Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 20048–20062.
DOI: 10.3934/math.20231021
Received: 15 April 2023
Revised: 31 May 2023
Accepted: 07 June 2023
Published: 16 June 2023

Research article

Shadowing properties and chaotic properties of non-autonomous product systems

Jingmin Pi[1,2], **Tianxiu Lu**[1,*] **and Jie Zhou**[1]

[1] College of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, 643000, China

[2] South Sichuan Applied Mathematics Research Center, Zigong, 643000, China

* **Correspondence:** Email: lubeeltx@163.com.

Abstract: This paper examines how properties such as shadowing properties, transitivity, and accessibility in non-autonomous discrete dynamical systems carry over to their product systems. The paper establishes a proof that the product system exhibits the pseudo-orbit shadowing property (PSP) if, and only if, both factor systems possess PSP. This relationship, which is both sufficient and necessary, also holds for the average shadowing property (ASP) and accessibility. Consequently, in practical problem scenarios, certain chaotic properties of two-dimensional systems can be simplified to those observed in one-dimensional systems. However, it should be noted that while the point-transitivity, transitivity, or mixing of the product system can be deduced from the factor systems, the reverse is not true. In particular, this paper constructs counterexamples to demonstrate that some of the theorems presented herein do not hold when considering their inverses.

Keywords: non-autonomous discrete dynamical systems; product map; shadowing properties; transitivity; accessibility
Mathematics Subject Classification: 37B45, 37B55, 54H20

1. Introduction

Non-autonomous discrete dynamical systems (NDDSs), also known as time-varying parametric dynamical systems, are generated by a sequence of time-varying mapping iterations, which is an important part of the study of topological dynamical systems.

In 1996, Kolyada [1] first proposed the concept of NDDSs. Let $\mathbb{N}$ represent the set of positive integers, X denotes a compact metric Hausdorff space equipped with metric d. $f_n : X \to X$ ($n \in \mathbb{N}$) is a continuous mapping sequence, and denoted by $f_{1,\infty} = (f_1, f_2, \cdots) = (f_n)_{n=1}^{\infty}$. This mapping sequence defines an NDDS $(X, f_{1,\infty})$. Under this sequence, the orbit of the point $x \in X$ is $\mathrm{Orb}_{f_{1,\infty}}(x) =$

$\{x, f_1(x), f_2 \circ f_1(x), \cdots, f_1^n(x), \cdots\}$ ($n \in \mathbb{N}$), where $f_1^n = f_n \circ \cdots \circ f_2 \circ f_1$. Likewise, $f_n^k = f_{n+k-1} \circ \cdots \circ f_{n+1} \circ f_n$ ($n, k \in \mathbb{N}$). Additionally, f_1^0 represents the identity mapping. If $f_i = f_j$ ($i, j \in \mathbb{N} : i \neq j$), $(X, f_{1,\infty})$ is referred to as an autonomous discrete dynamical systems (ADDSs).

Compared to the ADDSs, the NDDSs offer greater flexibility and convenience in describing various dynamic behaviors of a system. ADDSs may struggle to capture the complexity of problems in signal processing, biology, and physics, whereas NDDSs prove to be effective in their description. Obviously, the NDDSs are natural extensions of the ADDSs, which can solve more complex practical problems. However, the dynamic behavior of NDDSs is much more complex than that of ADDSs. In fact, there has been significant research on the chaotic behavior of mappings in ADDSs, yielding substantial results. The chaoticity of NDDSs has gradually become a hot research direction of many scholars in recent years.

In 2006, Tian [2] investigated Devaney chaos in NDDSs. In 2009, Shi [3] introduced the concept of several types of chaotic properties in NDDSs, such as transitivity, sensitivity, Li-Yorke chaos, etc. In 2011, C*á*novas [4] discussed the dynamic characteristics between topological entropy and Li-Yorke chaos on NDDSs. In 2012, Balibrea [5] examined the connection between topological entropy and weak mixing on NDDSs. Song [6] discussed the Ruelle-Takens chaotic properties of non-autonomous product dynamical systems. In 2013, Wu [7] proved that Li-Yorke sensitivity and sensitivity of sequences with the form $f_{1,\infty}$ are inherited under iterations. In 2015, Huang [8] extended some results of sensitivity or strong sensitivity from ADDSs to NDDSs. In 2018, Ma [9] studied the relations of sensitivity and transitivity between iterative function systems and their product systems. In 2020, Li [10] studied stronger forms of transitivity and sensitivity for NDDSs by using Furstenberg family. In 2022, Anwar [11] studied the relations of some sensitivity between iterative function systems and their product systems. Additionally, we studied the relations of some sensitivity between NDDSs and their product systems (see [12]). Some other research about chaotic properties of NDDSs are [13–16] and others.

In [17], we proved that under a specific metric, the product system having the $\mathscr{P}$-property is equivalent to its factor systems also having the property of $\mathscr{P}$-property, where $\mathscr{P}$-property represents one of the following five properties: $\overline{d}$ shadowing property, $\underline{d}$ shadowing property, $\mathscr{F}$-shadowing property, and ergodic shadowing property. Naturally, two questions arise: first, whether the conclusion still holds for other properties, and second, whether the conclusion still holds if other metrics are used. This paper will explore these two questions.

This paper aims to examine the relationship between accessibility, transitivity, or shadowing properties of product systems and their corresponding factor systems in NDDSs.

2. Preliminaries

Let $f_{1,\infty} = (f_n)_{n=1}^{\infty}$, $g_{1,\infty} = (g_n)_{n=1}^{\infty}$ be two continuous mapping sequences on compact metric spaces (X, d_1) and (Y, d_2), respectively. Define $f_{1,\infty} \times g_{1,\infty}$ on $X \times Y$ as follow

$$(f_1^n \times g_1^n)((x, y)) = (f_n \times g_n) \circ \cdots \circ (f_1 \times g_1)(x, y) = (f_1^n(x), g_1^n(y)),$$

for any $(x, y) \in X \times Y$, $n \in \mathbb{N}$. For any $(x_1, y_1), (x_2, y_2) \in X \times Y$, define

$$D((x_1, y_1), (x_2, y_2)) = \sqrt{d_1^2(x_1, x_2) + d_2^2(y_1, y_2)}$$

is the metric on $X\times Y$. $(X\times Y, D, f_{1,\infty}\times g_{1,\infty})$ is called the product system of $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$.

Let D_1 and D_2 be the metric on $X \times Y$. D_1 and D_2 are equivalent metrics if and only if there exist $b \geq a > 0$ such that

$$aD_1((x_1, y_1), (x_2, y_2)) \leq D_2((x_1, y_1), (x_2, y_2)) \leq bD_1((x_1, y_1), (x_2, y_2))$$

for any $(x_1, y_1), (x_2, y_2) \in X \times Y$. [18]

Example 2.1. *Let*

$$D_1((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2),$$

$$D_2((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2) + d_2(y_1, y_2)\},$$

for any $(x_1, y_1), (x_2, y_2) \in X \times Y$. It is easy to get that, D_1 and D_2 are equivalent metric of D.

Definition 2.1. *[3, 19, 20] Let $f_{1,\infty}$ be a continuous mapping sequences on X. $f_{1,\infty}$ or the NDDS $(X, d, f_{1,\infty}))$ is considered to be*

(1) transitive if for any two nonempty open sets U and V in X, there exists an $n \in \mathbb{N}$ such that

$$f_1^n(U) \cap V \neq \emptyset;$$

(2) point-transitive if there is an $x \in X$ such that $\overline{orb_{f_{1,\infty}}(x)} = X$;

(3) accessible if for any $\varepsilon > 0$ and two nonempty open sets $U, V \subset X$, there exists an $n \in \mathbb{N}$ and points $x \in U, y \in V$ such that $d(f_1^n(x), f_1^n(y)) < \varepsilon$;

(4) mixing if for any two nonempty open subsets $U, V \subset X$, there exists an $N \in \mathbb{N}$ such that the set $N_{f_{1,\infty}}(U, V) = \{n \in \mathbb{N} : f_1^n(U) \cap V \neq \emptyset\}$ contains any natural number $n \geq N$.

Definition 2.2. *[21, 22] Let $\delta > 0$ and $\{x_i\}_{i=0}^{+\infty} \subset X$.*

(1) The sequence $\{x_i\}_{i=0}^{+\infty} \subset X$ is a δ pseudo-orbit of $f_{1,\infty}$, if $d(f_{i+1}(x_i), x_{i+1}) < \delta$ for any $i \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$;

(2) The sequence $\{x_i\}_{i=0}^{+\infty} \subset X$ is a δ average pseudo-orbit of $f_{1,\infty}$, if there exists an $N \in \mathbb{N}$ such that

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f_{i+k+1}(x_{i+k}), x_{i+k+1}) < \delta$$

for any $k \in \mathbb{N}_0$, $n \geq N$.

Definition 2.3. *[20, 21] Let $f_{1,\infty}$ be continuous mapping sequences on X.*

(1) The system $(X, d, f_{1,\infty})$ has pseudo-orbit shadowing property (PSP) if for any $\varepsilon > 0$, there is a $\delta > 0$, for every δ pseudo-orbit $\{x_i\}_{i=0}^{+\infty}$ of $(X, f_{1,\infty})$, there exists a $z \in X$ such that $d\left(f_1^i(z), x_i\right) < \varepsilon$ for any $i \in \mathbb{N}_0$;

(2) The system $(X, d, f_{1,\infty})$ has average shadowing property (ASP) if for any $\varepsilon > 0$, there is a $\delta > 0$, for each δ average pseudo-orbits $\{x_i\}_{i=0}^{+\infty}$ of $f_{1,\infty}$, there exists a $z \in X$ such that

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f_1^i(z), x_i) < \varepsilon.$$

3. Main results

Theorem 3.1. *$(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ has PSP if and only if $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ have PSP, where ρ is any equivalent metric of D.*

Proof. Since measure ρ be an equivalent measure of measure D, there exist $c_2 \geq c_1 > 0$ such that

$$c_1 \cdot \rho((x_1, y_1), (x_2, y_2)) \leq D((x_1, y_1), (x_2, y_2)) \leq c_2 \cdot \rho((x_1, y_1), (x_2, y_2))$$

for any $(x_1, y_1), (x_2, y_2) \in X \times Y$.

(Necessity) Assume that $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ has PSP. Then for any $\varepsilon > 0$ such that every δ^* ($\delta^* > 0$) pseudo-orbit $\{(a_i^*, b_i^*)\}_{i=0}^{+\infty}$ of $(X \times Y, f_{1,\infty} \times g_{1,\infty})$, there is a $(a, b) \in X \times Y$ satisfying that

$$\rho(f_1^i \times g_1^i(a, b), (a_i^*, b_i^*)) < \varepsilon$$

for any $i \in \mathbb{N}_0$. □

Claim: There exist $\{a_i\}_{i=0}^{+\infty} \subset X$, $\{b_i\}_{i=0}^{+\infty} \subset Y$, for any $\varepsilon > 0$ such that $\{a_i\}_{i=0}^{+\infty} \subset X$, $\{b_i\}_{i=0}^{+\infty} \subset Y$ are ε pseudo-orbit of $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$, respectively.

Proof of claim. In fact, for any $a \in X$, $b \in Y$, and $\delta > 0$, take $a_0 = a, a_1 = f_1(a), a_2 = f_1^2(a), \cdots$; $b_0 = b, b_1 = g_1(b), b_2 = g_1^2(b), \cdots$. Obviously, $d_1(f_{i+1}(a_i), a_{i+1}) = 0 < \varepsilon$ and $d_2(g_{i+1}(b_i), b_{i+1}) = 0 < \varepsilon$ for any $i \in \mathbb{N}_0$. So, $\{a_i\}_{i=0}^{+\infty} \subset X$, $\{b_i\}_{i=0}^{+\infty} \subset Y$ are ε pseudo-orbit of $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$, respectively.

Take $\delta = \frac{\sqrt{2}c_1}{2}\delta^*$, let $\{a_i\}_{i=0}^{+\infty} \subset X$, $\{b_i\}_{i=0}^{+\infty} \subset Y$ be the δ pseudo-orbit of $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$, respectively. Then,

$$d_1(f_{i+1}(a_i), a_{i+1}) < \delta \quad \textit{and} \quad d_2(g_{i+1}(b_i), b_{i+1}) < \delta$$

for any $i \in \mathbb{N}_0$. Then,

$$\begin{aligned} & \rho(f_{i+1} \times g_{i+1}(a_i, b_i), (a_{i+1}, b_{i+1})) \\ \leq & \frac{1}{c_1} D(f_{i+1} \times g_{i+1}(a_i, b_i), (a_{i+1}, b_{i+1})) \\ = & \frac{1}{c_1} \sqrt{d_1^2(f_{i+1}(a_i), a_{i+1}) + d_2^2(g_{i+1}(b_i), b_{i+1})} \\ < & \frac{\sqrt{2}}{c_1}\delta = \delta^*, \end{aligned}$$

for any $i \in \mathbb{N}_0$. This means that $\{(a_i, b_i)\}_{i=0}^{+\infty} \subset X \times Y$ is the δ^* pseudo-orbit of $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$. So,

$$\frac{1}{c_2} D(f_1^i \times g_1^i(a, b), (a_i, b_i)) \leq \rho(f_1^i \times g_1^i(a, b), (a_i, b_i)) < \varepsilon.$$

It is evident that

$$d_1(f_1^i(a), a_i) < c_2\varepsilon \quad \textit{and} \quad d_2(g_1^i(b), b_i) < c_2\varepsilon.$$

Thus, $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ have PSP.

(Sufficiency) Assume that $f_{1,\infty}$ and $g_{1,\infty}$ have PSP, then for any $\varepsilon > 0$, there is a $\delta_1 > 0$ such that every δ_1 pseudo-orbit $\{a_i^*\}_{i=0}^{+\infty}$ of $(X, d_1, f_{1,\infty})$, there is a $a \in X$ satisfying that $d_1(f_1^i(a), a_i^*) < \varepsilon$; there is a $\delta_2 > 0$ such that every δ_2 pseudo-orbit $\{b_i^*\}_{i=0}^{+\infty}$ of $(Y, d_2, g_{1,\infty})$, there is a $b \in Y$ conforming that $d_2(g_1^i(b), b_i^*) < \varepsilon$.

Let $\{(a_i, b_i)\}_{i=0}^{+\infty} \subset X \times Y$ be the δ pseudo-orbit of $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$, where $\delta = \frac{1}{c_2}\min\{\delta_1, \delta_2\}$. Then for any $i \in \mathbb{N}_0$, it is obvious that

$$\rho(f_{i+1} \times g_{i+1}(a_i, b_i), (a_{i+1}, b_{i+1})) < \delta.$$

Then

$$\begin{aligned} & D(f_{i+1} \times g_{i+1}(a_i, b_i), (a_{i+1}, b_{i+1})) \\ & \leq c_2\rho(f_{i+1} \times g_{i+1}(a_i, b_i), (a_{i+1}, b_{i+1})) \\ & < c_2\delta. \end{aligned}$$

Subsequently,

$$d_1(f_1^n(a_i), a_{i+1}) < c_2\delta \quad and \quad d_2(g_1^n(b_i), b_{i+1}) < c_2\delta.$$

So, $\{a_i\}_{i=0}^{+\infty} \subset X$, $\{b_i\}_{i=0}^{+\infty} \subset Y$ are the δ^* ($\delta^* = c_2\delta$) pseudo-orbit of $f_{1,\infty}$ and $g_{1,\infty}$, respectively. According to the PSP of $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$, then for any $\varepsilon > 0$ and $i \in \mathbb{N}_0$ there exists $a \in X, B \in Y$, such that

$$d_1(f_1^i(a), a_i) < \varepsilon \quad and \quad d_2(g_1^i(b), b_i) < \varepsilon.$$

Then,

$$\begin{aligned} & \rho(f_1^i \times g_1^i(a, b), (a_i, b_i)) \\ & \leq \frac{1}{c_1} D(f_1^i \times g_1^i(a, b), (a_i, b_i)) \\ & = \frac{1}{c_1}\sqrt{d_1^2(f_1^i(a), a_i) + d_2^2(g_1^i(b), b_i)} \\ & < \frac{\sqrt{2}}{c_1}\varepsilon, \end{aligned}$$

for any $i \in \mathbb{N}_0$. Therefore, $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ has PSP.

This complete the proof. □

An illustratable example is provided below to demonstrate Theorem 3.1.

Example 3.1. *Let $X = [0, 1]$. The metric d is denoted by $d(a, b) = |a - b|$ ($\forall a, b \in X$). Three mappings $g_1(x)$, $g_2(x)$, and $g_3(x)$ are defined by $g_1(x) = \sqrt{1 - 4(x - \frac{1}{2})^2}$ for $x \in X$, $g_2(x) = -4x^2 + 4x$ for $x \in X$, and $g_3(x) = 0$ for $x \in X$. Let $(h_n)_{n=1}^{\infty} = \{g_1, g_3, g_1, g_3, \cdots\}$, $(\ell_n)_{n=1}^{\infty} = \{g_2, g_3, g_2, g_3, \cdots\}$.*

For any $x \in X$, $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{3}$. Let

$$\begin{aligned} x_1 &= \max\{h_1(x) - \frac{\varepsilon}{3}, 0\},\ x_2 = \min\{h_1(x) + \frac{\varepsilon}{3}, 1\}; \\ y_1 &= \max\{\ell_1(x) - \frac{\varepsilon}{3}, 0\},\ y_2 = \min\{\ell_1(x) + \frac{\varepsilon}{3}, 1\}. \end{aligned}$$

and

$$a_0 = x,\ a_1 \in (x_1, x_2),\ a_2 \in [0, \frac{\varepsilon}{3}),\ a_3 \in [0, \frac{\varepsilon}{3}), \cdots;$$

$$b_0 = x,\ b_1 \in (y_1, y_2),\ b_2 \in [0, \frac{\varepsilon}{3}),\ b_3 \in [0, \frac{\varepsilon}{3}), \cdots.$$

Obviously,

$$d(h_{i+1}(a_i), a_{i+1}) < \delta = \frac{\varepsilon}{3} \quad \text{and} \quad d(\ell_{i+1}(b_i), b_{i+1}) < \delta = \frac{\varepsilon}{3}$$

for any $i \in \mathbb{N}$. So, $\{a_i\}_{i=0}^{+\infty} \subset X$, $\{b_i\}_{i=0}^{+\infty} \subset Y$ are the δ pseudo-orbits of $h_{1,\infty}$ and $\ell_{1,\infty}$, respectively. Since

$$d(h_1^i(x), a_i) < \varepsilon \quad \text{and} \quad d(\ell_1^i(x), b_i) < \varepsilon$$

for any $i \in \mathbb{N}$. Thus, $(X, d, h_{1,\infty})$ and $(X, d, \ell_{1,\infty})$ have PSP.

In view of

$$D((h_{i+1} \times \ell_{i+1})(a_i, b_i), (a_{i+1}, b_{i+1})) = \sqrt{d^2(h_{i+1}(a_i), a_{i+1}) + d^2(\ell_{i+1}(b_i), b_{i+1})} < \sqrt{2}\delta$$

for any $i \in \mathbb{N}$, then $\{(a_i, b_i)\}_{i=0}^{+\infty} \subset X \times Y$ is a δ^* pseudo-orbit of $f_{1,\infty} \times g_{1,\infty}$, where $\delta^* = \sqrt{2}\delta$. Due to

$$D((f_1^i \times g_1^i)(a, b), (a_i, b_i)) = \sqrt{d^2(f_1^i(a), a_i) + d^2(g_1^i(b), b_i)} < \sqrt{2}\varepsilon$$

for any $i \in \mathbb{N}_0$. This indicates that $(X \times X, D, f_{1,\infty} \times g_{1,\infty})$ has PSP.

Theorem 3.2. *$(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ has ASP if and only if $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ have ASP.*

Proof. (Necessity) Since ρ is an equivalent metric of D, there exist $c_2 \geq c_1 > 0$ such that

$$c_1 \cdot \rho((x_1, y_1), (x_2, y_2)) \leq D((x_1, y_1), (x_2, y_2)) \leq c_2 \cdot \rho((x_1, y_1), (x_2, y_2))$$

for any $(x_1, y_1), (x_2, y_2) \in X \times Y$.

Assume that $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ has ASP, then, for any $\varepsilon > 0$ and every δ^* average pseudo-orbit $\{(a_i^*, b_i^*)\}_{i=0}^{+\infty}$ of $(X \times Y, f_{1,\infty} \times g_{1,\infty})$, there exists a $(a, b) \in X \times Y$ such that

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \rho(f_1^i \times g_1^i(a, b), (a_i^*, b_i^*)) < \varepsilon.$$

Take $\delta = \frac{c_1}{2}\delta^*$. Let $\{a_i\}_{i=0}^{+\infty} \subset X$, $\{b_i\}_{i=0}^{+\infty} \subset Y$ be δ average pseudo-orbits of $f_{1,\infty}$ and $g_{1,\infty}$, respectively. Then there exist $m_1, m_2 \in \mathbb{N}$ such that

$$\frac{1}{n_1} \sum_{i=0}^{n_1-1} d_1(f_{i+k+1}(a_{i+k}), a_{i+k+1}) < \delta \quad \text{and} \quad \frac{1}{n_2} \sum_{i=0}^{n_2-1} d_2(g_{i+k+1}(b_{i+k}), b_{i+k+1}) < \delta$$

for any $n_1 \geq m_1, n_2 \geq m_2$, and $k \in \mathbb{N}_0$. Then,

$$\begin{aligned}
&\frac{1}{n} \sum_{i=0}^{n-1} \rho(f_{i+k+1} \times g_{i+k+1}(a_{i+k}, b_{i+k}), (a_{i+k+1}, b_{i+k+1})) \\
\leq\ &\frac{1}{c_1 n} \sum_{i=0}^{n-1} D(f_{i+k+1} \times g_{i+k+1}(a_{i+k}, b_{i+k}), (a_{i+k+1}, b_{i+k+1}))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{c_1 n}\sum_{i=0}^{n-1} \sqrt{d_1^2(f_{i+k+1}(a_{i+k}), a_{i+k+1}) + d_2^2(g_{i+k+1}(b_{i+k}), b_{i+k+1})} \\
&\le \frac{1}{c_1 n}\sum_{i=0}^{n-1}(d_1(f_{i+k+1}(a_{i+k}), a_{i+k+1}) + d_2(g_{i+k+1}(b_{i+k}), b_{i+k+1})) \\
&= \frac{1}{c_1 n}\sum_{i=0}^{n-1} d_1(f_{i+k+1}(a_{i+k}), a_{i+k+1}) + \frac{1}{c_1 n}\sum_{i=0}^{n-1} d_2(g_{i+k+1}(b_{i+k}), b_{i+k+1}) \\
&< \frac{2}{c_1}\delta = \delta^*,
\end{aligned}
$$

for any $n > m = \max\{m_1, m_2\}$. Therefore, $\{(a_i, b_i)\}_{i=0}^{+\infty} \subset X \times Y$ is a δ^* average pseudo-orbit of $f_{1,\infty} \times g_{1,\infty}$. Since $f_{1,\infty} \times g_{1,\infty}$ has ASP, then, for any $\varepsilon > 0$ and the δ^* average pseudo-orbit $\{(a_i, b_i)\}_{i=0}^{+\infty}$ of $(X \times Y, f_{1,\infty} \times g_{1,\infty})$, there exists a $(a, b) \in X \times Y$ such that

$$
\frac{1}{c_2}\limsup_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1} D((f_1^i \times g_1^i)(a, b), (a_i, b_i)) \le \limsup_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1}\rho(f_1^i \times g_1^i(a, b), (a_i, b_i)) < \varepsilon.
$$

In consequence,

$$
\limsup_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1} d_1(f_1^i(a), a_i) < c_2\varepsilon \quad \text{and} \quad \limsup_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1} d_2(g_1^i(b), b_i) < c_2\varepsilon.
$$

Thus, $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ have ASP.

(Sufficiency) Since $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ have ASP, then for any $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that, for every δ_1 average pseudo-orbit $\{a_i^*\}_{i=0}^{+\infty}$ of $(X, f_{1,\infty})$ and every δ_2 average pseudo-orbit $\{b_i^*\}_{i=0}^{+\infty}$ of $(Y, g_{1,\infty})$, one can select $a \in X, b \in Y$ satisfying

$$
\limsup_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1} d_1(f_1^i(a), a_i^*) < \varepsilon \ \textit{and} \ \limsup_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1} d_2(g_1^i(b), b_i^*) < \varepsilon.
$$

Take $\delta = \frac{1}{c_2}\min\{\delta_1, \delta_2\}$. Let $\{(a_i, b_i)\}_{i=0}^{+\infty} \subset X \times Y$ be a δ average pseudo-orbit of $f_{1,\infty} \times g_{1,\infty}$. Then there exists an $m \in \mathbb{N}$ such that

$$
\begin{aligned}
&\frac{1}{c_2 n}\sum_{i=0}^{n-1} D((f_{i+k+1} \times g_{i+k+1})(a_{i+k}, b_{i+k}), (a_{i+k+1}, b_{i+k+1})) \\
\le&\frac{1}{n}\sum_{i=0}^{n-1}\rho((f_{i+k+1} \times g_{i+k+1})(a_{i+k}, b_{i+k}), (a_{i+k+1}, b_{i+k+1})) \\
<&\delta,
\end{aligned}
$$

for any $n \ge m$ and any $k \in \mathbb{N}_0$. This means that

$$
\frac{1}{n}\sum_{i=0}^{n-1} d_1(f_{i+k+1}(a_{i+k}), a_{i+k+1}) < c_2\delta \quad \text{and} \quad \frac{1}{n}\sum_{i=0}^{n-1} d_2(g_{i+k+1}(b_{i+k}), b_{i+k+1}) < c_2\delta.
$$

So, $\{a_i\}_{i=0}^{+\infty} \subset X$, $\{b_i\}_{i=0}^{+\infty} \subset Y$ are the δ^* average pseudo-orbits of $f_{1,\infty}$ and $g_{1,\infty}$, respectively, where $\delta^* = c_2\delta$.

By the ASP of $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$, for any $\varepsilon > 0$ and $i \in \mathbb{N}_0$, there exist $a \in X, b \in Y$ such that

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} d_1(f_1^i(a), a_i) < \varepsilon \quad \text{and} \quad \limsup_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} d_2(g_1^i(b), b_i) < \varepsilon.$$

Then,

$$\begin{aligned}
&\limsup_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} \rho((f_1^i \times g_1^i)(a,b), (a_i, b_i)) \\
\le\ & \frac{1}{c_1} \limsup_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} D((f_1^i \times g_1^i)(a,b), (a_i, b_i)) \\
=\ & \frac{1}{c_1} \limsup_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} \sqrt{d_1^2(f_1^i(a), a_i) + d_2^2(g_1^i(b), b_i)} \\
=\ & \frac{1}{c_1} \limsup_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} (d_1(f_1^i(a), a_i) + d_2(g_1^i(b), b_i)) \\
<\ & \frac{2}{c_1}\varepsilon.
\end{aligned}$$

Therefore, $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ has ASP.

This complete the proof. □

In 2022, [15] proved that some shadowing properties are sufficient conditions for being transitive or point-transitive for a NDDSs. Next, we will discuss the relations of point-transitivity between product systems and teir factor systems.

Theorem 3.3. *If $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ is point-transitive (resp. transitive, mixing), then $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ are point-transitive (resp. transitive, mixing).*

Proof. (1) Assume that $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ is point-transitive. Then there is a $(a, b) \in X \times Y$ such that $\overline{orb_{f_{1,\infty}\times g_{1,\infty}}(a,b)} = X \times Y$. That is, $\overline{\{(a,b), (f_1(a), g_1(b)), \cdots, (f_1^n(a), g_1^n(b)), \cdots\}} = X \times Y$. Obviously,

$$\overline{orb_{f_{1,\infty}}(a)} = X \quad \text{and} \quad \overline{orb_{g_{1,\infty}}(b)} = Y.$$

Thus $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ are point-transitive.

(2) Assume that $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ is transitive. Then for any two nonempty open sets $U_1 \times V_1$ and $U_2 \times V_2$ in $X \times Y$, there exists an $n \in \mathbb{N}$ such that $(f_1^n \times g_1^n)(U_1 \times V_1) \cap (U_2 \times V_2) \neq \emptyset$. To elaborate, there is a $(a_1, b_1) \in U_1 \times V_1$ such that $(f_1^n \times g_1^n)(a_1 \times b_1) = (f_1^n(a_1), g_1^n(b_1)) \in U_2 \times V_2$. Then,

$$f_1^n(a_1) \in U_2 \quad \text{and} \quad g_1^n(b_1) \in V_2.$$

So,

$$f_1^n(U_1) \cap U_2 \neq \emptyset \quad \text{and} \quad g_1^n(V_1) \cap V_2 \neq \emptyset.$$

By the arbitrariness of U_i, V_i $(i = 1, 2)$, $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ are transitive.

(3) The proof of mixing is omitted as it follows a similar line of reasoning as that of transitivity. This complete the proof. □

The inverse of Theorem 3.3 may not hold in all cases. This illustrated through Examples 3.2 and 3.3.

Example 3.2. *Let* $X = [0,1]$*, the set of all rational numbers in* X *is denoted by* $\{a_1, a_2, a_3, \cdots\}$*. For any* $x \in [0,1]$*, let* $f_1(x) = a_1$*,* $f_2(x) = a_2, \cdots f_n(x) = a_n, \cdots$*. For* $f_{1,\infty} = (f_n)_{n=1}^{\infty} = \{f_1, f_2, f_3, \cdots\}$*, it is obvious that* $\overline{orb_{f_{1,\infty}}(a)} = X$ *for any* $a \in X$*. So,* $(X, d, f_{1,\infty})$ *is point-transitive.*

However, for any $a, b \in X$*,* $(f_1^n \times f_1^n)(a, b) = (a_i, a_i)$ $(i \in \mathbb{N})$*. Then,* $\overline{orb_{f_{1,\infty}\times f_{1,\infty}}(a,b)} \neq X \times X$*. So,* $(X \times X, D, f_{1,\infty} \times f_{1,\infty})$ *is not point-transitive.*

Example 3.3. *Let* $X = [0,1]$*. Two mappings* $\phi(x)$*,* $\omega(x)$ *be defined by* $\phi(x) = x$ *for any* $x \in [0,1]$ *and*

$$\omega(x) = \begin{cases} 2x + \frac{1}{2} & \text{for } x \in \left[0, \frac{1}{4}\right]; \\ -2x + \frac{3}{2} & \text{for } x \in \left[\frac{1}{4}, \frac{1}{2}\right]; \\ -x + 1 & \text{for } x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Let $(g_n)_{n=1}^{\infty} = \{\omega, \phi, \omega, \phi, \omega, \phi, \cdots\}$*.*

The function images of g_1^2, g_1^3, g_1^4, and g_1^5 are given in Figure 1, and the image of $g_1^n (n > 5)$ can be inferred.

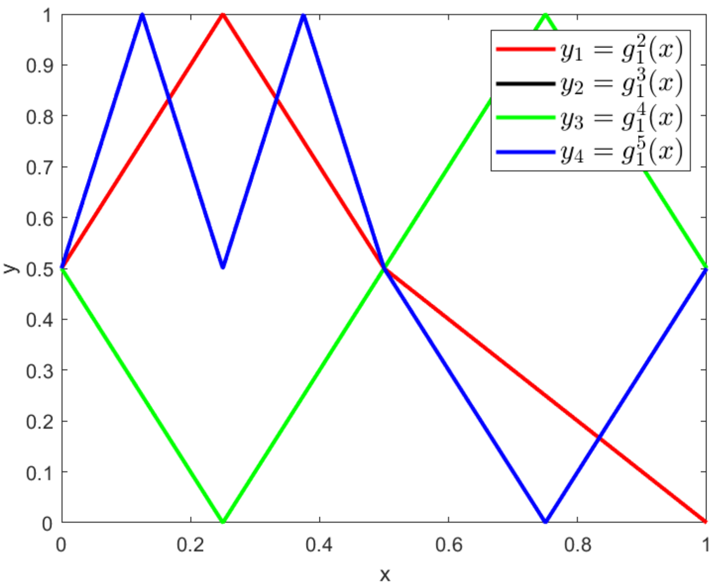

Figure 1. The function images of g_1^2, g_1^3, g_1^4, and g_1^5.

Figure 1 shows that, for any nonempty open set $U \in X$, there are large enough $m_1, m_2 \in \mathbb{N}$ such that $g_1^{m_1}(U) \supset (0, \frac{1}{2})$, $g_1^{m_2}(U) \supset (\frac{1}{2}, 1)$. Then, for any nonempty open set $V \in X$, there is an $n \in \mathbb{N}$ such that $f_1^n(U) \cap V \neq \emptyset$. So, $(X, d, g_{1,\infty})$ is transitive.

Take $U_1 = U_2 = (0, \frac{1}{2})$, $V_1 = (\frac{1}{8}, \frac{1}{4})$, $V_2 = (\frac{5}{8}, \frac{7}{8})$. Then $(g_1^n \times g_1^n)(U_1 \times U_2) \subset [(0, \frac{1}{2}) \times (0, \frac{1}{2})] \cup [(\frac{1}{2}, 1) \times (\frac{1}{2}, 1)]$ for any $n \in \mathbb{N}$. However, $(g_1^n \times g_1^n)(U_1 \times U_2) \cap (V_1 \times V_2) = \emptyset$. Thus $(X \times X, D, g_{1,\infty} \times g_{1,\infty})$ is not transitive.

Now, accessibility will be discussed by us.

Theorem 3.4. *If $(X, d, f_{1,\infty})$ is mixing, then $(X, d, f_{1,\infty})$ is accessible.*

Proof. Let any two nonempty open sets U_1 and U_2 be in X. For any $\varepsilon > 0$, there is a $V \subset X$ such that $diam(V) < \varepsilon$. Due to $(X, d, f_{1,\infty})$ is mixing, then there is an $N_1 > 0$ such that $f_1^n(U_1) \cap V \neq \emptyset$ for all $n \geq N_1$, and there is an $N_2 > 0$ such that $f_1^n(U_2) \cap V \neq \emptyset$ for all $n \geq N_2$.

Take $N = \max\{N_1, N_2\}$, then, for U_1, U_2, and V in X, $f_1^n(U_1) \cap V \neq \emptyset$ and $f_1^n(U_2) \cap V \neq \emptyset$ for all $n \geq N$. Then, there exist $a \in U_1, b \in U_2$ such that $f_1^n(a) \in V$ and $f_1^n(b) \in V$. Since $diam(V) < \varepsilon$, then, $d(f_1^n(a), f_1^n(b)) < \varepsilon$. So, $(X, d, f_{1,\infty})$ is accessible.

This complete the proof. □

The inverse of Theorem 3.4 is not necessarily held. This is illustrated through an counterexample.

Example 3.4. *Let $X = [0, 1]$,*

$$f_1(x) = \begin{cases} 2x & \text{for } x \in \left[0, \frac{1}{2}\right]; \\ -2x + 2 & \text{for } x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

$$f_2(x) = \begin{cases} x & \text{for } x \in \left[0, \frac{1}{2}\right]; \\ -x + 1 & \text{for } x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

$$f_3(x) = \begin{cases} \frac{1}{2}x & \text{for } x \in \left[0, \frac{1}{2}\right]; \\ -\frac{1}{2}x + \frac{1}{2} & \text{for } x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

$$f_4(x) = \begin{cases} \frac{1}{4}x & \text{for } x \in \left[0, \frac{1}{2}\right]; \\ -\frac{1}{4}x + \frac{1}{4} & \text{for } x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

$$\cdots$$

$$f_n(x) = \begin{cases} \frac{1}{2^{n-2}}x & \text{for } x \in \left[0, \frac{1}{2}\right]; \\ -\frac{1}{2^{n-2}}x + \frac{1}{2^{n-2}} & \text{for } x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

$$\cdots$$

Let $(f_n)_{n=1}^{\infty} = \{f_1, f_2, \cdots, f_n, \cdots\}$.

The function images of f_1^2, f_1^3, f_1^4, and f_1^5 are given in Figure 2, and the image of $f_1^n(n > 5)$ can be inferred.

Figure 2 shows that, $f_1^n(U) \to \{0\}(n \to \infty)$ for any $U \in X$. For any $\varepsilon > 0$ and any $a, b \in X$, there is an $m_1 \in \mathbb{N}$ such that $d(f_1^n(a), f_1^n(b)) < \varepsilon$ for any $n > m_1$. So, $(X, d, f_{1,\infty})$ is accessible.

Take $V = [\frac{1}{2}, 1]$, for any $U \in X$, there is an $m_2 \in \mathbb{N}$ such that $f_1^n(U) \cap V = \emptyset$ for any $n > m_2$. Thus $(X, d, f_{1,\infty})$ is not mixing.

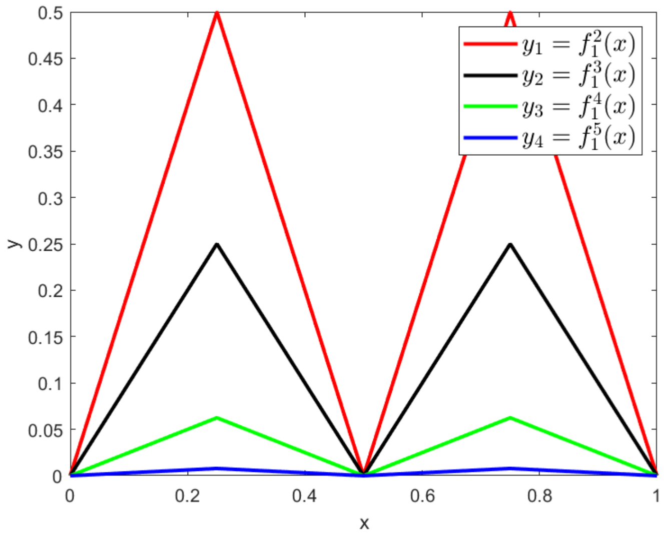

Figure 2. The function images of f_1^2, f_1^3, f_1^4, and f_1^5.

Theorem 3.5. *$(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ is accessible if and only if $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ are accessible.*

Proof. Since metric ρ is an equivalent metric of metric D, there exist $b \geq a > 0$ such that

$$a\rho(x_1, y_1), (x_2, y_2) \leq D(x_1, y_1), (x_2, y_2) \leq b\rho(x_1, y_1), (x_2, y_2)$$

for any $(x_1, y_1), (x_2, y_2) \in X \times Y$.

(Necessity) Assume that $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ is accessible. Then, for any $\varepsilon > 0$ and two nonempty open sets $U_1 \times V_1, U_2 \times V_2 \in X \times Y$, there exist points $(a_1, b_1) \in U_1 \times V_1, (a_2, b_2) \in U_2 \times V_2$, and $n \in \mathbb{N}$ such that

$$\rho((f_1^n \times g_1^n)(a_1, b_1), (f_1^n \times g_1^n)(a_2, b_2)) = \rho((f_1^n(a_1), g_1^n(b_1)), (f_1^n(a_2), g_1^n(b_2))) < \frac{1}{b}\varepsilon.$$

Then,

$$\begin{aligned}
&\sqrt{d_1^2(f_1^n(a_1), f_1^n(a_2)) + d_2^2(g_1^n(b_1), g_1^n(b_2))} \\
&= D((f_1^n \times g_1^n)(a_1, b_1), (f_1^n \times g_1^n)(a_2, b_2)) \\
&\leq b\rho((f_1^n(a_1), g_1^n(b_1)), (f_1^n(a_2), g_1^n(b_2))) \\
&< \varepsilon.
\end{aligned}$$

This means that $d_1(f_1^n(a_1), f_1^n(a_2))$ and $d_2(g_1^n(b_1), g_1^n(b_2))$ are less than ε. So, $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ are accessible.

(Sufficiency) By contradiction, if $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ is not accessible, then there exist a $\varepsilon_0 > 0$ and two nonempty open sets $U_1 \times V_1, U_2 \times V_2 \in X \times Y$, for any $n \in \mathbb{N}$ and points $(a_1, b_1) \in U_1 \times V_1, (a_2, b_2) \in U_2 \times V_2$, it is obvious that

$$\rho((f_1^n \times g_1^n)(a_1, b_1), (f_1^n \times g_1^n)(a_2, b_2)) > \varepsilon_0.$$

Then,

$$
\begin{aligned}
&\sqrt{d_1^2(f_1^n(a_1), f_1^n(a_2)) + d_2^2(g_1^n(b_1), g_1^n(b_2))} \\
&= D((f_1^n \times g_1^n)(a_1, b_1), (f_1^n \times g_1^n)(a_2, b_2)) \\
&\geq a\rho((f_1^n \times g_1^n)(a_1, b_1), (f_1^n \times g_1^n)(a_2, b_2)) \\
&> a\varepsilon_0.
\end{aligned}
$$

This means that at least one of $d_1(f_1^n(a_1), f_1^n(a_2))$ and $d_2(g_1^n(b_1), g_1^n(b_2))$ has to be greater than $\frac{\sqrt{2}}{2}a\varepsilon_0$. To be more specific, at least one of $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ is not accessible. Contradict to that $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ are accessible. So, $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ is accessible.

This complete the proof. □

The following example demonstrates that, if only one of the $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ is accessible, there is not necessarily follow that $(X \times X, \rho, f_{1,\infty} \times g_{1,\infty})$ is accessible.

Example 3.5. *Let $X = [0, 1]$, two mappings $h_1(x)$, $h_2(x)$ be defined by $h_1(x) = 1, h_2(x) = x$ for $x \in X$. Let $(f_n)_{n=1}^{\infty} = \{h_1, h_1, h_1, \cdots\}$, $(g_n)_{n=1}^{\infty} = \{h_2, h_2, h_2, \cdots\}$.*

Obviously, $d(f_1^n(x_1), f_1^n(x_2)) = 0$ for any $n \in \mathbb{N}$ and $x_1, x_2 \in X$. So, $(X, d, f_{1,\infty})$ is accessible. Let $U = (0, \frac{1}{8}), V = (\frac{7}{8}, 1)$, then for any $x_1 \in U, y_1 \in V$, $d(g_1^n(x_1), g_1^n(y_1)) > \frac{3}{4}$. Thus, $(X, d, g_{1,\infty})$ is not accessible. For $U \times U, V \times V \in X \times X$, any $(x_1, x_2) \in U \times U, (y_1, y_2) \in V \times V$, it is obvious that

$$
\begin{aligned}
&D((f_1^n \times g_1^n)(x_1, x_2), (f_1^n \times g_1^n)(y_1, y_2) \\
= &\sqrt{d^2(f_1^n(x_1), f_1^n(y_1)) + d^2(g_1^n(x_2), g_1^n(y_2))} > \frac{3}{4}.
\end{aligned}
$$

Therefore, $(X \times X, D, f_{1,\infty} \times g_{1,\infty})$ is not accessible.

Theorem 3.6. *Let $f_n (n \in \mathbb{N})$ be surjections. If the system $(X, d, f_{1,\infty})$ is accessible, then, for any $n \in \mathbb{N}$, the system $(X, d, f_{n,\infty} = (f_n, f_{n+1}, \cdots))$ is accessible.*

Proof. To illustrate this point, it is sufficient to consider the case when $n = 2$.

For any two nonempty open subsets $U, V \subset X$, taking an inverse image of each element in U and V under f_1, one can get two sets U^* and V^* in X, separately. Since f_1 is continuous surjective, then U^* and V^* are also nonempty open subsets of X. Due to $f_{1,\infty}$ is accessible, then for any $\varepsilon > 0$, there exist points $a^* \in U^*, b^* \in V^*$ and $n \in \mathbb{N}$ such that $d(f_1^{n+1}(a^*), f_1^{n+1}(b^*)) < \varepsilon$.

Given that there exist $a \in U, b \in V$ such that $f_1(a^*) = a, f_1(b^*) = b$, then $d(f_2^n(a), f_2^n(b)) = d(f_1^{n+1}(a^*), f_1^{n+1}(b^*)) < \varepsilon$. So, $(X, d, f_{2,\infty})$ is accessible.

This complete the proof. □

Theorem 3.7. *Let $f_n (n \in \mathbb{N})$ and $g_n (n \in \mathbb{N})$ be surjections. If $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ is accessible, then $(X, d_1, f_{n,\infty})$ and $(Y, d_2, g_{n,\infty})$ is accessible for any $n \in \mathbb{N}$.*

Proof. The result is evident by applying Theorems 3.5 and 3.6.

This complete the proof. □

An appropriate example that aligns with Theorem 3.7 is presented below.

Example 3.6. *Let $X = [0,1]$. Two mappings $\varphi(x)$, $\psi(x)$ be defined by*

$$\varphi(x) = \begin{cases} 3x & for\ x \in \left[0, \frac{1}{3}\right]; \\ -3x+2 & for\ x \in \left[\frac{1}{3}, \frac{2}{3}\right]; \\ 3x-2 & for\ x \in \left[\frac{2}{3}, 1\right] \end{cases} \quad and \quad \psi(x) = \begin{cases} 6x & for\ x \in \left[0, \frac{1}{6}\right]; \\ -6x+2 & for\ x \in \left[\frac{1}{6}, \frac{1}{3}\right]; \\ 6x-2 & for\ x \in \left[\frac{1}{3}, \frac{1}{2}\right]; \\ -6x+4 & for\ x \in \left[\frac{1}{2}, \frac{2}{3}\right]; \\ 6x-4 & for\ x \in \left[\frac{2}{3}, \frac{5}{6}\right]; \\ -6x+6 & for\ x \in \left[\frac{5}{6}, 1\right]. \end{cases}$$

Let $(f_n)_{n=1}^{\infty} = \{\varphi, \varphi, \varphi, \cdots\}$, $(g_n)_{n=1}^{\infty} = \{\psi, \psi, \psi, \cdots\}$.

Obviously, $\varphi(x)$ and $\psi(x)$ are triangle-tent map. Then there must exist large enough $n_1, n_2 \in \mathbb{N}$ such that $f_1^{n_1}(U) = X$ and $g_1^{n_2}(U) = X$ for any nonempty open set $U \in X$. Take $n = \max\{n_1, n_2\}$, then $f_1^n(U) \cap f_1^n(V) \neq \emptyset$ for any nonempty open set $V \in X$. So, there exist points $x \in U, y \in V$ such that $d(f_1^n(x), f_1^n(y)) = 0$. Thus, $(X, d, f_{1,\infty})$ and $(X, d, g_{1,\infty})$ are accessible.

For any two nonempty open sets $U_1 \times V_1$ and $U_2 \times V_2$ in $X \times X$, there exists an $n \in \mathbb{N}$ such that $(f_1^n \times g_1^n)(U_1 \times V_1) = X \times X$. Then, $(f_1^n \times g_1^n)(U_1 \times V_1) \cap (f_1^n \times g_1^n)(U_2 \times V_2) \neq \emptyset$. So, there exist points $(a_1, b_1) \in U_1 \times V_1, (a_2, b_2) \in U_2 \times V_2$ such that $D((f_1^n \times g_1^n)(a_1, b_1), (f_1^n \times g_1^n)(a_2, b_2)) = 0$ for any metric ρ of $X \times X$. Therefore, $(X \times X, D, f_{1,\infty} \times g_{1,\infty})$ is accessible.

Corollary 3.1. *If $(X \times Y, \rho, f_{1,\infty} \times g_{1,\infty})$ is mixing, then $(X, d_1, f_{1,\infty})$ and $(Y, d_2, g_{1,\infty})$ are accessible.*

4. Conclusions

Transitivity, mixing, and accessibility are chaotic properties related to Devaney chaos. They represent a kind of ergodic state of discrete dynamical systems. While shadowing properties are often used in computer simulation. The orbit obtained by numerical calculation approximately reflects the local dynamic behavior of the system. The premise is that the difference between each iteration point and the real orbit is small enough. This paper discusses the relationship between the above properties of non-autonomous product systems and that of corresponding factor systems. The results tell us that in practical problems, the method of decreasing (or increasing) dimension is feasible. The reasons for the infeasibility are explained by counterexamples. In the future, we can combine more specific applications in physics, electronic information, or computer technology to study the chaotic properties of high-dimensional systems or low-dimensional systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by Natural Science Foundation of Sichuan Province (No. 2023NSFSC0070), the Scientific Research Project of SUSE (No. 2020RC24), and the Graduate student Innovation Fund (No. y2022189).

Conflict of interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

1. S. Kolyada, L. Snoha, Topological entropy of nonautonomous dynamical systems, *Random Comput. Dynam.*, **4** (1996), 205–233.
2. C. J. Tian, G. R. Chen, Chaos of a sequence of maps in a metric space, *Chaos Solitons Fractals*, **28** (2006), 1067–1075. https://doi.org/10.1016/j.chaos.2005.08.127
3. Y. M. Shi, G. R. Chen, Chaos of time-varying discrete dynamical systems, *J. Differ. Equ. Appl.*, **15** (2009), 429–449. https://doi.org/10.1080/10236190802020879
4. J. S. Cánovas, Li-Yorke chaos in a class of non-autonomous discrete systems, *J. Differ. Equ. Appl.*, **17** (2011), 479–486. https://doi.org/10.1080/10236190903049025
5. F. BallBrea, P. Oprocha, Weak mixing and chaos in non-autonomous discrete systems, *Appl. Math. Lett.*, **25** (2012), 1135–1141. https://doi.org/10.1016/j.aml.2012.02.021
6. X. Q. Song, J. K. Liu, L. W. Wang, Ruelle-Takens chaos in non-autonomous dynamical systems, *Eng. Math. Lett.*, **1** (2012), 65–74.
7. X. X. Wu, P. Y. Zhu, Chaos in a class of non-autonomous discrete systems, *Appl. Math. Lett.*, **26** (2013), 431–436. https://doi.org/10.1016/j.aml.2012.11.003
8. Q. L. Huang, Y. M. Shi, L. J. Zhang, Sensitivity of non-autonomous discrete dynamical systems, *Appl. Math. Lett.*, **39** (2015), 31–34. https://doi.org/10.1016/j.aml.2014.08.007
9. C. N. Ma, P. Y. Zhu, A remark on sensitivity and Li-Yorke sensitivity of iterated function systems, *Qual. Theor. Dyn. Syst.*, **18** (2019), 1–9. https://doi.org/10.1007/s12346-018-0270-7
10. R. S. Li, Y. Zhao, H. Q. Wang, H. H. Liang, Stronger forms of transitivity and sensitivity for nonautonomous discrete dynamical systems and Furstenberg families, *J. Dyn. Control Syst.*, **26** (2020), 109–126. https://doi.org/10.1007/s10883-019-09437-6
11. W. Anwar, T. X. Lu, X. F. Yang, Sensitivity of iterated function systems under the product operation, *Results Math.*, **77** (2022), 185. https://doi.org/10.1007/s00025-022-01669-6
12. J. M. Pi, T. X. Lu, Y. L. Chen, Collective sensitivity and collective accessibility of non-autonomous discrete dynamical systems, *Fractal Fract.*, **6** (2022), 535. https://doi.org/10.3390/fractalfract6100535
13. Y. M. Shi, Chaos in non-autonomous discrete dynamical systems approached by their induced systems, *Internat. J. Bifur. Chaos*, **22** (2012), 1250284. https://doi.org/10.1142/S0218127412502847
14. H. Shao, Y. M. Shi, H. Zhu, On distributional chaos in non-autonomous discrete systems, *Chaos Solitons Fractals*, **107** (2018), 234–243. https://doi.org/10.1016/j.chaos.2018.01.005
15. J. M. Pi, T. X. Lu, Y. F. Xue, Transitivity and shadowing properties of non-autonomous discrete dynamical systems, *Int. J. Bifurcat. Chaos*, **32** (2022), 2250246. https://doi.org/10.1142/S0218127422502467

16. R. Vasisht, R. Das, Coneralizations of expansiveness in non-autonomous discrete systems, *B. Iran. Math. Soc.*, **48** (2022), 417–433. https://doi.org/10.1007/s41980-020-00525-z

17. Y. X. Jiang, T. X. Lu, J. M. Pi, W. Anwar, The retentivity of four kinds of shadowing properties in non-Autonomous discrete dynamical systems, *Entropy*, **24** (2022), 397. https://doi.org/10.3390/e24030397

18. X. Meng, F. Yi, Some equivalence determination and application of equivalent metric, *J. Shenyang Norm. Univ.*, 2012.

19. L. D. Wang, Y. N. Li, Y. L. Gao, H. Liu, Distributional chaos of time-varying discrete dynamical systems, *Ann. Pol. Math.*, **107** (2013), 49–57. http://doi.org/10.40642Fap107-1-3

20. X. F. Yang, T. X. Lu, W. Anwar, Chaotic properties of a class of coupled mapping lattice induced by fuzzy mapping in non-autonomous discrete systems, *Chaos Solitons Fractals*, **148** (2021), 110979. https://doi.org/10.1016/j.chaos.2021.110979

21. M. L. Blank, Metric properties of ε-trajectories of dynamical systems with stochastic behaviour, *Ergod. Theor. Dyn. Syst.*, **8** (1988), 365–378. https://doi.org/10.1017/S014338570000451X

22. D. Dastjerdi, M. Hosseini, Sub-shadowings, *Nonlinear Anal.*, **72** (2010), 3759–3766. https://doi.org/10.1016/j.na.2010.01.014