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Research article

Weighted variable Morrey-Herz space estimates for mth order commutators of $n-$dimensional fractional Hardy operators

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Abstract: In this paper, we establish the boundedness for mth order commutators of $n-$dimensional fractional Hardy operators and adjoint operators on weighted variable exponent Morrey-Herz space $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\omega)$.

Keywords: Morrey-Herz space; commutators; fractional Hardy operators; variable exponent function space; weights
Mathematics Subject Classification: 42B35, 26D10, 47B47, 47G10

1. Introduction

As we all know, the study of variable exponent function space inspired by nonlinear elasticity theory and nonstandard growth differential equations is one of the key contents of harmonic analysis in the past three decades, attracting extensive attention from many scholars. In [19], the theory of function spaces with variable exponent was progressed since some elementary properties were established by Kováčik and Rákosník, and they studied many basic properties of variable exponent Lebesgue spaces and Sobolev spaces on $\mathbb{R}^n$. Later, the Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ were extensively investigated; see [7, 8, 22]. In [14], Izuki first introduced the Herz spaces with variable exponent $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$, which are generalizations of the Herz spaces $\dot{K}^{\alpha,q}_{p}(\mathbb{R}^n)$, and considered the boundedness of commutators of fractional integrals on Herz spaces with variable exponent . In [13], Izuki introduced the Herz-Morrey spaces with variable exponent $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$, which are generalizations of the Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{q,p}(\mathbb{R}^n)$, and studied the boundedness of vector valued sublinear operators on Herz-Morrey spaces with variable exponent $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$. On the other hand, in the study of boundary value problems for the Laplace equation on Lipschitz domains, the classical theory of Muckenhoupt weights is a powerful tool in harmonic analysis; see [21]. Generalized Muckenhoupt weights with variable exponent have been intensively studied; see [4, 5].

In [11], Hardy defined the classical Hardy operators as:

$$\mathcal{P}(f)(x) := \frac{1}{x}\int_0^x f(t)\mathrm{d}t, \quad x > 0. \tag{1.1}$$

In [6], Christ and Grafakos defined the $n-$dimensional Hardy operators as:

$$\mathcal{H}(f)(x) := \frac{1}{|x|^n}\int_{|t|<|x|} f(t)\mathrm{d}t, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{1.2}$$

and established the boundedness of $\mathcal{P}(f)(x)$ in $L^p(\mathbb{R}^n)$, getting the best constants.

In [9], under the condition of $0 \le \beta < n$ and $|x| = \sqrt{\sum_{i=1}^n x_i^2}$, Fu et al. defined the $n-$dimensional fractional Hardy operators and its adjoint operators as:

$$\mathcal{H}_\beta f(x) := \frac{1}{|x|^{n-\beta}}\int_{|t|<|x|} f(t)\mathrm{d}t, \;\; \mathcal{H}_\beta^* f(x) := \int_{|t|\ge|x|} \frac{f(t)}{|t|^{n-\beta}}\mathrm{d}t, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{1.3}$$

and established the boundedness of their commutators in Lebesgue spaces and homogeneous Herz spaces.

Let $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$. Given a function $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $m \in \mathbb{N}$, Wang et al. [23] defined the mth order commutators of $n-$dimensional fractional Hardy operators and adjoint operators as:

$$\mathcal{H}^m_{\beta,b} f(x) := \frac{1}{|x|^{n-\beta}}\int_{|t|<|x|} (b(x)-b(t))^m f(t)\mathrm{d}t \tag{1.4}$$

and

$$\mathcal{H}^{*m}_{\beta,b} f(x) := \int_{|t|\ge|x|} \frac{(b(x)-b(t))^m f(t)}{|t|^{n-\beta}}\mathrm{d}t, \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{1.5}$$

Obviously, when $m = 0$, $\mathcal{H}^0_{\beta,b} = \mathcal{H}_\beta$, $\mathcal{H}^{*0}_{\beta,b} = \mathcal{H}^*_\beta$, and when $m = 1$, $\mathcal{H}^1_{\beta,b} = \mathcal{H}_{\beta,b}$, $\mathcal{H}^{*1}_{\beta,b} = \mathcal{H}^*_{\beta,b}$. More important results with regard to these commutators, see [20, 26, 27].

Due to the need of future calculation in this paper, let $0 < \beta < n$, and the fractional integral operator I_β is defined as:

$$I_\beta(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\beta}}\mathrm{d}y, \quad x \in \mathbb{R}^n. \tag{1.6}$$

Let $0 \le \beta < n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and the fractional maximal operator M_β is defined as:

$$M_\beta f(x) := \sup_{x\in B} \frac{1}{|B|^{1-\frac{\beta}{n}}}\int_B |f(y)|\mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{1.7}$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x. When $\beta = 0$, we simply write M instead of M_0, which is exactly the Hardy-Littlewood maximal function.

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ consist of all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $\mathrm{BMO}(\mathbb{R}^n) < \infty$. b is a bounded mean oscillation function if $||b||_{\mathrm{BMO}} < \infty$, and the $||b||_{\mathrm{BMO}}$ is defined as follow:

$$||b||_{\mathrm{BMO}} := \sup_B \int_B |b(x) - b_B|\mathrm{d}x, \tag{1.8}$$

where the supremum is taken all over the balls $B \in \mathbb{R}^n$ and $b_B := |B|^{-1} \int_B b(y)\mathrm{d}y$. For a comprehensive review of the bounded mean oscillation (BMO) space, please see the book [10].

Recently, Muhammad Asim et al. established the estimates of fractional Hardy operators on weighted variable exponent Morrey-Herz spaces in [1]. Amjad Hussain et al. established the boundedness of the commutators of the Fractional Hardy operators on weighted variable Herz-Morrey spaces in [12]. Motivated by the mentioned work, in this paper, we will give the boundedness of the mth order commutators of $n-$dimensional fractional Hardy operators $\mathcal{H}^m_{\beta,b}$ and its adjoint operators $\mathcal{H}^{*m}_{\beta,b}$ on weighted variable exponent Morrey-Herz space $\mathrm{M}\dot{\mathrm{K}}^{\alpha,\lambda}_{q,p(\cdot)}(\omega)$.

The paper is organized as follows. In Section 2, we provide some preliminary knowledge. The main results and their proofs are given in Section 3. In Section 4, we provide the conclusion of this paper. Throughout this paper, we use the following symbols and notations:

(1) For a constant $R > 0$ and a point $x \in \mathbb{R}^n$, we write $B(x, R) := \{y \in \mathbb{R}^n : |x - y| < R\}$.

(2) For any measurable set $E \subset \mathbb{R}^n$, $|E|$ denotes the Lebesgue measure, and χ_E means the characteristic function.

(3) Given $k \in \mathbb{Z}$, we write $B_k := \overline{B(0, 2^k)} = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$.

(4) We define a family $\{A_k\}_{k=-\infty}^{\infty}$ by $A_k := B_k \setminus B_{k-1} = \{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k\}$. Moreover χ_k denotes the characteristic function of A_k, namely, $\chi_k := \chi_{A_k}$.

(5) For any index $1 < p(x) < \infty$, $p'(x)$ denotes its conjugate index, namely, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

(6) If there exists a positive constant C independent of the main parameters such that $A \leq CB$, then we write $A \lesssim B$. Additionally, $A \approx B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

2. Preliminary knowledge

2.1. Some definitions that need to be used in this paper

Definition 2.1. ([7]) Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a measurable function.

(i) The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \Big\{f \text{ is measurable function} : \int_{\mathbb{R}^n} \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} \mathrm{d}x < \infty \text{ for some constant } \lambda > 0\Big\}.$$

(ii) The spaces with variable exponent $L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ are defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n) := \{f \text{ is measurable function} : f \in L^{p(\cdot)}(K) \text{ for all compact subsets } K \subset \mathbb{R}^n\}.$$

The Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} \mathrm{d}x \leq 1\Big\}.$$

Definition 2.2. ([7]) (i) The set $\mathcal{P}(\mathbb{R}^n)$ consists of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ satisfying

$$1 < p_- \leq p(x) \leq p_+ < \infty,$$

where

$$p_- := \operatorname{essinf}\{p(x) : x \in \mathbb{R}^n\} > 1, \quad p_+ := \operatorname{esssup}\{p(x) : x \in \mathbb{R}^n\} < \infty.$$

(ii) The set $\mathcal{B}(\mathbb{R}^n)$ consists of all measurable function $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Definition 2.3. ([7]) Suppose that $p(\cdot)$ is a real-valued function on $\mathbb{R}^n$. We say that

(i) $C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ is the set of all local log-Hölder continuous functions $p(\cdot)$ satisfying

$$|p(x) - p(y)| \le -\frac{C}{\log(|x-y|)}, \ |x-y| \le \frac{1}{2}, \quad x, y \in \mathbb{R}^n. \tag{2.1}$$

(ii) $C^{\log}_0(\mathbb{R}^n)$ is the set of all local log-Hölder continuous functions $p(\cdot)$ satisfying at origin

$$|p(x) - p_0| \le \frac{C}{\log(e + \frac{1}{|x|})}, \quad x \in \mathbb{R}^n. \tag{2.2}$$

(iii) $C^{\log}_\infty(\mathbb{R}^n)$ is the set of all local log-Hölder continuous functions satisfying at infinity

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + |x|)}, \quad x \in \mathbb{R}^n. \tag{2.3}$$

(iv) $C^{\log}(\mathbb{R}^n) = C^{\log}_\infty(\mathbb{R}^n) \cap C^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ denotes the set of all global log-Hölder continuous functions $p(\cdot)$.

It was proved in [7] that if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$, then the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Definition 2.4. ([21]) Given a non-negative, measure function ω, for $1 < p < \infty$, $\omega \in A_p$ if

$$[\omega]_{A_p} := \sup_B \Big(\frac{1}{|B|}\int_B \omega(x)\mathrm{d}x\Big)\Big(\frac{1}{|B|}\int_B \omega(x)^{1-p'}\mathrm{d}x\Big)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. Especially, we say $\omega \in A_1$ if

$$[\omega]_{A_1} := \sup_B \frac{\frac{1}{|B|}\int_B \omega(x)\mathrm{d}x}{\operatorname{essinf}\{\omega(x) : x \in B\}} < \infty.$$

These weights characterize the weighted norm inequalities for the Hardy-Littlewood maximal operator, that is, $\omega \in A_p$, $1 < p < \infty$, if and only if $M : L^p(\omega) \to L^p(\omega)$.

Definition 2.5. ([15]) Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A weight ω is in the class $A_{p(\cdot)}$ if

$$\sup_{B:\text{ball}} |B|^{-1} \|\omega^{\frac{1}{p(\cdot)}}\chi_B\|_{L^{p(\cdot)}} \|\omega^{-\frac{1}{p(\cdot)}}\chi_B\|_{L^{p'(\cdot)}} < \infty. \tag{2.4}$$

Obviously, if $p(\cdot) = p, 1 < p < \infty$, then the above definition reduces to the classical Muckenhoupt A_p class.

From [15], if $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $p(\cdot) \le q(\cdot)$, then $A_1 \subset A_{p(\cdot)} \subset A_{q(\cdot)}$.

Definition 2.6. ([15]) Let $0 < \beta < n$ and $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\beta}{n}$. A weight ω is said to be an $A(p_1(\cdot), p_2(\cdot))$ weight if

$$\|\chi_B\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \|\chi_B\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \le C|B|^{1-\frac{\beta}{n}}. \tag{2.5}$$

Definition 2.7. ([25]) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\omega \in A_{p(\cdot)}$. The weighted variable exponent Lebesgue space $L^{p(\cdot)}(\omega)$ denotes the set of all complex-valued measurable functions f satisfying

$$L^{p(\cdot)}(\omega) := \{f : f\omega^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}(\mathbb{R}^n)\}.$$

This is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\omega)} := \|f\omega^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Definition 2.8. ([1]) Let ω be a weight on $\mathbb{R}^n$, $0 \leq \lambda < \infty$, $0 < q < \infty$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The weighted variable exponent Morrey-Herz space $\mathrm{M}\dot{\mathrm{K}}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\omega)$ is the set of all measurable functions f given by

$$\mathrm{M}\dot{\mathrm{K}}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\omega) := \{f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}, \omega) : \|f\|_{\mathrm{M}\dot{\mathrm{K}}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\omega)} < \infty\},$$

where

$$\|f\|_{\mathrm{M}\dot{\mathrm{K}}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\omega)} := \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big\{\sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)q}\|f\chi_k\|^q_{L^{p(\cdot)}(\omega)}\Big\}^{\frac{1}{q}}.$$

It is noted that $\mathrm{M}\dot{\mathrm{K}}^{\alpha(\cdot),0}_{q,p(\cdot)}(\omega) = \dot{\mathrm{K}}^{\alpha(\cdot)}_{q,p(\cdot)}(\omega)$ is the variable exponent weighted Herz space defined in [2].

Definition 2.9. ([15]) Let $\mathcal{M}$ be the set of all complex-valued measurable functions defined on $\mathbb{R}^n$ and X be a linear subspace of $\mathcal{M}$.

(1) The space X is said to be a Banach function space if there exists a function $\|\cdot\|_X : \mathcal{M} \to [0, \infty]$ satisfying the following properties: Let $f, g, f_j \in \mathcal{M}$ $(j = 1, 2, \ldots)$. Then

(a) $f \in X$ holds if and only if $\|f\|_X < \infty$.

(b) Norm property:

i. Positivity: $\|f\|_X \geq 0$.

ii. Strict positivity: $\|f\|_X = 0$ holds if and only if $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.

iii. Homogeneity: $\|\lambda f\|_X = |\lambda| \cdot \|f\|_X$ holds for all $\lambda \in \mathbb{C}$.

iv. Triangle inequality: $\|f + g\|_X \leq \|f\|_X + \|g\|_X$.

(c) Symmetry: $\|f\|_X = \||f|\|_X$.

(d) Lattice property: If $0 \leq g(x) \leq f(x)$ for almost every $x \in \mathbb{R}^n$, then $\|g\|_X \leq \|f\|_X$.

(e) Fatou property: If $0 \leq f_j(x) \leq f_{j+1}(x)$ for all j, and $f_j(x) \to f(x)$ as $j \to \infty$ for almost every $x \in \mathbb{R}^n$, then $\lim\limits_{j\to\infty} \|f_j\|_X = \|f\|_X$.

(f) For every measurable set $F \subset \mathbb{R}^n$ such that $|F| < \infty$, $\|\chi_F\|_X$ is finite. Additionally, there exists a constant $C_F > 0$ depending only on F so that $\int_F |h(x)|\mathrm{d}x \leq C_F\|h\|_X$ holds for all $h \in X$.

(2) Suppose that X is a Banach function space equipped with a norm $\|\cdot\|_X$. The associated space X' is defined by

$$X' := \{f \in \mathcal{M} : \|f\|_{X'} < \infty\},$$

where

$$\|f\|_{X'} := \sup_g \Big\{\Big|\int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x\Big| : \|g\|_X \leq 1\Big\}.$$

2.2. Some lemmas that need to be used in this paper

Lemma 2.1. ([3]) Let X be a Banach function space, and then we have the following:
(i) The associated space X' is also a Banach function space.
(ii) $\|\cdot\|_{(X')'}$ and $\|\cdot\|_X$ are equivalent.
(iii) If $g \in X$ and $f \in X'$, then

$$\int_{\mathbb{R}^n} |f(x)g(x)| \mathrm{d}x \leq \|f\|_X \|g\|_{X'} \tag{2.6}$$

is the generalized Hölder inequality.

Lemma 2.2. ([15]) If X is a Banach function space, then we have, for all balls B,

$$1 \leq |B|^{-1} \|\chi_B\|_X \|\chi_B\|_{X'}. \tag{2.7}$$

Lemma 2.3. ([16]) Let X be a Banach function space. Suppose that the Hardy-Littlewood maximal operator M is weakly bounded on X, that is,

$$\|\chi_{\{Mf>\lambda\}}\|_X \lesssim \lambda^{-1} \|f\|_X$$

is true for all $f \in X$ and all $\lambda > 0$. Then, we have

$$\sup_{B:\text{ball}} \frac{1}{|B|} \|\chi_B\|_X \|\chi_B\|_{X'} < \infty. \tag{2.8}$$

Lemma 2.4. ([15]) Given a function W such that $0 < W(x) < \infty$ for almost every $x \in \mathbb{R}^n$, $W \in X_{\text{loc}}(\mathbb{R}^n)$ and $W^{-1} \in (X')_{\text{loc}}(\mathbb{R}^n)$,
(i) $X(\mathbb{R}^n, W)$ is Banach function space equipped with the norm

$$\|f\|_{X(\mathbb{R}^n,W)} := \|fW\|_X, \tag{2.9}$$

where

$$X(\mathbb{R}^n, W) := \{f \in \mathcal{M} : fW \in X\}. \tag{2.10}$$

(ii) The associated space $X'(\mathbb{R}^n, W^{-1})$ of $X(\mathbb{R}^n, W)$ is also a Banach function space.

Lemma 2.5. ([15]) Let X be a Banach function space and M be bounded on X'. Then, there exists a constant $\delta \in (0, 1)$ for all $B \subset \mathbb{R}^n$ and $E \subset B$,

$$\frac{\|\chi_E\|_X}{\|\chi_B\|_X} \leq \Big(\frac{|E|}{|B|}\Big)^{\delta}. \tag{2.11}$$

The paper [19] shows that $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space and the associated space $L^{p'(\cdot)}(\mathbb{R}^n)$ with equivalent norm.

Remark 2.6. ([1]) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and by comparing the $L^{p(\cdot)}(\omega^{p(\cdot)})$ and $L^{p'(\cdot)}(\omega^{-p'(\cdot)})$ with the definition of $X(\mathbb{R}^n, W)$, we have the following:
(1) If we take $W = \omega$ and $X = L^{p(\cdot)}(\mathbb{R}^n)$, then we get $L^{p(\cdot)}(\mathbb{R}^n, \omega) = L^{p(\cdot)}(\omega^{p(\cdot)})$.

(2) If we consider $W = \omega^{-1}$ and $X = L^{p'(\cdot)}(\mathbb{R}^n)$, then we get $L^{p'(\cdot)}(\mathbb{R}^n, \omega^{-1}) = L^{p'(\cdot)}(\omega^{-p'(\cdot)})$.
By virtue of Lemma 2.4, we get

$$(L^{p(\cdot)}(\mathbb{R}^n, \omega))' = (L^{p(\cdot)}(\omega^{p(\cdot)}))' = L^{p'(\cdot)}(\omega^{-p'(\cdot)}) = L^{p'(\cdot)}(\mathbb{R}^n, \omega^{-1}).$$

Lemma 2.7. ([17]) Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ be a log-Hölder continuous function both at infinity and at origin, if $\omega^{p_2(\cdot)} \in A_{p_2(\cdot)}$ implies $\omega^{-p_2'(\cdot)} \in A_{p_2'(\cdot)}$. Thus, the Hardy-Littlewood operator is bounded on $L^{p_2'(\cdot)}(\omega^{-p_2'(\cdot)})$, and there exist constants $\delta_1, \delta_2 \in (0, 1)$ such that

$$\frac{\|\chi_E\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}}{\|\chi_B\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}} = \frac{\|\chi_E\|_{(L^{p_2'(\cdot)}(\omega^{-p_2'(\cdot)}))'}}{\|\chi_B\|_{(L^{p_2'(\cdot)}(\omega^{-p_2'(\cdot)}))'}} \leq C\Big(\frac{|E|}{|B|}\Big)^{\delta_1}, \tag{2.12}$$

and

$$\frac{\|\chi_E\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))'}}{\|\chi_B\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))'}} \leq C\Big(\frac{|E|}{|B|}\Big)^{\delta_2}, \tag{2.13}$$

for all balls $B \subset \mathbb{R}^n$ and all measurable sets $E \subset B$.

Lemma 2.8. ([15]) Let $p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ and $0 < \beta < \frac{n}{p_1^+}$. Define $p_2(\cdot)$ by $\frac{1}{p_1(x)} - \frac{1}{p_2(\cdot)} = \frac{\beta}{n}$. If $\omega \in A(p_1(\cdot), p_2(\cdot))$, then I_β is bounded from $L^{p_1(\cdot)}(\omega^{p_1(\cdot)})$ to $L^{p_2(\cdot)}(\omega^{p_2(\cdot)})$.

Lemma 2.9. ([24, Corollary 3.11]) Let $b \in \mathrm{BMO}(\mathbb{R}^n), m \in \mathbb{N}$, and $k, j \in \mathbb{Z}$ with $k > j$. Then, we have

$$C^{-1}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)} \leq \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\omega)}} \|(b - b_B)^m \chi_B\|_{L^{p(\cdot)}(\omega)} \leq C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}. \tag{2.14}$$

$$\|(b - b_{B_j})^m \chi_{B_k}\|_{L^{p(\cdot)}(\omega)} \leq C(k - j)^m \|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p(\cdot)}(\omega)}. \tag{2.15}$$

3. Main results and their proofs

Proposition 3.1. ([12] Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < p < \infty$, and $0 \leq \lambda < \infty$. If $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$, then

$$\begin{aligned}\|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(\omega^{q(\cdot)})} &= \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda p} \sum_{j=-\infty}^{k_0} 2^{j\alpha(\cdot)p} \|f\chi_j\|^p_{L^{q(\cdot)}(\omega^{q(\cdot)})} \\ &\leq \max\Big\{ \sup_{\substack{k_0 \in \mathbb{Z} \\ k_0 < 0}} 2^{-k_0\lambda p}\Big(\sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p} \|f\chi_j\|^p_{L^{q(\cdot)}(\omega^{q(\cdot)})}\Big), \sup_{\substack{k_0 \in \mathbb{Z} \\ k_0 \geq 0}} \Big(2^{-k_0\lambda p}\Big(\sum_{j=-\infty}^{-1} 2^{j\alpha(0)p} \|f\chi_j\|^p_{L^{q(\cdot)}(\omega^{q(\cdot)})}\Big) \\ &\quad + 2^{-k_0\lambda p}\Big(\sum_{j=0}^{k_0} 2^{j\alpha(\infty)p} \|f\chi_j\|^p_{L^{q(\cdot)}(\omega^{q(\cdot)})}\Big)\Big)\Big\}.\end{aligned}$$

Theorem 3.1. Let $0 < q_1 \leq q_2 < \infty$, $p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ and $p_1(\cdot)$ be such that $\frac{1}{p_2(\cdot)} = \frac{1}{p_1(\cdot)} - \frac{\beta}{n}$. Also, let $\omega^{p_2(\cdot)} \in A_1$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, $\lambda > 0$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ be log-Hölder continuous at the origin, with $\alpha(0) \leq \alpha(\infty) < \lambda + n\delta_2 - \beta$, where $\delta_2 \in (0, 1)$ is the constant appearing in (2.13). Then,

$$\|\mathcal{H}^m_{\beta,b} f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_2,p_2(\cdot)}(\omega^{p_2(\cdot)})} \lesssim \|b\|^m_{\mathrm{BMO}} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}. \tag{3.1}$$

Proof. For arbitrary $f \in \mathrm{M}\dot{\mathrm{K}}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})$, let $f_j = f \cdot \chi_j = f \cdot \chi_{A_j}$ for every $j \in \mathbb{Z}$, and then

$$f(x) = \sum_{j=-\infty}^{\infty} f(x) \cdot \chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x). \tag{3.2}$$

By the inequality of C_p, it is not difficult to see that

$$\begin{aligned}
|\mathcal{H}^m_{\beta,b} f(x)\chi_k(x)| &\leq \frac{1}{|x|^{n-\beta}} \int_{|t|<|x|} |b(x) - b(t)|^m |f(t)| \mathrm{d}t \cdot \chi_k(x) \\
&\leq \frac{1}{|x|^{n-\beta}} \int_{B(0,|x|)} |b(x) - b(t)|^m |f(t)| \mathrm{d}t \cdot \chi_k(x) \\
&\leq \frac{1}{|x|^{n-\beta}} \int_{B_k} |b(x) - b(t)|^m |f(t)| \mathrm{d}t \cdot \chi_k(x) \\
&\leq 2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \int_{A_j} |b(x) - b(t)|^m |f(t)| \mathrm{d}t \cdot \chi_k(x) \\
&\lesssim 2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \int_{A_j} |b(x) - b_{A_j}|^m |f(t)| \mathrm{d}t \cdot \chi_k(x) \\
&\quad + 2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \int_{A_j} |b(t) - b_{A_j}|^m |f(t)| \mathrm{d}t \cdot \chi_k(x) = E_1 + E_2.
\end{aligned} \tag{3.3}$$

For E_1, by the generalized Hölder inequality, we have

$$\begin{aligned}
E_1 &= 2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \int_{A_j} |b(x) - b_{A_j}|^m |f(t)| \mathrm{d}t \cdot \chi_k(x) \\
&\leq 2^{-k(n-\beta)} \sum_{j=-\infty}^{k} |b(x) - b_{A_j}|^m \cdot \chi_k(x) \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}.
\end{aligned} \tag{3.4}$$

By taking the $(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))$-norm on both sides of (3.4) and using (2.15) of Lemma 2.9, we get

$$\begin{aligned}
\|E_1\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))} &\leq 2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \||b(x) - b_{A_j}|^m \cdot \chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \\
&\lesssim 2^{-k(n-\beta)} \sum_{j=-\infty}^{k} (k-j)^m \|b\|^m_{\mathrm{BMO}} \|\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}.
\end{aligned} \tag{3.5}$$

For E_2, by the generalized Hölder inequality, we have

$$\begin{aligned}
E_2 &= 2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \int_{A_j} |b(t) - b_{A_j}|^m |f(t)| \mathrm{d}t \cdot \chi_k(x) \\
&\leq 2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \||b(t) - b_{A_j}|^m \cdot \chi_j(x)\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \cdot \chi_k(x).
\end{aligned} \tag{3.6}$$

By taking the $(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))$-norm on both sides of (3.6) and using (2.14) of Lemma 2.9, we get

$$\begin{aligned}\|E_2\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))} &\le 2^{-k(n-\beta)}\sum_{j=-\infty}^{k}\||b(x)-b_{A_j}|^m\cdot\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\|\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))}\\ &\lesssim 2^{-k(n-\beta)}\sum_{j=-\infty}^{k}\|b\|_{\mathrm{BMO}}^m\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\|\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))}. \end{aligned}\tag{3.7}$$

Hence, from inequalities (3.3), (3.5) and (3.7), we get

$$\begin{aligned}&\|\mathcal{H}^m_{\beta,b}f(x)\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))}\\ &\lesssim 2^{-k(n-\beta)}\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\Big\{\sum_{j=-\infty}^{k}(k-j)^m\|b\|_{\mathrm{BMO}}^m\|\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))}\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}\\ &+\sum_{j=-\infty}^{k}\|b\|_{\mathrm{BMO}}^m\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}\|\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))}\Big\}\\ &\lesssim 2^{-k(n-\beta)}\|b\|_{\mathrm{BMO}}^m\sum_{j=-\infty}^{k}(k-j)^m\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}\|\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))}. \end{aligned}\tag{3.8}$$

By virtue of Lemma 2.5, we have

$$\frac{\|\chi_{B_k}\|_X}{\|\chi_k\|_X}\le\Big(\frac{|B_k|}{|A_k|}\Big)^{\delta}=C\implies\|\chi_{B_k}\|_X\le C\|\chi_k\|_X.\tag{3.9}$$

Note that $\|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}\le\|\chi_{B_j}\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}$ and $\chi_{B_j}(x)\lesssim 2^{-j\beta}I_\beta(\chi_{B_j})$ (see [18, p. 350]). By applying (2.8), (3.9) and Lemma 2.8, we obtain

$$\begin{aligned}\|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}&\le\|\chi_{B_j}\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}\lesssim 2^{-j\beta}\|I_\beta(\chi_{B_j})\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}\\ &\lesssim 2^{-j\beta}\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\lesssim 2^{-j\beta}\|\chi_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\lesssim 2^{j(n-\beta)}\|\chi_j\|^{-1}_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}. \end{aligned}\tag{3.10}$$

By virtue of (2.7) and (2.8), combining (2.13) and (3.10), we have

$$\begin{aligned}&2^{k(\beta-n)}\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}\|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}\\ &=2^{k\beta}\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}2^{-kn}\|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}\\ &\lesssim 2^{k\beta}\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}\|\chi_k\|^{-1}_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))'}\\ &=2^{k\beta}\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}\|\chi_j\|^{-1}_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))'}\frac{\|\chi_j\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))'}}{\|\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))'}}\\ &\lesssim 2^{k\beta}2^{n\delta_2(j-k)}\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}\|\chi_j\|^{-1}_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))'}\\ &\lesssim 2^{k\beta}2^{n\delta_2(j-k)}2^{j(n-\beta)}\|\chi_j\|^{-1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}\|\chi_j\|^{-1}_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))'}\\ &=2^{k\beta}2^{n\delta_2(j-k)}2^{-j\beta}\Big(2^{-jn}\|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}\|\chi_j\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))'}\Big)^{-1}\\ &\lesssim 2^{(\beta-n\delta_2)(k-j)}. \end{aligned}\tag{3.11}$$

Hence by virtue of (3.8) and (3.11), we have

$$\|\mathcal{H}^m_{\beta,b}f(x)\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))} \lesssim \|b\|^m_{\mathrm{BMO}} \sum_{j=-\infty}^{k} (k-j)^m 2^{(\beta-n\delta_2)(k-j)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}. \tag{3.12}$$

In order to estimate $\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}$, we consider two cases as below.

Case 1: For $j < 0$, we get

$$\begin{aligned}
\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} &= 2^{-j\alpha(0)}\Big(2^{j\alpha(0)q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\Big)^{\frac{1}{q_1}} \\
&\le 2^{-j\alpha(0)}\Big(\sum_{i=-\infty}^{j} 2^{i\alpha(0)q_1}\|f_i\|^{q_1}_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\Big)^{\frac{1}{q_1}} \\
&= 2^{j(\lambda-\alpha(0))}\Big\{2^{-j\lambda}\Big(\sum_{i=-\infty}^{j} 2^{i\alpha(\cdot)q_1}\|f_i\|^{q_1}_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\Big)^{\frac{1}{q_1}}\Big\} \\
&\lesssim 2^{j(\lambda-\alpha(0))}\|f\|_{\mathrm{M}\dot{\mathrm{K}}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}.
\end{aligned} \tag{3.13}$$

Case 2: For $j \ge 0$, we get

$$\begin{aligned}
\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} &= 2^{-j\alpha(\infty)}\Big(2^{j\alpha(\infty)q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\Big)^{\frac{1}{q_1}} \\
&\le 2^{-j\alpha(\infty)}\Big(\sum_{i=-\infty}^{j} 2^{i\alpha(\infty)q_1}\|f_i\|^{q_1}_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\Big)^{\frac{1}{q_1}} \\
&= 2^{j(\lambda-\alpha(\infty))}\Big\{2^{-j\lambda}\Big(\sum_{i=-\infty}^{j} 2^{i\alpha(\cdot)q_1}\|f_i\|^{q_1}_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\Big)^{\frac{1}{q_1}}\Big\} \\
&\lesssim 2^{j(\lambda-\alpha(\infty))}\|f\|_{\mathrm{M}\dot{\mathrm{K}}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}.
\end{aligned} \tag{3.14}$$

Now, by virtue of the condition $q_1 \le q_2$ and the definition of weighted variable exponent Morrey-Herz space along with the use of Proposition 3.1, we get

$$\begin{aligned}
\|\mathcal{H}^m_{\beta,b}f\|^{q_1}_{\mathrm{M}\dot{\mathrm{K}}^{\alpha(\cdot),\lambda}_{q_2,p_2(\cdot)}(\omega^{p_2(\cdot)})} &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)q_1}\|\mathcal{H}^m_{\beta,b}f\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\
&\le \max\Big\{\sup_{\substack{k_0\in\mathbb{Z}\\k_0<0}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1}\|\mathcal{H}^m_{\beta,b}f\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}, \\
&\qquad \sup_{\substack{k_0\in\mathbb{Z}\\k_0\ge0}} 2^{-k_0\lambda q_1}\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_1}\|\mathcal{H}^m_{\beta,b}f\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\
&\qquad\quad + \sum_{k=0}^{k_0} 2^{k\alpha(\infty)q_1}\|\mathcal{H}^m_{\beta,b}f\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}\Big)\Big\} \\
&= \max\{J_1, J_2+J_3\},
\end{aligned} \tag{3.15}$$

where

$$J_1=\sup_{\substack{k_0\in\mathbb{Z}\\k_0<0}}2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}2^{k\alpha(0)q_1}\|\mathcal{H}^m_{\beta,b}f\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})},$$

$$J_2=\sup_{\substack{k_0\in\mathbb{Z}\\k_0\geq0}}2^{-k_0\lambda q_1}\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_1}\|\mathcal{H}^m_{\beta,b}f\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})},$$

$$J_3=\sup_{\substack{k_0\in\mathbb{Z}\\k_0\geq0}}2^{-k_0\lambda q_1}\sum_{k=0}^{k_0}2^{k\alpha(\infty)q_1}\|\mathcal{H}^m_{\beta,b}f\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}.$$

First, we estimate J_1. Since $\alpha(0)\leq\alpha(\infty)<n\delta_2+\lambda-\beta$, combining (3.12) and (3.13), we get

$$\begin{aligned}
J_1&\lesssim\sup_{\substack{k_0\in\mathbb{Z}\\k_0<0}}2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}2^{k\alpha(0)q_1}\Big(\sum_{j=-\infty}^{k}(k-j)^m\|b\|^m_{\mathrm{BMO}}2^{(\beta-n\delta_2)(k-j)}\|f\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\Big)^{q_1}\\
&\lesssim\sup_{\substack{k_0\in\mathbb{Z}\\k_0<0}}2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}2^{k\alpha(0)q_1}\Big(\sum_{j=-\infty}^{k}(k-j)^m\|b\|^m_{\mathrm{BMO}}2^{(\beta-n\delta_2)(k-j)}2^{j(\lambda-\alpha(0))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}\Big)^{q_1}\\
&\lesssim\|b\|^{mq_1}_{\mathrm{BMO}}\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}\sup_{\substack{k_0\in\mathbb{Z}\\k_0<0}}2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}2^{k\alpha(0)q_1}\Big(\sum_{j=-\infty}^{k}(k-j)^m2^{(\beta-n\delta_2)(k-j)}2^{j(\lambda-\alpha(0))}\Big)^{q_1}\\
&\lesssim\|b\|^{mq_1}_{\mathrm{BMO}}\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}\sup_{\substack{k_0\in\mathbb{Z}\\k_0<0}}2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}2^{k\lambda q_1}\Big(\sum_{j=-\infty}^{k}(k-j)^m2^{(j-k)(n\delta_2+\lambda-\beta-\alpha(0))}\Big)^{q_1}\\
&\lesssim\|b\|^{mq_1}_{\mathrm{BMO}}\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}.
\end{aligned}$$

The estimate of J_2 is similar to that of J_1.
Lastly, we estimate J_3. Since $\alpha(0)\leq\alpha(\infty)<n\delta_2+\lambda-\beta$, combining (3.12) and (3.14), we get

$$\begin{aligned}
J_3&\lesssim\sup_{\substack{k_0\in\mathbb{Z}\\k_0\geq0}}2^{-k_0\lambda q_1}\sum_{k=0}^{k_0}2^{k\alpha(\infty)q_1}\Big(\sum_{j=-\infty}^{k}(k-j)^m\|b\|^m_{\mathrm{BMO}}2^{(\beta-n\delta_2)(k-j)}\|f\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\Big)^{q_1}\\
&\lesssim\sup_{\substack{k_0\in\mathbb{Z}\\k_0\geq0}}2^{-k_0\lambda q_1}\sum_{k=0}^{k_0}2^{k\alpha(\infty)q_1}\Big(\sum_{j=-\infty}^{k}(k-j)^m\|b\|^m_{\mathrm{BMO}}2^{(\beta-n\delta_2)(k-j)}2^{j(\lambda-\alpha(\infty))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}\Big)^{q_1}\\
&\lesssim\|b\|^{mq_1}_{\mathrm{BMO}}\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}\sup_{\substack{k_0\in\mathbb{Z}\\k_0\geq0}}2^{-k_0\lambda q_1}\sum_{k=0}^{k_0}2^{k\alpha(\infty)q_1}\Big(\sum_{j=-\infty}^{k}(k-j)^m2^{(\beta-n\delta_2)(k-j)}2^{j(\lambda-\alpha(\infty))}\Big)^{q_1}\\
&\lesssim\|b\|^{mq_1}_{\mathrm{BMO}}\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}\sup_{\substack{k_0\in\mathbb{Z}\\k_0\geq0}}2^{-k_0\lambda q_1}\sum_{k=0}^{k_0}2^{k\lambda q_1}\Big(\sum_{j=-\infty}^{k}(k-j)^m2^{(j-k)(n\delta_2+\lambda-\beta-\alpha(\infty))}\Big)^{q_1}\\
&\lesssim\|b\|^{mq_1}_{\mathrm{BMO}}\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}.
\end{aligned}$$

The desired result is obtained by combining the estimates of J_1, J_2 and J_3.

Theorem 3.2. Let $0 < q_1 \leq q_2 < \infty$, $p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ and $p_1(\cdot)$ be such that $\frac{1}{p_2(\cdot)} = \frac{1}{p_1(\cdot)} - \frac{\beta}{n}$. Also, let $\omega^{p_2(\cdot)} \in A_1$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, $\lambda > 0$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ be log-Hölder continuous at the origin, with $\lambda - n\delta_1 < \alpha(0) \leq \alpha(\infty)$, where $\delta_1 \in (0,1)$ is the constant appearing in (2.12). Then,

$$\|\mathcal{H}^{*m}_{\beta,b} f\|_{\mathrm{M}\dot{\mathrm{K}}^{\alpha(\cdot),\lambda}_{q_2,p_2(\cdot)}(\omega^{p_2(\cdot)})} \lesssim \|b\|^m_{\mathrm{BMO}} \|f\|_{\mathrm{M}\dot{\mathrm{K}}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}. \tag{3.16}$$

Proof. From an application of the inequality of C_p, it is not difficult to see that

$$\begin{aligned}
|\mathcal{H}^{*m}_{\beta,b} f(x)\chi_k(x)| &\leq \int_{\mathbb{R}^n\setminus B_k} |t|^{\beta-n}|b(x)-b(t)|^m|f(t)|\mathrm{d}t\cdot\chi_k(x)\\
&\leq \sum_{j=k+1}^{\infty}\int_{A_j} |t|^{\beta-n}|b(x)-b(t)|^m|f(t)|\mathrm{d}t\cdot\chi_k(x)\\
&\lesssim \sum_{j=k+1}^{\infty}\int_{A_j} |t|^{\beta-n}|b(x)-b_{A_j}|^m|f(t)|\mathrm{d}t\cdot\chi_k(x)\\
&\quad+\sum_{j=k+1}^{\infty}\int_{A_j} |t|^{\beta-n}|b(t)-b_{A_j}|^m|f(t)|\mathrm{d}t\cdot\chi_k(x)\\
&= F_1 + F_2.
\end{aligned} \tag{3.17}$$

For F_1, by the generalized Hölder inequality, we have

$$\begin{aligned}
F_1 &\leq \sum_{j=k+1}^{\infty} 2^{-j(n-\beta)}\int_{A_j}|b(x)-b_{A_j}|^m|f(t)|\mathrm{d}t\cdot\chi_k(x)\\
&\leq \sum_{j=k+1}^{\infty} 2^{-j(n-\beta)}|b(x)-b_{A_j}|^m\cdot\chi_k(x)\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}.
\end{aligned} \tag{3.18}$$

By taking the $(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))$-norm on both sides of (3.18) and using (2.15) of Lemma 2.9, we get

$$\begin{aligned}
\|F_1\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))} &\leq \sum_{j=k+1}^{\infty} 2^{-j(n-\beta)}\||b(x)-b_{A_j}|^m\cdot\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))}\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}\\
&\lesssim \sum_{j=k+1}^{\infty} 2^{-j(n-\beta)}(j-k)^m\|b\|^m_{\mathrm{BMO}}\|\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))}\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}.
\end{aligned} \tag{3.19}$$

For F_2, by the generalized Hölder inequality, we have

$$\begin{aligned}
F_2 &\leq \sum_{j=k+1}^{\infty} 2^{-j(n-\beta)}\int_{A_j}|b(t)-b_{A_j}|^m|f(t)|\mathrm{d}t\cdot\chi_k(x)\\
&\leq \sum_{j=k+1}^{\infty} 2^{-j(n-\beta)}\||b(t)-b_{A_j}|^m\cdot\chi_j(x)\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}\|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\cdot\chi_k(x).
\end{aligned} \tag{3.20}$$

By taking the $(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))$-norm on both sides of (3.20) and using (2.15) of Lemma 2.9, we get

$$\begin{aligned}\|F_2\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))} &\leq \sum_{j=k+1}^{\infty} 2^{-j(n-\beta)} \||b(t)-b_{A_j}|^m \cdot \chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))} \\ &\lesssim \sum_{j=k+1}^{\infty} 2^{-j(n-\beta)} \|b\|_{\mathrm{BMO}}^m \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))}. \end{aligned} \tag{3.21}$$

Hence, from inequalities (3.17), (3.19) and (3.21), we get

$$\begin{aligned}&\|\mathcal{H}^{*m}_{\beta,b} f(x)\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))} \\ &\quad\lesssim \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \Big\{ \sum_{j=k+1}^{\infty} 2^{-j(n-\beta)} (j-k)^m \|b\|_{\mathrm{BMO}}^m \|\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \\ &\qquad + \sum_{j=k+1}^{\infty} 2^{-j(n-\beta)} \|b\|_{\mathrm{BMO}}^m \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))} \Big\} \\ &\quad\lesssim \|b\|_{\mathrm{BMO}}^m \sum_{j=k+1}^{\infty} 2^{-j(n-\beta)} (j-k)^m \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \|\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))}. \end{aligned} \tag{3.22}$$

On the other hand, by (2.7) and (2.8), combining (2.12) and (3.10), we have

$$\begin{aligned}&2^{-j(n-\beta)} \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \\ &\quad= 2^{j\beta} \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} 2^{-jn} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \\ &\quad\lesssim 2^{j\beta} \|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \|\chi_j\|^{-1}_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \\ &\quad= 2^{j\beta} \|\chi_j\|^{-1}_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \frac{\|\chi_k\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}}{\|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}} \\ &\quad\lesssim 2^{j\beta} 2^{n\delta_1(k-j)} \|\chi_j\|^{-1}_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_j\|_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\ &\quad\lesssim 2^{j\beta} 2^{n\delta_1(k-j)} 2^{j(n-\beta)} \|\chi_j\|^{-1}_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_j\|^{-1}_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'} \\ &\quad= 2^{j\beta} 2^{n\delta_1(k-j)} 2^{-j\beta} \Big(2^{-jn} \|\chi_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(\omega^{p_1(\cdot)}))'}\Big)^{-1} \lesssim 2^{n\delta_1(k-j)}. \end{aligned} \tag{3.23}$$

Hence combining (3.22) and (3.23), we obtain

$$\|\mathcal{H}^{*m}_{\beta,b} f(x)\chi_k\|_{(L^{p_2(\cdot)}(\omega^{p_2(\cdot)}))} \lesssim \|b\|_{\mathrm{BMO}}^m \sum_{j=k+1}^{\infty} (j-k)^m 2^{n\delta_1(k-j)} \|f_j\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}. \tag{3.24}$$

Next, by virtue of the condition $q_1 \leq q_2$ and the definition of weighted variable exponent Morrey-Herz space along with the use of Proposition 3.1, we get

$$\begin{aligned}\|\mathcal{H}^{*m}_{\beta,b} f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_2,p_2(\cdot)}(\omega^{p_2(\cdot)})} &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)q_1} \|\mathcal{H}^{*m}_{\beta,b} f\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})} \\ &\leq \max\Big\{ \sup_{\substack{k_0\in\mathbb{Z}\\ k_0<0}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1} \|\mathcal{H}^{*m}_{\beta,b} f\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}, \end{aligned}$$

$$
\begin{aligned}
&\sup_{\substack{k_0\in\mathbb{Z}\\k_0\ge 0}} 2^{-k_0\lambda q_1}\Big(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_1}\|\mathcal{H}^{*m}_{\beta,b} f\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}\\
&\quad+\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q_1}\|\mathcal{H}^{*m}_{\beta,b} f\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}\Big)\Big\}\\
&=\max\{Y_1, Y_2+Y_3\},
\end{aligned}
\tag{3.25}
$$

where

$$
\begin{aligned}
Y_1 &= \sup_{\substack{k_0\in\mathbb{Z}\\k_0< 0}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1}\|\mathcal{H}^{*m}_{\beta,b} f\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})},\\
Y_2 &= \sup_{\substack{k_0\in\mathbb{Z}\\k_0\ge 0}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_1}\|\mathcal{H}^{*m}_{\beta,b} f\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})},\\
Y_3 &= \sup_{\substack{k_0\in\mathbb{Z}\\k_0\ge 0}} 2^{-k_0\lambda q_1}\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q_1}\|\mathcal{H}^{*m}_{\beta,b} f\chi_k\|^{q_1}_{L^{p_2(\cdot)}(\omega^{p_2(\cdot)})}.
\end{aligned}
$$

First, we estimate Y_1. Since $\lambda - n\delta_1 < \alpha(0) \le \alpha(\infty)$, combining (3.24) and (3.13), we get

$$
\begin{aligned}
Y_1 &\lesssim \sup_{\substack{k_0\in\mathbb{Z}\\k_0< 0}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1}\Big(\sum_{j=k+1}^{\infty}(j-k)^m\|b\|^m_{\mathrm{BMO}}2^{n\delta_1(k-j)}\|f\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\Big)^{q_1}\\
&\lesssim \sup_{\substack{k_0\in\mathbb{Z}\\k_0< 0}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1}\Big(\sum_{j=k+1}^{\infty}(j-k)^m\|b\|^m_{\mathrm{BMO}}2^{n\delta_1(k-j)}2^{j(\lambda-\alpha(0))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}\Big)^{q_1}\\
&\lesssim \|b\|^{mq_1}_{\mathrm{BMO}}\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}\sup_{\substack{k_0\in\mathbb{Z}\\k_0< 0}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1}\Big(\sum_{j=k+1}^{\infty}(j-k)^m 2^{n\delta_1(k-j)}2^{j(\lambda-\alpha(0))}\Big)^{q_1}\\
&\lesssim \|b\|^{mq_1}_{\mathrm{BMO}}\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}\sup_{\substack{k_0\in\mathbb{Z}\\k_0< 0}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{k\lambda q_1}\Big(\sum_{j=k+1}^{\infty}(j-k)^m 2^{(j-k)(\lambda-n\delta_1-\alpha(0))}\Big)^{q_1}\\
&\lesssim \|b\|^{mq_1}_{\mathrm{BMO}}\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}.
\end{aligned}
$$

The estimate of Y_2 is similar to that of Y_1.

Lastly, we estimate Y_3. Since $\lambda - n\delta_1 < \alpha(0) \le \alpha(\infty)$, combining (3.24) and (3.14), we get

$$
\begin{aligned}
Y_3 &\lesssim \sup_{\substack{k_0\in\mathbb{Z}\\k_0\ge 0}} 2^{-k_0\lambda q_1}\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q_1}\Big(\sum_{j=k+1}^{\infty}(j-k)^m\|b\|^m_{\mathrm{BMO}}2^{n\delta_1(k-j)}\|f\|_{L^{p_1(\cdot)}(\omega^{p_1(\cdot)})}\Big)^{q_1}\\
&\lesssim \sup_{\substack{k_0\in\mathbb{Z}\\k_0\ge 0}} 2^{-k_0\lambda q_1}\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q_1}\Big(\sum_{j=k+1}^{\infty}(j-k)^m\|b\|^m_{\mathrm{BMO}}2^{n\delta_1(k-j)}2^{j(\lambda-\alpha(\infty))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}\Big)^{q_1}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \|b\|_{\mathrm{BMO}}^{mq_1}\|f\|^{q_1}_{\mathrm{M\dot{K}}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})} \sup_{\substack{k_0\in\mathbb{Z}\\ k_0\geq 0}} 2^{-k_0\lambda q_1}\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q_1}\Big(\sum_{j=k+1}^{\infty}(j-k)^m 2^{n\delta_1(k-j)}2^{j(\lambda-\alpha(\infty))}\Big)^{q_1}\\
&\lesssim \|b\|_{\mathrm{BMO}}^{mq_1}\|f\|^{q_1}_{\mathrm{M\dot{K}}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})} \sup_{\substack{k_0\in\mathbb{Z}\\ k_0\geq 0}} 2^{-k_0\lambda q_1}\sum_{k=0}^{k_0} 2^{k\lambda q_1}\Big(\sum_{j=k+1}^{\infty}(j-k)^m 2^{(j-k)(\lambda-n\delta_1-\alpha(\infty))}\Big)^{q_1}\\
&\lesssim \|b\|_{\mathrm{BMO}}^{mq_1}\|f\|^{q_1}_{\mathrm{M\dot{K}}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(\omega^{p_1(\cdot)})}.
\end{aligned}
$$

The desired result is obtained by combining the estimates of Y_1, Y_2 and Y_3.

4. Conclusions

This paper considers the boundedness for mth order commutators of $n-$dimensional fractional Hardy operators $\mathcal{H}^m_{\beta,b}$ and adjoint operators $\mathcal{H}^{*m}_{\beta,b}$ on weighted variable exponent Morrey-Herz spaces $\mathrm{M\dot{K}}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\omega)$. When $m = 0$, our main result holds on weighted variable exponent Morrey-Herz space for fractional Hardy operators and generalizes the result of Asim et al. in [1, Theorems 4.2 and 4.3]. When $m = 1$, our main result holds on weighted variable exponent Morrey-Herz space for commutators of the fractional Hardy operators and generalizes the result of Hussain et al. in [12, Theorems 18 and 19].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare that they have no conflicts of interest.

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