)\sin^2\frac{1}{12}\pi, \quad \text{for } t \neq t_j, \ j = 1, \cdots, l.$$

Integrating both sides of the above inequality from 0 to T, we find

$$\theta(T) - \theta(0) = \sum_{j=1}^{l}\int_{t_{j-1}}^{t_j}\theta'(t)dt + \int_{t_l}^{T}\theta'(t)dt + \Delta_1\Theta \geq \eta B\sin^2\frac{1}{12}\pi - l\arcsin(\sqrt{2}\mu) > 0.$$

We take $\Theta_* = \mu^* - l\arcsin(\sqrt{2}\mu) > 0$ and obtain

$$\theta(T) - \theta(0) \geq \Theta_*. \tag{3.2}$$

Let $m^*(k) = \left[\frac{2\pi k}{\Theta_*}\right] + 1$. Thus, for a given positive integer $m \geq m^*(k)$, there exists a small circle $r = r_-(m, \delta_1)$ such that the solution of (1.1) starting from $r = r_-(m)$ completes k counterclockwise rotations on $[0, mT]$, that is,

$$\theta(mT; \theta_0, r_0) - \theta_0 > 2k\pi.$$

As is well-known, the global existence of solutions is a crucial requirement for applying the Poincaré-Birkhoff theorem. To this end, we will prove that the solution of (1.1) possesses a spiral property in the phase plane, which guarantees the global existence of the solution.

Lemma 3.2. *Assume that* (A_2) *and* (A_3), (H_1)–(H_3) *hold. For fixed* $m \geq m^*(k)$ *and sufficiently large* $R_* > 0$, *there exists* $R_m^\pm : [R_*, +\infty) \to \mathbb{R}$ *such that*

$$R_m^\pm(s) \to +\infty \Longleftrightarrow s \to +\infty.$$

Moreover, let $z(t)$ *be the solution of* (1.1) *with* $r_0 = |z(t_0)| \geq R_*$. *Then we have either*

$$R_m^-(r_0) \leq |z(t)| \leq R_m^+(r_0), \quad \forall\, t \in [t_0, t_0 + mT],$$

or there exists $t_* \in (t_0, t_0 + mT)$ *such that*

$$\theta(t_*) - \theta(t_0) = 2\pi,$$

and

$$R_m^-(r_0) \leq |z(t)| \leq R_m^+(r_0), \quad \forall\, t \in [t_0, t_*].$$

Proof. Let $z(t)$ be the solution of (1.1) satisfying the initial value $z(t_0)$, where $|z(t_0)|$ is sufficiently large. Notice that for impulsive Eq (1.1), the motion of the solution is same as the motion of the corresponding equation without impulses until it meets the next impulse time. By Lemma 2.1, we can find curves $\xi^{\pm}_{j,m}(r(t^+_{j-1})) := \xi^{\pm}_m(r(t^+_{j-1}))$, $j = 1, \cdots, ml + 1$, such that

$$\xi^{\pm}_{j,m}(r(t^+_{j-1})) \to +\infty \Longleftrightarrow r(t^+_{j-1}) \to +\infty,\ \ j = 1, \cdots, ml + 1.$$

Moreover,

$$\xi^-_{j,m}(r(t^+_{j-1})) \le r(t) \le \xi^+_{j,m}(r(t^+_{j-1})), \quad t \in (t^+_{j-1}, t^-_j),\ \ j = 1, \cdots, ml,$$

and

$$\xi^-_{ml+1,m}(r(t^+_{ml})) \le r(t) \le \xi^+_{ml+1,m}(r(t^+_{ml})), \quad t \in (t^+_{ml}, mT].$$

In particular,

$$\xi^-_{j,m}(r(t^+_{j-1})) \le r(t^-_j) \le \xi^+_{j,m}(r(t^+_{j-1})), \quad j = 1, \cdots, ml. \tag{3.3}$$

By (H_3), we get

$$|r(t^+_j) - r(t^-_j)| = \sqrt{I^2_j + J^2_j} < \sqrt{2}M_0, \quad j = 1, \cdots, ml.$$

Hence, for $j = 1, \cdots, ml$,

$$Q^-_j(r(t^+_j)) := \xi^-_{j,m}(r(t^+_{j-1})) - \sqrt{2}M_0 \le r(t^+_j) \le \xi^+_{j,m}(r(t^+_{j-1})) + \sqrt{2}M_0 =: Q_j(r(t^+_j)). \tag{3.4}$$

Let

$$R^{\pm}_m(r_0) := \xi^{\pm}_{ml+1,m} \circ Q^{\pm}_{ml} \circ \cdots \xi^{\pm}_{2,m} \circ Q^{\pm}_1 \circ \xi^{\pm}_{1,m}(r_0)$$

Combining (3.3) with (3.4), we have

$$R^{\pm}_m(r_0) \to +\infty \Longleftrightarrow r_0 \to +\infty,$$

and

$$R^-_m(r_0) \le r(t) \le R^+_m(r_0), \quad \forall\ t \in [0, mT].$$

Further, we prove that the solution $z(t)$ with sufficiently large $|z(t_0)|$, cannot complete one turn around the origin on $[0, mT]$, where m is given by Lemma 3.1.

Lemma 3.3. *Assume that (A_3), (H_1)–(H_3) hold. For any given $m \in \mathbb{N}$, there exists $r_+(m) > 0$ such that,*

$$\theta(mT; \theta_0, r_0) - \theta_0 < 2\pi, \quad \textit{for}\ \ \theta_0 \in \mathbb{R}\ \textit{and}\ r_0 = r_+(m).$$

Proof. Without loss of generality we assume that (H_3) holds with g satisfying (g_-). Then there exists $M_g > 0$ such that

$$\sup_{x \le 0} |g(x)| \le M_g.$$

Let $z(t) = (x(t), y(t))$ be a nontrivial solution of (1.1) with the initial value $z_0 = (x_0, y_0)$. There are two cases to discuss.

Case 3.1. Let $z_0 \in \mathcal{D}_1 \bigcup \mathcal{D}_3 \bigcup \mathcal{D}_4$. We claim that $z(t)$ cannot cross entirely $\mathcal{D}_2$ during $[0, mT]$, if $|z_0|$ is sufficiently large.

Proof by contradiction. Assume that there exists $[\tau_1, \tau_2] \subset [0, mT]$, $\tau_1, \tau_2 \neq t_j$, $j = 1, \cdots, ml$, such that

$$0 \geq x(\tau_1) \geq -M_0,\ y(\tau_1) > 0, \quad 0 > x(\tau_2),\ M_0 \geq y(\tau_2) \geq 0,$$

where M_0 is given by (H_5). Then,

$$\begin{aligned} |y(\tau_1)| &\leq |y(\tau_2)| + \left| M_g \int_{\tau_1}^{\tau_2} \beta(s)ds \right| + \sum_{\tau_1 < t_j < \tau_2} |J_j(x(t_j^-), y(t_j^-))| \\ &\leq M_0 + mM_gB + mlM_0 := R^*. \end{aligned}$$

By Lemma 3.2, we choose $r_+(m)$ such that for $r_0 = r_+(m)$, $|z(t)| \geq R_m^-(r_0) > R^* + M_0$, $t \in [0, mT]$. This leads to a contradiction. Hence,

$$\theta(mT; \theta_0, r_0) - \theta_0 < 2\pi.$$

Case 3.2. Let $z_0 \in \mathcal{D}_2$. We claim that $z(t)$ cannot cross entirely $\mathcal{D}_3$ during $[0, mT]$, if $|z_0|$ is sufficiently large.

Similarly, by contradiction, we assume that there exists $[\tau_3, \tau_4] \subset [0, mT]$, $\tau_3, \tau_4 \neq t_j$, $j = 1, \cdots, ml$, such that

$$0 > x(\tau_3),\ 0 \geq y(\tau_3) \geq -M_0, \quad 0 \geq x(\tau_4) \geq -M_0,\ 0 \geq y(\tau_4),$$

Thus,

$$\begin{aligned} |y(t)| &\leq |y(\tau_3)| + M \int_{\tau_3}^{\tau_4} \beta(s)ds + \sum_{\tau_3 < t_j < \tau_4} |J_j(x(t_j^-), y(t_j^-))| \\ &\leq M_0 + mM_gB + mlM_0 = R^*. \end{aligned}$$

Let

$$N^* := \max\{|f(y)| : |y| \leq R^*\},$$

then

$$\begin{aligned} |x(\tau_3)| &\leq |x(\tau_4)| + N^* \int_{\tau_3}^{\tau_4} \alpha(s)ds + \sum_{\tau_3 < t_j < \tau_4} |I_j(x(t_j^-), y(t_j^-))| \\ &\leq M_0 + mN^*A + mlM_0 := R^{**}. \end{aligned}$$

By Lemma 3.2, we choose $r_+(m)$ such that for $r_0 = r_+(m)$, $|z(t)| \geq R_m^-(r_0) > R^{**} + M_0$, $t \in [0, mT]$. This leads to a contradiction. Hence,

$$\theta(mT; \theta_0, r_0) - \theta_0 < 2\pi.$$

Now we are in a position to consider the Poincaré mapping associated with (1.1). According to Lemma 3.2, the solutions of (1.1) exists globally. Denote by $(x(t; x_0, y_0), y(t; x_0, y_0))$ the solution of (1.1) with the initial value $(x_0, y_0) = (x(0; x_0, y_0), y(0; x_0, y_0))$ and define

$$\begin{aligned} \mathcal{P}_0 &: (x_0, y_0) \to (x(t_1-; x_0, y_0), y(t_1-; x_0, y_0)), \\ \mathcal{P}_1 &: (x_1, y_1) \to (x(t_2-; x_1, y_1), y(t_2-; x_1, y_1)), \end{aligned}$$

$$\vdots$$
$$\mathcal{P}_j : (x_j, y_j) \to (x(t_{j+1}-; x_j, y_j), y(t_{j+1}-; x_j, y_j)),$$
$$\vdots$$
$$\mathcal{P}_{ml-1} : (x_{ml-1}, y_{ml-1}) \to \left(x\left(t_{ml}^-; x_{ml-1}, y_{ml-1}\right), y\left(t_{ml}^-; x_{ml-1}, y_{ml-1}\right)\right),$$
$$\mathcal{P}_{ml} : (x_{ml}, y_{ml}) \to \left(x\left(mT; x_{ml}, y_{ml}\right), y\left(mT; x_{ml}, y_{ml}\right)\right),$$

where $(x_j, y_j) = (x(t_j+; x_j, y_j), y(t_j+; x_j, y_j))$, $j = 1, 2, \cdots, ml$. The jumping map is defined by

$$\Phi_j : (x, y) \longmapsto (x + I_j(x, y), y + J_j(x, y)), \quad j = 1, 2, \cdots, ml.$$

Then the Poincaré mapping $\mathcal{P}$ associated with (1.1) can be written in the form

$$\mathcal{P} = \mathcal{P}_{ml} \circ \Phi_{ml} \circ \cdots \mathcal{P}_1 \circ \Phi_1 \circ \mathcal{P}_0.$$

Since

$$\begin{cases} x' = -\alpha(t) f(y), \\ y' = \beta(t) g(x), \end{cases}$$

is conservative, $\mathcal{P}_j$, $j = 0, 2, \cdots, ml$, are symplectic. And Φ_j, $j = 1, 2, \cdots, ml$, are area-preserving homeomorphisms. Hence, the Poincaré mapping $\mathcal{P}$ is area-preserving. Moreover, by Lemmas 3.2 and 3.3, $\mathcal{P}$ satisfies the boundary twist condition.

Finally, we apply the Poincaré-Birkhoff theorem to prove the existence of subharmonic solutions.

Applying Theorem 2.1 , $\mathcal{P}$ has at least two geometrically distinct fixed points

$$(x_i, y_i) = (r_i \cos\theta_i, r_i \sin\theta_i), \quad i = 1, 2,$$

which correspond to two mT-periodic solutions of system (1.1) with

$$\theta(mT; \theta_i, r_i) - \theta_i = 2k\pi, \quad i = 1, 2,$$

that is, $\mathrm{rot}(z_{i,0}; [0, mT]) = k$, $i = 1, 2$.

The proof of Theorem 1.1 is thus completed.

Remark 3.1. *For any positive integer $m' \geq m^*(k)$, there exist a new $r_\pm(m')$ such that $\mathcal{P}$ satisfies the boundary twist condition. Then, by Theorem 1.1, $\mathcal{P}$ has at least two geometrically distinct fixed points*

$$\left(x_i^{(m')}, y_i^{(m')}\right) = \left(r_i^{(m')} \cos\theta_i^{(m')}, r_i^{(m')} \sin\theta_i^{(m')}\right), \quad i = 1, 2,$$

which correspond to two mT-periodic solutions of system (1.1) *with* $\mathrm{rot}(z_{i,0}^{m'}; [0, mT]) = k$, *$i = 1, 2$. Therefore, system* (1.1) *has infinitely many geometrically distinct mT-periodic solutions.*

4. Biological interpretations and conclusions

Our system (1.1) covers many mathematical models with biological properties. We briefly illustrate our degenerate systems with examples.

A classical predator-prey model of Lotka-Volterra type [19–21] has the form

$$\begin{cases} x' = x(a(t) - b(t)y), \\ y' = y(-c(t) + d(t)x), \end{cases} \tag{4.1}$$

where $a, b, c, d : \mathbb{R} \to \mathbb{R}$ are T-periodic functions and $b(t) \gtrsim 0, d(t) \gtrsim 0$. If system (4.1) possesses a T-periodic coexistence state $(\tilde{x}(t), \tilde{y}(t))$, namely, a positive componentwise T-periodic solution of the system, then by change of variables

$$x(t) = u(t)\tilde{x}(t), \quad y(t) = v(t)\tilde{y}(t),$$

system (4.1) is changed into the equivalent system

$$\begin{cases} u' = b(t)\tilde{y}(t)u(1 - v), \\ v' = d(t)\tilde{x}(t)v(-1 + u). \end{cases} \tag{4.2}$$

Commonly, $(1, 1)$ is regarded as a trivial coexistence state of system (4.2). From a geometric point of view, nontrivial componentwise positive mT-periodic solutions are the trajectories inside the first quadrant wound around the equilibrium point.

For the convenience of study, we can move the equilibrium point to the origin. System (4.2) is changed, via the change of variables $x = \ln u$ and $y = \ln v$, into the system

$$\begin{cases} x' = -\alpha(t)(-1 + e^{y}), \\ y' = \beta(t)(-1 + e^{x}). \end{cases} \tag{4.3}$$

with $\alpha(t) \gtrsim 0$ and $\beta(t) \gtrsim 0$. Obviously, (4.3) satisfies (A_2) and (A_3) and is a particular case of (1.1).

Now, let's provide some biological explanations for (H_1). In nature, a predator or prey can maintain a fairly constant density despite fluctuations in the density of its prey. At certain periodic times (certain seasons) of the year, the predators hunt preys and collect them. At other periodic times, they stop hunting the preys and simply consume what they have collected, during which the prey is unaffected by the predator.

Introducing a simple prototype model for system (1.1). In Mediterranean pine forests, there is a species of moth called the pine processionary. Typically, in late summer or early autumn, they lay their eggs on pine trees. After a period of incubation, the larvae emerge, and the larval stage can last for several months, during which the caterpillars actively feed on pine needles. the caterpillars actively feed on pine needles, causing defoliation. During this period, the density of pine trees remains constant, while the density of pine processionary caterpillars fluctuates. Depending on the impact of defoliation, the population of the next generation of caterpillars can increase or decrease before the start of a new cycle. Coefficient functions α and β vanish on some subintervals to simulate this property, which makes our model more realistic.

On the other hand, models (4.1) and (4.2) are affected by short-term perturbations, which are usually considered as impulse terms. It is well known that many biological systems are disturbed by human activities, such as biological control, feeding, harvesting, and planting in fisheries and forest management, chemotherapy to cancer cells, and regular release of toxins from environmental pollution. In order to describe this system more accurately, continuous human activity is often removed from the model and replaced with impulses [22, 24–26].

A commonly employed method in biological pest control is the release of natural enemies to reduce the population of pests. For example, parasitic wasps are natural enemies of many pests and are commonly used in biological pest control. In greenhouse production, pests often experience outbreaks during the summer season. In this period of time, we can release parasitic wasps to control the growth of pests, and in some cases, even eliminate them.

In this paper, based on Poincaré-Birkhoff theorem, by using phase plane analysis, we show that for every integer $m \geq m^*(k)$, impulsive systems (1.1) has at least two mT-periodic solutions. From a biological perspective, Theorem 1.1 indicates that small bounded disturbances caused by human activities do not affect the existence of infinitely many equilibrium states.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. P. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, In: *CBMS Regional Conference Series in Mathematics*, 1986. https://dx.doi.org/10.1090/cbms/065
2. J. Mawhin, M. Willem, *Critical point theory and Hamiltonian systems*, New York: Springer, 1989. https://dx.doi.org/10.1007/978-1-4757-2061-7
3. Y. Long, Multiple solutions of perturbed superquadratic second order Hamiltonian systems, *Trans. Amer. Math. Soc.*, **311** (1989), 749–780. https://dx.doi.org/10.2307/2001151
4. A. Fonda, *Playing around resonance: An invitation to the search of periodic solutions for second order ordinary differential equations*, Cham: Springer, 2016. https://dx.doi.org/10.1007/978-3-319-47090-0
5. H. Jacobowitz, Periodic solutions of $x'' + f(x, t) = 0$ via the Poincaré-Birkhoff theorem, *J. Differ. Equations*, **20** (1976), 37–52. https://dx.doi.org/10.1016/0022-0396(76)90094-2
6. A. Margheri, C. Rebelo, F. Zanolin, Maslov index, Poincaré-Birkhoff theorem and periodic solutions of asymptotically linear planar Hamiltonian systems, *J. Differ. Equations*, **183** (2002), 342–367. http://dx.doi.org/10.1006/jdeq.2001.4122

7. A. Fonda, Positively homogeneous Hamiltonian systems in the plane, *J. Differ. Equations*, **200** (2004), 162–184. http://dx.doi.org/10.1016/j.jde.2004.02.001

8. D. Qian, P. J. Torres, Periodic motions of linear impact oscillators via the successor map, *SIAM J. Math. Anal.*, **36** (2005), 1707–1725. http://dx.doi.org/10.1137/S003614100343771X

9. D. Qian, P. J. Torres, P. Wang, Periodic solutions of second order equations via rotation numbers, *J. Differ. Equations*, **266** (2019), 4746–4768. http://dx.doi.org/10.1016/j.jde.2018.10.010

10. F. Jiang, J. Shen, Y. Zeng, Applications of the Poincaré-Birkhoff theorem to impulsive Duffing equations at resonance, *Nonlinear Anal. Real*, **13** (2012), 1292–1305. http://dx.doi.org/10.1016/j.nonrwa.2011.10.006

11. D. Qian, L. Chen, X. Sun, Periodic solutions of superlinear impulsive differential equations: A geometric approach, *J. Differ. Equations*, **258** (2015), 3088–3106. http://dx.doi.org/10.1016/j.jde.2015.01.003

12. Y. Niu, X. Li, Periodic solutions of semilinear Duffing equations with impulsive effects, *J. Math. Anal. Appl.*, **467** (2018), 349–370. http://dx.doi.org/10.1016/j.jmaa.2018.07.008

13. J. Shen, L. Chen, X. Yuan, Lagrange stability for impulsive Duffing equations, *J. Differ. Equations*, **266** (2019), 6924–6962. http://dx.doi.org/10.1016/j.jde.2018.11.022

14. A. Hausrath, R. Manásevich, Periodic solutions of a periodically perturbed Lotka-Volterra equation using the Poincaré-Birkhoff theorem, *J. Math. Anal. Appl.*, **157** (1991), 1–9. http://dx.doi.org/10.1016/0022-247X(91)90132-J

15. T. Ding, F. Zanolin, Harmonic solutions and subharmonic solutions for periodic Lotka-Volterra systems, In: *Dynamical systems: Proceedings Of the special program at Nankai institute of mathematics*, World Scientific, 1993.

16. T. Ding, F. Zanolin, Periodic solutions and subharmonic solutions for a class of planar systems of Lotka-Volterra type, In: World congress of nonlinear analysts'92, Boston: De Gruyter, 1996. http://dx.doi.org/10.1515/9783110883237.395

17. A. Fonda, R. Toader, Subharmonic solutions of Hamiltonian systems displaying some kind of sublinear growth, *Adv. Nonlinear Anal.*, **8** (2019), 583–602. http://dx.doi.org/10.1515/anona-2017-0040

18. T. Dondé, F. Zanolin, Multiple periodic solutions for one-sided sublinear systems: A refinement of the Poincaré-Birkhoff approach, *Topol. Methods Nonlinear Anal.*, **55** (2020), 565–581. http://dx.doi.org/10.12775/TMNA.2019.104

19. J. López-Gómez, E. Muñoz-Hernández, F. Zanolin, On the applicability of the poincaré-Birkhoff twist theorem to a class of planar periodic predator-prey models, *Discrete Contin. Dyn. Syst.*, **40** (2020), 2393–2419. http://dx.doi.org/10.3934/dcds.2020119

20. J. López-Gómez, E. Muñoz-Hernández, Global structure of subharmonics in a class of periodic predator-prey models, *Nonlinearity*, **33** (2020), 34. http://dx.doi.org/10.1088/1361-6544/ab49e1

21. J. López-Gómez, E. Muñoz-Hernández, F. Zanolin, The Poincaré-Birkhoff theorem for a class of degenerate planar Hamiltonian systems, *Adv. Nonlinear Stud.*, **21** (2021), 489–499. http://dx.doi.org/10.1515/ans-2021-2137

22. S. Tang, L. Chen, The periodic predator-prey Lotka-Volterra model with impulsive effect, *J. Mech. Med. Biol.*, **2** (2002), 267–296. http://dx.doi.org/10.1142/S021951940200040X

23. D. Bainov, P. Simenov, *Impulsive differential equations: Periodic solutions and applications*, CRC Press, 1993.

24. Q. Liu, Q. Chen, Dynamics of stochastic delay Lotka-Volterra systems with impulsive toxicant input and Lévy noise in polluted environments, *Appl. Math. Comput.*, **256** (2015), 52–67. http://dx.doi.org/10.1016/j.amc.2015.01.009

25. L. Chen, J. Sun, F. Chen, L. Zhao, Extinction in a Lotka-Volterra competitive system with impulse and the effect of toxic substances, *Appl. Math. Model.*, **40** (2016), 2015–2024. http://dx.doi.org/10.1016/j.apm.2015.09.057

26. R. Tuladhar, F. Santamaria, I. Stamova, Fractional Lotka-Volterra-type cooperation models: Impulsive control on their stability, *Entropy*, **22** (2020), 970. https://doi.org/10.3390/e22090970