Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(9): 19971–19992.
DOI:10.3934/math.20231018
Received: 25 March 2023
Revised: 13 May 2023
Accepted: 21 May 2023
Published: 15 June 2023

Research article

On a nonlinear system of plate equations with variable exponent nonlinearity and logarithmic source terms: Existence and stability results

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Abstract: In this paper, we consider a coupling non-linear system of two plate equations with logarithmic source terms. First, we study the local existence of solutions of the system using the Faedo-Galerkin method and Banach fixed point theorem. Second, we prove the global existence of solutions of the system by using the potential wells. Finally, using the multiplier method, we establish an exponential decay result for the energy of solutions of the system. Some conditions on the variable exponents that appear in the coupling functions and the involved constants that appear in the source terms are determined to ensure the existence and stability of solutions of the system. A series of lemmas and theorems have been proved and used to overcome the difficulties caused by the variable exponent and the logarithmic nonlinearities. Our result generalizes some earlier related results in the literature from the case of only constant exponent of the nonlinear internal forcing terms to the case of variable exponent and logarithmic source terms, which is more useful from the physical point of view and needed in several applications.

Keywords: plate equations; variable-exponent; logarithmic nonlinearity; logarithmic Sobolev inequality; Galerkin method; Banach fixed point theorem; multiplier method; exponential decay
Mathematics Subject Classification: 35B37, 35L55, 74D05, 93D15, 93D20

1. Introduction

In this paper, we deal with the existence and asymptotic behavior of solutions of the following system:

$$\begin{cases} \psi_{tt} + \Delta^2\psi + \psi + \psi_t + h_1(\psi, \varphi) = \alpha\psi \ln|\psi|, & \text{in } \Omega,\ t > 0, \\ \varphi_{tt} + \Delta^2\varphi + \varphi + \varphi_t + h_2(\psi, \varphi) = \alpha\varphi \ln|\varphi|, & \text{in } \Omega,\ t > 0, \\ \psi(\cdot, t) = \varphi(\cdot, t) = \frac{\partial\psi}{\partial\nu} = \frac{\partial\varphi}{\partial\nu} = 0, & \text{on } \partial\Omega, t \geq 0, \\ (\psi(0), \varphi(0)) = (\psi_0, \varphi_0), (\psi_t(0), \varphi_t(0)) = (\psi_1, \varphi_1), & \text{in } \Omega, \end{cases} \tag{P}$$

where Ω is a bounded and regular domain of $\mathbb{R}^2$, with smooth boundary $\partial\Omega$. The vector ν is the unit outer normal to $\partial\Omega$ and the constant α is a small positive real number satisfying some conditions. The coupling functions h_1, h_2 are of the form

$$\begin{aligned} h_1(\psi, \varphi) &= c_1|\psi + \varphi|^{2(p(\cdot)+1)}(\psi + \varphi) + c_2|\psi|^{p(\cdot)}\psi|\varphi|^{p(\cdot)+2}, \\ h_2(\psi, \varphi) &= c_1|\psi + \varphi|^{2(p(\cdot)+1)}(\psi + \varphi) + c_2|\psi|^{p(\cdot)+2}|\varphi|^{p(\cdot)}\varphi, \end{aligned} \tag{1.1}$$

where $c_1, c_2 > 0$ are constants, and p is a continuous function on $\overline{\Omega}$ satisfying some conditions to be mentioned later.

We study the existence and asymptotic behavior of solutions for the nonlinear coupling system of two plate equations with logarithmic source terms (P). For this purpose, we use the well-known logarithmic Sobolev inequality and logarithmic Gronwall inequality to treat the terms involving logarithms. We also use the embedding properties to treat the terms involving variable exponents. In addition, the main methods used to achieve our results are the Faedo-Galerkin method, the Banach fixed point theorem and the multiplier method.

From both the theoretical point of view and the application point of view, it is of great importance to have an idea about the existence and asymptotic behavior of solutions for coupling systems with logarithmic source terms in which the coupling functions are nonlinear with variable exponents. The significance of studying our system (P) is important in many fields. For example, from the logarithmic point of view, the logarithmic nonlinearity appears naturally in inflation cosmology and supersymmetric field theories, quantum mechanics and many other branches of physics, such as nuclear physics, optics and geophysics [1–3] and [4]. These specific applications in physics and other fields attract a lot of mathematical scientists to work with these problems.

Regarding problems with logarithmic source terms, we refer to the works of [5–11]. From the variable exponent non-linearity point of view, there has been an increasing interest in treating equations with variable exponents of nonlinearity. This great interest is motivated by the applications to the mathematical modeling of non-Newtonian fluids. These fluids include electro-rheological fluids, which have the ability to drastically change when applying some external electromagnetic field. The variable exponents of non-linearity is a given function of density, temperature, saturation, electric field, etc.

As a consequence, the topic of long-time behavior of solutions for non-linear equations with source terms has attracted many researchers. For example, there is an extensive literature on the existence, asymptotic behavior and nonexistence of solutions for the following wave equation:

$$\begin{cases} \psi_{tt} - \Delta\psi + g(\psi_t) = f(\psi), \ x \in \Omega, \ t > 0, \\ \psi(x,t) = 0, \ x \in \partial\Omega, \ t \geq 0, \\ \psi(x,0) = \psi_0(x), \ \psi_t(x,0) = \psi_1(x), \ x \in \Omega. \end{cases} \tag{1.2}$$

For when the source function $f(\psi)$ is a polynomial type function, we refer the reader to see [12–21]. Antontsev et al. [22] studied the following Petrovsky equation:

$$\psi_{tt} + \Delta^2\psi - \Delta\psi_t + |\psi_t|^{m(x)-2}\,\psi_t = |\psi|^{p(x)-2}\,\psi. \tag{1.3}$$

They proved the existence of local weak solutions by using the Banach fixed-point theorem, and gave a blow-up result for negative-initial-energy solutions, under suitable assumptions. In [23], Liao and Tan treated the following similar problem:

$$\psi_{tt} + \Delta^2\psi - M\left(\|\nabla\psi\|_2^2\right) - \Delta\psi_t + |\psi_t|^{m(x)-2}\,\psi_t = |\psi|^{p(x)-2}\,\psi,$$

where $M(s) = a + bs^\gamma$ is a positive C^1-function, $a > 0, b > 0, \gamma \geq 1$, and m, p are given measurable functions. They established some uniform decay estimates and the upper and lower bounds of the blow-up time.

Concerning the existence, asymptotic behavior and nonexistence of solutions of coupled systems, Andrade and Mognon [24] treated the following problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + f_1(u,v) = 0, \quad \text{in } [0,T]\times\Omega, \\ v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta v(s)ds + f_2(u,v) = 0, \quad \text{in } [0,T]\times\Omega, \end{cases} \tag{1.4}$$

with

$$f_1(u,v) = |u|^{p-2}u|v|^p \quad \text{and} \quad f_2(u,v) = |v|^{p-2}v(t)|u|^p,$$

where $p > 1$ if $N = 1, 2$ and $1 < p \leq \frac{N-1}{N-2}$ if $N = 3$. They proved the well posedness under some assumptions on the relaxation functions. Also, we point out the work of Agre and Rammaha [25], where they considered a system of wave equations of the form

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1}u_t = f_1(u,v), \\ v_{tt} - \Delta v + |v_t|^{r-1}v_t = f_2(u,v), \end{cases} \tag{1.5}$$

in $\Omega \times (0,T)$, with initial and boundary conditions of Dirichlet type, and the nonlinear functions f_1 and f_2 are given by

$$\begin{cases} f_1(u,v) = a|u+v|^{2(\rho+1)}\,(u+v) + b|u|^\rho u|v|^{\rho+2}, \\ f_2(u,v) = a|u+v|^{2(\rho+1)}\,(u+v) + b|v|^\rho v|u|^{\rho+2}. \end{cases} \tag{1.6}$$

They proved, under some appropriate conditions on f_1, f_2 and the initial data, several results on local and global existence. They also showed that any weak solution with negative initial energy blows up in finite time. Wang et al. [26] considered the following system:

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta u + u_t = |u|^{k-2}u\ln u \\ v_{tt} - M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta v + v_t = |v|^{k-2}v\ln v, \end{cases} \tag{1.7}$$

where $k \geq 2$, $M(s) = \alpha + \beta s^{\gamma}$ for any $\alpha \geq 1, \beta \geq 0$ and $\gamma > 0$. By employing the potential well method, the concavity method and the unstable invariant set, they proved the global existence and a finite time blow up. In [27], Bouhoufani and Hamchi discussed the following coupled system of two nonlinear hyperbolic equations with variable-exponents:

$$\begin{cases} \psi_{tt} - div(A\nabla\psi) + |\psi_t|^{m(x)-2}\,\psi_t = h_1\,(\psi,\varphi) & \text{in } \Omega \times (0,T)\,, \\ \varphi_{tt} - div(B\nabla\varphi) + |\varphi_t|^{r(x)-2}\,\varphi_t = h_2\,(\psi,\varphi) & \text{in } \Omega \times (0,T)\,, \end{cases} \tag{1.8}$$

with initial and Dirichlet-boundary conditions, where h_1 and h_2 are the coupling terms introduced in (1.1). Under suitable assumptions on the variable exponents m, r and p, the authors proved the global existence of a weak solution and established decay rates of the solution in a bounded domain. In [28], Messaoudi et al. considered the following system:

$$\begin{cases} \psi_{tt} - \Delta\psi + |\psi_t|^{m(x)-2}\psi_t + h_1(\psi,\varphi) = 0 & \text{in } \Omega \times (0,T)\,, \\ \varphi_{tt} - \Delta\varphi + |\varphi_t|^{r(x)-2}\varphi_t + h_2(\psi,\varphi) = 0 & \text{in } \Omega \times (0,T)\,, \end{cases} \tag{1.9}$$

with initial and Dirichlet-boundary conditions (here, h_1 and h_2 are the coupling terms introduced in (1.1)). The authors proved the existence of global solutions, obtained explicit decay rate estimates, under suitable assumptions on the variable exponents m, r and p, and presented some numerical tests. For more studies on the existence and asymptotic behavior of solutions for other nonlinear coupling systems, we refer to [29–34].

However, to the best of our knowledge, there are no investigations on the existence and asymptotic behavior of solutions for a nonlinear coupling system of two plate equations of type (P). Therefore, our aim in the present work is to prove the local and global existence of the solutions of this problem and study the long-time behavior of the energy associated with this problem. So, the originating motivation in the study of problem (P) is twofold:

- On the one hand, we consider the non-linear system of plate equations with a nonstandard internal forcing terms caused by the smart nature of the medium.
- On the other hand, we investigate the effect of replacing the classical power-type nonlinearity with the logarithmic nonlinearity, which is a natural extension done for many problems.

The following remark states the main difference of our result with the present ones in the literature. This will clarify our main contributions in the present work.

Remark 1.1. Notice that our work is an extension of all the above works. For example, the works of [24, 25] only treated nonlinear systems of two wave equations where the coupling functions are polynomials of constant exponents. The work of [26] only treated nonlinear systems where the coupling functions are only a polynomial of constant type. The works of [27, 28, 33] only treated nonlinear systems of two wave equations without logarithmic source terms. So, in the present work, we treated the nonlinear system of two plate equations with logarithmic source terms, and the coupling functions are nonlinear polynomials of variable exponents type, which are more general than the ones in the literature. We note here that though the logarithmic nonlinearity is somehow weaker than the polynomial nonlinearity, both the existence and stability result are not obtained by straightforward application of the method used for polynomial nonlinearity.

The rest of this paper is organized as follows: In Section 2, we present some definitions and basic properties of the logarithmic nonlinearity and the variable-exponent Lebesgue and Sobolev spaces. The local and global existence results are given in Section 3. In Section 4, we prove the stability result. A conclusion is given in Section 5.

2. Preliminaries

In this section, we present some notations and material needed in the proof of our results. We use the standard Lebesgue space $L^2(\Omega)$ and Sobolev space $H_0^2(\Omega)$ with their usual scalar products and norms. Throughout this paper, c is used to denote a generic positive constant, and we assume the following hypotheses:

(H_1) $p(\cdot)$ is a given continuous function on $\overline{\Omega}$ satisfying the log-Hölder continuity condition:

$$|p(x) - p(y)| \leq -\frac{M}{\log|x-y|}, \text{ for all } x, y \in \Omega, \text{ with } |x-y| < \delta, \tag{2.1}$$

where $M > 0$, $0 < \delta < 1$, and

$$0 < p_1 = \text{ess inf}_{x\in\Omega} p(x) \leq p(x) \leq p_2 = \text{ess sup}_{x\in\Omega} p(x). \tag{2.2}$$

(H_2) The constant α in (P) satisfies $0 < \alpha < \alpha_0$, where α_0 is the positive real number satisfying

$$\sqrt{\frac{2\pi}{c_p\alpha_0}} = e^{-\frac{3}{2}-\frac{1}{\alpha_0}}, \tag{2.3}$$

and c_p is the smallest positive number satisfying

$$\|\nabla u\|_2^2 \leq c_p\|\Delta u\|_2^2, \quad \forall u \in H_0^2(\Omega),$$

where $\|.\|_2 = \|.\|_{L^2(\Omega)}$.

Lemma 2.1. *[35, 36] (**Logarithmic Sobolev inequality**) Let v be any function in $H_0^1(\Omega)$ and $a > 0$ be any number. Then,*

$$\int_\Omega v^2 \ln|v| dx \leq \frac{1}{2}\|v\|_2^2 \ln\|v\|_2^2 + \frac{a^2}{2\pi}\|\nabla v\|_2^2 - (1 + \ln a)\|v\|_2^2. \tag{2.4}$$

Corollary 2.2. *Let u be any function in $H_0^2(\Omega)$ and a be any positive real number. Then,*

$$\int_\Omega u^2 \ln|u| dx \leq \frac{1}{2}\|u\|_2^2 \ln\|u\|_2^2 + \frac{c_p a^2}{2\pi}\|\Delta u\|_2^2 - (1 + \ln a)\|u\|_2^2. \tag{2.5}$$

Remark 2.1. The function $f(s) = \sqrt{\frac{2\pi}{c_p s}} - e^{-\frac{3}{2}-\frac{1}{s}}$ is continuous and decreasing on $(0, \infty)$, with

$$\lim_{s\to 0^+} f(s) = \infty \text{ and } \lim_{s\to\infty} f(s) = -e^{-\frac{3}{2}}.$$

Therefore, there exists a unique $\alpha_0 > 0$ such that $f(\alpha_0) = 0$; that is,

$$\sqrt{\frac{2\pi}{\alpha_0 c_p}} = e^{-\frac{3}{2}-\frac{1}{\alpha_0}}. \tag{2.6}$$

Moreover,

$$e^{-\frac{3}{2}-\frac{1}{s}} < \sqrt{\frac{2\pi}{c_p s}},\ \forall s \in (0, \alpha_0). \tag{2.7}$$

Because $f\left(\frac{2\pi e^3}{c_p}\right) > 0$, $f(\alpha) > 0$, and so (2.7) holds for $s = \alpha$.

Lemma 2.3. *[37] **(Young's inequality)** Let* $p, s : \Omega \longrightarrow [1, \infty)$ *be a measurable functions, such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)},\ \text{for } a.e.\ y \in \Omega.$$

Then, for all $a, b \geq 0$, we have

$$\frac{(ab)^{s(.)}}{s(.)} \leq \frac{a^{p(.)}}{p(.)} + \frac{b^{q(.)}}{q(.)}.$$

By taking $s = 1$ and $1 < p, q < +\infty$, it follows that for any $\varepsilon > 0$,

$$ab \leq \varepsilon a^p + C_\varepsilon b^q,\ \text{where } C_\varepsilon = 1/q(\varepsilon p)^{\frac{q}{p}}. \tag{2.8}$$

Lemma 2.4. *[37] If* $1 < q^- \leq q(x) \leq q^+ < \infty$ *holds, then*

$$\int_\Omega |v|^{q(\cdot)} dx \leq \|v\|_{q^-}^{q^-} + \|v\|_{q^+}^{q^+}, \tag{2.9}$$

for any $v \in L^{q(\cdot)}(\Omega)$.

Lemma 2.5. *[37] **(Embedding property)** Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with a smooth boundary* $\partial\Omega$*. If* $q \in C(\overline{\Omega})$*, and* $p : \Omega \longrightarrow (1, \infty)$ *is a continuous function such that*

$$essinf_{x\in\Omega}(q^*(x) - p(x)) > 0 \text{ with } q^*(x) = \begin{cases} \frac{nq(x)}{esssup_{x\in\Omega}(n-q(x))}, & \text{if } q^+ < n, \\ \infty, & \text{if } q^+ \geq n, \end{cases}$$

then the embedding $W^{1,q(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous and compact.

Lemma 2.6. *[5] (**Logarithmic Gronwall inequality**) Let* $c > 0$, $u \in L^1(0, T; \mathbb{R}^+)$*, and assume that the function* $v : [0, T] \to [1, \infty)$ *satisfies*

$$v(t) \leq c\left(1 + \int_0^t u(s)v(s) \ln v(s) ds\right),\quad 0 \leq t \leq T. \tag{2.10}$$

Then,

$$v(t) \leq c \exp\left(c \int_0^t u(s) ds\right),\quad 0 \leq t \leq T. \tag{2.11}$$

Remark 2.2. By recalling the definitions of $h_1(\psi,\varphi)$ and $h_2(\psi,\varphi)$ in (1.1), it is easily seen that

$$\psi h_1(\psi,\varphi)+\varphi h_2(\psi,\varphi)=2(p(x)+2)H(\psi,\varphi),\quad \forall(\psi,\varphi)\in\mathbb{R}^2, \tag{2.12}$$

where

$$H(\psi,\varphi)=\frac{1}{2(p(x)+2)}\Big[c_1|\psi+\varphi|^{2(p(x)+2)}+2c_2|\psi\varphi|^{p(x)+2}\Big].$$

We define the energy functional $E(t)$ associated to System (P) as follows:

$$\begin{aligned}E(t):=&\frac{1}{2}\Big[\|\psi_t\|_2^2+\|\varphi_t\|_2^2\Big]+\frac{1}{2}\Big[\|\Delta\psi\|_2^2+\|\Delta\varphi\|_2^2\Big]+\frac{\alpha+2}{4}\Big[\|\psi\|_2^2+\|\varphi\|_2^2\Big]\\&+\int_\Omega H(\psi,\varphi)dx-\frac{1}{2}\int_\Omega\psi^2\ln|\psi|dx-\frac{1}{2}\int_\Omega\varphi^2\ln|\varphi|dx.\end{aligned} \tag{2.13}$$

By multiplying the two equations in (P) by ψ_t and φ_t, respectively, integrating over Ω, using integration by parts and adding results together, we get

$$\frac{d}{dt}E(t)=-\int_\Omega|\psi_t(t)|^2dx-\int_\Omega|\varphi_t(t)|^2dx\le 0. \tag{2.14}$$

3. Existence

In this section, we state and prove the local and global existence results of system (P).

3.1. Local existence

In this subsection, we state and prove the local existence of the solutions of system (P) using the Faedo-Galerkin method and Banach fixed point theorem.

Definition 3.1. Let X be a Banach space and Y be its dual space. Then,

$$C_w([0,T],X)=\{z(t):[0,T]\to X: t\to\langle y,z(t)\rangle \text{ is continuous on } [0,T],\forall y\in Y\}.$$

Definition 3.2. Let $(\psi_0,\psi_1),(\varphi_0,\varphi_1)\in H_0^2(\Omega)\times L^2(\Omega)$. Any pair of functions

$$(\psi,\varphi)\in C_w([0,T],H_0^2(\Omega)),\ (\psi_t,\varphi_t)\in C_w([0,T],L^2(\Omega))$$

is called a weak solution of (P) if

$$\begin{cases}\frac{d}{dt}\int_\Omega\psi_t(x,t)\overline{\psi}(x)dx+\int_\Omega\Delta\psi(x,t)\Delta\overline{\psi}(x)dx\\ \quad+\int_\Omega\psi_t(x,t)\overline{\psi}(x)dx+\int_\Omega\psi(x,t)\overline{\psi}(x)dx+\int_\Omega\overline{\psi}(x)h_1dx=\int_\Omega\alpha\overline{\psi}(x)\psi\ln|\psi|dx\\ \frac{d}{dt}\int_\Omega\varphi_t(x,t)\overline{\varphi}(x)dx+\int_\Omega\Delta\varphi(x,t)\Delta\overline{\varphi}(x)dx\\ \quad+\int_\Omega\varphi_t(x,t)\overline{\varphi}(x)dx+\int_\Omega\varphi(x,t)\overline{\varphi}(x)dx+\int_\Omega\overline{\varphi}(x)h_2dx=\int_\Omega\alpha\overline{\varphi}(x)\varphi\ln|\varphi|dx\\ \psi(0)=\psi_0,\ \psi_t(0)=\psi_1,\ \varphi(0)=\varphi_0,\ \varphi_t(0)=\varphi_1,\end{cases} \tag{3.1}$$

for a.e. $t\in[0,T]$ and all test functions $\overline{\psi},\overline{\varphi}\in H_0^2(\Omega)$.

In order to establish an existence result of a local weak solution for system (P), we, first, consider the following initial-boundary-value problem:

$$\begin{cases} \psi_{tt} + \Delta^2\psi + \psi + \psi_t + \tilde{h}(x,t) = \psi \ln|\psi|^{\alpha} & \text{in } \Omega \times (0,T), \\ \varphi_{tt} + \Delta^2\varphi + \varphi + \varphi_t + \tilde{k}(x,t) = \varphi \ln|\varphi|^{\alpha} & \text{in } \Omega \times (0,T), \\ \psi = \varphi = \frac{\partial\psi}{\partial\nu} = \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ \psi(0) = \psi_0, \psi_t(0) = \psi_1, \varphi(0) = \varphi_0, \varphi_t(0) = \varphi_1 & \text{in } \Omega, \end{cases} \tag{S}$$

for given $\tilde{h}, \tilde{k} \in L^2(\Omega \times (0,T))$ and $T > 0$.

We have the following theorem of existence for problem (S).

Theorem 3.1. *Let $(\psi_0, \psi_1), (\varphi_0, \varphi_1) \in H_0^2(\Omega) \times L^2(\Omega)$. Assume that assumptions (H_1)–(H_2) hold. Then, problem (S) admits a weak solution on $[0,T)$.*

The proof of Theorem 3.1 will be carried out through several steps and lemmas. We use the standard Faedo-Galerkin method to prove this theorem.

Step 1: (Approximate solution) Consider $\{w_j\}_{j=1}^{\infty}$ an orthonormal basis of $H_0^2(\Omega)$. Let $V_k = span\{w_1, w_2, ..., w_k\}$, and the projections of initial data on the finite-dimensional subspace V_k are given by

$$\psi_0^k = \sum_{j=1}^{k} a_j w_j, \quad \varphi_0^k = \sum_{j=1}^{k} b_j w_j, \quad \psi_1^k = \sum_{j=1}^{k} c_j w_j, \quad \varphi_1^k = \sum_{j=1}^{k} d_j w_j,$$

where,

$$\begin{cases} \psi_0^k \to \psi_0 \text{ and } \varphi_0^k \to \varphi_0 \text{ in } H_0^2(\Omega) \\ \text{and} \\ \psi_1^k \to \psi_1 \text{ and } \varphi_1^k \to \varphi_1 \text{in } L^2(\Omega). \end{cases} \tag{3.2}$$

We search for solutions of the form

$$\psi^k(x) = \sum_{j=1}^{k} r_j(t) w_j(x) \text{ and } \varphi^k(x) = \sum_{j=1}^{k} g_j(t) w_j(x)$$

for the following approximate system in V_k:

$$\begin{cases} \langle \psi_{tt}^k, w_j \rangle_{L^2(\Omega)} + \langle \Delta\psi^k, \Delta w_j \rangle_{L^2(\Omega)} + \langle \psi^k, w_j \rangle_{L^2(\Omega)} + \langle \psi_t^k, w_j \rangle_{L^2(\Omega)} \\ \quad + \langle \tilde{h}(x,t), w_j \rangle_{L^2(\Omega)} = \langle \alpha\psi^k \ln|\psi^k|, w_j \rangle_{L^2(\Omega)}, \quad j = 1, 2, ..., k, \\ \langle \varphi_{tt}^k, w_j \rangle_{L^2(\Omega)} + \langle \Delta\varphi^k, \Delta w_j \rangle_{L^2(\Omega)} + \langle \varphi^k, w_j \rangle_{L^2(\Omega)} + \langle \varphi_t^k, w_j \rangle_{L^2(\Omega)} \\ \quad + \langle \tilde{k}(x,t), w_j \rangle_{L^2(\Omega)} = \langle \alpha\varphi^k \ln|\varphi^k|, w_j \rangle_{L^2(\Omega)}, \quad j = 1, 2, ..., k, \\ \psi^k(0) = \psi_0^k, \ \psi_t^k(0) = \psi_1^k, \ \varphi^k(0) = \varphi_0^k, \ \varphi_t^k(0) = \varphi_1^k. \end{cases} \tag{3.3}$$

This leads to a system of ODE's for unknown functions r_j and g_j. Based on standard existence theory for ODEs, system (3.3) admits a solution (ψ^k, φ^k) on a maximal time interval $[0, t_k)$, $0 < t_k < T$, for each $k \in \mathbb{N}$.

Lemma 3.2. *There exists a constant $T > 0$ such that the approximate solutions (ψ^k, φ^k) satisfy, for all $k \geq 1$,*

$$\begin{cases} (\psi^k) \text{ and } (\varphi^k) \text{ are bounded sequences in } L^\infty(0,T; H_0^2(\Omega)), \\ (\psi_t^k) \text{ and } (\varphi_t^k) \text{ are bounded sequences in } L^\infty(0,T; L^2(\Omega)) \cap L^2(\Omega \times (0,T)). \end{cases} \tag{3.4}$$

Proof. We multiply the first equation by r_j' and the second equation by g_j' in (3.3), sum over $j = 1, 2, ...k$ and add the two equations to obtain

$$\frac{1}{2}\frac{d}{dt}\Bigg(\|\psi_t^k\|_2^2 + \|\varphi_t^k\|_2^2 + \|\Delta\psi^k\|_2^2 + \|\Delta\varphi^k\|_2^2 + \frac{\alpha+2}{2}\left(\|\psi^k\|_2^2 + \|\varphi^k\|_2^2\right) - \int_\Omega (\psi^k)^2 \ln|\psi^k| dx - \int_\Omega (\varphi^k)^2 \ln|\varphi^k| dx\Bigg) = -\int_\Omega |\psi_t^k(s)|^2 dx - \int_\Omega |\varphi_t^k(s)|^2 dx - \int_\Omega \left(\psi_t^k \tilde{h}(x,t) + \varphi_t^k \tilde{k}(x,y)\right) dx. \tag{3.5}$$

The integration of (3.5), over $(0, t)$, leads to

$$\begin{aligned}&\frac{1}{2}\left(\|\psi_t^k\|_2^2 + \|\Delta\psi^k\|_2^2 + \|\varphi_t^k\|_2^2 + \|\Delta\varphi^k\|_2^2 + \frac{\alpha+2}{2}\left(\|\psi^k\|_2^2 + \|\varphi^k\|_2^2\right) - \int_\Omega (\psi^k)^2 \ln\psi^k dx\right)\\ &\quad - \frac{1}{2}\int_\Omega (\varphi^k)^2 \ln\varphi^k dx + \int_0^t \int_\Omega |\psi_t^k(s)|^2 dxds + \int_0^t \int_\Omega |\varphi_t^k(s)|^2 dxds\\ &= \frac{1}{2}\left(\|\Delta\psi_0^k\|_2^2 + \|\psi_1^k\|_2^2 + \|\Delta\varphi_0^k\|_2^2 + \|\varphi_1^k\|_2^2\right) + \frac{\alpha+2}{4}\left(\|\psi_0^k\|_2^2 + \|\varphi_0^k\|_2^2\right)\\ &\quad - \int_0^t \int_\Omega \left(\psi_t^k \tilde{h}(x,s) + \varphi_t^k \tilde{k}(x,s)\right) dxds + \frac{1}{2}\int_\Omega (\psi_0^k)^2 \ln|\psi_0^k| dx + \frac{1}{2}\int_\Omega (\varphi_0^k)^2 \ln|\varphi_0^k| dx.\end{aligned} \tag{3.6}$$

Combining (3.6) and convergence (3.2) implies

$$\begin{aligned}&\frac{1}{2}\left(\|\psi_t^k\|_2^2 + \|\Delta\psi^k\|_2^2 + \|\varphi_t^k\|_2^2 + \|\Delta\varphi^k\|_2^2 + \frac{\alpha+2}{2}\left(\|\psi^k\|_2^2 + \|\varphi^k\|_2^2\right) - \int_\Omega (\psi^k)^2 \ln|\psi^k| dx\right)\\ &\quad - \frac{1}{2}\int_\Omega (\varphi^k)^2 \ln|\varphi^k| dx + \int_0^t \int_\Omega |\psi_t^k(s)|^2 dxds + \int_0^t \int_\Omega |\varphi_t^k(s)|^2 dxds\\ &\le C_0 - \int_0^t \int_\Omega \left(\psi_t^k \tilde{h}(x,s) + \varphi_t^k \tilde{k}(x,s)\right) dxds.\end{aligned} \tag{3.7}$$

Applying the logarithmic Sobolev inequality to (3.7), we obtain

$$\begin{aligned}&\|\psi_t^k\|_2^2 + \|\varphi_t^k\|_2^2 + \left(1 - \frac{\alpha a^2}{2\pi}\right)\left(\|\Delta\psi^k\|_2^2 + \|\Delta\varphi^k\|_2^2\right) + \left[\frac{\alpha+2}{2} + \alpha(1+\ln a)\right]\left(\|\psi^k\|_2^2 + \|\psi^k\|_2^2\right)\\ &\quad + \int_0^t \int_\Omega |\psi_t^k(s)|^2 dxds + \int_0^t \int_\Omega |\varphi_t^k(s)|^2 dxds\\ &\le C_0 - \int_0^t \int_\Omega \left(\psi_t^k \tilde{h}(x,s) + \varphi_t^k \tilde{k}(x,s)\right) dxds + \frac{\alpha}{2}\left(\|\psi^k\|_2^2 \ln\|\psi^k\|_2^2 + \|\varphi^k\|_2^2 \ln\|\varphi^k\|_2^2\right).\end{aligned} \tag{3.8}$$

Now, we select

$$e^{-\frac{3}{2}-\frac{1}{\alpha}} < a < \sqrt{\frac{2\pi}{\alpha c_p}}, \tag{3.9}$$

to make

$$1 - \frac{c_p \alpha a^2}{2\pi} > 0, \text{ and } \frac{\alpha+2}{2} + \alpha(1+\ln a) > 0. \tag{3.10}$$

Exploiting Young's inequality and using (3.10), (3.8) becomes, for some $C > 0$,

$$\begin{aligned}&\|\psi_t^k\|_2^2 + \|\varphi_t^k\|_2^2 + \|\Delta\psi^k\|_2^2 + \|\Delta\varphi^k\|_2^2 + \|\psi^k\|_2^2 + \|\psi^k\|_2^2 \\ &\leq C + (1+\varepsilon)\int_0^{T_k}\left(\left\|u_t^k(s)\right\|_2^2 + \left\|v_t^k(s)\right\|_2^2\right)ds \\ &+ \frac{\alpha}{2}\left(\|\psi^k\|_2^2 \ln \|\psi^k\|_2^2 + \|\varphi^k\|_2^2 \ln \|\varphi^k\|_2^2\right) + C_\varepsilon \int_0^T \int_\Omega \left(\left|\tilde{h}(x,s)\right|^2 + \left|\tilde{k}(x,s)\right|^2\right)dxds.\end{aligned}$$

Using the fact that $\tilde{h}, \tilde{k} \in L^2(\Omega \times (0,T))$, we infer that

$$\begin{aligned}&\|\psi_t^k\|_2^2 + \|\varphi_t^k\|_2^2 + \|\Delta\psi^k\|_2^2 + \|\Delta\varphi^k\|_2^2 + \|\psi^k\|_2^2 + \|\psi^k\|_2^2 \\ &\leq C\left(1 + \int_0^{T_k}\left(\left\|u_t^k(s)\right\|_2^2 + \left\|v_t^k(s)\right\|_2^2\right)ds + \|\psi^k\|_2^2 \ln \|\psi^k\|_2^2 + \|\varphi^k\|_2^2 \ln \|\varphi^k\|_2^2\right).\end{aligned} \tag{3.11}$$

Let us note that

$$\psi^k(.,t) = \psi^k(.,0) + \int_0^t \frac{\partial \psi^k}{\partial s}(.,s)ds, \text{ and } \varphi^k(.,t) = \varphi^k(.,0) + \int_0^t \frac{\partial \varphi^k}{\partial s}(.,s)ds.$$

Then, applying the Cauchy-Schwarz' inequality, we get

$$\begin{aligned}\|\psi^k(t)\|_2^2 &\leq 2\|\psi^k(0)\|_2^2 + 2\left\|\int_0^t \frac{\partial\psi^k}{\partial s}(s)ds\right\|_2^2 \leq 2\|\psi^k(0)\|_2^2 + 2T\int_0^t \|\psi_t^k(s)\|_2^2 ds, \\ \|\varphi^k(t)\|_2^2 &\leq 2\|\varphi^k(0)\|_2^2 + 2\left\|\int_0^t \frac{\partial\varphi^k}{\partial s}(s)ds\right\|_2^2 \leq 2\|\varphi^k(0)\|_2^2 + 2T\int_0^t \|\varphi_t^k(s)\|_2^2 ds.\end{aligned} \tag{3.12}$$

Hence, inequality (3.11) leads to

$$\begin{aligned}\|\psi^k\|_2^2 + \|\varphi^k\|_2^2 &\leq 2\|\psi^k(0)\|_2^2 + 2\|\varphi^k(0)\|_2^2 + 2cT\left(1 + \int_0^t \|\psi^k\|_2^2 \ln\|\psi^k\|_2^2 ds + \int_0^t \|\varphi^k\|_2^2 \ln\|\varphi^k\|_2^2 ds\right) \\ &\leq 2C\left(1 + \int_0^t \|\psi^k\|_2^2 \ln\|\psi^k\|_2^2 ds + \int_0^t \|\varphi^k\|_2^2 \ln\|\varphi^k\|_2^2 ds\right) \\ &\leq 2C_1\Bigg(1 + \int_0^t \left(C_1 + \|\psi^k\|_2^2\right)\ln\left(C_1 + \|\psi^k\|_2^2\right)ds \\ &\qquad + \int_0^t \left(C_1 + \|\varphi^k\|_2^2\right)\ln\left(C_1 + \|\varphi^k\|_2^2\right)ds\Bigg),\end{aligned} \tag{3.13}$$

where, without loss of generality, $C_1 \geq 1$. The logarithmic Gronwall inequality implies that

$$\|\psi^k\|_2^2 + \|\varphi^k\|_2^2 \leq 2C_1 e^{2C_1 T} := C_2. \tag{3.14}$$

$$\|\psi^k\|_2^2 \ln\|\psi^k\|_2^2 + \|\varphi^k\|_2^2 \ln\|\varphi^k\|_2^2 \leq C.$$

After combining (3.13) and (3.14), we obtain

$$\sup_{(0,T_k)}\left[\|\psi_t^k\|_2^2 + \|\varphi_t^k\|_2^2 + \|\Delta\psi^k\|_2^2 + \|\Delta\varphi^k\|_2^2 + \|\psi^k\|_2^2 + \|\psi^k\|_2^2\right]$$

$$\leq C_\varepsilon + T\varepsilon \sup_{(0,T_k)} \left(\|u_t^k\|_2^2 + \|v_t^k\|_2^2 \right). \tag{3.15}$$

Choosing $\varepsilon < \frac{1}{2T}$, estimate (3.15) yields, for all $T_k \leq T$,

$$\sup_{(0,T_k)} \left[\|\psi_t^k\|_2^2 + \|\varphi_t^k\|_2^2 + \|\Delta\psi^k\|_2^2 + \|\Delta\varphi^k\|_2^2 + \|\psi^k\|_2^2 + \|\psi^k\|_2^2 \right] \leq C_T,$$

which completes the proof of (3.4). □

Lemma 3.3. *The approximate solutions (ψ^k, φ^k) satisfy, for all $k \geq 1$,*

$$\begin{aligned} &\psi^k \ln |\psi^k|^\alpha \to \psi \ln |\psi|^\alpha \textit{ strongly in } L^2(0,T; L^2(\Omega)), \\ &\varphi^k \ln |\varphi^k|^\alpha \to \varphi \ln |\varphi|^\alpha \textit{ strongly in } L^2(0,T; L^2(\Omega)). \end{aligned} \tag{3.16}$$

Proof. The arguments in (3.4) imply that there exist subsequences of (ψ^k) and (φ^k), still denoted by (ψ^k) and (φ^k), such that

$$\begin{cases} \psi^k \rightharpoonup^* \psi \text{ and } \varphi^k \rightharpoonup^* \varphi \ \text{ in } L^\infty(0,T; H_0^2(\Omega)), \\ \psi_t^k \rightharpoonup^* \psi_t \text{ and } \varphi_t^k \rightharpoonup^* \varphi_t \ \text{ in } L^\infty(0,T; L^2(\Omega)). \end{cases} \tag{3.17}$$

Making use of the Aubin-Lions theorem, we find, up to subsequences, that

$$\psi^k \to \psi \text{ and } \varphi^k \to \varphi \text{ strongly in } L^2(0,T; L^2(\Omega)),$$

and

$$\psi^k \to \psi \text{ and } \varphi^k \to \varphi \text{ a.e. in } \Omega \times (0,T). \tag{3.18}$$

We use (3.18) and the fact that the map $s \to s \ln |s|^\alpha$ is continuous on $\mathbb{R}$, then, we have the convergence

$$\psi^k \ln |\psi^k|^\alpha \to \psi \ln |\psi|^\alpha \text{ a.e. in } \Omega \times (0,T).$$

Using the embedding of $H_0^2(\Omega)$ in $L^\infty(\Omega)$ (since $\Omega \subset \mathbb{R}^2$), it is clear that $\psi^k \ln |\psi^k|^\alpha$ is bounded in $L^\infty(\Omega \times (0,T))$. Next, taking into account the Lebesgue bounded convergence theorem (Ω is bounded), we get

$$\psi^k \ln |\psi^k|^\alpha \to \psi \ln |\psi|^\alpha \text{ strongly in } L^2(0,T; L^2(\Omega)). \tag{3.19}$$

Similarly, we can establish the second argument of (3.17). □

Step 2: (Limiting process) Integrate (3.3) over $(0,t)$ to obtain

$$\begin{aligned} &\int_\Omega \psi_t^k(t) w_j dx - \int_\Omega \psi_1^k w_j dx + \int_\Omega \psi^k(t) w_j dx + \int_0^t \int_\Omega \Delta\psi^k(t) \Delta w_j dx ds \\ &+ \int_0^t \int_\Omega \psi_t^k(t) w_j dx ds + \int_0^t \int_\Omega w_j \tilde{h}(x,s) dx ds = \int_0^t \int_\Omega \alpha w_j \psi^k \ln |\psi^k| ds \\ &\int_\Omega \varphi_t^k(t) w_j dx - \int_\Omega \varphi_1^k w_j dx + \int_\Omega \varphi^k(t) w_j dx + \int_0^t \int_\Omega \Delta\varphi^k(t) \Delta w_j dx ds \\ &+ \int_0^t \int_\Omega \varphi_t^k(t) w_j dx ds + \int_0^t \int_\Omega w_j \tilde{k}(x,s) dx ds = \int_0^t \int_\Omega \alpha w_j \varphi^k \ln |\varphi^k| ds, \quad \forall\, j = 1, 2, ..., k. \end{aligned} \tag{3.20}$$

Convergence (3.2) and (3.17) allow us to pass to the limit in both equations in (3.20), as $k \to +\infty$, and get

$$\begin{aligned}
&\int_\Omega \psi_t(t) w_j dx - \int_\Omega \psi_1 w_j dx + \int_\Omega \psi(t) w_j dx + \int_0^t \int_\Omega \Delta\psi(t)\Delta w_j dxds \\
&+ \int_0^t \int_\Omega \psi_t(t) w_j dxds + \int_0^t \int_\Omega w_j \tilde{h}(x,s) dxds = \int_0^t \int_\Omega \alpha w_j \psi \ln|\psi| ds \\
&\int_\Omega \varphi_t(t) w_j dx - \int_\Omega \varphi_1 w_j dx + \int_\Omega \varphi(t) w_j dx + \int_0^t \int_\Omega \Delta\varphi(t)\Delta w_j dxds \\
&\int_0^t \int_\Omega \varphi_t(t) w_j dxds + \int_0^t \int_\Omega w_j \tilde{k}(x,s) dxds = \int_0^t \int_\Omega \alpha w_j \varphi \ln|\varphi| ds, \quad \forall\ j = 1,2,...,k,
\end{aligned} \tag{3.21}$$

which implies that (3.21) is valid for any $w \in H_0^2(\Omega)$. Using the fact that the left hand sides of both equations in (3.21) are absolutely continuous functions, they are differentiable for a.e. $t \in (0,\infty)$. Therefore, for a.e. $t \in [0,T]$, the equations in (3.21) become

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \psi_t(x,t) w(x) dx + \int_\Omega \Delta\psi(x,t)\Delta w(x) dx + \int_\Omega \psi w(x) dx \\
&+ \int_\Omega \psi_t w(x) dx + \int_\Omega w\tilde{h}(x,s) dx = \int_\Omega \alpha w(x)\psi(x,t) \ln|\psi(x,t)| dx, \quad \forall w \in H_0^2(\Omega) \\
&\frac{d}{dt}\int_\Omega \varphi_t(x,t) w(x) dx + \int_\Omega \Delta\varphi(x,t)\Delta w(x) dx + \int_\Omega \varphi w(x) dx \\
&+ \int_\Omega \varphi_t w(x) dx + \int_\Omega w\tilde{k}(x,s) dx = \int_\Omega \alpha w(x)\varphi(x,t) \ln|\varphi(x,t)| dx, \quad \forall w \in H_0^2(\Omega).
\end{aligned}$$

Step 3: (Initial conditions) To handle the initial conditions, we note that

$$\begin{aligned}
\psi^k \rightharpoonup \psi \text{ and } \varphi^k \rightharpoonup \varphi \quad &\text{weakly in } L^2(0,T;H_0^2(\Omega)), \\
\psi_t^k \rightharpoonup \psi_t \text{ and } \varphi_t^k \rightharpoonup \varphi_t \quad &\text{weakly in } L^2(0,T;L^2(\Omega)).
\end{aligned} \tag{3.22}$$

Thus, using the Lions lemma and (3.2), we easily obtain

$$\psi(x,0) = \psi_0(x) \text{ and } \varphi(x,0) = \varphi_0(x).$$

As in [38], multiply (3.3) by $\tilde{\psi} \in C_0^\infty(0,T)$ and integrate over $(0,T)$, and we obtain for any $w \in V_k$

$$\begin{aligned}
&-\int_0^T \int_\Omega \psi_t^k w \tilde{\psi}'(t) dxdt = -\int_0^T \int_\Omega \Delta\psi^k \Delta w \tilde{\psi} dxdt - \int_0^T \int_\Omega \psi^k w \tilde{\psi} dxdt \\
&+ \int_0^T \int_\Omega \alpha w \psi^k \tilde{\psi} \ln|\psi^k| dxdt - \int_0^T \int_\Omega \psi_t^k w \tilde{\psi} dxdt - \int_0^T \int_\Omega w \tilde{\psi} \tilde{h}(x,s) dxdt \\
&-\int_0^T \int_\Omega \varphi_t^k w \tilde{\psi}'(t) dxdt = -\int_0^T \int_\Omega \Delta\varphi^k \Delta w \tilde{\psi} dxdt - \int_0^T \int_\Omega \varphi^k w \tilde{\psi} dxdt \\
&+ \int_0^T \int_\Omega \alpha w \varphi^k \tilde{\psi} \ln|\varphi^k| dxdt - \int_0^T \int_\Omega \varphi_t^k w \tilde{\psi} dxdt - \int_0^T \int_\Omega w \tilde{\psi} \tilde{k}(x,s) dxdt.
\end{aligned} \tag{3.23}$$

As $k \to +\infty$, we have, for any $w \in H_0^2(\Omega)$ and any $\tilde{\psi} \in C_0^\infty((0,T))$,

$$\begin{aligned}
&-\int_0^T\int_\Omega \psi_t w\tilde{\psi}'(t)dxdt = -\int_0^T\int_\Omega \Delta\psi\Delta w\tilde{\psi}dxdt - \int_0^T\int_\Omega \psi w\tilde{\psi}dxdt \\
&+\int_0^T\int_\Omega \alpha w\psi\tilde{\psi}\ln|\psi|dxdt - \int_0^T\int_\Omega w\tilde{\psi}\tilde{h}(x,s)dxdt - \int_0^T\int_\Omega \psi_t w\tilde{\psi}dxdt \\
&-\int_0^T\int_\Omega \psi_t w\tilde{\psi}'(t)dxdt = -\int_0^T\int_\Omega \Delta\psi\Delta w\tilde{\psi}dxdt - \int_0^T\int_\Omega \varphi w\tilde{\psi}dxdt \\
&+\int_0^T\int_\Omega \alpha w\varphi\tilde{\psi}\ln|\varphi|dxdt - \int_0^T\int_\Omega w\tilde{\psi}\tilde{k}(x,s)dxdt - \int_0^T\int_\Omega \varphi_t w\tilde{\psi}dxdt.
\end{aligned} \tag{3.24}$$

This means (see [38])

$$\psi_{tt}, \varphi_{tt} \in L^2([0,T), H^{-2}(\Omega)).$$

Recalling that $\psi_t, \varphi_t \in L^2((0,T), L^2(\Omega))$, we obtain

$$\psi_t, \varphi_t \in C([0,T), H^{-2}(\Omega)).$$

So, $\psi_t^k(x,0)$ makes sense, and

$$\psi_t^k(x,0) \to \psi_t(x,0) \text{ and } \varphi_t^k(x,0) \to \varphi_t(x,0) \text{ in } H^{-2}(\Omega).$$

However,

$$\psi_t^k(x,0) = \psi_1^k(x) \to \psi_1(x) \text{ and } \varphi_t^k(x,0) = \varphi_1^k(x) \to \varphi_1(x) \text{ in } L^2(\Omega).$$

Hence,

$$\psi_t(x,0) = \psi_1(x) \text{ and } \varphi_t(x,0) = \varphi_1(x).$$

This completes the proof. Therefore, (ψ, φ) is a local solution of (S).
Now, we state and the existence result related to system (P).

Theorem 3.4. *System* (P) *admits a weak solution* (ψ, φ), *in the sense of Definition* (3.2)*, for* T *small enough.*

Proof. By recalling the definition of h_1 and h_2 in (1.1) and using Lemma 2.4 and Young's inequality, we have

$$\begin{aligned}
&\int_\Omega |h_1(y,z)|^2\,dx \le 2\left[c_1^2\int_\Omega |y+z|^{2(2p(x)+3)}\,dx + c_2^2\int_\Omega |y|^{2p(x)+2}\,|z|^{2p(x)+4}\,dx\right] \\
&\le C_0\left[\int_\Omega |y+z|^{2(2p_2+3)}\,dx + \int_\Omega |y+z|^{2(2p_1+3)}\,dx + \int_\Omega |y|^{3(2p(x)+2)}\,dx + \int_\Omega |z|^{3(p(x)+2)}\,dx\right],
\end{aligned} \tag{3.25}$$

where $C_0 = 2\max\left\{c_1^2, 3c_2^2\right\} > 0$. By the embedding, we have, for $n = 2$,

$$1 < 3(p_1+2) \le 3(p_2+2) \le 2(2p_2+3) \le 3(2p_2+2) < \infty,$$

since $3 \le 2p_1 + 3 \le 2p(x) + 3 \le 2p_2 + 3 < \infty$. Therefore, estimate (3.25) and Lemma 2.4 lead to

$$\int_\Omega |h_1(y,z)|^2\,dx$$

$$\leq C_1\left[\|\Delta(y+z)\|_2^{2(2p_2+3)}+\|\Delta(y+z)\|_2^{2(2p_1+3)}+\|\Delta y\|_2^{3(2p_2+2)}+\|\Delta y\|_2^{3(2p_1+2)}\right]$$
$$+\,C_1\left[\|\Delta z\|_2^{3(p_2+2)}+\|\Delta z\|_2^{3(p_1+2)}\right]<+\infty, \tag{3.26}$$

where $C_1 = C_0C_e$. Consequently, under the assumption (2.2), we have,

$$\int_\Omega |h_1(y,z)|^2\,dx<\infty$$

and, similarly,

$$\int_\Omega |h_2(y,z)|^2\,dx<\infty.$$

Thus,

$$h_1(y,z), h_2(y,z)\in L^2(\Omega\times(0,T)),\ \forall\, y,z\in H_0^2(\Omega).$$

Now, let

$$W_T=\left\{w\in L^\infty((0,T),H_0^2(\Omega))/w_t\in L^\infty((0,T),L^2(\Omega))\right\},$$

and define the map $K : W_T\times W_T :\longrightarrow W_T\times W_T$ by $K(y,z)=(\psi,\varphi)$, where (ψ,φ) is the solution of

$$\begin{cases}\psi_{tt}+\Delta^2\psi+\psi+\psi_t+h_1(y,z)=\alpha\psi\ln|\psi|, & \text{in }\Omega,\ t>0,\\ \varphi_{tt}+\Delta^2\varphi+\varphi+\varphi_t+h_2(y,z)=\alpha\varphi\ln|\varphi|, & \text{in }\Omega,\ t>0,\\ \psi(\cdot,t)=\varphi(\cdot,t)=\frac{\partial\psi}{\partial\nu}=\frac{\partial\varphi}{\partial\nu}=0, & \text{on }\partial\Omega, t\geq 0,\\ (\psi(0),\varphi(0))=(\psi_0,\varphi_0),(\psi_t(0),\varphi_t(0))=(\psi_1,\varphi_1), & \text{in }\Omega.\end{cases} \tag{3.27}$$

We note that W_T is a Banach space with respect to the norm

$$\|w\|_{W_T}^2=\sup_{(0,T)}\int_\Omega|\Delta w|^2dx+\sup_{(0,T)}\int_\Omega|w_t|^2dx,$$

and K is well defined by virtue of Theorem (3.1). As in [39] and [33], it is a routine work to prove that K is a contraction mapping from a bounded ball $B(0,d)$ into itself, where

$$B(0,d)=\left\{(y,z)\in W_T\times W_T/\,\|(y,z)\|_{W_{T_0}\times W_{T_0}}\leq d\right\},$$

for $d>1$ and some $T_0>0$. Then, the Banach-fixed-point theorem guarantees the existence of a solution $(\psi,\varphi)\in B(0,d)$, such that $K(\psi,\varphi)=(\psi,\varphi)$, which is a local weak solution of (P).

□

3.2. Global existence

In this subsection, we state and prove a global existence result using the potential wells corresponding to the logarithmic nonlinearity. For this purpose, we define the following functionals:

$$J(\psi,\varphi)=\frac{1}{2}\left(\|\Delta\psi\|_2^2+\|\Delta\varphi\|_2^2-\int_\Omega\psi^2\ln|\psi|^\alpha dx-\int_\Omega\varphi^2\ln|\varphi|^\alpha dx\right)$$
$$+\frac{\alpha+2}{4}\left(\|\psi\|_2^2+\|\varphi\|_2^2\right). \tag{3.28}$$

$$I(\psi,\varphi)=\|\Delta\psi\|_2^2+\|\Delta\phi\|_2^2+\|\psi\|_2^2+\|\phi\|_2^2-\int_\Omega\psi^2\ln|\psi|^\alpha dx-\int_\Omega\varphi^2\ln|\phi|^\alpha dx. \tag{3.29}$$

Remark 3.1. 1) From the above definitions, it is clear that

$$J(\psi,\varphi)=\frac{1}{2}I(\psi,\varphi)+\frac{\alpha}{4}\left(\|\psi\|_2^2+\|\varphi\|_2^2\right), \tag{3.30}$$

$$E(t)=\frac{1}{2}\left(\|\psi_t\|_2^2+\|\varphi_t\|_2^2\right)+\int_\Omega H(\psi,\phi)dx+J(\psi,\varphi). \tag{3.31}$$

2) According to the logarithmic Sobolev inequality, $J(\psi,\varphi)$ and $I(\psi,\varphi)$ are well defined.

We define the potential well (stable set)

$$\mathcal{W}=\{(\psi,\phi)\in H_0^2(\Omega)\times H_0^2(\Omega), I(\psi,\phi)>0\}\cup\{(0,0)\}.$$

The potential well depth is defined by

$$0<d=\inf_{(\psi,\varphi)}\{\sup_{\lambda\geq 0} J(\lambda\psi,\lambda\varphi):(\psi,\varphi)\in H_0^2(\Omega)\times H_0^2(\Omega), \|\Delta\psi\|_2\neq 0 \text{ and } \|\Delta\varphi\|_2\neq 0\}, \tag{3.32}$$

and the well-known Nehari manifold

$$\mathcal{N}=\{(\psi,\varphi):(\psi,\varphi)\in H_0^2(\Omega)\times H_0^2(\Omega)/I(\psi,\varphi)=0, \|\Delta\psi\|_2\neq 0 \text{ and } \|\Delta\varphi\|_2\neq 0\}. \tag{3.33}$$

Proceeding as in [21, 40], one has

$$0<d=\inf_{(\psi,\varphi)\in\mathcal{N}} J(\psi,\varphi). \tag{3.34}$$

Lemma 3.5. *For any* $(\psi,\varphi)\in H_0^2(\Omega)\times H_0^2(\Omega)$, $\|\psi\|_2\neq 0$ *and* $\|\varphi\|_2\neq 0$, *let* $g(\lambda)=J(\lambda\psi,\lambda\varphi)$. *Then, we have*

$$I(\lambda\psi,\lambda\varphi)=\lambda g'(\lambda)\begin{cases}>0, & 0\leq\lambda<\lambda^*,\\ =0, & \lambda=\lambda^*,\\ <0, & \lambda^*<\lambda<+\infty,\end{cases}$$

where

$$\lambda^*=\exp\left(\frac{\|\Delta\psi\|_2^2+\|\Delta\varphi\|_2^2-\int_\Omega\psi^2\ln|\psi|^\alpha dx-\int_\Omega\varphi^2\ln|\varphi|^\alpha dx}{\alpha(\|\psi\|_2^2+\|\varphi\|_2^2)}\right)$$

Proof.

$$\begin{aligned}g(\lambda)=J(\lambda\psi,\lambda\varphi)=&\frac{1}{2}\lambda^2\left(\|\Delta\psi\|_2^2+\|\Delta\varphi\|_2^2\right)-\frac{1}{2}\lambda^2\left(\int_\Omega\psi^2\ln|\psi|^\alpha dx+\int_\Omega\varphi^2\ln|\varphi|^\alpha dx\right)\\&+\lambda^2\left(\frac{\alpha+2}{4}-\frac{\alpha}{2}\ln|\lambda|\right)\left(\|\psi\|_2^2+\|\varphi\|_2^2\right).\end{aligned}$$

Since $\|\psi\|_2\neq 0$ and $\|\varphi\|_2\neq 0$, $g(0)=0$, $g(+\infty)=-\infty$, and

$$\begin{aligned}I(\lambda\psi,\lambda\varphi)=\lambda\frac{dJ(\lambda\psi,\lambda\varphi)}{d\lambda}=\lambda g'(\lambda)=&\lambda^2\left(\|\Delta\psi\|_2^2+\|\Delta\varphi\|_2^2\right)-\lambda^2\left(\int_\Omega\psi^2\ln|\psi|^\alpha dx+\int_\Omega\varphi^2\ln|\phi|^\alpha dx\right)\\&+\lambda^2(1-\alpha\ln|\lambda|)\left(\|\psi\|_2^2+\|\varphi\|_2^2\right).\end{aligned}$$

This implies that $\frac{d}{d\lambda}J(\lambda u)_{\lambda=\lambda^*}=0$, and $J(\lambda u)$ is increasing on $0<\lambda\leq\lambda^*$, decreasing on $\lambda^*\leq\lambda<\infty$ and takes the maximum at $\lambda=\lambda^*$. In other words, there exists a unique $\lambda^*\in(0,\infty)$ such that $I(\lambda^*u)=0$, and so, we have the desired result. □

Lemma 3.6. *Let* $(\psi, \varphi) \in H_0^2(\Omega) \times H_0^2(\Omega)$ *and* $\beta_0 = \sqrt{\frac{2\pi}{\alpha}} e^{\frac{1}{\alpha}+1}$. *If* $0 < \|\psi\|_2 \le \beta_0$ *and* $0 < \|\varphi\|_2 \le \beta_0$, *then* $I(\psi, \varphi) \ge 0$.

Proof. Using the logarithmic Sobolev inequality (2.5), for any $a > 0$, we have

$$\begin{aligned} I(\psi, \phi) &= \|\Delta\psi\|_2^2 + \|\Delta\varphi\|_2^2 - \int_\Omega \psi^2 \ln|\psi|^\alpha dx - \int_\Omega \varphi^2 \ln|\varphi|^\alpha dx \\ &\ge \left(1 - \frac{c_p \alpha a^2}{2\pi}\right)\left(\|\Delta\psi\|_2^2 + \|\Delta\varphi\|_2^2\right) + \alpha(1+\ln a)\|\psi\|_2^2 - \frac{\alpha}{2}\|\psi\|_2^2 \ln\|\psi\|_2^2 \\ &\quad + \alpha(1+\ln a)\|\phi\|_2^2 - \frac{\alpha}{2}\|\phi\|_2^2 \ln\|\phi\|_2^2. \end{aligned} \tag{3.35}$$

Taking $a = \sqrt{\frac{2\pi}{c_p \alpha}}$ in (3.35), we obtain

$$\begin{aligned} I(\psi, \phi) \ge &\left(1 + \alpha\left(1 + \ln\sqrt{\frac{2\pi}{\alpha}}\right) - \frac{\alpha}{2}\ln\|\psi\|_2^2\right)\|\psi\|_2^2 \\ &+ \left(1 + \alpha\left(1 + \ln\sqrt{\frac{2\pi}{\alpha}}\right) - \frac{\alpha}{2}\ln\|\varphi\|_2^2\right)\|\varphi\|_2^2. \end{aligned} \tag{3.36}$$

If $0 < \|\psi\|_2 \le \beta_0$ and $0 < \|\varphi\|_2 \le \beta_0$, then

$$1 + \alpha\left(1 + \ln\sqrt{\frac{2\pi}{\alpha}}\right) - \frac{\alpha}{2}\ln\|\psi\|_2^2 \ge 0 \text{ and } 1 + \alpha\left(1 + \ln\sqrt{\frac{2\pi}{\alpha}}\right) - \frac{\alpha}{2}\ln\|\varphi\|_2^2 \ge 0,$$

which gives $I(\psi, \varphi) \ge 0$. □

Lemma 3.7. *The potential well depth d satisfies*

$$d \ge \frac{\pi}{\alpha} e^{2+\frac{2}{\alpha}}. \tag{3.37}$$

Proof. The proof of this lemma is similar to the proof of Lemma 4.3. in [10]. □

Lemma 3.8. *Let* $(\psi_0, \psi_1), (\varphi_0, \varphi_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *such that* $0 < E(0) < d$ *and* $I(\psi_0, \varphi_0) > 0$. *Then, any solution of* (P), $(\psi, \varphi) \in \mathcal{W}$.

Proof. Let T be the maximal existence time of a weak solution of (ψ, φ). From (2.14) and (3.31), we have

$$\frac{1}{2}\left(\|\psi_t\|^2 + \|\varphi_t\|^2\right) + J(\psi, \varphi) \le \frac{1}{2}\left(\|\psi_1\|^2 + \|\varphi_1\|^2\right) + J(\psi_0, \varphi_0) < d, \text{ for any } t \in [0, T). \tag{3.38}$$

Then, we claim that $(\psi(t), \varphi(t)) \in \mathcal{W}$ for all $t \in [0, T)$. If not, then there is a $t_0 \in (0, T)$ such that $I(\psi(t_0), \varphi(t_0)) < 0$. Using the continuity of $I(\psi(t), \varphi(t))$ in t, we deduce that there exists a $t_* \in (0, T)$ such that $I(\psi(t_*), \varphi(t_*)) = 0$. Then, using the definition of d in (3.32) gives

$$d \le J(\psi(t_*), \varphi(t_*)) \le E(\psi(t_*), \varphi(t_*)) \le E(0) < d,$$

which is a contradiction. □

4. Stability

In this section, we discuss the decay of the solutions of system (P).

Lemma 4.1. *The functional*

$$L(t) = E(t) + \varepsilon \int_\Omega \psi\psi_t dx + \varepsilon \int_\Omega \varphi\varphi_t dx + \frac{\varepsilon}{2}\int_\Omega \psi^2 dx + \frac{\varepsilon}{2}\int_\Omega \varphi^2 dx$$

satisfies, along the solutions of (P),

$$L \sim E, \tag{4.1}$$

and

$$\begin{aligned} L'(t) \le& -(1-\varepsilon)\int_\Omega \left(|\psi_t|^2 + |\varphi_t|^2\right) dx - \varepsilon \int_\Omega \left(|\Delta\psi|^2 + |\Delta\varphi|^2\right) dx - \varepsilon \int_\Omega 2(p(x)+2)H(\psi,\varphi)dx \\ &+ \varepsilon\alpha \int_\Omega \psi^2 \ln|\psi| dx + \varepsilon\alpha \int_\Omega \varphi^2 \ln|\varphi| dx - \varepsilon \int_\Omega \psi^2 dx - \varepsilon \int_\Omega \varphi^2 dx. \end{aligned} \tag{4.2}$$

Proof. We differentiate $L(t)$ and use (P) to get

$$\begin{aligned} L'(t) =& -\int_\Omega \psi_t^2 dx - \int_\Omega \varphi_t^2 dx + \varepsilon \int_\Omega \psi\psi_t dx + \varepsilon \int_\Omega \varphi\varphi_t dx \\ &+ \varepsilon \int_\Omega \psi_t^2 dx + \varepsilon \int_\Omega \psi\Big[-\Delta^2\psi - \psi_t - \psi - h_1(\psi,\varphi) + \alpha\psi\ln|\psi|\Big]dx \\ &+ \varepsilon \int_\Omega \varphi_t^2 dx + \varepsilon \int_\Omega \varphi\Big[-\Delta^2\varphi - \varphi - \varphi_t - h_2(\psi,\varphi) + \alpha\varphi\ln|\varphi|\Big]dx \\ =& -(1-\varepsilon)\int_\Omega \left(|\psi_t|^2 + |\varphi_t|^2\right) dx - \varepsilon \int_\Omega \left(|\Delta\psi|^2 + |\Delta\varphi|^2\right) dx \\ &- \varepsilon \int_\Omega \psi h_1(\psi,\varphi)dx - \varepsilon \int_\Omega \varphi h_2(\psi,\varphi)dx - \varepsilon \int_\Omega \psi^2 dx - \varepsilon \int_\Omega \varphi^2 dx \\ &- \varepsilon \int_\Omega \psi\psi_t dx - \varepsilon \int_\Omega \varphi\varphi_t dx + \varepsilon \int_\Omega \psi\psi_t dx + \varepsilon \int_\Omega \varphi\varphi_t dx \\ &+ \varepsilon\alpha \int_\Omega \psi^2 \ln|\psi| dx + \varepsilon\alpha \int_\Omega \varphi^2 \ln|\varphi| dx. \end{aligned}$$

Recalling the definition of H, (4.2) is established. □

Theorem 4.2. *Assume that* (H_1) *and* (H_2) *hold and let* $(\psi_0,\psi_1),(\varphi_0,\varphi_1) \in H_0^2(\Omega)\times L^2(\Omega)$. *Assume further that* $0 < E(0) < \ell\tau < d$*, where*

$$\tau = \frac{\pi}{\alpha}e^{2+\frac{2}{\alpha}},\ \ 0 < \sqrt{\frac{2\ell}{\alpha}}e^{\frac{1}{\alpha}} < 1. \tag{4.3}$$

Then there exist two positive constants κ_1 and κ_2 such that the energy defined in (2.13) satisfies

$$0 < E(t) \le \kappa_1 e^{-\kappa_2 t},\ t \ge 0. \tag{4.4}$$

Proof. By adding $(\pm\varepsilon\sigma E)$ to the right hand side of (4.2), we have, for a positive constant σ,

$$
\begin{aligned}
L'(t) = & -\varepsilon\sigma E(t) + \left(\frac{\sigma\varepsilon}{2} + \varepsilon - 1\right)\int_\Omega \left(|\psi_t|^2 + |\varphi_t|^2\right)dx + \varepsilon\left(\frac{\sigma}{2} - 1\right)\int_\Omega \left(|\Delta\psi|^2 + |\Delta\varphi|^2\right)dx \\
& + \varepsilon\sigma\int_\Omega H(\psi,\varphi)dxdx - \varepsilon\int_\Omega 2(p(x)+2)H(\psi,\varphi)dx \\
& + \frac{\varepsilon\sigma(\alpha+2)}{4}\left[\|\psi\|_2^2 + \|\varphi\|_2^2\right] - \varepsilon\int_\Omega \psi^2 dx - \varepsilon\int_\Omega \varphi^2 dx \\
& + \varepsilon(1-\frac{\sigma}{2})\int_\Omega \psi^2 \ln|\psi|^\alpha dx + \varepsilon(1-\frac{\sigma}{2})\int_\Omega \varphi^2\ln|\varphi|^\alpha dx.
\end{aligned}
$$

Using the logarithmic Sobolev inequality, we get

$$
\begin{aligned}
L'(t) \le & -\sigma\varepsilon E(t) + \left(\frac{\sigma\varepsilon}{2} + \varepsilon - 1\right)\int_\Omega \left(|\psi_t|^2 + |\varphi_t|^2\right)dx + \varepsilon\left(\frac{\sigma}{2} - 1\right)\int_\Omega \left(|\Delta\psi|^2 + |\Delta\varphi|^2\right)dx \\
& + \varepsilon\left[\sigma - 2(p(x)+2)\right]\int_\Omega H(\psi,\varphi)dx \\
& + \frac{\varepsilon\sigma(\alpha+2)}{4}\left[\|\psi\|_2^2 + \|\varphi\|_2^2\right] - \varepsilon\int_\Omega \psi^2 dx - \varepsilon\int_\Omega \varphi^2 dx \\
& + \varepsilon\alpha(1-\frac{\sigma}{2})\left[\frac{1}{2}\|\psi\|_2^2\ln\|\psi\|_2^2 + \frac{a^2}{2\pi}\|\Delta\psi\|_2^2 - (1+\ln a)\|\psi\|_2^2\right] \\
& + \varepsilon\alpha(1-\frac{\sigma}{2})\left[\frac{1}{2}\|\varphi\|_2^2\ln\|\varphi\|_2^2 + \frac{a^2}{2\pi}\|\Delta\varphi\|_2^2 - (1+\ln a)\|\varphi\|_2^2\right] \\
\le & -\sigma\varepsilon E(t) + \left(\frac{\sigma\varepsilon}{2} + \varepsilon - 1\right)\int_\Omega \left(|\psi_t|^2 + |\varphi_t|^2\right)dx \\
& + \varepsilon\left[\sigma - 2(p_1+2)\right]\int_\Omega H(\psi,\varphi)dx \\
& - \varepsilon\left(1-\frac{\sigma}{2}\right)\left(1-\frac{\alpha a^2}{2\pi}\right)\int_\Omega \left(|\Delta\psi|^2 + |\Delta\varphi|^2\right)dx \\
& + \varepsilon\left[\frac{\sigma(\alpha+2)}{4} - 1 + \alpha\left(1-\frac{\sigma}{2}\right)\left(\frac{1}{2}\ln\|\psi\|_2^2 - (1+\ln a)\right)\right]\|\psi\|_2^2 \\
& + \varepsilon\left[\frac{\sigma(\alpha+2)}{4} - 1 + \alpha\left(1-\frac{\sigma}{2}\right)\left(\frac{1}{2}\ln\|\varphi\|_2^2 - (1+\ln a)\right)\right]\|\varphi\|_2^2.
\end{aligned}
\tag{4.5}
$$

Using (2.13), (2.14) and the fact that $u \in W$, we find that

$$
\begin{aligned}
\ln\|\psi\|_2^2 & < \ln\left(\frac{4E(t)}{\alpha}\right) < \ln\left(\frac{4E(0)}{\alpha}\right) \\
& < \ln\frac{4\ell\tau}{\alpha} = \ln\frac{4\pi\ell e^{2+\frac{2}{\alpha}}}{\alpha^2}.
\end{aligned}
\tag{4.6}
$$

Similarly, we obtain

$$
\ln\|\varphi\|_2^2 < \ln\frac{4\pi\ell e^{2+\frac{2}{\alpha}}}{\alpha^2}. \tag{4.7}
$$

By picking $0 < \sigma < \min\{2(p_1 + 2), \frac{4}{\alpha+2}\}$ and taking a such that

$$\frac{2\sqrt{\pi\ell}}{\alpha} e^{\frac{1}{\alpha}} < a < \sqrt{\frac{2\pi}{\alpha}}, \tag{4.8}$$

we guarantee the following:

$$\frac{\sigma(\alpha + 2)}{4} - 1 < 0,$$

$$\left(1 - \frac{\sigma}{2}\right)\left[\frac{1}{2} \ln \|\psi\|_2^2 - (1 + \ln a)\right] < 0,$$

and

$$\left(1 - \frac{\sigma}{2}\right)\left[\frac{1}{2} \ln \|\varphi\|_2^2 - (1 + \ln a)\right] < 0.$$

Now, selecting $\varepsilon > 0$ small enough so that (4.1) remains true, and

$$\left(\frac{\sigma\varepsilon}{2} + \varepsilon - 1\right) < 0.$$

Using all the above inequalities, we see that

$$L'(t) \leq -\sigma\varepsilon E(t). \tag{4.9}$$

Now, using (4.1) and integrating (4.9) over $(0, t)$, the proof of Theorem 4.2 is completed. □

5. Conclusions

This paper has proved the local and global existence and established an exponential decay estimate for a nonlinear system with nonlinear forcing terms of variable exponent type and logarithmic source terms. These results are new and generalize many related problems in the literature. In addition, the results in this paper have shown how to overcome the difficulties coming from the variable exponent and logarithmic nonlinearities.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to acknowledge the support provided by King Fahd University of Petroleum & Minerals (KFUPM), Saudi Arabia. The support provided by the Interdisciplinary Research Center for Construction & Building Materials (IRC-CBM) at King Fahd University of Petroleum & Minerals (KFUPM), Saudi Arabia, for funding this work through Project (No. INCB2311), is also greatly acknowledged.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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