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Research article

Measure of non-compactness for nonlocal boundary value problems via (k,ψ)-Riemann-Liouville derivative on unbounded domain

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Abstract: In this paper, we investigate the existence result for (k,ψ)-Riemann-Liouville fractional differential equations via nonlocal conditions on unbounded domain. The main result is proved by applying a fixed-point theorem for Meir-Keeler condensing operators with a measure of noncompactness. Finally, two numerical examples are also demonstrated to confirm the usefulness and applicability of our theoretical results.

Keywords: boundary value problem; fixed point theorem; (k,ψ)-Riemann-Liouville fractional derivative; measure of noncompactness; Meir-Keeler condensing operators
Mathematics Subject Classification: 26A33, 34A08, 34B40, 47H08, 47H10

1. Introduction

In this paper, we aim to establish the existence of solutions for the following nonlinear boundary value problems ($\mathbb{BVP}$s) of (k,ψ)-Riemann-Liouville ($\mathbb{RL}$) fractional differential equation with nonlocal conditions on an unbounded domain (short (k,ψ)-$\mathbb{RL}$-$\mathbb{NBVP}$) as:

$$\begin{cases} {}_k\mathfrak{D}_{a+}^{\alpha;\psi}u(t) = f(t,u(t)), \quad \alpha \in (1,2], \quad t \in (a,\infty), \quad a > 0, \\ {}_k\mathcal{I}_{a+}^{2k-\alpha;\psi}u(a^+) = \mathcal{A} + \sum_{i=1}^{m}\lambda_i u(\xi_i), \quad \lim_{t\to\infty} {}_k\mathfrak{D}_{a+}^{\alpha-k;\psi}u(t) = \mathcal{B} + \sum_{j=1}^{n}\mu_j\, {}_k\mathcal{I}_{a+}^{\sigma_j;\psi}u(\eta_j), \end{cases} \tag{1.1}$$

where ${}_k\mathfrak{D}^{q;\psi}_{a+}$ denotes the (k,ψ)-$\mathbb{RL}$-fractional derivative of order $q \in \{\alpha, \alpha - k\}$ with $k > 0$, ${}_k\mathcal{I}^{p;\psi}_{a+}$ is the (k,ψ)-$\mathbb{RL}$-fractional integral of order $p \in \{2k-\alpha, \sigma_j\} > 0$, $j = 1,2,\ldots,n$, $\mathcal{E}$ is a Banach space, $f \in C((a,\infty)\times\mathcal{E},\mathcal{E})$, the given constants $\mathcal{A}$, $\mathcal{B}$, λ_i, $\mu_j \in \mathbb{R}$, and the points ξ_i, $\eta_j \in (a,\infty)$, for $i = 1,2,\ldots,m$, and $j = 1,2,\ldots,n$.

The (k,ψ)-$\mathbb{RL}$-$\mathbb{NBVP}$ problem (1.1) in this work was inspired by the following important histories: hundreds of years ago, fractional calculus ($\mathbb{FC}$) was first constructed and later developed by many researchers in the branches of applied sciences and engineering such as biology, control theory, economics, electromagnetic, electrochemistry, finance, mechanics, and other fields which have been established in the last few decades. It has become a better tool for understanding some physical phenomena in the last decade, especially when dealing with processes with memory [1]. Modeling viscoelastic and viscoplastic materials [2], bioengineering [3], and a variety of sciences and engineering problems are a few examples of applications. It is non-integer calculus and deals with fractional integral and derivative operators ($\mathbb{FIO}$/$\mathbb{FDO}$) such as $\mathbb{RL}$, Caputo, Hadamard, Erdélyi-Kober, Katugampola, Hilfer, and others; we refer to read the details of applications on $\mathbb{FC}$ [2–10]. Especially, the $\mathbb{FDO}$ of Hilfer's type, which uses the notation ${}^{\mathbb{H}}\mathfrak{D}^{\alpha,\beta}x(t)$. It is a generalization of $\mathbb{RL}$'s type when $\beta = 0$ and Caputo's type when $\beta = 1$ ([5]). In a more profound, there is another type of Hilfer derivative. It is called (k,ψ)-Hilfer-$\mathbb{FDO}$ [11, 12], which uses the notation ${}^{\mathbb{H}}_{k}D^{\alpha,\beta;\psi}_{a^+}u(t)$, and it is a generalized form of the Hilfer-$\mathbb{FDO}$ that covers the ψ-Hilfer-$\mathbb{FDO}$ when $k = 1$. Particularly, it can be reduced to (k,ψ)-$\mathbb{RL}$-$\mathbb{FDO}$ with $k > 0$ and $\beta = 0$. Many researchers use differential equations ($\mathbb{DE}$s) for real-world problems simulation and solving difficulties employing some powerful techniques. Moreover, fractional differential equations ($\mathbb{FDE}$s) which are the combination between $\mathbb{FC}$ and $\mathbb{DE}$s, have been applied to model the problems since fractional-order has more additional degrees of freedom than integer-order and also to describes the memory and the hereditary properties that allow for more accurate and realistic solutions. Plenty of great pieces of literature have been produced to study nonlinear $\mathbb{FDE}$s dealing with initial/boundary conditions under various kinds of fractional derivatives that discuss the qualitative properties such as existence, uniqueness and stability of solutions utilizing a variety of fixed-point theorem. We refer readers to some works [13–20] for contemporary papers on $\mathbb{FDE}$s.

Nonlocal conditions were introduced by Byszewaski [21] in 1991, who established the existence and uniqueness of mild and classical solutions to nonlocal Cauchy problems. As remarked by Byszewski [22, 23], nonlocal conditions can be more useful than the standard condition to explain some physical phenomena. In 1930, Kuratowski [24] initially recommended the idea of a measure of noncompactness. It is significant in fixed-point theory and has several capitalizations in nonlinear analysis techniques, such as $\mathbb{DE}$s, integro-differential equations ($\mathbb{IDE}$s), optimization, and so on. In conclusion, it is a function determined in a class of nonempty and bounded subsets of satisfaction of some metric space that is equal to zero in all families of relatively compact sets. In 1955, Darbo [25] applied the Kuratowski measure to study a family of operators known as condensing operators, whose qualities are middle between those of contraction and compact mappings. For more details on a variety of measures of noncompactness, see; [26]. Besides, many researches have been produced involving establishing the qualitative properties of solutions on finite intervals. However, fewer research works studied the solutions on unbounded domain as $[0,\infty)$; for example, in 2010, Arara et al. [27] utilized a fixed-point theory of Schauder's type under the diagonalization procedure to discuss the existence results of $\mathbb{BVP}$s of $\mathbb{FDE}$s on an unbounded domain under Caputo-$\mathbb{FDO}$ as:

$$\begin{cases} {}^{C}\mathfrak{D}_{0^+}^{\alpha}u(t) = f(t,u(t)), \quad \alpha \in (1,2], \quad t \in [0,\infty), \\ u(0) = u_0, \quad u \text{ is bounded on } [0,\infty), \end{cases}$$

where ${}^{C}\mathfrak{D}_{0^+}^{\alpha}$ denotes the Caputo-$\mathbb{FDO}$ of order α, $f \in C([0,\infty)\times\mathbb{R},\mathbb{R})$ and $u_0 \in \mathbb{R}$. Later, in 2011, Su [28] used Darbo's fixed-point theorem to investigate an existence of solutions for $\mathbb{BVP}$ under $\mathbb{RL}$-$\mathbb{FDO}$ in Banach space of the form

$$\begin{cases} \mathfrak{D}_{0^+}^{\alpha}u(t) = f(t,u(t)), \quad \alpha \in (1,2], \quad t \in [0,\infty), \\ u(0) = 0, \quad \mathfrak{D}_{0^+}^{\alpha-1}u(\infty) = u_\infty, \end{cases}$$

where $\mathfrak{D}_{0^+}^{q}$ denotes the $\mathbb{RL}$-$\mathbb{FDO}$ of order $q \in \{\alpha,\alpha-1\}$, $f \in C([0,\infty)\times\mathbb{R},\mathbb{R})$, $u_\infty \in \mathbb{R}$, and $\mathfrak{D}_{0^+}^{\alpha-1}u(\infty) = \lim_{t\to\infty}\mathfrak{D}_{0^+}^{\alpha-1}u(t)$. Recently, Beddani and Hedia [29] employed Tykhonoff's fixed-point theorem to obtain the existence results of solutions on an unbounded domain to $\mathbb{BVP}$ of $\mathbb{FDE}$s under $\mathbb{RL}$-$\mathbb{FDO}$ in a Fréchet space of the form:

$$\begin{cases} \mathfrak{D}_{a+}^{\alpha}u(t) = f(t,u(t)), \quad \alpha \in (1,2], \quad t \in (0,\infty), \\ u(0) = 0, \quad \mathfrak{D}_{0+}^{\alpha-1}u(\infty) = u_\infty, \end{cases}$$

where $\mathfrak{D}_{0+}^{q}$ denotes the $\mathbb{RL}$-$\mathbb{FDO}$ of order $q \in \{\alpha,\alpha-1\}$, $f \in C((0,\infty)\times\mathbb{R},\mathbb{R})$, and $u_\infty \in \mathbb{R}$. After that, in 2022, Benia et al. [30] studied $\mathbb{FDE}$s with ψ-$\mathbb{RL}$-$\mathbb{FDO}$ on an unbounded domain in Banach space and investigated its existence of solution sets and topological structure as follows:

$$\begin{cases} \mathfrak{D}_{a+}^{\alpha;\psi}u(t) = f(t,u(t)), \quad \alpha \in (1,2], \quad t \in (0,\infty), \\ \mathcal{I}_{0+}^{2-\alpha;\psi}u(0^+) = \mathcal{A}, \quad \mathfrak{D}_{0+}^{\alpha-1;\psi}u(\infty) = \mathcal{B}, \end{cases}$$

where $\mathfrak{D}_{0+}^{q;\psi}$ denotes the ψ-$\mathbb{RL}$-$\mathbb{FDO}$ of order $q \in \{\alpha,\alpha-1\}$, $\mathcal{I}_{0+}^{2-\alpha;\psi}$ is the ψ-$\mathbb{RL}$-$\mathbb{FDO}$ of order $2-\alpha > 0$, $f \in C((0,\infty)\times\mathbb{R},\mathbb{R})$, and $\mathcal{A}$, $\mathcal{B} \in \mathbb{R}$. Several prices of research studied $\mathbb{FDE}$s on unbounded domain, we recommended readers to good works; [31–42] and references therein.

Motivated by the aforementioned works, to make it different and novel, the spotlight in this study is to establish the existence result of solution sets for the (k,ψ)-$\mathbb{RL}$-$\mathbb{NBVP}$ problem (1.1) which will be introduced later in the main Theorem (3.1). Next, some consequences of the main theorem are listed. Finally, numerical examples of applications are pointed out to support our theoretical results. This work is collected as follows: the basic definition of a measure of noncompactness under Kuratowski and its essential qualities are mentioned in Section 2. The existence result of the considered problem is established using a fixed-point theorem for Meir-Keeler condensing operators in Section 3. Numerical examples are provided in Section 4, and the conclusion is shown in the last section.

2. Fundamental concepts

Next, we define the basic definition of the measure of noncompactness under Kuratowski and its essential qualities. For any $\mathcal{D} \subseteq \mathcal{E}$, the set of all bounded subsets of $\mathcal{D}$ is denoted by $\mathcal{S}_b(\mathcal{D})$. Next, we give the Kuratowski measure of noncompactness and provide some of its important properties.

Definition 2.1. *([26,43]) The Kuratowski measure of noncompactness $\mathfrak{G}$ is defined on each bounded subset $\mathcal{D}$ of Banach space $\mathcal{E}$ by*

$$\mathfrak{G}(\mathcal{D}) = \inf\{\epsilon > 0 \mid \mathcal{D} \text{ admits a finite cover by sets of diameter } \leq \epsilon\}.$$

The following properties about the Kuratowski measure of noncompactness are well known.

Lemma 2.1. *([26, 43]) Let $\mathcal{E}$ be a Banach space and $\mathcal{H}$, $\mathcal{K} \in \mathcal{S}_b(\mathcal{E})$. The following properties true:*

($\mathbb{P}_1$) $\mathfrak{G}(\mathcal{H}) = 0$ *if and only if* $\mathcal{H}$ *is relatively compact.*
($\mathbb{P}_2$) $\mathfrak{G}(\mathcal{H}) = \mathfrak{G}(\overline{\mathcal{H}})$, *where* $\overline{\mathcal{H}}$ *is the closure of* $\mathcal{H}$.
($\mathbb{P}_3$) $\mathfrak{G}(\mathcal{H} + \mathcal{K}) \leq \mathfrak{G}(\mathcal{H}) + \mathfrak{G}(\mathcal{K})$.
($\mathbb{P}_4$) $\mathcal{H} \subset \mathcal{K}$ *implies* $\mathfrak{G}(\mathcal{H}) \leq \mathfrak{G}(\mathcal{K})$.
($\mathbb{P}_5$) $\mathfrak{G}(\alpha\mathcal{H}) = \|\alpha\|\mathfrak{G}(\mathcal{H})$ *for all* $\alpha \in \mathcal{E}$.
($\mathbb{P}_6$) $\mathfrak{G}(\{\alpha\} \cup \mathcal{H}) = \mathfrak{G}(\mathcal{H})$ *for all* $\alpha \in \mathcal{E}$.
($\mathbb{P}_7$) $\mathfrak{G}(\mathcal{H}) = \mathfrak{G}(Conv(\mathcal{H}))$, *where* $Conv(\mathcal{H})$ *is the smallest convex that contains* $\mathcal{H}$.

The following lemmas are need in our argument.

Lemma 2.2. *([44]) Let $\mathcal{D} \in \mathcal{S}_b(\mathcal{E})$ and $\epsilon > 0$. Then, there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \subset \mathcal{D}$, so that*

$$\mathfrak{G}(\mathcal{D}) \leq 2\mathfrak{G}(\{z_n\}; n \in \mathbb{N}) + \epsilon.$$

Lemma 2.3. *([43]) Let $\mathcal{E}$ be a Banach space. If $\mathcal{D}$ is an equicontinuous and bounded subset of $C([a, b], \mathcal{E})$, then $\mathfrak{G}(\mathcal{D}(\cdot)) \in C([a, b], \mathbb{R}^+)$*

$$\mathfrak{G}_C(\mathcal{D}) = \max_{s\in[a,b]} \mathfrak{G}(\mathcal{D}(s)), \quad \mathfrak{G}\Big(\Big\{ \int_a^t z(s)ds : z \in \mathcal{D}\Big\}\Big) \leq \int_a^b \mathfrak{G}(\mathcal{D}(s))ds,$$

where $\mathcal{D}(s) = \{z(s) : s \in \mathcal{D}\}$ and $\mathfrak{G}_C$ is the noncompactness measure on the space $C([a, b], \mathcal{E})$.

Definition 2.2. *([45]) Assume that κ is an arbitrary measure of noncompactness on $\mathcal{E}$ and $\mathcal{D}$ is a nonempty subset of $\mathcal{E}$. Assume that $Q : \mathcal{D} \to \mathcal{D}$ is an operator. The operator Q is said to be Meir-Keeler condensing operator if*

$$\forall\epsilon > 0, \exists\mathcal{N}(\epsilon) > 0, \forall\mathcal{D} \in \mathcal{S}_b(\mathcal{D}) : \epsilon \leq \kappa(\mathcal{D}) < \epsilon + \mathcal{N} \implies \kappa(Q\mathcal{D}) < \epsilon.$$

Theorem 2.1. *([45]) Assume that κ is an arbitrary measure of noncompactness on $\mathcal{E}$ and $\mathcal{D}$ is a nonempty subset of $\mathcal{E}$. Assume that $Q : \mathcal{D} \to \mathcal{D}$ is an operator, and Q is a Meir-Keeler condensing operator and continuous, then the set $\{z \in \mathcal{D} : Q(z) = z\}$ is nonempty and compact.*

Definition 2.3. *([46]) For $Re(z) > 0$ and $k > 0$, the k-gamma function $\Gamma_k(\cdot)$ is defined by*

$$\Gamma_k(z) = \int_0^\infty s^{z-1} e^{\frac{-s^k}{k}} ds,$$

which satisfies the following properties: $\Gamma_k(z + k) = z\Gamma_k(z)$, $\Gamma_k(k) = 1$, and $\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma_k(\frac{z}{k})$.

Definition 2.4. *([46]) For $Re(a)$, $Re(b)$, $k > 0$. Then, the k-beta function $B_k(a, b)$ is defined by*

$$B_k(a, b) = \frac{1}{k}\int_0^1 s^{\frac{a}{k}-1}(1 - s)^{\frac{b}{k}-1} ds,$$

which has the following relations: $B_k(a, b) = \frac{1}{k}B(\frac{a}{k}, \frac{b}{k})$, and $B_k(a, b) = \frac{\Gamma_k(a)\Gamma_k(b)}{\Gamma_k(a+b)}$.

Definition 2.5. *([47]) Assume that $h \in \mathcal{L}^1[a,b]$ and $k > 0$. Then, the (k,ψ)-$\mathbb{RL}$-$\mathbb{FIO}$ of order $\alpha > 0$ ($\alpha \in \mathbb{R}$) of h is provided by*

$$ {}_k\mathcal{I}_{a+}^{\alpha;\psi}h(t) = \frac{1}{k\Gamma_k(\alpha)}\int_a^t \psi'(s)\Psi^{\frac{\alpha}{k}-1}(t,s)h(s)ds, \quad \Psi^{\frac{\alpha}{k}-1}(t,s) = (\psi(t)-\psi(s))^{\frac{\alpha}{k}-1}. $$

Definition 2.6. *([47]) Assume that $\alpha, k > 0$, $\psi \in C^n[a,b] (n \in \mathbb{N})$, $\psi'(t) \neq 0$, $t \in [a,b]$ and $h \in C^n[a,b]$. Then, the (k,ψ)-$\mathbb{RL}$-$\mathbb{FDO}$ of order $\alpha > 0$ ($\alpha \in \mathbb{R}$) of h is provided by*

$$ {}_k\mathfrak{D}_{a+}^{\alpha;\psi}h(t) = \left(\frac{k}{\psi'(t)}\cdot\frac{d}{dt}\right)^n {}_k\mathcal{I}_{a+}^{nk-\alpha;\psi}h(t), \quad n = \left\lceil\frac{\alpha}{k}\right\rceil. $$

Throughout this work, assume $\mathcal{J} \subset (0,\infty)$ is a compact interval and $C(\mathcal{J},\mathcal{E})$ is a Banach space of continuous functions $u : \mathcal{J} \to \mathcal{E}$ equipped with the usual supremum norm $\|u\|_C = \sup_{t\in\mathcal{J}}\{\|u(t)\|\}$. By $\mathcal{L}^1(\mathcal{J},\mathcal{E})$, we denote the space of Bochner integrable functions $u : \mathcal{J} \to \mathcal{E}$ with the norm $\|u\|_{\mathcal{L}^1} = \int_a^{+\infty}\|u(s)\|\,ds$. Consider the following Banach space

$$ C_\psi^{\frac{\alpha}{k}} = \left\{u \in C(\mathcal{J},\mathcal{E}) : \lim_{t\to 0}\Psi^{2-\frac{\alpha}{k}}(t,a)u(t) < +\infty \text{ and } \lim_{t\to\infty}\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)u(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)} < +\infty\right\}, $$

equipped with the norm

$$ \|u\|_\psi^{\frac{\alpha}{k}} = \sup_{t\in\mathcal{J}}\left\{\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)\,\|u(t)\|}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right\}. $$

For $u \in C_\psi^{\frac{\alpha}{k}}$, we define $u_\psi^{\frac{\alpha}{k}}$ by

$$ u_\psi^{\frac{\alpha}{k}}(t) = \begin{cases} \dfrac{\Psi^{2-\frac{\alpha}{k}}(t,a)u(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}, & \text{if} \quad t \in (a,\infty),\ a > 0, \\ \lim\limits_{t\to a}\Psi^{2-\frac{\alpha}{k}}(t,a)u(t), & \text{if} \quad t = a > 0. \end{cases} $$

It is clear that $u_\psi^{\frac{\alpha}{k}} \in C_\psi^{\frac{\alpha}{k}}$.

The following are some significant basic properties that will be used throughout this paper.

Lemma 2.4. *([47]) Assume that α, $k > 0$ and $\frac{\mu}{k} > -1$. Then, we have*

(i) ${}_k\mathcal{I}_{a^+}^{\alpha;\psi}\Big(\Psi^{\frac{\mu}{k}}(t,a)\Big) = \frac{\Gamma_k(\mu+k)\Psi^{\frac{\mu+\alpha}{k}}(t,a)}{\Gamma_k(\mu+k+\alpha)}$.

(ii) ${}_k\mathfrak{D}_{a^+}^{\alpha;\psi}\Big(\Psi^{\frac{\mu}{k}}(t,a)\Big) = \frac{\Gamma_k(\mu+k)\Psi^{\frac{\mu-\alpha}{k}}(t,a)}{\Gamma_k(\mu+k-\alpha)}$.

(iii) ${}_k\mathcal{I}_{a^+}^{\alpha;\psi}\left({}_k\mathcal{I}_{a^+}^{\beta;\psi}f\right)(t) = {}_k\mathcal{I}_{a^+}^{\alpha+\beta;\psi}f(t) = {}_k\mathcal{I}_{a^+}^{\beta;\psi}\left({}_k\mathcal{I}_{a^+}^{\alpha;\psi}f\right)(t)$.

Lemma 2.5. *([47]) Assume that α, $k > 0$ and $n = \left\lceil\frac{\alpha}{k}\right\rceil$ where $n \in \mathbb{N}$. If $h \in C^n([a,b],\mathbb{R})$ and ${}_k\mathcal{I}_{a^+}^{kn-\alpha;\psi}h \in C^n([a,b],\mathbb{R})$, then*

$$ {}_k\mathcal{I}_{a+}^{\alpha;\psi}\left({}_k\mathfrak{D}_{a+}^{\alpha;\psi}h(t)\right) = h(t) - \sum_{i=1}^n \frac{\Psi^{\frac{\alpha}{k}-i}(t,a)}{\Gamma_k(\alpha-ik+k)}\left[\left(\frac{1}{\psi'(t)}\cdot\frac{d}{dt}\right)^{n-i}k^{n-i}\left({}_k\mathcal{I}_{a^+}^{nk-\alpha;\psi}h(a)\right)\right]. $$

Next, we require the following auxiliary result.

Lemma 2.6. *Let $\nu \in (m-1, m)$, $\alpha \in (n-1, n)$, n, $m \in \mathbb{N}$, $n \leq m$, and $k > 0$. If $h \in C^n([a,b], \mathbb{R})$, then*

$$ {}_k\mathfrak{D}_{a^+}^{\alpha;\psi}\Big[{}_k\mathcal{I}_{a^+}^{\nu;\psi}h(t)\Big] = k^{n-1}{}_k\mathcal{I}_{a^+}^{\nu-\alpha;\psi}h(t). \tag{2.1}$$

Proof. By applying Definition 2.6 and (iii) of Lemma 2.4, we have

$$ {}_k\mathfrak{D}_{a^+}^{\alpha;\psi}\Big[{}_k\mathcal{I}_{a^+}^{\nu;\psi}h(t)\Big] = \left(\frac{1}{\psi'(t)}\cdot\frac{d}{dt}\right)^n k^n\Big[{}_k\mathcal{I}_{a^+}^{kn-\alpha+\nu;\psi}h(t)\Big]. \tag{2.2}$$

By using Definition 2.5, for $n = 1$, we obtain

$$\begin{aligned}\left(\frac{1}{\psi'(t)}\cdot\frac{d}{dt}\right)k\,{}_k\mathcal{I}_{a^+}^{kn-\alpha+\nu;\psi}h(t) &= \frac{k}{\psi'(t)}\cdot\frac{d}{dt}\left(\frac{1}{k\Gamma_k(kn-\alpha+\nu)}\int_a^t\Psi^{\frac{kn-\alpha+\nu}{k}-1}(t,s)\psi'(s)h(s)ds\right)\\ &= \frac{1}{k\Gamma_k(kn-\alpha+\nu-k)}\int_a^t\Psi^{\frac{kn-\alpha+\nu-k}{k}-1}(t,s)\psi'(s)h(s)ds\\ &= {}_k\mathcal{I}_{a^+}^{kn-\alpha+\nu-k;\psi}h(t).\end{aligned}$$

In the same way, for $n = 2$, we have

$$\begin{aligned}\left(\frac{1}{\psi'(t)}\cdot\frac{d}{dt}\right)^2 k^2{}_k\mathcal{I}_{a^+}^{kn-\alpha+\nu;\psi}h(t) &= \frac{k^2}{\psi'(t)}\cdot\frac{d}{dt}\left(\frac{1}{k\Gamma_k(kn-\alpha+\nu-k)}\int_a^t\Psi^{\frac{kn-\alpha+\nu-k}{k}-1}(t,s)\psi'(s)h(s)ds\right)\\ &= \frac{k}{k\Gamma_k(kn-\alpha+\nu-2k)}\int_a^t\Psi^{\frac{kn-\alpha+\nu-2k}{k}-1}(t,s)\psi'(s)h(s)ds\\ &= k{}_k\mathcal{I}_{a^+}^{kn-\alpha+\nu-2k;\psi}h(t).\end{aligned}$$

Repeating the above method, we obtain

$$\begin{aligned}\left(\frac{1}{\psi'(t)}\cdot\frac{d}{dt}\right)^n k^n{}_k\mathcal{I}_{a^+}^{kn-\alpha+\nu;\psi}h(t) &= \frac{k^n}{\psi'(t)}\cdot\frac{d}{dt}\left(\frac{1}{k\Gamma_k(kn-\alpha+\nu-(n-1)k)}\int_a^t\Psi^{\frac{kn-\alpha+\nu-(n-1)k}{k}-1}(t,s)\psi'(s)h(s)ds\right)\\ &= \frac{k^{n-1}}{k\Gamma_k(\nu-\alpha)}\int_a^t\Psi^{\frac{\nu-\alpha}{k}-1}(t,s)\psi'(s)h(s)ds\\ &= k^{n-1}{}_k\mathcal{I}_{a^+}^{\nu-\alpha;\psi}h(t).\end{aligned}$$

The proof is done.

Lemma 2.7. *Let $\Omega \neq 0$, $k, \sigma_j > 0$, $\mathcal{A}$, $\mathcal{B}$, λ_i, $\mu_j \in \mathbb{R}$ and ξ_i, $\eta_j \in (a, \infty)$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Then the problem*

$$\begin{cases} {}_k\mathfrak{D}_{a+}^{\alpha;\psi}u(t) = h(t), \quad \alpha\in(1,2], \quad t\in(a,\infty), \quad a>0,\\ {}_k\mathcal{I}_{a+}^{2k-\alpha;\psi}u(a^+) = \mathcal{A} + \sum_{i=1}^m\lambda_i u(\xi_i), \quad \lim_{t\to\infty}{}_k\mathfrak{D}_{a+}^{\alpha-k;\psi}u(t) = \mathcal{B} + \sum_{j=1}^n\mu_{j\,k}\mathcal{I}_{a+}^{\sigma_j;\psi}u(\eta_j), \end{cases} \tag{2.3}$$

has a unique solution provided by

$$u(t) = \frac{\Psi^{\frac{\alpha}{k}-1}(t,a)}{\Omega\Gamma_k(\alpha)}\Bigg[\omega_4\left(\mathcal{A} + \sum_{i=1}^m\frac{\lambda_i}{k\Gamma_k(\alpha)}\int_a^{\xi_i}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,s)h(s)ds\right)$$

$$+(1-\omega_1)\left(\mathcal{B}+\sum_{j=1}^{n}\frac{\mu_j}{k\Gamma_k(\alpha+\sigma_j)}\int_a^{\eta_j}\psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,s)h(s)ds-\int_a^{\infty}\psi'(s)h(s)ds\right)\Bigg]$$
$$+\frac{\Psi^{\frac{\alpha}{k}-2}(t,a)}{\Omega\Gamma_k(\alpha-k)}\Bigg[(1-\omega_3)\left(\mathcal{A}+\sum_{i=1}^{m}\frac{\lambda_i}{k\Gamma_k(\alpha)}\int_a^{\xi_i}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,s)h(s)ds\right)$$
$$+\omega_2\left(\mathcal{B}+\sum_{j=1}^{n}\frac{\mu_j}{k\Gamma_k(\alpha+\sigma_j)}\int_a^{\eta_j}\psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,s)h(s)ds-\int_a^{\infty}\psi'(s)h(s)ds\right)\Bigg]$$
$$+\frac{1}{k\Gamma_k(\alpha)}\int_a^{t}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(t,s)h(s)ds, \tag{2.4}$$

where

$$\omega_1:=\sum_{i=1}^{m}\frac{\lambda_i\Psi^{\frac{\alpha}{k}-2}(\xi_i,a)}{\Gamma_k(\alpha-k)},\qquad \omega_2:=\sum_{i=1}^{m}\frac{\lambda_i\Psi^{\frac{\alpha}{k}-1}(\xi_i,a)}{\Gamma_k(\alpha)}, \tag{2.5}$$
$$\omega_3:=\sum_{j=1}^{n}\frac{\mu_j\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j)},\quad \omega_4:=\sum_{j=1}^{n}\frac{\mu_j\Psi^{\frac{\alpha+\sigma_j}{k}-2}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j-k)}, \tag{2.6}$$
$$\Omega:=(1-\omega_1)(1-\omega_3)-\omega_2\omega_4. \tag{2.7}$$

Proof. Using Lemma 2.5, the problem (2.3) can be rewritten as

$$u(t)=\frac{c_1\Psi^{\frac{\alpha}{k}-1}(t,a)}{\Gamma_k(\alpha)}+\frac{c_2\Psi^{\frac{\alpha}{k}-2}(t,a)}{\Gamma_k(\alpha-k)}+\frac{1}{\Gamma_k(\alpha)}\int_a^{t}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(t,a)h(s)ds, \tag{2.8}$$

where $c_1,\ c_2\in\mathbb{R}$. Taking the (k,ψ)-$\mathbb{RL}$-$\mathbb{FIO}$ of order $2k-\alpha$ and the (k,ψ)-$\mathbb{RL}$-$\mathbb{FDO}$ of order $\alpha-k$ into (2.8), respectively, we have

$${}_k\mathcal{I}_{a+}^{2k-\alpha;\psi}u(t) = \frac{c_1\Psi(t,a)}{\Gamma_k(\alpha)}+c_2+\frac{1}{k\Gamma_k(2k)}\int_a^{t}\psi'(s)\Psi(t,a)h(s)ds, \tag{2.9}$$
$${}_k\mathfrak{D}_{a+}^{\alpha-k;\psi}u(t) = c_1+\frac{c_2}{\Psi(t,a)}+\int_a^{t}\psi'(s)h(s)ds. \tag{2.10}$$

From the nonlocal conditions of (2.3), it follows that

$$c_2\left(1-\sum_{i=1}^{m}\frac{\lambda_i\Psi^{\frac{\alpha}{k}-2}(\xi_i,a)}{\Gamma_k(\alpha-k)}\right)-c_1\sum_{i=1}^{m}\frac{\lambda_i\Psi^{\frac{\alpha}{k}-1}(\xi_i,a)}{\Gamma_k(\alpha)}$$
$$=\mathcal{A}+\sum_{i=1}^{m}\frac{\lambda_i}{k\Gamma_k(\alpha)}\int_a^{\xi_i}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,a)h(s)ds, \tag{2.11}$$
$$c_1\left(1-\sum_{j=1}^{n}\frac{\mu_j\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j)}\right)-c_2\sum_{j=1}^{n}\frac{\mu_j\Psi^{\frac{\alpha+\sigma_j}{k}-2}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j-k)}$$
$$=\mathcal{B}+\sum_{j=1}^{n}\frac{\mu_j}{k\Gamma_k(\alpha+\sigma_j)}\int_a^{\eta_j}\psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,a)h(s)ds-\int_a^{\infty}\psi'(s)h(s)ds. \tag{2.12}$$

Solving the systems (2.11) and (2.12), we obtain

$$
\begin{aligned}
c_1 &= \frac{1}{\Omega}\Bigg[\omega_4\Bigg(\mathcal{A}+\sum_{i=1}^{m}\frac{\lambda_i}{k\Gamma_k(\alpha)}\int_a^{\xi_i}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,a)h(s)ds\Bigg)\\
&\quad+(1-\omega_1)\Bigg(\mathcal{B}+\sum_{j=1}^{n}\frac{\mu_j}{k\Gamma_k(\alpha+\sigma_j)}\int_a^{\eta_j}\psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,a)h(s)ds-\int_a^{\infty}\psi'(s)h(s)ds\Bigg)\Bigg],\\
c_2 &= \frac{1}{\Omega}\Bigg[(1-\omega_3)\Bigg(\mathcal{A}+\sum_{i=1}^{m}\frac{\lambda_i}{k\Gamma_k(\alpha)}\int_a^{\xi_i}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,a)h(s)ds\Bigg)\\
&\quad+\omega_2\Bigg(\mathcal{B}+\sum_{j=1}^{n}\frac{\mu_j}{k\Gamma_k(\alpha+\sigma_j)}\int_a^{\eta_j}\psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,a)h(s)ds-\int_a^{\infty}\psi'(s)h(s)ds\Bigg)\Bigg].
\end{aligned}
$$

Substituting the constants c_1 and c_2 into (2.8), we achieve (2.4). The converse follows by direct calculation. The proof is done.

3. Existence result

This section establishes the main theorem for the existence result utilizing the concept of the Meir-Keeler contraction fixed point theorem with condensing operators via a measure of . Let $\mathbb{B}_{\mathcal{R}} = \{u \in C_{\psi}^{\frac{\alpha}{k}} : \|u\|_C \leq \mathcal{R}\}$, where $\mathcal{R}$ is set by (3.3). From Lemma 2.7, we set an operator $\mathfrak{F} : \mathbb{B}_{\mathcal{R}} \to \mathbb{B}_{\mathcal{R}}$ by

$$
\begin{aligned}
&(\mathfrak{F}u)(t)\\
=\;& \frac{\Psi^{\frac{\alpha}{k}-1}(t,a)}{\Omega\Gamma_k(\alpha)}\Bigg[\omega_4\Bigg(\mathcal{A}+\sum_{i=1}^{m}\frac{\lambda_i}{k\Gamma_k(\alpha)}\int_a^{\xi_i}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,s)f(s,u(s))ds\Bigg)\\
&+(1-\omega_1)\Bigg(\mathcal{B}+\sum_{j=1}^{n}\frac{\mu_j}{k\Gamma_k(\alpha+\sigma_j)}\int_a^{\eta_j}\psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,s)f(s,u(s))ds-\int_a^{\infty}\psi'(s)f(s,u(s))ds\Bigg)\Bigg]\\
&+\frac{\Psi^{\frac{\alpha}{k}-2}(t,a)}{\Omega\Gamma_k(\alpha-k)}\Bigg[(1-\omega_3)\Bigg(\mathcal{A}+\sum_{i=1}^{m}\frac{\lambda_i}{k\Gamma_k(\alpha)}\int_a^{\xi_i}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,s)f(s,u(s))ds\Bigg)\\
&+\omega_2\Bigg(\mathcal{B}+\sum_{j=1}^{n}\frac{\mu_j}{k\Gamma_k(\alpha+\sigma_j)}\int_a^{\eta_j}\psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,s)f(s,u(s))ds-\int_a^{\infty}\psi'(s)f(s,u(s))ds\Bigg)\Bigg]\\
&+\frac{1}{k\Gamma_k(\alpha)}\int_a^{t}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(t,s)f(s,u(s))ds. \qquad (3.1)
\end{aligned}
$$

It should be noted that $\mathfrak{F}$ has a fixed-point if the problem (1.1) has a solution.

Theorem 3.1. *Let $f \in C(\mathcal{J}\times\mathcal{E},\mathcal{E})$. Suppose that the following assumptions*

($\mathbb{H}_1$) *There exists a constant $\mathcal{L} > 0$ so that*

$$\|f(t,u)-f(t,v)\| \leq \mathcal{L}\Psi^{2-\frac{\alpha}{k}}(t,a)\|u-v\|,$$

for each $u, v \in \mathcal{E}$, $t \in (a,T] \subset (a,\infty)$, $T > a > 0$, with the following conditions

$$\mathcal{P}^* := \int_a^{\infty}\psi'(s)ds < +\infty, \qquad \mathcal{Q}^* := \int_a^{\infty}\psi'(s)[1+\Psi^{\frac{\alpha}{k}}(s,a)]ds < \frac{k\Gamma_k(\alpha)|\Omega|}{\mathcal{L}}.$$

$(\mathbb{H}_2)$ *There is a function $g \in C([a,\infty),\mathbb{R}^+)$, for each nonempty, bounded set $\Lambda \subset C_\psi^{\frac{\alpha}{k}}((a,\infty),\mathcal{E})$ so that*

$$\mathfrak{G}(f,\Lambda(t)) \le g(t)\Psi^{2-\frac{\alpha}{k}}(t,a)\mathfrak{G}(\Lambda(t)), \quad t \in (a,\infty), \tag{3.2}$$

with the following condition

$$\max\left\{\frac{\mathcal{Q}^*\mathcal{L}}{|\Omega|}, 2\mathcal{G}^*\right\} < k\Gamma_k(\alpha) \quad where \quad \mathcal{G}^* := \int_a^\infty \psi'(s)[1+\Psi^{\frac{\alpha}{k}}(s,a)]g(s)ds.$$

$(\mathbb{H}_3)$ *There is a strictly real constant $\mathcal{R} > 0$ so that*

$$\mathcal{R} \ge \frac{\zeta_1|\mathcal{A}| + \zeta_2|\mathcal{B}| + \mathcal{F}^*(\frac{\zeta_3\mathcal{P}^*}{k} + \gamma_1)}{1-\mathcal{Y}}, \qquad \mathcal{Y} := \mathcal{L}\left([\gamma_1+\gamma_2] + \frac{\zeta_3\mathcal{Q}^*}{k}\right) < 1, \tag{3.3}$$

where

$$\begin{aligned}
\zeta_1 &:= \frac{|\omega_4| + |\alpha-k||1+\omega_3|}{|\Omega|\Gamma_k(\alpha)}, \quad \zeta_2 := \frac{|1+\omega_1| + |\alpha-k||\omega_2|}{|\Omega|\Gamma_k(\alpha)}, \quad \zeta_3 := \frac{|1+\omega_1| + |\alpha-k||\omega_2|+1}{|\Omega|\Gamma_k(\alpha)},\\
\gamma_1 &:= \sum_{i=1}^m \frac{\zeta_1|\lambda_i|\Psi^{\frac{\alpha}{k}}(\xi_i,a)}{\Gamma_k(\alpha+k)} + \sum_{j=1}^n \frac{\zeta_2|\mu_j|\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j+k)},\\
\gamma_2 &:= \sum_{i=1}^m \frac{\zeta_1|\lambda_i|\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha}{k}}(\xi_i,a)}{\Gamma_k(2\alpha+k)} + \sum_{j=1}^n \frac{\zeta_2|\mu_j|\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(2\alpha+\sigma_j+k)}.
\end{aligned}$$

Then the (k,ψ)-$\mathbb{RL}$-$\mathbb{NBVP}$ problem (1.1) *has at least one solution.*

Proof. We transform the (k,ψ)-$\mathbb{RL}$-$\mathbb{NBVP}$ problem (1.1) into a fixed-point problem, that is $u = \mathfrak{F}u$, where $\mathfrak{F}$ is given by (3.1). It is easily to see that the fixed-points of $\mathfrak{F}$ are the solution to the (k,ψ)-$\mathbb{RL}$-$\mathbb{NBVP}$ problem (1.1). Let $\sup_{t\in[a,\infty)} \|f(t,0)\| := \mathcal{F}^* < \infty$ and define $u \in C_\psi^{\frac{\alpha}{k}}$. According to assumption $(\mathbb{H}_1)$. we get

$$\|f(t,u)\| \le \|f(t,0)\| + \|f(t,u)-f(t,0)\| \le \mathcal{F}^* + \mathcal{L}\Psi^{2-\frac{\alpha}{k}}(t,a)\|u(t)\| \le \mathcal{F}^* + \mathcal{L}\Big[1+\Psi^{\frac{\alpha}{k}}(t,a)\Big]\|u\|_\psi^{\frac{\alpha}{k}}. \tag{3.4}$$

The process of the procedure is finished in four steps.
Step 1. We show that $\mathfrak{F}\mathbb{B}_\mathcal{R} \subset \mathbb{B}_\mathcal{R}$.
By applying $(\mathbb{H}_1)$ with (3.4), for each $u \in \mathbb{B}_\mathcal{R}$ and $t \in (a,\infty)$, we obtain

$$\begin{aligned}
&\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}u)(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right\|\\
\le\ & \frac{|\omega_4| + |\alpha-k||1+\omega_3|}{|\Omega|\Gamma_k(\alpha)}\left(|\mathcal{A}| + \sum_{i=1}^m \frac{|\lambda_i|}{k\Gamma_k(\alpha)}\int_a^{\xi_i}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,s)\|f(s,u(s))\|ds\right)\\
&+\frac{|1+\omega_1| + |\alpha-k||\omega_2|}{|\Omega|\Gamma_k(\alpha)}\left(|\mathcal{B}| + \sum_{j=1}^n \frac{|\mu_j|}{k\Gamma_k(\alpha+\sigma_j)}\int_a^{\eta_j}\psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,s)\|f(s,u(s))\|ds\right)\\
&+\frac{|1+\omega_1| + |\alpha-k||\omega_2|+1}{|\Omega|k\Gamma_k(\alpha)}\int_a^\infty \psi'(s)\Psi^{\frac{\alpha}{k}-1}(t,s)\|f(s,u(s))\|ds
\end{aligned}$$

$$\leq \zeta_1|\mathcal{A}| + \zeta_2|\mathcal{B}| + \frac{\zeta_3\mathcal{P}^*\mathcal{F}^*}{k} + \frac{\zeta_3\mathcal{Q}^*\mathcal{L}}{k}\|u\|_\psi^{\frac{\alpha}{k}} + \mathcal{F}^*\left(\zeta_1\sum_{i=1}^m \frac{|\lambda_i|\Psi^{\frac{\alpha}{k}}(\xi_i,a)}{\Gamma_k(\alpha+k)} + \zeta_2\sum_{j=1}^n \frac{|\mu_j|\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j+k)}\right)$$
$$+\mathcal{L}\left(\zeta_1\sum_{i=1}^m \frac{|\lambda_i|\Psi^{\frac{\alpha}{k}}(\xi_i,a)}{\Gamma_k(\alpha+k)} + \zeta_2\sum_{j=1}^n \frac{|\mu_j|\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j+k)}\right)\|u\|_\psi^{\frac{\alpha}{k}}$$
$$+\mathcal{L}\left(\zeta_1\sum_{i=1}^m \frac{|\lambda_i|\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha}{k}}(\xi_i,a)}{\Gamma_k(2\alpha+k)} + \zeta_2\sum_{j=1}^n \frac{|\mu_j|\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(2\alpha+\sigma_j+k)}\right)\|u\|_\psi^{\frac{\alpha}{k}}$$
$$= \zeta_1|\mathcal{A}| + \zeta_2|\mathcal{B}| + \frac{\zeta_3\mathcal{P}^*\mathcal{F}^*}{k} + \mathcal{F}^*\gamma_1 + \left(\mathcal{L}[\gamma_1+\gamma_2] + \frac{\zeta_3\mathcal{Q}^*\mathcal{L}}{k}\right)\|u\|_\psi^{\frac{\alpha}{k}} \leq \mathcal{R}.$$

Then, we obtain $\|\mathfrak{F}u\|_\psi^{\frac{\alpha}{k}} \leq \mathcal{R}$. Hence, $\mathfrak{F}$ is bounded on $\mathbb{B}_\mathcal{R}$.

Step 2. We show that $\mathfrak{F}$ is continuous.

Now, we rewrite $\mathfrak{F}$ as

$$(\mathfrak{F}u)(t)$$
$$= \frac{\Psi^{\frac{\alpha}{k}-1}(t,a)}{\Omega\Gamma_k(\alpha)}\Bigg[\omega_4\left(\mathcal{A} + \sum_{i=1}^m \frac{\lambda_i}{k\Gamma_k(\alpha)}\int_a^{\xi_i}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,s)f(s,u(s))ds\right)$$
$$+(1-\omega_1)\left(\mathcal{B} + \sum_{j=1}^n \frac{\mu_j}{k\Gamma_k(\alpha+\sigma_j)}\int_a^{\eta_j}\psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,s)f(s,u(s))ds - \int_t^\infty \psi'(s)f(s,u(s))ds\right)\Bigg]$$
$$+\frac{\Psi^{\frac{\alpha}{k}-2}(t,a)}{\Omega\Gamma_k(\alpha-k)}\Bigg[(1-\omega_3)\left(\mathcal{A} + \sum_{i=1}^m \frac{\lambda_i}{k\Gamma_k(\alpha)}\int_a^{\xi_i}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,s)f(s,u(s))ds\right)$$
$$+\omega_2\left(\mathcal{B} + \sum_{j=1}^n \frac{\mu_j}{k\Gamma_k(\alpha+\sigma_j)}\int_a^{\eta_j}\psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,s)f(s,u(s))ds - \int_t^\infty \psi'(s)f(s,u(s))ds\right)\Bigg]$$
$$+\frac{1}{k\Gamma_k(\alpha)}\int_a^t \psi'(s)\left[\Psi^{\frac{\alpha}{k}-1}(t,s) - \frac{(1-\omega_1)}{\Omega}\Psi^{\frac{\alpha}{k}-1}(t,a) - \frac{(\alpha-k)\omega_2}{\Omega}\Psi^{\frac{\alpha}{k}-1}(t,a)\right]f(s,u(s))ds.$$

Assume that $\{u_n\}_{n=1}^\infty$ is a sequence so that $u_n \to u \in C_\psi^{\frac{\alpha}{k}}$. Assume that $T > a > 0$ and $\epsilon > 0$, by noticing that the function u_n for $n \in \mathbb{N}$ and u are bounded, which verifies that there is a constant $M > 0$ so that $\|u_n\|_\psi^{\frac{\alpha}{k}} \leq M$, for $n \in \mathbb{N}$, and $\|u\|_\psi^{\frac{\alpha}{k}} \leq M$. From (3.4), there is $\tau^* > a > 0$ with $\tau^* > T$, so that

$$\int_{\tau^*}^\infty \psi'(s)ds \leq \frac{\epsilon k}{5\mathcal{F}^*\zeta_2}, \qquad \int_{\tau^*}^\infty \psi'(s)\left(1+\Psi^{\frac{\alpha}{k}}(t,a)\right)ds \leq \frac{\epsilon k}{10\mathcal{L}M\zeta_2}.$$

From $(\mathbb{H}_1)$, there is $\mathcal{N} \in \mathbb{N}$ so that, for all $n \geq \mathcal{N}$ and $t \in (a,\tau^*]$, we obtain

$$\|f(s,u(s)) - f(s,u_n(s))\| \leq \frac{\epsilon k}{5}\max\left\{\frac{\Gamma_k(\alpha)}{m\zeta_1|\lambda_i|\Psi^{\frac{\alpha}{k}}(\xi_i,a)}, \frac{\Gamma_k(\alpha+\sigma_j)}{n\zeta_2|\mu_j|\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}, \frac{1}{\zeta_3\Psi(\tau^*,a)}\right\}.$$

Hence, for each $t \in (a,\tau^*)$ and $n > \mathcal{N}$, it follows that

$$\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)((\mathfrak{F}u_n)(t) - (\mathfrak{F}u)(t))}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right\|$$

$$
\begin{aligned}
&\leq \frac{|\omega_4|+|\alpha-k||1+\omega_3|}{|\Omega|\Gamma_k(\alpha)}\Bigg(\sum_{i=1}^{m}\frac{|\lambda_i|}{k\Gamma_k(\alpha)}\int_a^{\xi_i}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,s)\|f(s,u_n(s))-f(s,u(s))\|ds\Bigg)\\
&\quad+\frac{|1+\omega_1|+|\alpha-k||\omega_2|}{|\Omega|\Gamma_k(\alpha)}\Bigg(\sum_{j=1}^{n}\frac{|\mu_j|}{k\Gamma_k(\alpha+\sigma_j)}\int_a^{\eta_j}\psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,s)\|f(s,u_n(s))-f(s,u(s))\|ds\Bigg)\\
&\quad+\frac{|1+\omega_1|+|\alpha-k||\omega_2|}{|\Omega|k\Gamma_k(\alpha)}\int_t^{\infty}\psi'(s)\|f(s,u_n(s))-f(s,u(s))\|ds\\
&\quad+\frac{|1+\omega_1|+|\alpha-k||\omega_2|+1}{|\Omega|k\Gamma_k(\alpha)}\int_a^{t}\psi'(s)\|f(s,u_n(s))-f(s,u(s))\|ds\\
&= \zeta_1\Bigg(\sum_{i=1}^{m}\frac{|\lambda_i|}{k\Gamma_k(\alpha)}\int_a^{\xi_i}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,s)\|f(s,u_n(s))-f(s,u(s))\|ds\Bigg)\\
&\quad+\zeta_2\Bigg(\sum_{j=1}^{n}\frac{|\mu_j|}{k\Gamma_k(\alpha+\sigma_j)}\int_a^{\eta_j}\psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,s)\|f(s,u_n(s))-f(s,u(s))\|ds\Bigg)\\
&\quad+\frac{\zeta_2}{k}\int_t^{\tau^*}\psi'(s)\|f(s,u_n(s))-f(s,u(s))\|ds+\frac{\zeta_2}{k}\int_{\tau^*}^{\infty}\psi'(s)\|f(s,u_n(s))-f(s,u(s))\|ds\\
&\quad+\frac{\zeta_3}{k}\int_a^{t}\psi'(s)\|f(s,u_n(s))-f(s,u(s))\|ds\\
&\leq \zeta_1\Bigg(\sum_{i=1}^{m}\frac{|\lambda_i|}{k\Gamma_k(\alpha)}\int_a^{\xi_i}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,s)\|f(s,u_n(s))-f(s,u(s))\|ds\Bigg)\\
&\quad+\zeta_2\Bigg(\sum_{j=1}^{n}\frac{|\mu_j|}{k\Gamma_k(\alpha+\sigma_j)}\int_a^{\eta_j}\psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,s)\|f(s,u_n(s))-f(s,u(s))\|ds\Bigg)\\
&\quad+\frac{\zeta_3}{k}\int_a^{\tau^*}\psi'(s)\|f(s,u_n(s))-f(s,u(s))\|ds+\frac{\zeta_2}{k}\int_{\tau^*}^{\infty}\psi'(s)\|f(s,u_n(s))-f(s,u(s))\|ds\\
&\quad+\frac{\zeta_3\Psi(L,a)\epsilon k}{5k\zeta_3\Psi(L,a)}+\frac{\zeta_2\mathcal{F}^*}{k}\int_{\tau^*}^{\infty}\psi'(s)ds+\frac{2M\zeta_2\mathcal{L}}{k}\int_{\tau^*}^{\infty}\psi'(s)\Big(1+\Psi^{\frac{\alpha}{k}}(s,a)\Big)ds\\
&\leq \frac{\epsilon}{5}+\frac{\epsilon}{5}+\frac{\epsilon}{5}+\frac{\epsilon}{5}+\frac{\epsilon}{5}=\epsilon.
\end{aligned}
$$

Then, we can conclude that $\|\mathfrak{F}u_n-\mathfrak{F}u\|_{\psi}^{\frac{\alpha}{k}}\to 0$, as $n\to\infty$, namely, $\mathfrak{F}$ is continuous.

Step 3. We show the following results:

($\mathbb{P}_1$) $\mathfrak{F}\mathbb{B}_{\mathcal{R}}=\{(\mathfrak{F}u)_{\mathcal{R}}: u\in\mathbb{B}_{\mathcal{R}}\}$ is equicontinuous on any compact $[a,T]$ of (a,∞), $T>a>0$.

($\mathbb{P}_2$) For given $\epsilon>0$, there exists a positive constant $\mathcal{N}_1$ such that

$$
\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t_2,a)(\mathfrak{F}u)(t_2)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)}-\frac{\Psi^{2-\frac{\alpha}{k}}(t_1,a)(\mathfrak{F}u)(t_1)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right\|<\epsilon,
$$

for any $t_1,t_2\geq\mathcal{N}_1$ and $u\in\mathbb{B}_{\mathcal{R}}$.

Let us show the equicontinuity of $\mathfrak{F}\mathbb{B}_{\mathcal{R}}$ on any compact $[a,T]$. Indeed, let $u\in\mathbb{B}_{\mathcal{R}}$, $t_1,t_2\in[a,T]$, where $t_1>t_2$. Then

$$
\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t_2,a)(\mathfrak{F}u)(t_2)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)}-\frac{\Psi^{2-\frac{\alpha}{k}}(t_1,a)(\mathfrak{F}u)(t_1)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right\|
$$

$$
\begin{aligned}
\leq\ & \Bigg[|\omega_4|\Bigg(|\mathcal{A}| + \sum_{i=1}^{m} \frac{|\lambda_i|}{k\Gamma_k(\alpha)} \int_a^{\xi_i} \psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i, s)\|f(s,u(s))\|ds\Bigg) \\
& +|1+\omega_1|\Bigg(|\mathcal{B}| + \sum_{j=1}^{n} \frac{|\mu_j|}{k\Gamma_k(\alpha+\sigma_j)} \int_a^{\eta_j} \psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j, s)\|f(s,u(s))\|ds \\
& + \int_a^{\infty} \psi'(s)\|f(s,u(s))\|ds\Bigg)\Bigg]\frac{1}{|\Omega|\Gamma_k(\alpha)}\left|\frac{\Psi(t_2,a)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)} - \frac{\Psi(t_1,a)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right| \\
& +\Bigg[|1+\omega_3|\Bigg(|\mathcal{A}| + \sum_{i=1}^{m} \frac{|\lambda_i|}{k\Gamma_k(\alpha)} \int_a^{\xi_i} \psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i, s)\|f(s,u(s))\|ds\Bigg) \\
& +|\omega_2|\Bigg(|\mathcal{B}| + \sum_{j=1}^{n} \frac{|\mu_j|}{k\Gamma_k(\alpha+\sigma_j)} \int_a^{\eta_j} \psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j, s)\|f(s,u(s))\|ds \\
& + \int_a^{\infty} \psi'(s)\|f(s,u(s))\|ds\Bigg)\Bigg]\frac{1}{|\Omega|\Gamma_k(\alpha-k)}\left|\frac{1}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)} - \frac{1}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right| \\
& +\frac{1}{k\Gamma_k(\alpha)}\left\|\int_a^{t_2} \psi'(s)\Psi^{\frac{\alpha}{k}-1}(t_2, s)f(s,u(s))ds - \int_a^{t_1} \psi'(s)\Psi^{\frac{\alpha}{k}-1}(t_1, s)f(s,u(s))ds\right\| \\
\leq\ & \Bigg[|\omega_4|\Bigg(|\mathcal{A}| + \sum_{i=1}^{m} \frac{|\lambda_i|}{k\Gamma_k(\alpha)} \int_a^{\xi_i} \psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i, s)\|f(s,u(s))\|ds\Bigg) \\
& +|1+\omega_1|\Bigg(|\mathcal{B}| + \sum_{j=1}^{n} \frac{|\mu_j|}{k\Gamma_k(\alpha+\sigma_j)} \int_a^{\eta_j} \psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j, s)\|f(s,u(s))\|ds \\
& + \int_a^{\infty} \psi'(s)\|f(s,u(s))\|ds\Bigg)\Bigg]\frac{1}{|\Omega|\Gamma_k(\alpha)}\left|\frac{\Psi(t_2,a)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)} - \frac{\Psi(t_1,a)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right| \\
& +\Bigg[|1+\omega_3|\Bigg(|\mathcal{A}| + \sum_{i=1}^{m} \frac{|\lambda_i|}{k\Gamma_k(\alpha)} \int_a^{\xi_i} \psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i, s)\|f(s,u(s))\|ds\Bigg) \\
& +|\omega_2|\Bigg(|\mathcal{B}| + \sum_{j=1}^{n} \frac{|\mu_j|}{k\Gamma_k(\alpha+\sigma_j)} \int_a^{\eta_j} \psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j, s)\|f(s,u(s))\|ds \\
& + \int_a^{\infty} \psi'(s)\|f(s,u(s))\|ds\Bigg)\Bigg]\frac{1}{|\Omega|\Gamma_k(\alpha-k)}\left|\frac{1}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)} - \frac{1}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right| \\
& +\frac{1}{k\Gamma_k(\alpha)}\int_a^{t_1} \psi'(s)[\Psi^{\frac{\alpha}{k}-1}(t_2, s) - \Psi^{\frac{\alpha}{k}-1}(t_1, s)]\|f(s,u(s))\|ds \\
& +\frac{1}{k\Gamma_k(\alpha)}\int_{t_1}^{t_2} \psi'(s)\Psi^{\frac{\alpha}{k}-1}(t_2, s)\|f(s,u(s))\|ds \\
\leq\ & \Bigg[|\omega_4|\Bigg(|\mathcal{A}| + \sum_{i=1}^{m} \frac{|\lambda_i|}{k\Gamma_k(\alpha)}\Bigg[\frac{\mathcal{F}^*\Psi^{\frac{\alpha}{k}}(\xi_i,a)}{\Gamma_k(\alpha+k)} + \mathcal{L}\Bigg(\frac{\Psi^{\frac{\alpha}{k}}(\xi_i,a)}{\Gamma_k(\alpha+k)} + \frac{\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha}{k}}(\xi_i,a)}{\Gamma_k(2\alpha+k)}\Bigg)\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\Bigg) \\
& +|1+\omega_1|\Bigg(|\mathcal{B}| + \sum_{j=1}^{n} \frac{|\mu_j|}{k\Gamma_k(\alpha+\sigma_j)}\Bigg[\frac{\mathcal{F}^*\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j+k)} + \mathcal{L}\Bigg(\frac{\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j+k)}
\end{aligned}
$$

$$+\frac{\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(2\alpha+\sigma_j+k)}\Bigg)\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\Bigg)+\frac{\mathcal{P}^*\mathcal{F}^*}{k}+\frac{Q^*\mathcal{L}}{k}\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\frac{1}{|\Omega|\Gamma_k(\alpha)}\left|\frac{\Psi(t_2,a)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)}-\frac{\Psi(t_1,a)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right|$$
$$+\Bigg[|1+\omega_3|\Bigg(|\mathcal{A}|+\sum_{i=1}^{m}\frac{|\lambda_i|}{k\Gamma_k(\alpha)}\Bigg[\frac{\mathcal{F}^*\Psi^{\frac{\alpha}{k}}(\xi_i,a)}{\Gamma_k(\alpha+k)}+\mathcal{L}\Bigg(\frac{\Psi^{\frac{\alpha}{k}}(\xi_i,a)}{\Gamma_k(\alpha+k)}+\frac{\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha}{k}}(\xi_i,a)}{\Gamma_k(2\alpha+k)}\Bigg)\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\Bigg)$$
$$+|\omega_2|\Bigg(|\mathcal{B}|+\sum_{j=1}^{n}\frac{|\mu_j|}{k\Gamma_k(\alpha+\sigma_j)}\Bigg[\frac{\mathcal{F}^*\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j+k)}+\mathcal{L}\Bigg(\frac{\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j+k)}$$
$$+\frac{\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(2\alpha+\sigma_j+k)}\Bigg)\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\Bigg)+\frac{\mathcal{P}^*\mathcal{F}^*}{k}+\frac{Q^*\mathcal{L}}{k}\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\frac{1}{|\Omega|\Gamma_k(\alpha-k)}\left|\frac{1}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)}-\frac{1}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right|$$
$$+\frac{(\mathcal{F}^*+\mathcal{L}\mathcal{R})}{\Gamma_k(\alpha+k)}(\Psi^{\frac{\alpha}{k}}(t_2,a)-\Psi^{\frac{\alpha}{k}}(t_2,t_1)-\Psi^{\frac{\alpha}{k}}(t_1,a))+\frac{\mathcal{L}\mathcal{R}}{\Gamma_k(2\alpha+k)}(\Psi^{\frac{2\alpha}{k}}(t_2,a)-\Psi^{\frac{2\alpha}{k}}(t_1,a))$$
$$+\frac{\mathcal{F}^*\Psi^{\frac{\alpha}{k}}(t_2,t_1)}{\Gamma_k(\alpha+k)}+\mathcal{L}\mathcal{R}\Bigg(\frac{\Psi^{\frac{\alpha}{k}}(t_2,t_1)}{\Gamma_k(\alpha+k)}+\frac{\Gamma_k(\alpha+k)\Psi^{\frac{\alpha}{k}}(t_2,t_1)}{\Gamma_k(2\alpha+k)}\Bigg).$$

As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero. Then $\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}\mathbb{B}_{\mathcal{R}})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}$ is equicontinuous on any compact $[a,T]$ of (a,∞).

Next, let us show the equiconvergence of $\mathfrak{F}\mathbb{B}_{\mathcal{R}}$. From $(\mathbb{H}_1)$, (3.4) and the boundedness of $\mathbb{B}_{\mathcal{R}}$, there exists a positive real constant $\mathbb{A}$ such that

$$\int_a^\infty \psi'(s)\|f(s,u(s))\|ds \le \mathbb{A}, \quad \text{for each} \quad u\in\mathbb{B}_{\mathcal{R}}. \tag{3.5}$$

Let $\epsilon>0$, we obtain

$$\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t_2,a)(\mathfrak{F}u)(t_2)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)}-\frac{\Psi^{2-\frac{\alpha}{k}}(t_1,a)(\mathfrak{F}u)(t_1)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right\|$$
$$\le \Bigg[|\omega_4|\Bigg(|\mathcal{A}|+\sum_{i=1}^{m}\frac{|\lambda_i|}{k\Gamma_k(\alpha)}\Bigg[\frac{\mathcal{F}^*\Psi^{\frac{\alpha}{k}}(\xi_i,a)}{\Gamma_k(\alpha+k)}+\mathcal{L}\Bigg(\frac{\Psi^{\frac{\alpha}{k}}(\xi_i,a)}{\Gamma_k(\alpha+k)}+\frac{\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha}{k}}(\xi_i,a)}{\Gamma_k(2\alpha+k)}\Bigg)\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\Bigg)$$
$$+|1+\omega_1|\Bigg(|\mathcal{B}|+\sum_{j=1}^{n}\frac{|\mu_j|}{k\Gamma_k(\alpha+\sigma_j)}\Bigg[\frac{\mathcal{F}^*\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j+k)}+\mathcal{L}\Bigg(\frac{\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j+k)}$$
$$+\frac{\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(2\alpha+\sigma_j+k)}\Bigg)\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\Bigg)+\frac{\mathcal{P}^*\mathcal{F}^*}{k}+\frac{Q^*\mathcal{L}}{k}\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\frac{1}{|\Omega|\Gamma_k(\alpha)}\left|\frac{\Psi(t_2,a)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)}-\frac{\Psi(t_1,a)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right|$$
$$+\Bigg[|1+\omega_3|\Bigg(|\mathcal{A}|+\sum_{i=1}^{m}\frac{|\lambda_i|}{k\Gamma_k(\alpha)}\Bigg[\frac{\mathcal{F}^*\Psi^{\frac{\alpha}{k}}(\xi_i,a)}{\Gamma_k(\alpha+k)}+\mathcal{L}\Bigg(\frac{\Psi^{\frac{\alpha}{k}}(\xi_i,a)}{\Gamma_k(\alpha+k)}+\frac{\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha}{k}}(\xi_i,a)}{\Gamma_k(2\alpha+k)}\Bigg)\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\Bigg)$$
$$+|\omega_2|\Bigg(|\mathcal{B}|+\sum_{j=1}^{n}\frac{|\mu_j|}{k\Gamma_k(\alpha+\sigma_j)}\Bigg[\frac{\mathcal{F}^*\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j+k)}+\mathcal{L}\Bigg(\frac{\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j+k)}$$
$$+\frac{\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(2\alpha+\sigma_j+k)}\Bigg)\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\Bigg)+\frac{\mathcal{P}^*\mathcal{F}^*}{k}+\frac{Q^*\mathcal{L}}{k}\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\frac{1}{|\Omega|\Gamma_k(\alpha-k)}\left|\frac{1}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)}-\frac{1}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right|$$

$$+\frac{1}{k\Gamma_k(\alpha)}\left\|\int_a^{t_2}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(t_2,s)f(s,u(s))ds-\int_a^{t_1}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(t_1,s)f(s,u(s))ds\right\|.$$

which yields that

$$\left\|\int_a^{t_2}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(t_2,s)f(s,u(s))ds-\int_a^{t_1}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(t_1,s)f(s,u(s))ds\right\|<\epsilon.$$

From (3.5), it follows that there exits a positive number $\mathcal{N}_0$ such that

$$\int_{\mathcal{N}_0}^{\infty}\psi'(s)\|f(s,u(s))\|ds\leq\frac{\epsilon}{3},\quad \text{for any}\quad u\in\mathbb{B}_{\mathcal{R}}. \tag{3.6}$$

On the other hand, since $\lim_{t\to\infty}\Psi^{\frac{\alpha}{k}-1}(t,\mathcal{N}_0)=0$, there exists $\mathcal{N}_1>\mathcal{N}_0$ such that, for any $t_1,t_2\geq\mathcal{N}_1$ and $s\in[0,\mathcal{N}_0]$, we get that

$$\left|\Psi^{\frac{\alpha}{k}-1}(t_1,s)-\Psi^{\frac{\alpha}{k}-1}(t_2,s)\right|<\frac{\epsilon}{3\mathbb{A}}. \tag{3.7}$$

By applying (3.6) and (3.7), for $t_1,t_2\geq\mathcal{N}_1$, one has

$$\begin{aligned}
&\left\|\int_a^{t_2}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(t_2,s)f(s,u(s))ds-\int_a^{t_1}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(t_1,s)f(s,u(s))ds\right\|\\
\leq\ &\int_a^{\mathcal{N}_1}\psi'(s)\left|\Psi^{\frac{\alpha}{k}-1}(t_1,s)-\Psi^{\frac{\alpha}{k}-1}(t_2,s)\right|\|f(s,u(s))\|ds\\
&+\int_{\mathcal{N}_1}^{t_2}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(t_2,s)\|f(s,u(s))\|ds+\int_{\mathcal{N}_1}^{t_1}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(t_1,s)\|f(s,u(s))\|ds\\
<\ &\frac{\epsilon}{3\mathbb{A}}\int_a^{\infty}\psi'(s)\|f(s,u(s))\|ds+2\int_{\mathcal{N}_1}^{\infty}\psi'(s)\|f(s,u(s))\|ds<\epsilon.
\end{aligned}$$

Hence, $\mathfrak{F}\mathbb{B}_{\mathcal{R}}$ is equiconvergent.

Step 4. We show that $\mathfrak{F}$ verifies the assumptions of Theorem 2.1.

Firstly, we will show that $\mathfrak{F}$ is given from $\mathbb{B}_{\mathcal{R}}$ to $\mathbb{B}_{\mathcal{R}}$. So, for each $u\in\mathbb{B}_{\mathcal{R}}$, by using $(\mathbb{H}_1)$, $(\mathbb{H}_3)$, (3.4) and by according to a few computation, we obtain

$$\begin{aligned}
&\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}u)(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right\|\\
\leq\ &\frac{|\omega_4|+|\alpha-k||1+\omega_3|}{|\Omega|\Gamma_k(\alpha)}\left(|\mathcal{A}|+\sum_{i=1}^{m}\frac{\lambda_i}{k\Gamma_k(\alpha)}\int_a^{\xi_i}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,s)\|f(s,u(s))\|ds\right)\\
&+\frac{|1+\omega_1|+|\alpha-k||\omega_2|}{|\Omega|\Gamma_k(\alpha)}\left(|\mathcal{B}|+\sum_{j=1}^{n}\frac{\mu_j}{k\Gamma_k(\alpha+\sigma_j)}\int_a^{\eta_j}\psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,s)\|f(s,u(s))\|ds\right)\\
&+\frac{|1+\omega_1|+|\alpha-k||\omega_2|+1}{|\Omega|k\Gamma_k(\alpha)}\int_a^{\infty}\psi'(s)\Psi^{\frac{\alpha}{k}-1}(t,s)\|f(s,u(s))\|ds\\
\leq\ &\zeta_1|\mathcal{A}|+\zeta_2|\mathcal{B}|+\frac{\zeta_3\mathcal{P}^*\mathcal{F}^*}{k}+\mathcal{F}^*\gamma_1+\left(\mathcal{L}[\gamma_1+\gamma_2]+\frac{\zeta_3\mathcal{Q}^*\mathcal{L}}{k}\right)\|u\|_{\psi}^{\frac{\alpha}{k}}\leq\mathcal{R}.
\end{aligned}$$

We set $\mathcal{D} = conv(\mathfrak{F}\mathbb{B}_{\mathcal{R}})$, Clearly, $\mathcal{D}$ is a closed, bounded and convex subset of $\mathbb{B}_{\mathcal{R}}$. Since, $\mathfrak{F}\mathcal{D} \subset \mathfrak{F}\mathbb{B}_{\mathcal{R}} \subset \mathcal{D}$, then $\mathfrak{F}$ keeps given from $\mathcal{D}$ to $\mathcal{D}$. We denote $\vartheta_{(\frac{\alpha}{k},\psi)}$ by the Kuratowski measure of noncompactness on $C_{\psi}^{\frac{\alpha}{k}}$, we will show the following equality

$$\vartheta_{(\frac{\alpha}{k},\psi)}(\mathfrak{F}\mathcal{V}) = \sup\left\{\vartheta\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right), t \in (a,\infty)\right\}, \quad \text{for all} \quad \mathcal{V} \subset \mathcal{D}. \tag{3.8}$$

Let us first present that for all $\epsilon > 0$, there exists a positive real number T_∞ so that, for each $t_1, t_2 \geq T_\infty$ and $u \in \mathcal{V}$, we have

$$\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t_2,a)(\mathfrak{F}u)(t_2)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)} - \frac{\Psi^{2-\frac{\alpha}{k}}(t_1,a)(\mathfrak{F}u)(t_1)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right\| < \epsilon. \tag{3.9}$$

Then, we obtain

$$\begin{aligned}
&\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t_2,a)(\mathfrak{F}u)(t_2)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)} - \frac{\Psi^{2-\frac{\alpha}{k}}(t_1,a)(\mathfrak{F}u)(t_1)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right\| \\
\leq\ & \Bigg[|\omega_4|\Bigg(|\mathcal{A}| + \sum_{i=1}^{m}\frac{|\lambda_i|}{k\Gamma_k(\alpha)}\Bigg[\frac{\mathcal{F}^*\Psi^{\frac{\alpha}{k}}(\xi_i,a)}{\Gamma_k(\alpha+k)} + \mathcal{L}\Bigg(\frac{\Psi^{\frac{\alpha}{k}}(\xi_i,a)}{\Gamma_k(\alpha+k)} + \frac{\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha}{k}}(\xi_i,a)}{\Gamma_k(2\alpha+k)}\Bigg)\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\Bigg) \\
&+|1+\omega_1|\Bigg(|\mathcal{B}| + \sum_{j=1}^{n}\frac{|\mu_j|}{k\Gamma_k(\alpha+\sigma_j)}\Bigg[\frac{\mathcal{F}^*\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j+k)} + \mathcal{L}\Bigg(\frac{\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j+k)} \\
&+\frac{\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(2\alpha+\sigma_j+k)}\Bigg)\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\Bigg) + \frac{\mathcal{P}^*\mathcal{F}^*}{k} + \frac{\mathcal{Q}^*\mathcal{L}}{k}\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\frac{1}{|\Omega|\Gamma_k(\alpha)}\left|\frac{\Psi(t_2,a)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)} - \frac{\Psi(t_1,a)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right| \\
&+\Bigg[|1+\omega_3|\Bigg(|\mathcal{A}| + \sum_{i=1}^{m}\frac{|\lambda_i|}{k\Gamma_k(\alpha)}\Bigg[\frac{\mathcal{F}^*\Psi^{\frac{\alpha}{k}}(\xi_i,a)}{\Gamma_k(\alpha+k)} + \mathcal{L}\Bigg(\frac{\Psi^{\frac{\alpha}{k}}(\xi_i,a)}{\Gamma_k(\alpha+k)} + \frac{\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha}{k}}(\xi_i,a)}{\Gamma_k(2\alpha+k)}\Bigg)\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\Bigg) \\
&+|\omega_2|\Bigg(|\mathcal{B}| + \sum_{j=1}^{n}\frac{|\mu_j|}{k\Gamma_k(\alpha+\sigma_j)}\Bigg[\frac{\mathcal{F}^*\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j+k)} + \mathcal{L}\Bigg(\frac{\Psi^{\frac{\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(\alpha+\sigma_j+k)} \\
&+\frac{\Gamma_k(\alpha+k)\Psi^{\frac{2\alpha+\sigma_j}{k}}(\eta_j,a)}{\Gamma_k(2\alpha+\sigma_j+k)}\Bigg)\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\Bigg) + \frac{\mathcal{P}^*\mathcal{F}^*}{k} + \frac{\mathcal{Q}^*\mathcal{L}}{k}\|u\|_{\psi}^{\frac{\alpha}{k}}\Bigg]\frac{1}{|\Omega|\Gamma_k(\alpha-k)}\left|\frac{1}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)} - \frac{1}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right| \\
&+\left|\frac{\Psi(t_2,a)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)} - \frac{\Psi(t_1,a)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right|\frac{1}{k\Gamma_k(\alpha)}\int_a^\infty \psi'(s)\|f(s,u(s))\|ds.
\end{aligned}$$

For this, we will separated into two cases.

Case I. If $\lim\limits_{t\to\infty}\Psi(t,a) = +\infty$ we obtain $\lim\limits_{t\to\infty}\frac{\Psi(t,a)}{1+\Psi^{\frac{\alpha}{k}}(t,a)} = 0$ and $\lim\limits_{t\to\infty}\frac{1}{1+\Psi^{\frac{\alpha}{k}}(t,a)} = 0$ then,

$$\begin{aligned}
&\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t_2,a)(\mathfrak{F}u)(t_2)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)} - \frac{\Psi^{2-\frac{\alpha}{k}}(t_1,a)(\mathfrak{F}u)(t_1)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right\| \\
=\ & \left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t_2,a)(\mathfrak{F}u)(t_2)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)} - \frac{\Psi^{2-\frac{\alpha}{k}}(t_2,a)(\mathfrak{F}u)(t_1)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)} + \frac{\Psi^{2-\frac{\alpha}{k}}(t_2,a)(\mathfrak{F}u)(t_1)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)} - \frac{\Psi^{2-\frac{\alpha}{k}}(t_1,a)(\mathfrak{F}u)(t_1)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right\| \\
=\ & \left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t_2,a)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)}\Big((\mathfrak{F}u)(t_2) - (\mathfrak{F}u)(t_1)\Big) + \left(\frac{\Psi^{2-\frac{\alpha}{k}}(t_2,a)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)} - \frac{\Psi^{2-\frac{\alpha}{k}}(t_1,a)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right)(\mathfrak{F}u)(t_1)\right\|.
\end{aligned}$$

This implies that

$$\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t_2,a)(\mathfrak{F}u)(t_2)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)}-\frac{\Psi^{2-\frac{\alpha}{k}}(t_1,a)(\mathfrak{F}u)(t_1)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right\|\to 0 \quad \text{as} \quad t_1,t_2\to\infty. \tag{3.10}$$

Case II. If $\lim_{t\to\infty}\Psi(t,a)=\mathfrak{L}<+\infty$ by observing the inequality

$$\left\|\frac{\Psi(t_2,a)(\mathfrak{F}u)(t_2)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)}-\frac{\Psi(t_1,a)(\mathfrak{F}u)(t_1)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right\|\leq\left\|\frac{\Psi(t_2,a)(\mathfrak{F}u)(t_2)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)}-\frac{\mathfrak{L}}{1+\mathfrak{L}^{\frac{\alpha}{k}}}\right\|+\left\|\frac{\Psi(t_1,a)(\mathfrak{F}u)(t_1)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}-\frac{\mathfrak{L}}{1+\mathfrak{L}^{\frac{\alpha}{k}}}\right\|.$$

It is easily to reach the estimate (3.10). In the same way, we verify that for all $\epsilon>0$, there exists a positive real number $0<T_0<<T_\infty$ so that, for each $t_1,t_2\leq T_\infty$ and $u\in\mathcal{V}$, we have

$$\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t_2,a)(\mathfrak{F}u)(t_2)}{1+\Psi^{\frac{\alpha}{k}}(t_2,a)}-\frac{\Psi^{2-\frac{\alpha}{k}}(t_1,a)(\mathfrak{F}u)(t_1)}{1+\Psi^{\frac{\alpha}{k}}(t_1,a)}\right\|<\epsilon. \tag{3.11}$$

Next, we will show that the equality (3.8), we have

$$\vartheta_{(\frac{\alpha}{k},\psi)}(\mathfrak{F}\mathcal{V})\leq\sup_{t\in(a,\infty)}\vartheta\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right).$$

Let $\mathfrak{F}\mathcal{V}|_{\mathcal{K}}$ be the restriction of $\mathfrak{F}\mathcal{V}$ on $\mathcal{K}=[T_0,T_\infty]$. Assume that ϵ is a strictly positive real number, by applying Lemma 2.3 and the third step, it follows that

$$\vartheta_{(\frac{\alpha}{k},\psi)}(\mathfrak{F}\mathcal{V}|_{\mathcal{K}})=\sup_{t\in\mathcal{K}}\vartheta\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right)\leq\sup_{t\in(a,\infty)}\vartheta\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right),$$

which implies that there is a finite partition $\mathfrak{F}\mathcal{V}_i$ of $\mathfrak{F}\mathcal{V}$ such that $\mathfrak{F}\mathcal{V}=\bigcup_i\mathfrak{F}\mathcal{V}_i$ and

$$diam(\mathfrak{F}\mathcal{V}_i|_{\mathcal{K}})\leq\sup_{t\in(a,\infty)}\vartheta\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right)+\epsilon, \quad i=0,1,\dots,k. \tag{3.12}$$

Consequently, using inequalities (3.9) and (3.12), we get, for all $\mathfrak{F}u_1,\mathfrak{F}u_2$ of $\mathfrak{F}\mathcal{V}_i$ and $t\geq T_\infty$,

$$\begin{aligned}\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}u_2)(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}-\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}u_1)(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right\| \leq & \left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}u_2)(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}-\frac{\Psi^{2-\frac{\alpha}{k}}(T_\infty,a)(\mathfrak{F}u_2)(T_\infty)}{1+\Psi^{\frac{\alpha}{k}}(T_\infty,a)}\right\|\\ &+\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(T_\infty,a)(\mathfrak{F}u_2)(T_\infty)}{1+\Psi^{\frac{\alpha}{k}}(T_\infty,a)}-\frac{\Psi^{2-\frac{\alpha}{k}}(T_\infty,a)(\mathfrak{F}u_1)(T_\infty)}{1+\Psi^{\frac{\alpha}{k}}(T_\infty,a)}\right\|\\ &+\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}u_1)(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}-\frac{\Psi^{2-\frac{\alpha}{k}}(T_\infty,a)(\mathfrak{F}u_1)(T_\infty)}{1+\Psi^{\frac{\alpha}{k}}(T_\infty,a)}\right\|\\ < & \ 3\epsilon+\sup_{t\in(a,\infty)}\vartheta\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right).\end{aligned}$$

So,

$$\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}u_2)(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}-\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}u_1)(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right\|\leq 3\epsilon+\sup_{t\in(a,\infty)}\vartheta\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right). \tag{3.13}$$

By the same procedure and using inequalities (3.11) and (3.12), it is not difficult to show that the inequality (3.13) is also true for all $\mathfrak{F}u_1, \mathfrak{F}u_2$ of $\mathfrak{F}\mathcal{V}_i$-$t \le T_0$. Then, from (3.12) and (3.13), we obtain

$$diam(\mathfrak{F}\mathcal{V}_i) < \sup_{t\in(a,\infty)} \vartheta\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right) + 3\epsilon.$$

Thus,

$$\vartheta_{(\frac{\alpha}{k},\psi)}(\mathfrak{F}\mathcal{V}) < \sup_{t\in(a,\infty)} \vartheta\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right) + 3\epsilon.$$

Since ϵ is arbitrary, which leads us to the result. Conversely, we show that

$$\sup_{t\in(a,\infty)} \vartheta\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right) < \vartheta_{(\frac{\alpha}{k},\psi)}(\mathfrak{F}\mathcal{V}).$$

According to the definition of Kuratowski measure of noncompactness, we obtain, for all $\epsilon > 0$ we will find a finite partition $\mathfrak{F}\mathcal{V} = \bigcup_i \mathfrak{F}\mathcal{V}_i$ so that $diam(\mathfrak{F}\mathcal{V}_i) < \vartheta_{(\frac{\alpha}{k},\psi)}(\mathfrak{F}\mathcal{V}) + \epsilon$, then for any $u_1, u_2 \in \mathcal{V}$ and $t \in (a,\infty)$, we have

$$\left\|\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}u_2)(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)} - \frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}u_1)(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right\| \le \|\mathfrak{F}u_2 - \mathfrak{F}u_1\|_\alpha^\psi < \vartheta_{(\frac{\alpha}{k},\psi)}(\mathfrak{F}\mathcal{V}) + \epsilon.$$

Since $\mathfrak{F}\mathcal{V}(t) = \bigcup_i \mathfrak{F}\mathcal{V}_i(t)$, we get $\vartheta\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right) < \vartheta_{(\frac{\alpha}{k},\psi)}(\mathfrak{F}\mathcal{V}) + \epsilon$ and ϵ is arbitrary, then $\vartheta\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right) < \vartheta_{(\frac{\alpha}{k},\psi)}(\mathfrak{F}\mathcal{V})$. So,

$$\sup_{t\in(a,\infty)} \vartheta\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right) < \vartheta_{(\frac{\alpha}{k},\psi)}(\mathfrak{F}\mathcal{V}).$$

Next, it remains to show that $\mathfrak{F}$ is a Meir-Keeler condensing operator via a measure of noncompactness $\vartheta_{(\frac{\alpha}{k},\psi)}$, this is equivalent to demonstrating the following implication

$$\forall\epsilon > 0, \exists\rho(\epsilon) : \epsilon \le \vartheta_{(\frac{\alpha}{k},\psi)}(\mathcal{V}) < \epsilon + \rho \implies \vartheta_{(\frac{\alpha}{k},\psi)}(\mathfrak{F}\mathcal{V}) < \epsilon \quad \text{for any} \mathcal{V} \subset \mathcal{D}. \tag{3.14}$$

Assume that ϵ is a strictly positive real, $\mathcal{V} \subset \mathcal{D}$ and $t \in (a,\infty)$, for all $p, q \in \mathbb{R}^+$ verifying $0 < p \le t \le q$, we define the auxiliary operator $\mathfrak{F}_{p,q}$ by

$$\begin{aligned}(\mathfrak{F}_{p,q}u)(t) \;=\; & \frac{\Psi^{\frac{\alpha}{k}-1}(t,a)}{\Omega\Gamma_k(\alpha)}\Bigg[\omega_4\Bigg(\mathcal{A} + \sum_{i=1}^{m} \frac{\lambda_i}{k\Gamma_k(\alpha)} \int_a^{\xi_i} \psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,s)f(s,u(s))ds\Bigg) \\ & +(1-\omega_1)\Bigg(\mathcal{B} + \sum_{j=1}^{n} \frac{\mu_j}{k\Gamma_k(\alpha+\sigma_j)} \int_a^{\eta_j} \psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,s)f(s,u(s))ds - \int_t^q \psi'(s)f(s,u(s))ds\Bigg)\Bigg] \\ & +\frac{\Psi^{\frac{\alpha}{k}-2}(t,a)}{\Omega\Gamma_k(\alpha-k)}\Bigg[(1-\omega_3)\Bigg(\mathcal{A} + \sum_{i=1}^{m} \frac{\lambda_i}{k\Gamma_k(\alpha)} \int_a^{\xi_i} \psi'(s)\Psi^{\frac{\alpha}{k}-1}(\xi_i,s)f(s,u(s))ds\Bigg) \\ & +\omega_2\Bigg(\mathcal{B} + \sum_{j=1}^{n} \frac{\mu_j}{k\Gamma_k(\alpha+\sigma_j)} \int_a^{\eta_j} \psi'(s)\Psi^{\frac{\alpha+\sigma_j}{k}-1}(\eta_j,s)f(s,u(s))ds - \int_t^q \psi'(s)f(s,u(s))ds\Bigg)\Bigg]\end{aligned}$$

$$+\frac{1}{k\Gamma_k(\alpha)}\int_p^t \psi'(s)\left[\Psi^{\frac{\alpha}{k}-1}(t,s)-\frac{(1-\omega_1)\Psi^{\frac{\alpha}{k}-1}(t,a)}{\Omega}-\frac{(\alpha-k)\omega_2\Psi^{\frac{\alpha}{k}-1}(t,a)}{\Omega}\right]f(s,u(s))ds.$$

By using the properties of ϑ, we get

$$\vartheta\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}_{p,q}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right)\rightarrow\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right) \quad \text{as} \quad p\rightarrow a \quad \text{and} \quad q\rightarrow\infty. \tag{3.15}$$

An argument is similar to show that of the third step, we show that the $\mathfrak{F}_{p,q}\mathcal{V}$ is equicontinuous and bounded on $[p,q]$. From Lemmas 2.1, 2.3 and 2.7, $(\mathbb{H}_2)$ and the previous steps, we get, there is a sequence $\{u_n\}_{n=0}^{\infty}\subset\mathcal{V}$ so that

$$\begin{aligned}
&\vartheta\left(\frac{\Psi^{2-\frac{\alpha}{k}}(t,a)(\mathfrak{F}_{p,q}\mathcal{V})(t)}{1+\Psi^{\frac{\alpha}{k}}(t,a)}\right)\\
\leq\;& \frac{\epsilon}{2}+\frac{1}{k\Gamma_k(\alpha)}\vartheta\left\{\int_t^q \psi'(s)f(s,u_n(s))ds, n\in\mathbb{N}\right\}+\frac{1}{k\Gamma_k(\alpha)}\vartheta\left\{\int_p^t \psi'(s)f(s,u_n(s))ds, n\in\mathbb{N}\right\}\\
\leq\;& \frac{\epsilon}{2}+\frac{1}{k\Gamma_k(\alpha)}\int_p^q \psi'(s)\vartheta\left\{f(s,u_n(s)), n\in\mathbb{N}\right\}ds\\
\leq\;& \frac{\epsilon}{2}+\frac{\mathcal{G}^*\vartheta_{(\frac{\alpha}{k},\psi)}(\mathcal{V})}{k\Gamma_k(\alpha)}.
\end{aligned}$$

From (3.15), we know that

$$\vartheta_{(\frac{\alpha}{k},\psi)}(\mathfrak{F}\mathcal{V})\leq\frac{\epsilon}{2}+\frac{\mathcal{G}^*\vartheta_{(\frac{\alpha}{k},\psi)}(\mathcal{V})}{k\Gamma_k(\alpha)}.$$

If

$$\vartheta_{(\frac{\alpha}{k},\psi)}(\mathfrak{F}\mathcal{V})\leq\frac{\epsilon}{2}+\frac{\mathcal{G}^*\vartheta_{(\frac{\alpha}{k},\psi)}(\mathcal{V})}{k\Gamma_k(\alpha)}<\epsilon,$$

which implies that

$$\vartheta_{(\frac{\alpha}{k},\psi)}(\mathcal{V})\leq\frac{k\Gamma_k(\alpha)\epsilon}{2\mathcal{G}^*},$$

so that implication (3.14) is fulfilled, we take

$$\rho=\frac{(k\Gamma_k(\alpha)-2\mathcal{G}^*)\epsilon}{2\mathcal{G}^*}.$$

So, $\mathfrak{F}$ is a Meir-Keeler condensing operator via $\vartheta_{(\frac{\alpha}{k},\psi)}$. Finally, all the conditions of Theorem 2.1 are satisfied, which implies that the solution sets of problem (1.1) are nonempty and compact.

4. Numerical illustrations

This section presents two numerical examples to show the application of our theoretical results.

Example 4.1. *Consider the following designed problem of the form:*

$$\begin{cases} {}_{\frac{3}{2}}\mathfrak{D}_{0^+}^{\frac{12}{7};\psi}u(t) = \left(\dfrac{4e^{-4t}}{6t+7} + \dfrac{4e^{-t}\Psi^{\frac{6}{7}}(t,0)u_n(t)}{2t+5}\right)_{n=1}^{\infty}, \quad t \in (0,+\infty), \\ {}_{\frac{3}{2}}\mathcal{I}_{0^+}^{\frac{9}{7};\psi}u(0) = 2 + u\Big(\dfrac{15}{2}\Big) + 3u(2) + 4u\Big(\dfrac{13}{2}\Big), \\ \lim\limits_{t\to\infty} {}_{\frac{3}{2}}\mathfrak{D}_{0^+}^{\frac{3}{14};\psi}u(\infty) = 3 + 4\Big({}_{\frac{3}{2}}\mathcal{I}_{0^+}^{\frac{1}{9};\psi}u\Big(\dfrac{7}{2}\Big)\Big) + 8\pi\Big({}_{\frac{3}{2}}\mathcal{I}_{0^+}^{\frac{1}{7};\psi}u\Big(\dfrac{11}{\pi}\Big)\Big) + \dfrac{2}{5}\Big({}_{\frac{3}{2}}\mathcal{I}_{0^+}^{\frac{1}{8};\psi}u\Big(\dfrac{9}{5}\Big)\Big). \end{cases} \tag{4.1}$$

Here $\psi(t) = -e^{-2t}/8$, $\alpha = 12/7$, $k = 3/2$, $a = 0$, $\mathcal{A} = 2$, $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = 4$, $\xi_1 = 15/2$, $\xi_2 = 2$, $\xi_3 = 13/2$, $\mathcal{B} = 3$, $\mu_1 = 4$, $\mu_2 = 8\pi$, $\mu_3 = 2/5$, $\eta_1 = 7/2$, $\eta_2 = 11/\pi$, $\eta_3 = 9/5$, $\sigma_1 = 1/9$, $\sigma_2 = 1/7$, $\sigma_3 = 1/8$. From the previous data, we obtain $\omega_1 \approx 10.34150$, $\omega_2 \approx 5.99069$, $\omega_3 \approx 18.10022$, $\omega_4 \approx 51.09593$, and $\Omega \approx -146.35805 \neq 0$. Assume that ℓ_∞ consisting of all bounded sequences of real numbers equipped with the norm $\|u\|_{\ell_\infty} := \sup_{n\in\mathbb{N}}\{|u_n|\}$ and $u = \{u_n\} \in \ell_\infty$. It is easy to see that $(\ell_\infty, \|\cdot\|)_{\ell_\infty}$ includes a Banach space. By the considered problem (4.1), we get $f(t,u(t)) = \{f_n(t,u(t)); n \in \mathbb{N}\}$, where

$$f(t,u) = \frac{4e^{-4t}}{6t+7}\{1,1,1,\ldots\} + \frac{4e^{-t}\Psi^{\frac{6}{7}}(t,0)}{2t+5}u.$$

Since, for any $u, v \in \ell_\infty$, we have

$$\|f(t,u) - f(t,v)\|_{\ell_\infty} \leq \sup_{n\in\mathbb{N}}\{|f_n(t,u(t)) - f_n(t,v(t))|\} \leq \frac{4}{5}\Psi^{\frac{6}{7}}(t,0)\|u-v\|_{\ell_\infty}.$$

The assumption $(\mathbb{H}_1)$ is satisfied with $\mathcal{L} = 4/5$. Since $f(t,u) : \mathcal{J} \times \ell_\infty \to \ell_\infty$ with the following inequality $\|f(t,u)\|_{\ell_\infty} \leq \frac{4}{7}e^{-4t} + \frac{4}{5}e^{-t}\Psi^{\frac{6}{7}}(t,0)\|u(t)\|_{\ell_\infty}$. Then

$$\mathcal{P}^* \approx 0.12500 < \infty, \qquad \mathcal{Q}^* \approx 0.13042 < 272.01233 \approx \frac{15\Gamma_{\frac{3}{2}}\left(\frac{12}{7}\right)|\Omega|}{8}.$$

For each bounded set $\Lambda \subset C_\psi^{\frac{\alpha}{k}}$, it follows that

$$f(t,\Lambda(t)) = \left\{\frac{4e^{-4t}}{6t+7}\right\} + \frac{4e^{-t}\Psi^{\frac{6}{7}}(t,0)}{2t+5}\Lambda(t).$$

Then,

$$\vartheta_{(\frac{\alpha}{k},\psi)}(f(t,\Lambda(t))) \leq \frac{4e^{-t}\Psi^{\frac{6}{7}}(t,0)}{2t+5}\vartheta_{(\frac{\alpha}{k},\psi)}(\Lambda(t)).$$

It is easily to calculate that

$$\int_a^\infty \psi'(s)[1 + \Psi^{\frac{\alpha}{k}}(s,a)]g(s)ds \approx 0.63973 < 0.74342 \approx \frac{\frac{3}{2}\Gamma_{\frac{3}{2}}(\frac{12}{7})}{2},$$

which the assumption $(\mathbb{H}_2)$ is verified. Finally, we will show that the assumption $(\mathbb{H}_3)$ is true. Since

$$\mathcal{Y} \;:=\; \mathcal{L}\left([\gamma_1 + \gamma_2] + \frac{\zeta_3\mathcal{Q}^*}{k}\right) \approx 0.23199 < 1,$$

$$\mathcal{R}^* \quad := \quad \frac{\zeta_1|\mathcal{A}| + \zeta_2|\mathcal{B}| + \mathcal{F}^*(\frac{\zeta_3\mathcal{P}^*}{k} + \gamma_1)}{1 - \mathcal{Y}} \approx 1.52910.$$

Since the all conditions of Theorem 3.1 are true. Hence, the considered problem (4.1) has at least one solution on $[0, \infty)$. Moreover, We will represent the relationship between the constant values such as ω_1, ω_2, ω_3, ω_4, Ω, ζ_1, ζ_2, ζ_3, γ_1, γ_2, $\mathcal{Y}$, and $\mathcal{R}^*$, which are shown in Tables 1 and 2 for vary values $\alpha \in [1.5, 2.0]$ and $k \in [1.0, 1.4]$.

Table 1. The numerical values of α, k, ω_1, ω_2, ω_3, ω_4, and Ω in Example 4.1.

α	k	ω_1	ω_2	ω_3	ω_4	Ω
1.50000	1.00000	12.81061	3.18052	8.71232	44.52751	-50.53357
1.52500	1.02000	12.94636	3.18240	8.75187	45.08040	-50.85736
1.55000	1.04000	13.08167	3.18413	8.79009	45.62960	-51.17354
1.57500	1.06000	13.21655	3.18573	8.82705	46.17524	-51.48238
1.60000	1.08000	13.35099	3.18719	8.86280	46.71744	-51.78419
1.62500	1.10000	13.48502	3.18853	8.89738	47.25631	-52.07920
1.65000	1.12000	13.61864	3.18974	8.93084	47.79196	-52.36768
1.67500	1.14000	13.75185	3.19084	8.96323	48.32450	-52.64986
1.70000	1.16000	13.88466	3.19182	8.99459	48.85403	-52.92598
1.72500	1.18000	14.01708	3.19270	9.02495	49.38063	-53.19625
1.75000	1.20000	14.14912	3.19348	9.05437	49.90441	-53.46088
1.77500	1.22000	14.28078	3.19415	9.08286	50.42544	-53.72006
1.80000	1.24000	14.41207	3.19474	9.11047	50.94380	-53.97400
1.82500	1.26000	14.54300	3.19523	9.13723	51.45958	-54.22286
1.85000	1.28000	14.67356	3.19564	9.16317	51.97283	-54.46683
1.87500	1.30000	14.80378	3.19596	9.18831	52.48365	-54.70607
1.90000	1.32000	14.93364	3.19620	9.21270	52.99208	-54.94074
1.92500	1.34000	15.06317	3.19637	9.23634	53.49820	-55.17099
1.95000	1.36000	15.19236	3.19646	9.25928	54.00207	-55.39697
1.97500	1.38000	15.32121	3.19648	9.28153	54.50374	-55.61882
2.00000	1.40000	15.44974	3.19644	9.30312	55.00328	-55.83667

Table 2. The numerical values of α, k, ζ_1, ζ_2, ζ_3, γ_1, γ_2, $\mathcal{Y}$, and $\mathcal{R}^*$ in Example 4.1.

α	k	ζ_1	ζ_2	ζ_3	γ_1	γ_2	$\mathcal{Y}$	$\mathcal{R}^*$
1.50000	1.00000	1.10270	0.34389	0.36622	0.51862	0.006494	0.45736	6.37350
1.52500	1.02000	1.09882	0.34177	0.36375	0.50922	0.006480	0.44886	6.24630
1.55000	1.04000	1.09530	0.33981	0.36145	0.50025	0.006466	0.44076	6.12916
1.57500	1.06000	1.09209	0.33799	0.35931	0.49167	0.006451	0.43301	6.02102
1.60000	1.08000	1.08919	0.33630	0.35731	0.48345	0.006436	0.42561	5.92093
1.62500	1.10000	1.08656	0.33473	0.35545	0.47558	0.006420	0.41851	5.82811
1.65000	1.12000	1.08419	0.33328	0.35371	0.46802	0.006403	0.41172	5.74183
1.67500	1.14000	1.08206	0.33193	0.35209	0.46077	0.006386	0.40520	5.66150
1.70000	1.16000	1.08016	0.33067	0.35058	0.45379	0.006368	0.39893	5.58656
1.72500	1.18000	1.07847	0.32951	0.34918	0.44708	0.006350	0.39291	5.51654
1.75000	1.20000	1.07698	0.32844	0.34787	0.44062	0.006332	0.38712	5.45101
1.77500	1.22000	1.07567	0.32744	0.34664	0.43440	0.006313	0.38154	5.38961
1.80000	1.24000	1.07454	0.32652	0.34551	0.42839	0.006295	0.37617	5.33198
1.82500	1.26000	1.07357	0.32568	0.34445	0.42259	0.006275	0.37098	5.27784
1.85000	1.28000	1.07275	0.32489	0.34346	0.41699	0.006256	0.36598	5.22691
1.87500	1.30000	1.07208	0.32417	0.34255	0.41158	0.006236	0.36114	5.17894
1.90000	1.32000	1.07154	0.32351	0.34170	0.40635	0.006217	0.35647	5.13372
1.92500	1.34000	1.07113	0.32290	0.34091	0.40128	0.006197	0.35195	5.09105
1.95000	1.36000	1.07084	0.32235	0.34018	0.39637	0.006177	0.34757	5.05075
1.97500	1.38000	1.07067	0.32185	0.33951	0.39161	0.006156	0.34333	5.01265
2.00000	1.40000	1.07060	0.32139	0.33889	0.38700	0.006136	0.33923	4.97660

Figure 1 (1a–1d) shows the graphical representations of ω_i for $i = 1, 2, 3, 4$, under $k \in \{1.00, 1.05, \ldots, 1.40\}$. Figure 2 (2a-2c) shows the graphical representations of ζ_i for $i = 1, 2, 3$, under $k \in \{1.00, 1.05, \ldots, 1.40\}$. Figure 3 (3a–3b) shows the graphical representations of γ_i for $i = 1, 2$, under $k \in \{1.00, 1.05, \ldots, 1.40\}$. Figures 4 and 5 show the graphical representations of $\mathcal{Y}$ and $\mathcal{R}^*$.

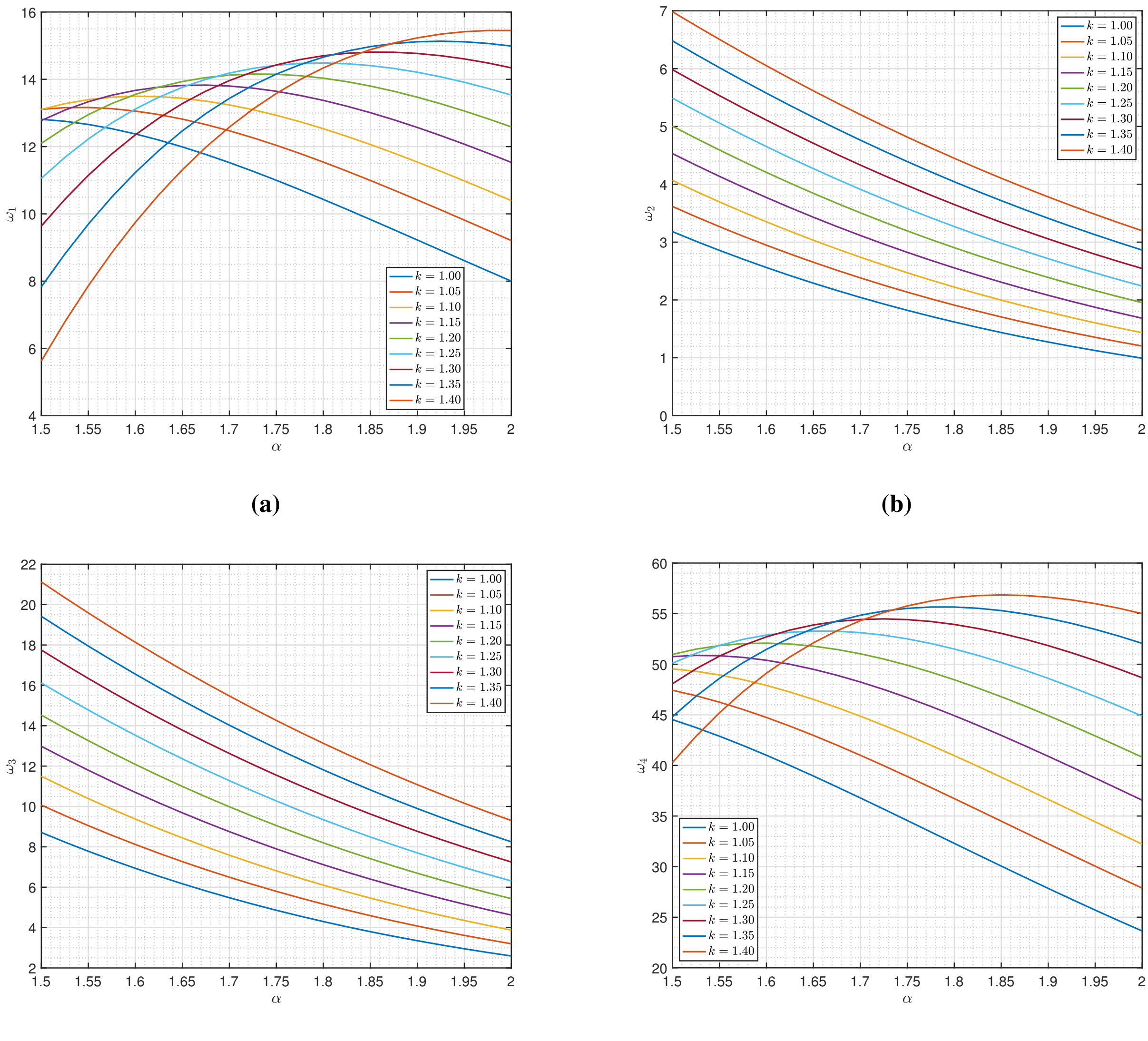
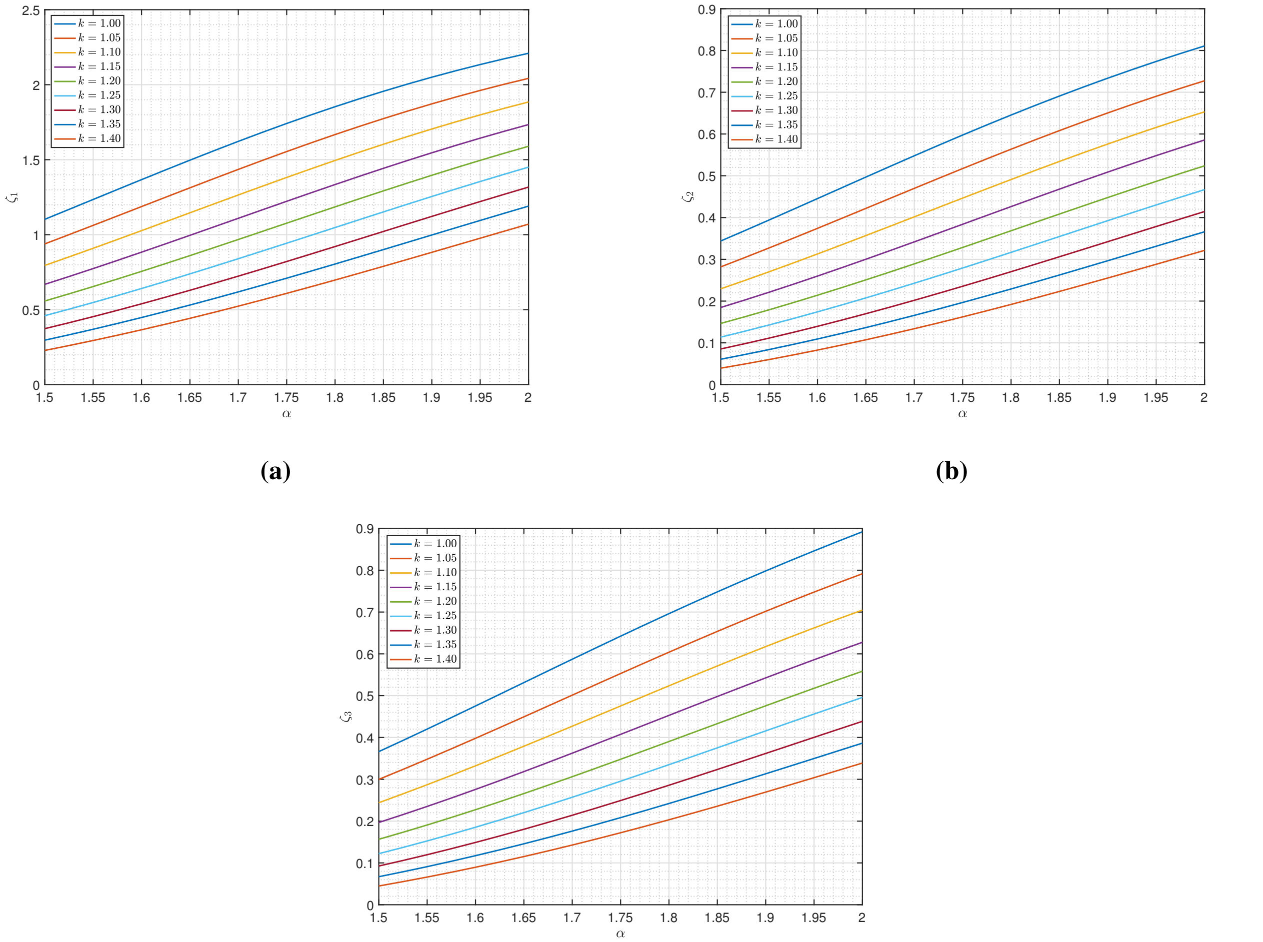
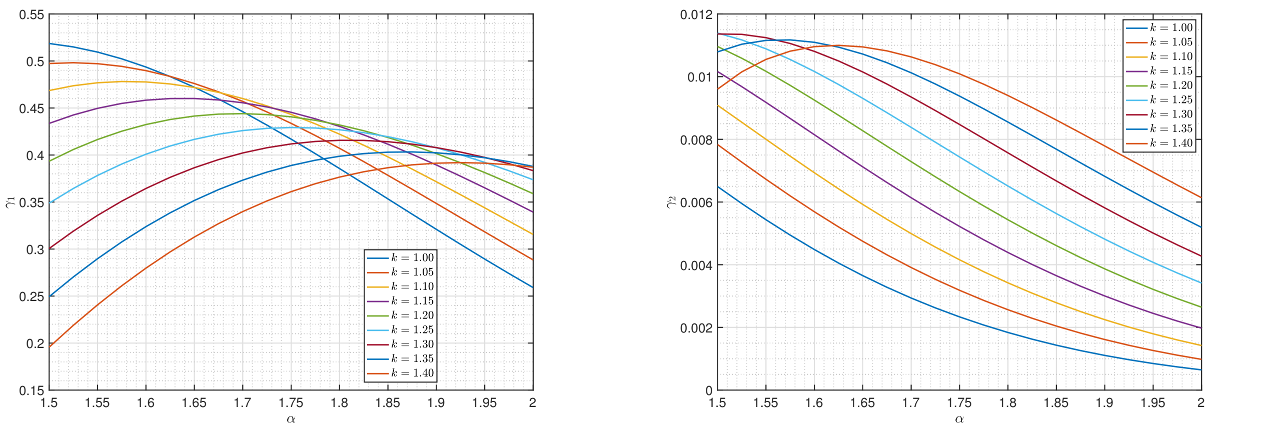


Figure 1. Graphical representation of ω_i for $i = 1, 2, 3, 4$ in Example 4.1.



(a) **(b)** **(c)**

Figure 2. Graphical representation of ζ_i for $i = 1, 2, 3$ in Example 4.1.



(a) **(b)**

Figure 3. Graphical representation of γ_i for $i = 1, 2$ in Example 4.1.

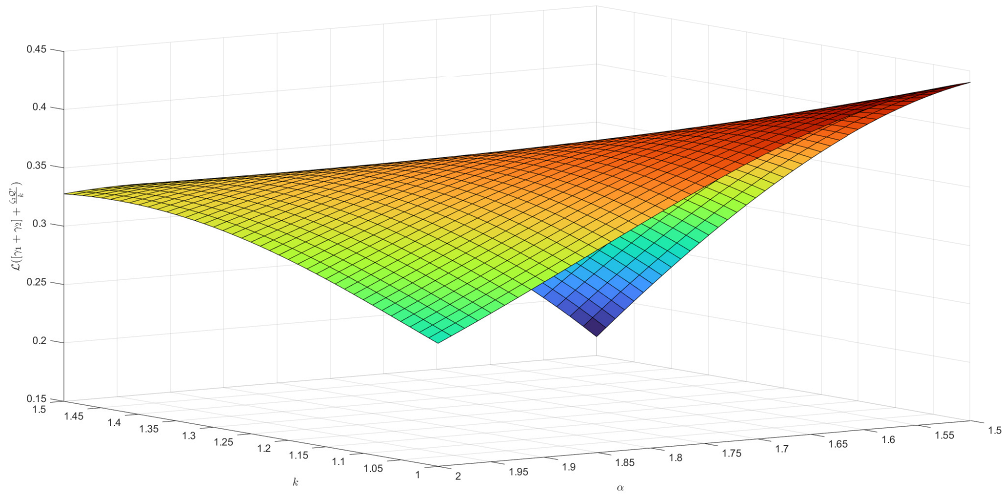


Figure 4. Graphical representation of $\mathcal{Y}$ in Example 4.1.

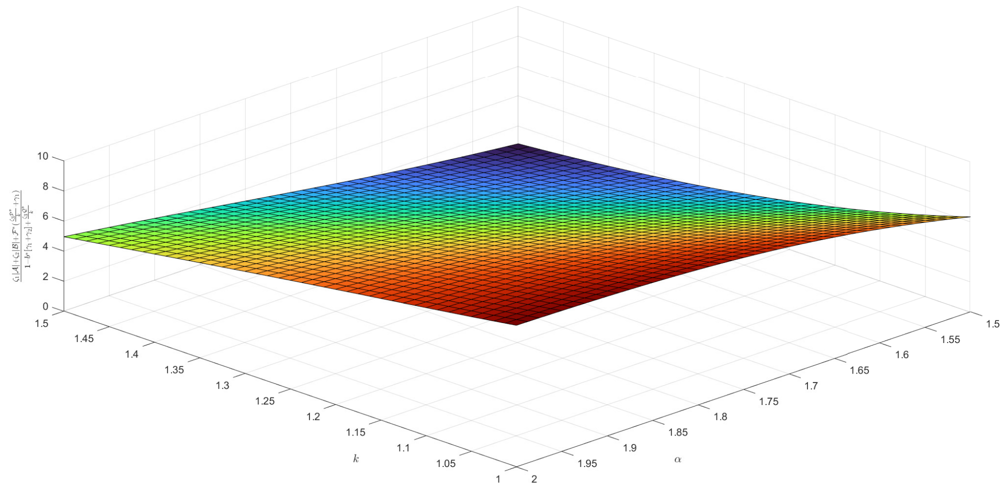


Figure 5. Graphical representation of $\mathcal{R}^*$ in Example 4.1.

Example 4.2. *Consider the following linear designed problem of the form:*

$$\begin{cases} {}_{\frac{3}{2}}\mathfrak{D}_{0+}^{\frac{12}{7};\psi}u(t) = \Psi^{\frac{\pi}{2}}(t,0), \quad t \in (0,+\infty), \\ {}_{\frac{3}{2}}\mathcal{I}_{0+}^{\frac{9}{7};\psi}u(0) = 2 + u\Big(\frac{15}{2}\Big) + 3u(2) + 4u\Big(\frac{13}{2}\Big), \\ {}_{\frac{3}{2}}\mathfrak{D}_{0+}^{\frac{3}{14};\psi}u(\infty) = 3 + 4\Big({}_{\frac{3}{2}}\mathcal{I}_{0+}^{\frac{1}{9};\psi}u\Big(\frac{7}{2}\Big)\Big) + 8\pi\Big({}_{\frac{3}{2}}\mathcal{I}_{0+}^{\frac{1}{7};\psi}u\Big(\frac{11}{\pi}\Big)\Big) + \frac{2}{5}\Big({}_{\frac{3}{2}}\mathcal{I}_{0+}^{\frac{1}{8};\psi}u\Big(\frac{9}{5}\Big)\Big). \end{cases} \tag{4.2}$$

By applying Lemma 2.5 under the function $f(t,u(t)) = \Psi^{\frac{\pi}{2}}(t,0)$, the solution of considered problem (4.2) is can be rewritten as

$$\begin{aligned} u(t) &= \frac{\Psi^{\frac{\alpha}{k}-1}(t,0)}{\Omega\Gamma_k(\alpha)}\Bigg[\omega_4\Bigg(\mathcal{A}+\sum_{i=1}^{m}\frac{\lambda_i\Gamma_k((\frac{\pi}{2}+1)k)\Psi^{\frac{\alpha}{k}+\frac{\pi}{2}}(\xi_i,0)}{\Gamma_k(\alpha+(\frac{\pi}{2}+1)k)}\Bigg) \\ &\quad +(1-\omega_1)\Bigg(\mathcal{B}+\sum_{j=1}^{n}\frac{\mu_j\Gamma_k((\frac{\pi}{2}+1)k)\Psi^{\frac{\alpha+\sigma_j}{k}+\frac{\pi}{2}}(\eta_j,0)}{\Gamma_k(\alpha+\sigma_j+(\frac{\pi}{2}+1)k)}-\frac{1}{\frac{\pi}{2}+1}\lim_{t\to\infty}\Big[\Psi^{\frac{\pi}{2}+1}(t,0)\Big]\Bigg)\Bigg] \\ &\quad +\frac{\Psi^{\frac{\alpha}{k}-2}(t,0)}{\Omega\Gamma_k(\alpha-k)}\Bigg[(1-\omega_3)\Bigg(\mathcal{A}+\sum_{i=1}^{m}\frac{\lambda_i\Gamma_k((\frac{\pi}{2}+1)k)\Psi^{\frac{\alpha}{k}+\frac{\pi}{2}}(\xi_i,0)}{\Gamma_k(\alpha+(\frac{\pi}{2}+1)k)}\Bigg) \\ &\quad +\omega_2\Bigg(\mathcal{B}+\sum_{j=1}^{n}\frac{\mu_j\Gamma_k((\frac{\pi}{2}+1)k)\Psi^{\frac{\alpha+\sigma_j}{k}+\frac{\pi}{2}}(\eta_j,0)}{\Gamma_k(\alpha+\sigma_j+(\frac{\pi}{2}+1)k)}-\frac{1}{\frac{\pi}{2}+1}\lim_{t\to\infty}\Big[\Psi^{\frac{\pi}{2}+1}(t,0)\Big]\Bigg)\Bigg] \\ &\quad +\frac{\Gamma_k((\frac{\pi}{2}+1)k)\Psi^{\frac{\alpha}{k}+\frac{\pi}{2}}(t,0)}{\Gamma_k(\alpha+(\frac{\pi}{2}+1)k)}. \end{aligned} \tag{4.3}$$

The solution $u(t)$ of the considered problem (4.2) via $\alpha = 1.5, 1.6, 1.7, 1.8, 1.9, 2.0$ with various functions $\psi(t)$ and the constants $k > 0$ (see; Table 3), is presented in Figures 6–9.

Table 3. The various functions $\psi(t)$ and the constants $k > 0$ in Example 4.2.

$k>0$	0.70	0.35	1.10	0.65
$\psi(t)$	$k\cosh\left(\frac{2\pi t}{2t+1}\right)$	$-2k-\frac{k^t+3}{2kt^k+1}$	$\log_k\left(\frac{3\sqrt{t+1}}{2\sqrt{t+1}}\right)$	$5-e^{-t^k}$

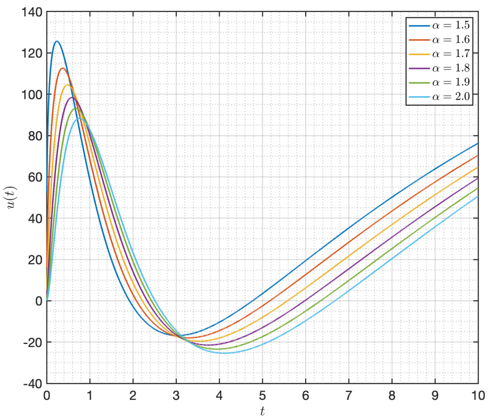


Figure 6. The solution $u(t)$ of Example 4.2 under $\psi(t) = k\cosh\left(\frac{2\pi t}{2t+1}\right)$ with $k = 0.70$.

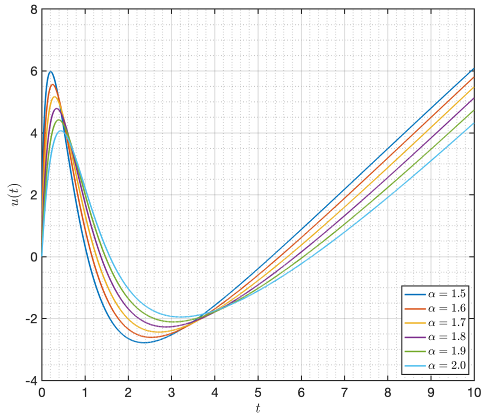


Figure 7. The solution $u(t)$ of Example 4.2 under $\psi(t) = -2k - \frac{k^t+3}{2kt^k+1}$ with $k = 0.35$.

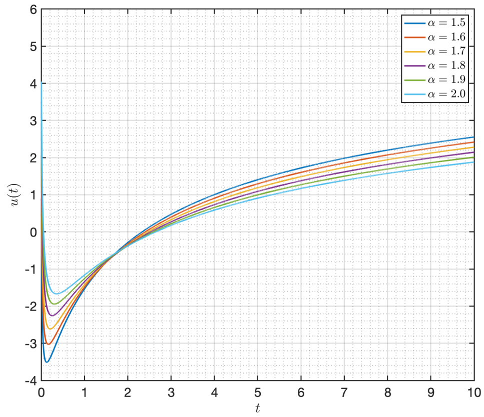


Figure 8. The solution $u(t)$ of Example 4.2 under $\psi(t) = \log_k\left(\frac{3\sqrt{t}+1}{2\sqrt{t}+1}\right)$ with $k = 1.10$.

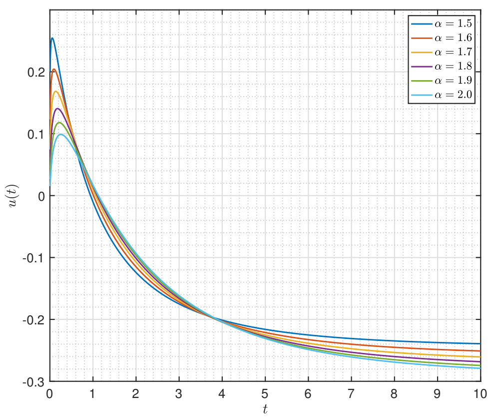


Figure 9. The solution $u(t)$ of Example 4.2 under $\psi(t) = 5 - e^{-t^k}$ with $k = 0.65$.

5. Conclusions

In conclusion, an outstanding qualitative analysis is accomplished in this work. We have proved the existence and uniqueness of solution sets as well as their topological structure for (k,ψ)-Riemann-Liouville fractional differential boundary value problem with nonlocal conditions on an unbounded domain. The Arrangement of the proof began with the provision of the measure of noncompactness in the sense of Kuratowski and its necessary properties. Later, a fixed point theorem for the Meir-Keeler condensing operators with a measure of noncompactness was applied to the proposed nonlinear problem (1.1) until the crucial outcome was finally achieved. Furthermore, suitable examples were illustrated to support the accuracy of the theoretical results. It was shown in Example 4.1 that the existence of the solution satisfied the conditions of Theorem 3.1 with varing values α and k. Numerical values of all parameters were calculated and given as in Tables 1 and 2. While, Example 4.2 presented the solution sets of the specific problem for different values k and various functions $\psi(t)$ in the case of polynomial, trigonometric, exponential, and logarithmic functions as seen in Table 3.

This research would be a great work to enrich the qualitative theory literature for the problem of nonlinear fractional differential with nonlocal boundary conditions on an unbounded domain involving a particular function. It probably extends future works to study the existence of solutions for the nonlinear differential and integral equations in the context of the other fractional operators and/or boundary conditions. In future work areas, we recommend working on nonlinear fractional integro-differential equations involving a special function, stability, or the algorithms to solve the (k,ψ)-Hilfer fractional differential equations in mathematical software. For some possible future works, researchers can provide applications to sciences and engineering using our obtained results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

References

1. S. Abbas, Y. Xia, Existence and attractivity of k-almost automorphic solutions of model of cellular neutral network with delay, *Acta. Math. Sci.*, **33** (2013), 290–302. https://doi.org/10.1016/S0252-9602(12)60211-2
2. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, 2006.

3. R. Magin, Fractional calculus in bioengineering, *Crit. Rev. Biomed. Eng.*, **32** (2004), 1–104. https://doi.org/10.1615/critrevbiomedeng.v32.i1.10
4. I. Podlubny, *Fractional differential equations*, New York: Academic Press, 1999.
5. R. Hilfer, *Applications of fractional calculus in physics*, Singapore: World Scientific, 2000.
6. L. Debanath, Recent applications of fractional calculus to science and engineering, *Int. J. Math. Math. Sci.*, **2003** (2003), 753601. https://doi.org/10.1155/S0161171203301486
7. D. Baleanu, J. A. Machado, A. C. Luo, *Fractional dynamics and control*, New York: Springer, 2012. https://doi.org/10.1007/978-1-4614-0457-6
8. U. N. Katugampola, A new approach to generalized fractional derivatives, *Bull. Math. Anal. Appl.*, **6** (2014), 1–15. https://doi.org/10.48550/arXiv.1106.0965
9. F. Jarad, T. Abdeljawad, D. Baleanu, On the generalized fractional derivatives and their Caputo modification, *J. Nonlinear Sci. Appl.*, **10** (2017), 2607–2619. https://doi.org/10.22436/jnsa.010.05.27
10. U. N. Katugampola, New approach to a generalized fractional integral, *Appl. Math. Comput.*, **218** (2011), 860–865. https://doi.org/10.1016/j.amc.2011.03.062
11. K. D. Kucche, A. D. Mali, On the nonlinear (k, ψ)-Hilfer fractional differential equations, *Chaos Soliton. Fract.*, **152** (2021), 111335. https://doi.org/10.1016/j.chaos.2021.111335
12. A. Salim, M. Benchohra, J. E. Lazreg, J. Henderson, On k-generalized ψ-Hilfer boundary value problems with retardation and anticipation, *Adv. Theor. Nonlinear Anal. Appl.*, **6** (2022), 173–190. https://doi.org/10.31197/atnaa.973992
13. K. Kotsamran, W. Sudsutad, C. Thaiprayoon, J. Kongson, J. Alzabut, Analysis of a nonlinear ψ-Hilfer fractional integro-differential equation describing cantilever beam model with nonlinear boundary conditions, *Fractal Fract.*, **5** (2021), 177. https://doi.org/10.3390/fractalfract5040177
14. A. Boutiara, M. S. Abdo, M. A. Almalahi, H. Ahmad, A. Ishan, Implicit hybrid fractional boundary value problem via generalized Hilfer derivative, *Symmetry*, **13** (2021), 1937. https://doi.org/10.3390/sym13101937
15. D. Baleanu, A. Mousalou, S. Rezapour, On the existence of solutions for some infinite coefficient-symmetric Caputo-Fabrizio fractional integro-differential equations, *Bound. Value Probl.*, **2017** (2017), 145. https://doi.org/10.1186/s13661-017-0867-9
16. A. Boutiara, K. Guerbati, M. Benbachir, Caputo-Hadamard fractional differential equation with three-point boundary conditions in Banach spaces, *AIMS Math.*, **5** (2020), 259–272. https://doi.org/10.3934/math.2020017
17. M. A. Almalahi, M. S. Abdo, S. K. Panchal, Existence and Ulam-Hyers-Mittag-Leffler stability results of ψ-Hilfer nonlocal Cauchy problem, *Rend. Circ. Mat. Palerm.*, **70** (2021), 57–77. https://doi.org/10.1007/s12215-020-00484-8
18. S. Pleumpreedaporn, W. Sudsutad, C. Thaiprayoon, J. E. Nápoles, J. Kongson, A study of ψ-Hilfer fractional boundary value problem via nonlinear integral conditions describing Navier Model, *Mathematics*, **9** (2021), 3292. https://doi.org/10.3390/math9243292
19. B. Ahmad, S. K. Ntouyas, Hilfer-Hadamard fractional boundary value problems with nonlocal mixed boundary conditions, *Fractal Fract.*, **5** (2021), 195. https://doi.org/10.3390/fractalfract5040195

20. S. Pleumpreedaporn, C. Pleumpreedaporn, W. Sudsutad, J. Kongson, C. Thaiprayoon, J. Alzabut, On a novel impulsive boundary value pantograph problem under Caputo proportional fractional derivative operator with respect to another function, *AIMS Math.*, **7** (2022), 7817–7846. https://doi.org/10.3934/math.2022438

21. L. Byszewski, V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a non-local abstract Cauchy problem in a Banach space, *Appl. Anal.*, **40** (1991), 11–19. https://doi.org/10.1080/00036819008839989

22. L. Byszewski, Existence and uniqueness of mild and classical solutions of semilinear functional differential evolution nonlocal Cauchy problem, In: *Selected Problems in Mathematics*, Cracow University of Technology, 1995, 25–33.

23. S. Muthaiaha, M. Murugesana, N. G. Thangaraja, Existence of solutions for nonlocal boundary value problem of Hadamard fractional differential equations, *Adv. Theor. Nonlinear Anal. Appl.*, **3** (2019), 162–173. https://doi.org/10.31197/atnaa.579701

24. C. Kuratowski, Sur les espaces complets, *Fund. Math.*, **15** (1930), 301–309. https://doi.org/10.4064/FM-15-1-301-309

25. G. Darbo, Punti uniti in trasformazioni a codominio non compatto, *Rend. Semin. Mat. U. Pad.*, **24** (1955), 84–92.

26. J. Banaś, Measures of noncompactness in Banach spaces, 1980.

27. A. Arara, M. Benchohra, N. Hamidi, J. J. Nieto, Fractional order differential equations on an unbounded domain, *Nonlinear Anal. Theor.*, **72** (2010), 580–586. https://doi.org/10.1016/j.na.2009.06.106

28. X. Su, Solutions to boundary value problem of fractional order on unbounded domains in a Banach space, *Nonlinear Anal. Theor.*, **74** (2011), 2844–2852. https://doi.org/10.1016/j.na.2011.01.006

29. M. Beddani, B. Hedia, Existence result for fractional differential equation on unbounded domain, *Kragujev. J. Math.*, **48** (2024), 755–766.

30. K. Benia, M. Beddani, M. Fečkan, B. Hedia, Existence result for a problem involving ψ-Riemann-Liouville Fractional Derivative on Unbounded domain, *Differ. Equ. Appl.*, **14** (2022), 83–97. https://doi.org/10.7153/dea-2022-14-06

31. R. P. Agarwal, M. Benchohra, S. Hamani, S. Pinelas, Boundary value problem for differential equations involving Riemann-Liouville fractional derivative on the half line, *Dyn. Contin. Discrete Impuls. Syst. Ser. A*, **18** (2011), 235–244.

32. S. Liang, J. Zhang, Existence of multiple positive solutions for m-point fractional boundary value problems on an infinite interval, *Math. Comput. Model.*, **54** (2011), 1334–1346. https://doi.org/10.1016/j.mcm.2011.04.004

33. L. Zhang, A. Bashir, G. Wang, Successive iterations for positive extremal solutions of nonlinear fractional differential equations on a half-line, *Bull. Aust. Math. Soc.*, **91** (2015), 116–128. https://doi.org/10.1017/S0004972714000550

34. G. Wang, Explicit iteration and unbounded solutions for fractional integral boundary value problem on an infinite interval, *Appl. Math. Lett.*, **47** (2015), 1–7. https://doi.org/10.1016/j.aml.2015.03.003

35. L. Zhang, B. Ahmad, G. Wang, Monotone iterative method for a class of nonlinear fractional differential equations on unbounded domains in Banach spaces, *Filomat*, **31** (2017), 1331–1338. https://doi.org/10.2298/FIL1705331Z
36. T. S. Cerdik, F. Y. Deren, N. A. Hamal, Unbounded solutions for boundary value problems of Riemann Liouville fractional differential equations on the half-line, *Fixed Point Theor.*, **19** (2018), 93–106. https://doi.org/10.24193/fpt-ro.2018.1.08
37. C. Zhai, L. Wei, The unique positive solution for fractional integro-differential equations of infinite intervals, *Sci. Asia*, **44** (2018), 118–124. https://doi.org/10.2306/scienceasia1513-1874.2018.44.118
38. F. Wang, Y. Cui, Unbounded solutions to abstract boundary value problems of fractional differential equations on a half line, *Math. Method. Appl. Sci.*, **44** (2021), 8166–8176. https://doi.org/10.1002/mma.5819
39. W. Zhang, W. Liu, Existence of solutions for fractional multi-point boundary value problems on an infinite interval at resonance, *Mathematics*, **8** (2020), 126. https://doi.org/10.3390/math8010126
40. A. Boutiara, M. Benbachir, M. K. Kaabar, F. Martinez, M. E. Samei, M. Kaplan, Explicit iteration and unbounded solutions for fractional q-difference equations with boundary conditions on an infinite interval, *J. Inequal. Appl.*, **2022** (2022), 29. https://doi.org/10.1186/s13660-022-02764-6
41. H. A. Salem, M. Cichon, Analysis of tempered fractional calculus in Hölder and Orlicz spaces, *Symmetry*, **14** (2022), 1581. https://doi.org/10.3390/sym14081581
42. J. Banaŝ, M. Mursaleen, *Sequence spaces and measures of noncompactness with applications to differential and integral equations*, New Delhi: Springer, 2014. https://doi.org/10.1007/978-81-322-1886-9
43. D. J. Guo, V. Lakshmikantham, X. Liu, *Nonlinear integral equations in abstract spaces*, Dordrecht: Springer Science & Business Media, 1996. https://doi.org/10.1007/978-1-4613-1281-9
44. C. Derbazi, Z. Baitiche, M. Benchohra, Cauchy problem with ψ-Caputo fractional derivative in Banach spaces, *Adv. Theor. Nonlinear Anal. Appl.*, **4** (2020), 349–360. https://doi.org/10.31197/atnaa.706292
45. A. Aghajani, M. Mursaleen, A. Shole Haghighi, Fixed point theorems for Meir-Keeler condensing operators via measure of noncompactness, *Acta Math. Sci.*, **35** (2015), 552–566. https://doi.org/10.1016/S0252-9602(15)30003-5
46. R. Díaz, E. Pariguan, On hypergeometric functions and Pochhammer k-symbol, *Divulgaciones Mat.*, **15** (2007), 179–192. https://doi.org/10.48550/arXiv.math/0405596
47. Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, S. M. Kang, Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities, *IEEE Access*, **6** (2018), 64946–64953. https://doi.org/10.1109/ACCESS.2018.2878266