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Research article

Permutabitity of principal MS-algebras

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Abstract: In this paper, we continue to introduce new properties of principal MS-algebras deal with congruence relations via MS-congruence pairs. Necessary and sufficient conditions for a pair of congruences $(\theta_1, \theta_2) \in Con(L^{\circ\circ}) \times Con_{lat}(D(L))$ to become an MS-congruence pair of a principal MS-algebra (principal Stone algebra) L are obtained. We describe the lattice of all MS-congruence pairs of a principal MS-algebra L which induced by the Boolean elements of L. We introduce certain special congruence Ψ on a principal MS-algebra and its related properties which are useful for the topic of this paper. A characterization of 2-permutable congruences using MS-congruence pairs of principal MS-algebras is established. Finally, a characterization of n-permutability of congruences of principal MS-algebras is given, which is a generalization of the characterization of 2-permutability of congruences of such algebras.

Keywords: MS-algebras; principal MS-algebras; congruence relation; MS-congruence pairs; 2-permutability of congruences; n-permutability of congruences
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1. Introduction

MS-algebras were considered by T. S. Blyth and J. C. Varlet [15] as a common properties of de Morgan algebras and Stone algebras. T. S. Blyth and J. C. Varlet [16] described the lattice $\Lambda(MS)$ of all subclasses of the class MS of all MS-algebras by identities.

A. Badawy, D. Guffová and M. Haviar [10] introduced and characterized the class of principal MS-algebras and the class of decomposable MS-algebras by means of principal MS-triples and decomposable MS-triples, respectively. They obtained a one-to-one correspondence between principal MS-algebras (decomposable MS-algebras) and principal MS-triples (decomposable MS-triples). Moreover, they proved that the class of principal MS-algebras is a subclass of the class of decomposable MS-algebras.

A. Badawy [6] established the relationship between de Morgan filters and congruences of a decomposable MS-algebra. A. Badawy [3] introduced the notion of d_L-filters of a principal MS-algebra and characterized certain congruences in terms of d_L-filters. Recently, A. Badawy, M. Haviar and M. Ploščica [11] introduced and described the notions of congruence pairs and perfect extensions of principal MS-algebras. They characterized the lattice of congruences of a principal MS-algebra in terms of congruence pairs. In 2021, S. El-Assar and A. Badawy [19] characterized permutability of congruences and strong extensions of decomposable MS-algebras by means of congruence pairs.

We review in Section 2 many basic concepts and results that we are using throughout this article. We give an example (Example 2.1) to determine the principsl MS-triple $(L^{\circ\circ}, D(L), \varphi_L)$ assosiated with a certain principal MS-algebra L. In Section 3, we give equivalent conditions for a pair of congruence $(\theta_1, \theta_2) \in Con(L^{\circ\circ}) \times Con_{lat}(D(L))$ to become an MS-congruence pair of a principal MS-algebra L, where θ_1 is a congruence on a de Morgan algebra $L^{\circ\circ}$ and θ_2 is a lattice congruence on the lattice $D(L)$. Also, we characterize the lattice of all MS- congruence pairs which are induced by the Boolean elements of a principal MS-algebra. In Section 4, we discuss many properties of the congruence relation Ψ which defined on a principal MS-algebra L by

$$(a, b) \in \Psi \Leftrightarrow a^\circ = b^\circ \Leftrightarrow a^{\circ\circ} = b^{\circ\circ}.$$

We prove that Ψ permutes with each congruence relation θ on L. Also, we characterize 2-permutability of congruences (briefly permutability) by means of MS-congruence pairs. We illustrate Examples 3.1 and 4.1 to clarify Theorems 3.1 and 4.4, respectively. Also, Example 4.2 introduces certain principal MS-algebra L, which has not 2-permutable congruences as well as $D(L)$ has not also 2-permutable congruences. In Section 5, A characterization of n-permutability of congruences of a principal MS-algebra is given, which is a generalization of the characterization of 2-permutability of congruences of such algebras.

2. Preliminaries

In this section, we recall some specific definitions and remarkable results which are discussed in previous articles [10, 14–17, 19].

Definition 2.1. *[16] A de Morgan algebra is an algebra* $(L; \vee, \wedge, ^-, 0, 1)$ *of type (2,2,1,0,0), where* $(L; \vee, \wedge, 0, 1)$ *is a bounded distributive lattice and the unary operation* $^-$ *satisfying:*

$$(1)\ \ \overline{\overline{a}} = a, \quad (2)\ \ \overline{(a \vee b)} = \overline{a} \wedge \overline{b}, \quad (3)\ \ \overline{1} = 0.$$

Definition 2.2. *[15] A Stone algebra is an algebra* $(L; \vee, \wedge, ^*, 0, 1)$ *of type* $(2, 2, 1, 0, 0)$*, where* $(L; \vee, \wedge, 0, 1)$ *is a bounded distributive lattice and the unary operation* * *satisfying:*

$$(1)\ \ (a \wedge b)^* = a^* \vee b^*, \quad (2)\ \ a^* \vee a^{**} = 1, \quad (3)\ \ 1^* = 0.$$

Definition 2.3. *[15] An MS-algebra is an algebra* $(L; \vee, \wedge, ^\circ, 0, 1)$ *of type (2,2,1,0,0), where* $(L; \vee, \wedge, 0, 1)$ *is a bounded distributive lattice and a unary operation* $^\circ$ *satisfying:*

$$(1)\ \ a \leq a^{\circ\circ}, \quad (2)\ \ (x \wedge y)^\circ = x^\circ \vee y^\circ, \quad (3)\ \ 1^\circ = 0.$$

The class MS of all MS-algebras is equational. A de Morgan algebra is an MS-algebra satisfying the identity, $a = a^{\circ\circ}$. The class S of a Stone algebra is a subclass of MS satisfying $a \wedge a^\circ = 0$.

The basic properties of MS-algebras which were shown in [15] are given in the following theorem.

Theorem 2.1. *Let a, b be any two elements of an MS-algebra L. Then*

(1) $0^{\circ\circ} = 0$ *and* $1^{\circ\circ} = 1$,

(2) $a \le b \Rightarrow b^{\circ} \le a^{\circ}$,

(3) $a^{\circ\circ\circ} = a^{\circ}$,

(4) $a^{\circ\circ\circ\circ} = a^{\circ\circ}$,

(5) $(a \vee b)^{\circ} = a^{\circ} \wedge b^{\circ}$,

(6) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$,

(7) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$.

An element a of an MS-algebra L is called a closed element of L if $a = a^{\circ\circ}$ and an element $d \in L$ is called a dense element of L if $d^{\circ} = 0$.

Theorem 2.2. *Let L be an MS-algebra. Then*

(1) the set $L^{\circ\circ} = \{a \in L : a = a^{\circ\circ}\}$ of all closed elements of L is a de Morgan subalgebra of L, see [17],

(2) the set $D(L) = \{a \in L : a^{\circ} = 0\}$ of all dense elements of L is a principal filter of L, see [10].

Definition 2.4. *[10] An MS-algebra $(L; \vee, \wedge, ^{\circ}, 0, 1)$ is called a principal MS-algebra if it satisfies the following conditions:*

(1) the filter $D(L)$ is principal, that is, there exists an element $d_L \in L$ such that $D(L) = [d_L)$,

(2) $a = a^{\circ\circ} \wedge (a \vee d_L)$, for all $a \in L$.

A principal MS-algebra L is a principal Stone algebra if $x^{\circ} \vee x^{\circ\circ} = 1$, for all $x \in L$.

Definition 2.5. *[8] A principal MS-triple is (M, D, φ), where*

(1) M is a de Morgan algebra,

(2) D is a bounded distributive lattice,

(3) φ is a $(0, 1)$-lattice homomorphism from M into D.

Lemma 2.1. *[10] Let L be a principal MS-algebra. Define a map $\varphi_L : L^{\circ\circ} \to [d_L)$ by the following rule*

$$\varphi_L(a) = a \vee d_L, \text{ for all } a \in L^{\circ\circ}.$$

Then φ_L is a $(0, 1)$-lattice homomorphism.

Example 2.1. *Consider the MS-algebra L in Figure 1.*

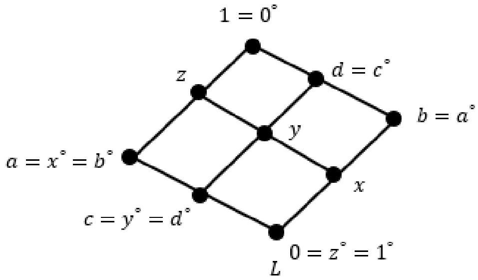

Figure 1. L is a principal MS-algebra.

It is clear that L is a principal MS-algebra with the smallest dense element z. Then the principal triple $(L^{\circ\circ}, D(L) = [z), \varphi_L)$ *which associated with L is given in Figure 2.*

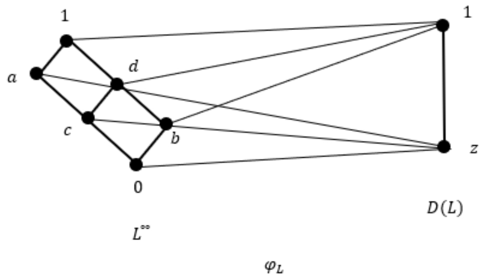

Figure 2. $(L^{\circ\circ}, D(L), \varphi_L)$ is a principal MS-triple.

Definition 2.6. *[22] An equivalence relation* θ *on a lattice L is called a lattice congruence if* $(a, b) \in \theta$ *and* $(c, d) \in \theta$ *imply* $(a \vee c, b \vee d) \in \theta$ *and* $(a \wedge c, b \wedge d) \in \theta$.

Definition 2.7. *An equivalence relation* θ *on an MS-algebra L is called a congruence on L if*

(1) θ *is a lattice congruence,*

(2) $(a, b) \in \theta$ *implies* $(a^\circ, b^\circ) \in \theta$.

We use $Con_{lat}(L)$ for the lattice of all lattice congruences of a lattice $(L; \vee, \wedge)$ and $Con(L)$ for the lattice of all congruences of an MS-algebra $(L; \vee, \wedge, ^\circ, 0, 1)$.

Definition 2.8. *Let L be a lattice and* $\theta \in Con_{lat}(L)$*. Then we define the principal congruence generated by* (a, b) *which denoted by* $\theta(a, b)$ *as follows:*

$$\theta(a, b) = \bigwedge\{\theta \in Con_{lat}(L) : (a, b) \in \theta\}.$$

If L is an MS-algebra and $\theta \in Con(L)$, then

$$\theta(a, b) = \bigwedge\{\theta \in Con(L) : (a, b) \in \theta\}.$$

Let L be an MS-algebra. Then $(\theta_{L^{\circ\circ}}, \theta_{D(L)}) \in Con(L^{\circ\circ}) \times Con_{lat}(D(L))$, where $\theta_{L^{\circ\circ}}$ and $\theta_{D(L)}$ are the restrictions of $\theta \in Con(L)$ to $L^{\circ\circ}$ and $D(L)$, respectively. It is clear that $\theta_{L^{\circ\circ}}$ is a congruence relation on a de Morgan algebra $L^{\circ\circ}$ and $\theta_{D(L)}$ is a lattice congruence on a lattice $D(L)$.

The symbols $\triangledown_L$ and $\triangle_L$ will be used for the universal congruence $L \times L$ and the equality congruence on L, respectively.

The concept of MS-congruence pairs is given as follows:

Definition 2.9. *[11] Let L be a principal MS-algebra with a smallest dense element* d_L*. A pair of congruences* $(\theta_1, \theta_2) \in Con(L^{\circ\circ}) \times Con_{lat}(D(L))$ *will be called an MS-congruence pairs if*

$$(a, b) \in \theta_1 \text{ implies } (a \vee d_L, b \vee d_L) \in \theta_2.$$

A characterization of congruences on a principal MS-algebra via MS-congruence pairs is given in the following theorem.

Theorem 2.3. *[11] Let L be a principal MS-algebra with the smallest dense element* d_L*. For every congruence* θ *on L, the restrictions of* θ *to* $L^{\circ\circ}$ *and* $D(L)$ *determine the MS-congruence pair* $(\theta_{L^{\circ\circ}}, \theta_{D(L)})$.

Conversely, every MS-congruence pair (θ_1, θ_2) *uniquely determines a congruence* θ *on L satisfying* $\theta_{L^{\circ\circ}} = \theta_1$ *and* $\theta_{D(L)} = \theta_2$*. Such congruence can be defined by the rule*

$$(x, y) \in \theta \text{ if and only if } (x^\circ, y^\circ) \in \theta_1 \text{ and } (x \vee d_L, y \vee d_L) \in \theta_2.$$

Throughout this paper, d_L denotes to the smallest dense element of a principal MS-algebra L.

For extra information of MS-algebras, principal MS-algebras and decomposable MS-algebras, we refer the reader to [1–9, 12–18, 20–24].

3. Characterization of MS-congruence pairs

In this section, we characterize MS-congruence pairs of a principal MS-algebra and a principal Stone algebra. Also, we describe the lattice of all MS-congruence pairs which induced by the Boolean elements of a principal MS-algebra, in fact such lattice forms a Boolean algebra on it is own.

Lemma 3.1. *Let L be a principal MS-algebra and (θ_1, θ_2) be an MS-congruence pair. Then*

$$(a, b) \in \theta_1 \text{ and } (x, y) \in \theta_2 \text{ imply } (a \vee x, b \vee y) \in \theta_2.$$

Proof. Let $(a, b) \in \theta_1$ and $(x, y) \in \theta_2$. Then $(a \vee d_L, b \vee d_L) \in \theta_2$ (by Definition 2.9) and $(x, y) \in \theta_2$ imply that $(a \vee x \vee d_L, b \vee y \vee d_L) \in \theta_2$. Therefore $(a \vee x, b \vee y) \in \theta_2$ as $x, y \geq d_L$. □

Now, we give an important characterization of MS-congruence pairs of a principal MS-algebra.

Theorem 3.1. *Let L be a principal MS-algebra with the smallest dense element d_L. Then the following statements are equivalent:*

(1) (θ_1, θ_2) is an MS-congruence pair of L,

(2) $Con_{\varphi_L}(L) \subseteq \theta_2$, where

$$Con_{\varphi_L}(\theta_1) = \{(\varphi_L(a), \varphi_L(b)) : (a, b) \in \theta_1\} \text{ and } \varphi_L(a) = a \vee d_L, \forall a \in L^{\circ\circ}.$$

Proof. (1 ⇒ 2) Let (θ_1, θ_2) be an MS-congruence pair of L and $(x, y) \in Con_{\varphi_L}(\theta_1)$. Then $(x, y) = (\varphi_L(a), \varphi_L(b)) = (a \vee d_L, b \vee d_L)$, where $(a, b) \in \theta_1$. Since $(a, b) \in \theta_1$, then $(a \vee d_L, b \vee d_L) \in \theta_2$, by (1). Thus $(x, y) \in \theta_2$. So $Con_{\varphi_L}(\theta_1) \subseteq \theta_2$.
(2 ⇒ 1) Let $Con(\theta_1) \subseteq \theta_2$. We prove that (θ_1, θ_2) is an MS-congruence pair of L. Let $(a, b) \in \theta_1$. Then by (2), $(a \vee d_L, b \vee d_L) = (\varphi_L(a), \varphi_L(b)) \in \theta_2$. □

A characterization of congruence pairs of a principal Stone algebra is given in the following.

Theorem 3.2. *Let L be a principal Stone algebra. Let $(\theta_1, \theta_2) \in Con(L^{\circ\circ}) \times Con_{lat}(D(L))$. Then the following statements are equivalent:*

(1) (θ_1, θ_2) is an MS-congruence pair of L,

(2) $Con_{\varphi_L}(\theta_1) \subseteq \theta_2$,

(3) $(a, 1) \in \theta_1$ and $u \geq a, u \in D(L)$ imply $(u, 1) \in \theta_2$.

Proof. In Theorem 3.1, we proved that (1) and (2) are equivalent.
(2 ⇒ 3) Let $Con_{\varphi_L}(\theta_1) \subseteq \theta_2$. Let $(a, 1) \in \theta_1$ and $u \geq a, u \in D(L)$. Then $(\varphi_L(a), \varphi_L(1)) \in \theta_2$ by (2). Thus $(a \vee d_L, 1) \in \theta_2$.
Now $(a \vee d_L, 1) \in \theta_2$ and $(u, u) \in \theta_2, u \geq a, u \in D(L)$ imply $(a \vee u \vee d_L, 1 \vee u) \in \theta_2$. Thus $(u, 1) \in \theta_2$ as $u \geq a, d_L$.

(3 ⇒ 1) Since L is a Stone algebra then $L^{\circ\circ}$ is a Boolean subalgebra of L and hence $a \vee a^{\circ} = 1$ for each $a \in L^{\circ\circ}$. Suppose (3) holds and $(a,b) \in \theta_1$. Then $(b^{\circ}, b^{\circ}) \in \theta_1$ implies $(a \vee b^{\circ}, b \vee b^{\circ}) = (a \vee b^{\circ}, 1) \in \theta_1$ and $(a \vee a^{\circ}, b \vee a^{\circ}) = (b \vee a^{\circ}, 1) \in \theta_1$. Therefore $(\beta, 1) \in \theta_1$, where $\beta = (a \vee b^{\circ}) \wedge (a^{\circ} \vee b)$. It is clear that $\beta \in L^{\circ\circ}$ and $\beta \wedge a = \beta \wedge b = a \wedge b$. Since $\beta \leq \beta \vee d_L \in D(L)$ and $(\beta, 1) \in \theta_1$ then $(\beta \vee d_L, 1) \in \theta_2$ by (3). Thus $(a \vee d_L, a \vee d_L) \wedge (1, \beta \vee d_L) \in \theta_2$ implies $(a \vee d_L, (a \wedge \beta) \vee d_L) = (a \vee d_L, (a \wedge b) \vee d_L) \in \theta_2$. Similarly, we can get $(b \vee d_L, (b \wedge \beta) \vee d_L) = (b \vee d_L, (a \wedge b) \vee d_L) \in \theta_2$. Then $(b \vee d_L, b \vee d_L) \in \theta_2$. □

Example 3.1. *Consider the principal MS-algebra L in Example 2.1 (Figure 1). The lattices Con*(L) *and* $A(L)$ *of all congruences of L and all MS-congruence pairs of L are given in Figure 3, respectively.*

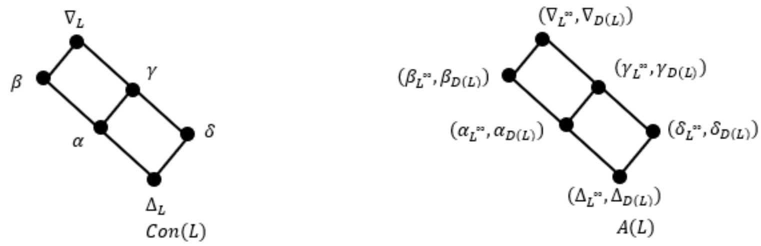

Figure 3. The congruence lattices $Con(L)$ and $A(L)$.

Where,

$$\begin{aligned} \triangle_L &= \{(x,x) : x \in L\}, \\ \alpha &= \{\{0,c,a\},\{x,y,z\},\{b,d,1\}\}, \\ \beta &= \{\{0,c,a\},\{x,b,y,d,z,1\}\}, \\ \gamma &= \{\{0,x,b\},\{c,y,d\},\{a,z,1\}\}, \\ \delta &= \{\{0\},\{c\},\{a\},\{x,b\},\{y,d\},\{z,1\}\}, \\ \nabla_L &= L \times L. \end{aligned}$$

And

$$\begin{aligned} A(L) =& \{(\triangle_{L^{\circ\circ}}, \triangle_{D_{(L)}}), (\alpha_{L^{\circ\circ}}, \alpha_{D(L)}), (\beta_{L^{\circ\circ}}, \beta_{D(L)}), (\gamma_{L^{\circ\circ}}, \gamma_{D(L)}), (\delta_{L^{\circ\circ}}, \delta_{D(L)}), (\nabla_{L^{\circ\circ}}, \nabla_{D(L)})\} \\ =& \{(\triangle_{L^{\circ\circ}}, \triangle_{D_{(L)}}), (\{\{0,c,a\},\{b,d,1\}\}, \triangle_{D_{(L)}}), (\{\{0,c,a\},\{b,d,1\}\}, \nabla_{D(L)}), \\ & (\{\{a,1\},\{c,d\},\{0,b\}\}, \nabla_{D(L)}), (\triangle_{L^{\circ\circ}}, \nabla_{D(L)}), (\nabla_{L^{\circ\circ}}, \nabla_{D(L)})\}. \end{aligned}$$

It is clear that $Con(L)$ *isomorphic to* $A(L)$ *under the isomorphism* $\theta \longmapsto (\theta_{L^{\circ\circ}}, \theta_{D(L)})$.

A subset I of a lattice L with 0 is called an ideal of L if

(1) $0 \in I$, (2) $a \vee b \in I, \ \forall a, b \in I$,
(3) I is an down-set, that is, if $x \leq y, \ y \in I$ and $x \in L$, then $x \in I$.

A principal ideal of L generated by $a \in L$ is defined by

$$(a] = \{x \in L : x \leq a\}.$$

A subset F of a lattice L with 1 is called a filter of L if

$$(1)\ 1 \in F, \qquad (2)\ a \wedge b \in F,\ \forall a, b \in F,$$
$$(3)\ F \text{ is an up-set, that is, if } x \geq y,\ y \in F \text{ and } x \in L, \text{then } x \in F.$$

A principal filter of L generated by $a \in L$ is defined by

$$[a) = \{x \in L : x \geq a\}.$$

Definition 3.1. *Let θ be a lattice congruence on a bounded lattice L. Then we have the following important subsets*

(i) The Kernel of θ ($Ker\theta$) is the set $\{x \in L : (x, 0) \in \theta\}$, which is an ideal of L,

(ii) The Cokernel of θ ($Coker\theta$) is the set $\{x \in L : (x, 1) \in \theta\}$, which is a filter of L.

Definition 3.2. *[7] An element c of an MS-algebra L is called a Boolean element of L if $c \vee c^{\circ} = 1$.*

It is ready seen that the set $B(L) = \{c : c \vee c^{\circ} = 1\}$ of all Boolean elements of L forms a Boolean subalgebra of $L^{\circ\circ}$.

Lemma 3.2. *Let c be a Boolean element of a principal MS-algebra L. Then*

(1) $\theta(0, c)$ is a principal congruence relation on $L^{\circ\circ}$ with $Ker\ \theta(0, c) = (c]$, where

$$(a, b) \in \theta(0, c) \Leftrightarrow a \wedge c^{\circ} = b \wedge c^{\circ},$$

(2) $\theta(\varphi_L(c), 1)$ is a principal congruence relation on $D(L)$ with $Coker\ \theta(\varphi_L(c), 1) = [\varphi_L(c))$, where

$$(x, y) \in \theta(\varphi_L(c), 1) \Leftrightarrow x \wedge (c \vee d_L) = y \wedge (c \vee d_L).$$

Proof.

(1) It is easy to check that $\theta(0, c)$ is a principal congruence relation on $L^{\circ\circ}$ with $Ker\ \theta(0, c) = (c]$.

(2) It is easy to show that $\theta(\varphi_L(c), 1)$ is an equivalence relation on $D(L)$. Let $(x, y), (m, n) \in \theta(\varphi_L(c), 1)$. Thus $x \wedge (c \vee d_L) = y \wedge (c \vee d_L)$ and $m \wedge (c \vee d_L) = n \wedge (c \vee d_L)$. Then, we get $(x \wedge m, y \wedge n) \in \theta(\varphi_L(c), 1)$ and $(x \vee m, y \vee n) \in \theta(\varphi_L(c), 1)$. Therefore $\theta(\varphi_L(c), 1)$ is a principal congruence on the lattice $D(L)$. Also, we have

$$\begin{aligned} Coker\ (\theta(\varphi_L(c), 1)) =& \{x \in D(L) : (x, 1) \in \theta(\varphi_L(c), 1)\} \\ =& \{x \in D(L) : x \wedge (c \vee d_L) = 1 \wedge (c \vee d_L)\} \\ =& \{x \in D(L) : x \wedge (c \vee d_L) = c \vee d_L\} \\ =& \{x \in D(L) : x \geq c \vee d_L\} \\ =& [c \vee d_L) = [\varphi_L(c)). \end{aligned}$$

□

Now, we observe that every Boolean element c of a principal MS-algebra L associated with the MS-congruence pair of the form $(\theta(0, c), \theta(\varphi_L(c^{\circ}), 1))$.

Theorem 3.3. *Let L be a principal MS-algebra and $c \in L^{\circ\circ}$. Then c is a Boolean element of L if and only if $(\theta(0, c), \theta(\varphi_L(c^{\circ}), 1))$ is an MS-congruence pair of L.*

Proof. Let c be a Boolean element of L. We proved that $\theta(0, c)$ and $\theta(\varphi_L(c^\circ), 1)$ are MS-congruence on $L^{\circ\circ}$ and lattice congruence on $D(L)$, respectively (Lemma 3.2). To show that $(\theta(0, c), \theta(\varphi_L(c^\circ), 1))$ is an MS-congruence pair, let $(x, y) \in \theta(0, c)$. Then

$$\begin{aligned}(x, y) \in \theta(0, c) &\Rightarrow x \wedge c^\circ = y \wedge c^\circ \\ &\Rightarrow (x \wedge c^\circ) \vee d_L = (y \wedge c^\circ) \vee d_L \\ &\Rightarrow (x \vee d_L) \wedge (c^\circ \vee d_L) = (y \vee d_L) \wedge (c^\circ \vee d_L) \\ &\Rightarrow (x \vee d_L, y \vee d_L) \in \theta(\varphi_L(c^\circ), 1).\end{aligned}$$

Conversely, let $(\theta(0, c), \theta(\varphi_L(c^\circ), 1))$ be an MS-congruence pair. Since $(0, c) \in \theta(0, c)$ then $c \wedge c^\circ = 0 \wedge c^\circ = 0$. Now, $c \vee c^\circ = (c^\circ \wedge c)^\circ = 1$. Therefore c is a Boolean element. □

The basic properties of principal congruence relations $\theta(0, a)$ and $\theta(\varphi_L(a), 1), \forall a \in B(L)$ are given in the following:

Lemma 3.3. *Let a, b be Boolean elements of a principal MS-algebra L. Then*

(1) $a \leq b$ if and only if $\theta(0, a) \subseteq \theta(0, b)$,

(2) $a = b$ if and only if $\theta(0, a) = \theta(0, b)$,

(3) $\theta(0, 0) = \triangle_{L^{\circ\circ}}$ and $\theta(0, 1) = \nabla_{L^{\circ\circ}}$,

(4) $\theta(0, a) \vee \theta(0, b) = \theta(0, a \vee b)$,

(5) $\theta(0, a) \cap \theta(0, b) = \theta(0, a \wedge b)$.

Lemma 3.4. *Let L be a principal MS-algebra. Then for every $a, b \in B(L)$, we have*

(1) $a \leq b$ implies $\theta(\varphi_L(a^\circ), 1) \subseteq \theta(\varphi_L(b^\circ), 1)$,

(2) $\theta(\varphi_L(a^\circ), 1) \vee \theta(\varphi_L(b^\circ), 1) = \theta(\varphi_L(a \vee b)^\circ, 1)$,

(3) $\theta(\varphi_L(a^\circ), 1) \wedge \theta(\varphi_L(b^\circ), 1) = \theta(\varphi_L(a \wedge b)^\circ, 1)$,

(4) $\theta(\varphi_L(0), 1) = \nabla_{D(L)}$ and $\theta(\varphi_L(1), 1) = \triangle_{D(L)}$.

Proof.

(1) Let $a \leq b$. Then $a^\circ \geq b^\circ$. Let $(x, y) \in \theta(\varphi_L(a^\circ), 1)$ then $x \wedge (a^\circ \vee d_L) = y \wedge (a^\circ \vee d_L)$. Now

$$\begin{aligned}x \wedge (b^\circ \vee d_L) &= x \wedge ((a^\circ \wedge b^\circ) \vee d_L) \\ &= x \wedge \{(a^\circ \vee d_L) \wedge (b^\circ \vee d_L)\} \\ &= x \wedge (a^\circ \vee d_L) \wedge (b^\circ \vee d_L) \\ &= y \wedge (a^\circ \vee d_L) \wedge (b^\circ \vee d_L) \\ &= y \wedge \{(a^\circ \wedge b^\circ) \vee d_L\} \\ &= y \wedge (b^\circ \vee d_L).\end{aligned}$$

Then $(x, y) \in \theta(\varphi_L(b^\circ), 1)$. Therefore $\theta(\varphi_L(a^\circ), 1) \subseteq \theta(\varphi_L(b^\circ), 1)$.

(2) Since $a, b \leq a \vee b$ then by (1), we have

$$\theta(\varphi_L(a^\circ), 1),\ \theta(\varphi_L(b^\circ), 1) \subseteq \theta(\varphi_L(a \vee b)^\circ, 1).$$

Then $\theta(\varphi_L(a \vee b)^\circ, 1)$ is an upper bound of $\theta(\varphi_L(a^\circ), 1)$ and $\theta(\varphi_L(b^\circ), 1)$. Let $\theta(\varphi_L(c^\circ), 1)$ be an upper bound of $\theta(\varphi_L(a^\circ), 1)$ and $\theta(\varphi_L(b^\circ), 1)$.
Then $\theta(\varphi_L(a^\circ), 1), \theta(\varphi_L(b^\circ), 1) \subseteq \theta(\varphi_L(c^\circ), 1)$ implies $\varphi_L(c^\circ) \leq \varphi_L(a^\circ), \varphi_L(b^\circ)$. Thus $\varphi_L(c^\circ) \leq \varphi_L(a^\circ \wedge b^\circ) = \varphi_L(a \vee b)^\circ$ and hence $\theta(\varphi_L(a \vee b)^\circ, 1) \subseteq \theta(\varphi_L(c^\circ), 1)$. Therefore $\theta(\varphi_L(a \vee b)^\circ, 1)$ is the least upper bound of $\theta(\varphi_L(a^\circ), 1)$ and $\theta(\varphi_L(b^\circ), 1)$.

(3) Since $a \wedge b \leq a,\ b$ then by (1) $\theta(\varphi_L(a \wedge b)^\circ, 1) \subseteq \theta(\varphi_L(a^\circ), 1), \theta(\varphi_L(b^\circ), 1)$. Then $\theta(\varphi_L(a \wedge b)^\circ, 1)$ is a lower bound of $\theta(\varphi_L(a^\circ), 1)$ and $\theta(\varphi_L(b^\circ), 1)$. Let $\theta(\varphi_L(c^\circ), 1)$ be a lower bound of $\theta(\varphi_L(a^\circ), 1)$ and $\theta(\varphi_L(b^\circ), 1)$. Then $\theta(\varphi_L(c^\circ), 1) \subseteq \theta(\varphi_L(a^\circ), 1), \theta(\varphi_L(b^\circ), 1)$ implies $\varphi_L(a^\circ), \varphi_L(b^\circ) \leq \varphi_L(c^\circ)$. Thus $\varphi_L(a \wedge b)^\circ = \varphi_L(a^\circ) \vee \varphi_L(b^\circ) \leq \varphi_L(c^\circ)$ and hence $\theta(\varphi_L(c^\circ), 1) \subseteq \theta(\varphi(a \wedge b)^\circ, 1)$. Therefore $\theta(\varphi_L(a \wedge b)^\circ, 1)$ is the greatest lower bound of $\theta(\varphi_L(a^\circ), 1)$ and $\theta(\varphi_L(b^\circ), 1)$.

(4) Since $d_L, 1 \in \theta(d_L, 1)$, then $\theta(d_L, 1) = D(L) \times D(L) = \triangledown_{D(L)}$. Also, let $(x, y) \in \theta(\varphi_L(1), 1)$ then $x \wedge (1 \vee d_L) = y \wedge (1 \vee d_L)$. It follows, $x = y$ and hence $\theta(\varphi_L(1), 1) = \triangle_{D(L)}$. □

Theorem 3.4. *Let L be a principal MS-algebra. Consider the subsets H and G of $Con(L^{\circ\circ})$ and $Con(D(L))$, respectively as follows:*

$$H = \{\theta(0, a) : a \in B(L)\}, \quad G = \{\theta(\varphi_L(a), 1) : a \in B(L)\}.$$

Then we have

(1) $(H; \vee, \wedge, ', \triangle_{L^{\circ\circ}}, \triangledown_{L^{\circ\circ}})$ is a Boolean algebra, where $(\theta(0, a))' = \theta(a^\circ, 0)$,

(2) $(G; \vee, \wedge, ', \triangle_{D(L)}, \triangledown_{D_L})$ is a Boolean algebra, where $(\theta(\varphi_L(a), 1))' = \theta(\varphi_L(a^\circ), 1)$.

Proof.

(1) Since $\theta(0, 0) = \triangle_{L^{\circ\circ}}$ and $\theta(0, 1) = \triangledown_{L^{\circ\circ}}$. Then $\triangledown_{L^{\circ\circ}},\ \triangle_{L^{\circ\circ}} \in H$.Let $\theta(0, a), \theta(0, b) \in H$. Then we get

$$\theta(0, a) \vee \theta(0, b) = \theta(0, a \vee b),$$

and

$$\theta(0, a) \wedge \theta(0, b) = \theta(0, a \wedge b).$$

Therefore H is a bounded lattice. Since $B(L)$ is a Boolean algebra, then $a^\circ \in B(L)$ for all $a \in B(L)$. Thus $\theta(0, a^\circ) \in H$. Then we have

$$\theta(0, a) \vee [\theta(0, a)]' = \theta(0, a) \vee \theta(0, a^\circ) = \theta(0, a \vee a^\circ) = \theta(0, 1) = \triangledown_{L^{\circ\circ}},$$

and

$$\theta(0, a) \wedge [(0, a)]' = \theta(0, a) \wedge \theta(0, a^\circ) = \theta(\varphi_L(0, a \wedge a^\circ) = \theta(0, 0) = \triangle_{L^{\circ\circ}}.$$

Therefore $(H; \vee, \wedge,', \triangle_{L^{\circ\circ}}, \triangledown_{L^{\circ\circ}})$ is a Boolean algebra.

(2) We have $\triangle_{D(L)} = \theta(\varphi_L(1), 1) \in G$ and $\triangledown_{D(L)} = \theta(\varphi_L(0), 1) = \triangledown_{D(L)} \in G$. Let $\theta(\varphi_L(a), 1), \theta(\varphi_L(b), 1) \in G$. Then we get

$$\theta(\varphi_L(a), 1) \vee \theta(\varphi_L(b), 1) = \theta(\varphi_L(a \wedge b), 1),$$

and

$$\theta(\varphi_L(a), 1) \wedge \theta(\varphi_L(b), 1) = \theta(\varphi_L(a \vee b), 1).$$

Therefore G is a bounded lattice. Since $B(L)$ is a Boolean algebra, then $a^\circ \in B(L)$ for all $a \in B(L)$. Thus $\theta(\varphi_L(a^\circ), 1) \in G$. Then we have

$$\theta(\varphi_L(a), 1) \vee [\theta(\varphi_L(a), 1)]' = \theta(\varphi_L(a), 1) \vee \theta(\varphi_L(a^\circ), 1) = \theta(\varphi_L(a \wedge a^\circ), 1) = \theta(\varphi_L(0), 1) = \nabla_{D(L)},$$

and

$$\theta(\varphi_L(a), 1) \wedge [\theta(\varphi_L(a), 1)]' = \theta(\varphi_L(a), 1) \wedge \theta(\varphi_L(a^\circ), 1) = \theta(\varphi_L(a \vee a^\circ), 1) = \theta(\varphi_L(1), 1) = \triangle_{D(L)}.$$

Therefore $(G; \vee, \wedge, ', \triangle_{D_L}, \nabla_{D_L})$ is a Boolean algebra. □

Let L be a principal MS-algebra. Let $A(L)$ be the lattice of all MS-congruence pairs of L. We consider a subset $A'(L)$ of $A(L)$ as follows:

$$A'(L) = \{(\theta(0, a), \theta(\varphi_L(a^\circ), 1)) : a \in B(L)\}.$$

From the above results, we observe that the set $A'(L)$ of all MS-congruence pairs induced by the Boolean elements of a principal MS-algebra forms bounded sublattice of the lattice $A(L)$. Moreover $A'(L)$ is a Boolean algebra on its own.

Theorem 3.5. *Let L be a principal MS-algebra. Then $(A'(L); \vee, \wedge, ', 0_{A'(L)}, 1_{A'(L)})$ is a Boolean algebra, where*

$$\begin{aligned}
(\theta(0, a), \theta(\varphi_L(a^\circ), 1)) \vee (\theta(0, b), \theta(\varphi_L(b^\circ), 1)) &= (\theta(0, a \vee b), \theta(\varphi_L(a \vee b)^\circ, 1)),\\
(\theta(0, a), \theta(\varphi_L(a^\circ), 1) \wedge (\theta(0, b), \theta(\varphi_L(b^\circ), 1)) &= (\theta(0, a \wedge b), \theta(\varphi_L(a \wedge b)^\circ, 1)),\\
[(\theta(0, a), \theta(\varphi_L(a^\circ), 1))]' &= (\theta(0, a^\circ), \theta(\varphi_L(a), 1)),\\
1_{A'(L)} &= (\nabla_{L^{\circ\circ}}, \triangle_{D_{(L)}}),\\
0_{A'(L)} &= (\triangle_{L^{\circ\circ}}, \triangle_{D_{(L)}}).
\end{aligned}$$

Example 3.2. *Consider the principal MS-algebra L in Example 2.1. The lattices $B(L)$ and $A'(L)$ of all Boolean elements of L and all MS-congruence pairs of L of the form $(\theta(0, a), \theta(\varphi_L(a^\circ), 1))$, $a \in B(L)$ are given in the following Figure 4, respectively.*

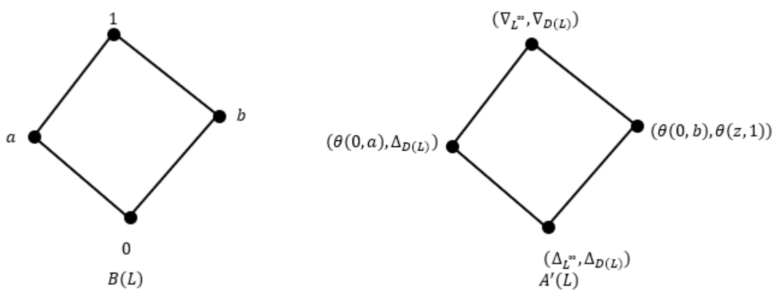

Figure 4. The lattices $B(L)$ and $A'(L)$.

It is clear that $B(L)$ and $A(L)$ are isomorphic Boolean algebras under the isomorphism $a \longmapsto (\theta(0, a), \theta(a^\circ \vee z, 1))$. Also, $A'(L)$ is a bounded sublattice of $A(L)$.

4. 2-permutability of principal MS-algebras

In this section, we characterize the notion of 2-permutability of congruences of a principal MS-algebra by means of MS-congruence pairs.

Let L be an algebra. We say the congruences $\theta, \Phi \in Con(L)$ are 2-permutable if $\theta \circ \Phi = \Phi \circ \theta$, that is, $(x, a) \in \theta$ and $(a, y) \in \Phi$, $x, y, a \in L$ imply $(x, b) \in \Phi$ and $(b, y) \in \theta$ for some $b \in L$. We call the algebra L has 2-permutable congruences (briefly permutable) if every pair of congruences of L permute.

Lemma 4.1. *A principal MS-algebra L has 2-permutable congruences if and only if every pair of principle congruences of L permute.*

Proof. Assume that every pair of principal congruences on L permute. Let θ and Φ be arbitrary congruences on L and $(a, b) \in (\theta \circ \Phi)$, $a, b \in L$. Then $(a, t) \in \theta$ and $(t, b) \in \Phi$ for some $t \in L$. Then $(a, t) \in \theta(a, t)$ and $(t, b) \in \theta(t, b)$ imply that $(a, b) \in \theta(a, t) \circ \theta(t, b)$. Thus $(a, b) \in \theta(t, b) \circ \theta(a, t)$. Since $\theta(a, t) \subseteq \theta$ and $\theta(t, b) \subseteq \Phi$, then $(a, b) \in (\Phi \circ \theta)$. Therefore θ and Φ permute. The second implication is obvious. □

Define a relation Ψ on a principal MS-algebra L as follows:

$$(a, b) \in \Psi \Leftrightarrow a^\circ = b^\circ \Leftrightarrow a^{\circ\circ} = b^{\circ\circ}.$$

Theorem 4.1. *Let L be a principal MS-algebra with the smallest dense element d_L. Then*

(1) Ψ is a congruence relation on L with Ker $\Psi = \{0\}$ and Coker $\Psi = D(L)$,

(2) Ψ is closed congruence on L, that is, $(x, x^{\circ\circ}) \in \Psi, \forall x \in L$,

(3) max $[x]_\Psi = x^{\circ\circ}, x \in L$, where $[x]_\Psi = \{y \in L : y^{\circ\circ} = x^{\circ\circ}\}$ is the congruence class of x modulo Ψ.

Proof.

(1) One can check that Ψ is a congruence relation on L. Now, we have

$$Ker\Psi = \{a \in L : (a, 0) \in \Psi\} = \{a \in L : a^\circ = 1\} = \{0\},$$

and

$$Coker\Psi = \{z \in L : (z, 1) \in \Psi\} = \{z \in L : z^\circ = 1\} = D(L) = [d_L).$$

(2) Since $x^{\circ\circ\circ\circ} = x^{\circ\circ}$ then $x^{\circ\circ} \in [x]_\Psi$. Thus $(x^{\circ\circ}, x) \in \Psi$.

(3) Let $y \in [x]_\Psi$. Then $y \leq y^{\circ\circ} = x^{\circ\circ}$ and $x^{\circ\circ} \in [x]_\Psi$. Thus $x^{\circ\circ}$ is the greatest member of the congruence class $[x]_\Psi$. □

Theorem 4.2. *Let L be a principal MS-algebra. Then*

(1) L/Ψ is a de Morgan algebra,

(2) L/Ψ and $L^{\circ\circ}$ are isomorphic de Morgan algebras.

Proof.

(1) It is ready seen that $(L/\Psi; \vee, \wedge, \{0\}, D(L))$ is a bounded distributive lattice, where $L/\Psi = \{[a]_\Psi : a \in L\}$ is the set of all congruence classes module Ψ and

$$[a]_\Psi \vee [b]_\Psi = [a \vee b]_\Psi,$$
$$[a]_\Psi \wedge [b]_\Psi = [a \wedge b]_\Psi.$$

Define $^\square$ on L/Ψ by $([a]_\Psi)^\square = [a^\circ]_\Psi, \forall a \in L$. We have

$$([0]_\Psi)^\square = [1]_\Psi,$$
$$([a]_\Psi)^{\square\square} = [a^{\circ\circ}]_\Psi = [a]_\Psi,$$
$$([a]_\Psi \wedge [b]_\Psi)^\square = ([a]_\Psi)^\square \vee ([b]_\Psi)^\square.$$

Then L/Ψ is a de Morgan algebra.

(2) Define $g : L^{\circ\circ} \to L/\Psi$ by

$$g(a) = [a]_\Psi, \forall a \in L^{\circ\circ}.$$

Let $a, b \in L^{\circ\circ}$ and $a = b$. Then $a^{\circ\circ} = b^{\circ\circ}$ implies $[a]_\Psi = [b]_\Psi$. It follows that g is well defined map of $L^{\circ\circ}$ into L/Ψ. Let $a, b \in L^{\circ\circ}$, we get

$$g(a \vee b) = [a \vee b]_\Psi = [a]_\Psi \vee [b]_\Psi = g(a) \vee g(b),$$

$$g(a \wedge b) = [a \wedge b]_\Psi = [a]_\Psi \wedge [b]_\Psi = g(a) \wedge g(b),$$

and

$$g(a^\circ) = [a^\circ]_\Psi = ([a]_\Psi)^\square = (g(a))^\square.$$

Let $[a]_\Psi = [b]_\Psi$. Then $a^{\circ\circ} = b^{\circ\circ}$ implies $a = b$. Therefore g is an injective map. Let $[a]_\Psi \in L/\Psi$. Then $[x]_\Psi = [x^{\circ\circ}]_\Psi = g(x)$. Then g is a surjective map. This deduce that g is an isomorphism of de Morgan algebras. □

Now, we observe that Ψ satisfies the following property,

$$\Psi \circ \theta = \theta \circ \Psi \text{ for all } \theta \in Con(L).$$

Theorem 4.3. *Let L be a principal MS-algebra. Then Ψ permutes with each congruence of L.*

Proof. We prove that $\Psi \circ \theta = \theta \circ \Psi$ for all $\theta \in Con(L)$. Let $(a, b) \in \Psi \circ \theta$. Then $(a, z) \in \Psi$ and $(z, b) \in \theta$ for some $z \in L$. It follows that $a^{\circ\circ} = z^{\circ\circ}$ and $(z, b) \in \theta$. Now

$$\begin{aligned}(z, b) \in \theta &\Rightarrow (z^{\circ\circ}, b^{\circ\circ}) \in \theta \text{ and } (a \vee d_L, a \vee d_L) \in \theta \\ &\Rightarrow (a^{\circ\circ} \wedge (a \vee d_L), b^{\circ\circ} \wedge (a \vee d_L)) \in \theta \\ &\Rightarrow (a, b^{\circ\circ} \wedge (a \vee d_L)) \in \theta \qquad (\text{as } a = a^{\circ\circ} \wedge (a \vee d_L))\end{aligned}$$

Since $[b^{\circ\circ} \wedge (a \vee d_L)]^{\circ\circ} = b^{\circ\circ}$, then $(b^{\circ\circ} \wedge (a \vee d_L), b) \in \Psi$. Therefore $(a, b^{\circ\circ} \wedge (a \vee d_L)) \in \theta$ and $(b^{\circ\circ} \wedge (a \vee d_L), b) \in \Psi$. imply $(a, b) \in \theta \circ \Psi$. Then $\Psi \circ \theta = \theta \circ \Psi,\ \forall \theta \in Con(L)$. □

Lemma 4.2. *Let L be a principal MS-algebra. Then*

(1) $\triangle_L$ permutes with every congruence on L,

(2) $\triangledown_L$ permutes with every congruence on L.

Proof.

(1) Let $(a, b) \in \theta \circ \triangle_L$. Then $(a, t) \in \theta$ and $(t, b) \in \triangle_L$ for some $t \in L$. Thus $(a, t) \in \theta$ and $t = b$. Then $(a, a) \in \triangle_L$ and $(a, b) \in \theta$ imply $(a, b) \in \triangle_L \circ \theta$. Therefore $\theta \circ \triangle_L = \triangle_L \circ \theta$.

(2) It is obvious. □

In the following theorem, we characterize 2-permutability of congruences of a principal MS-algebra L using MS-congruence pairs.

Theorem 4.4. *Let L be a principal MS-algebra. Then the following conditions are equivalent:*

(1) L has 2-permutable congruences.

(2) $L^{\circ\circ}$ and $D(L)$ have 2-permutable congruences.

Proof. To prove the conditions (1) and (2) are equivalent, we show that the two congruences θ , Φ $\in$ $Con(L)$ are 2-permutable if and only if their restrictions $\theta_{L^{\circ\circ}}$, $\Phi_{L^{\circ\circ}}$ and $\theta_{D(L)}$, $\Phi_{D(L)}$ have 2-permutable congruences on $L^{\circ\circ}$ and $D(L)$, respectively. Firstly, suppose that θ , Φ have 2-permutable congruences on L. Let $(x, z) \in \theta_{L^{\circ\circ}} \circ \Phi_{L^{\circ\circ}}$. Then there exist $y \in L^{\circ\circ}$ such that $(x, y) \in \theta_{L^{\circ\circ}}$ and $(y, z) \in \Phi_{L^{\circ\circ}}$. Then we have $(x, y) \in \theta$ and $(y, z) \in \Phi$. Since θ, Φ are 2-permutable, then there exists $a \in L$ such that $(x, a) \in \Phi$ and $(a, y) \in \theta$. We get $(x, a^{\circ\circ}) = (x^{\circ\circ}, a^{\circ\circ}) \in \Phi_{L^{\circ\circ}}$ and $(a^{\circ\circ}, z^{\circ\circ}) \in \theta_{L^{\circ\circ}}$. Therefore $\theta_{L^{\circ\circ}}$, $\Phi_{L^{\circ\circ}}$ are 2-permutable. Also, to prove that $\theta_{D(L)}$, $\Phi_{D(L)}$ are 2-permutable, let $(x, z) \in \theta_{D(L)} \circ \Phi_{D(L)}$. Then there exist $y \in D(L)$ such that $(x, y) \in \theta_{D(L)}$ and $(y, z) \in \Phi_{D(L)}$. Then we have $(x, y) \in \theta$ and $(y, z) \in \Phi$. Since θ, Φ are 2-permutable, there exists $a \in L$ such that

$$
\begin{aligned}
(x, a) \in \Phi \text{ and } (a, z) \in \theta &\Rightarrow (x, a \vee d_L) = (x \vee d_L, a \vee d_L) \in \Phi \text{ and } (a \vee d_L, z \vee d_L) \in \theta \\
&\Rightarrow (x, a \vee d_L) \in \Phi \text{ and } (a \vee d_L, z) \in \theta, \\
&\Rightarrow (x, a \vee d_L) \in \Phi_{D(L)} \text{ and } (a \vee d_L, z) \in \theta_{D(L)}, \text{ as } x, a \vee d_L, z \in D(L) \\
&\Rightarrow (x, z) \in \Phi_{D(L)} \circ \theta_{D(L)}.
\end{aligned}
$$

Thus $\theta_{D(L)}$, $\Phi_{D(L)}$ have 2-permutable congruences on $D(L)$.

Conversely, let θ , Φ $\in$ $Con(L)$. Assume that $\theta_{L^{\circ\circ}}$, $\Phi_{L^{\circ\circ}}$ and $\theta_{D(L)}$, $\Phi_{D(L)}$, have 2-permutable congruences on $L^{\circ\circ}$ and $D(L)$, respectively. Let $(x, z) \in \theta \circ \Phi$. Suppose that x, y, $z \in L$ with $(x, y) \in \theta$ and $(y, z) \in \Phi$. Then, by using Theorem 2.3, we get the following statements.

$$(x^{\circ\circ}, y^{\circ\circ}), (y^{\circ\circ}, z^{\circ\circ}) \in \Phi_{L^{\circ\circ}}, \ (x \vee d_L, y \vee d_L), \ (y \vee d_L, z \vee d_L) \ \Phi_{D(L)}.$$

Since $\theta_{L^{\circ\circ}}$, $\Phi_{L^{\circ\circ}}$ have 2-permutable congruences on $L^{\circ\circ}$, there exists an element $a \in L^{\circ\circ}$ such that

$$(x^{\circ\circ}, a) \in \Phi_{L^{\circ\circ}} \text{ and } (a, z^{\circ\circ}) \in \theta_{L^{\circ\circ}}.$$

Since $\theta_{D(L)}, \Phi_{D(L)}$ have 2-permutable congruences on $D(L)$, there exists $e \in D(L)$ such that $(x \vee d_L, e) \in \Phi_{D_L}$ and $(e, z \vee d_L) \in \theta_{D_L})$. It follows that $(x^{\circ\circ}, a) \in \Phi, (a, z^{\circ\circ}) \in \theta$ and $(x \vee d_L, e) \in \Phi, (e, z \vee d_L) \in \theta$. Since L is a principal MS-algebra we have

$$x = x^{\circ\circ} \wedge (x \vee d_L) \text{ and } z = z^{\circ\circ} \wedge (z \vee d_L).$$

Since θ, Φ are compatible with the $\wedge$ operation, then $(x^{\circ\circ}, a) \in \Phi$ and $(x \vee d_L, e) \in \Phi$ imply $(x, a \wedge e) = (x^{\circ\circ} \wedge (x \vee d_L), a \wedge e) \in \Phi$. Also $(a, z^{\circ\circ}) \in \theta$ and $(e, z \vee d_L) \in \theta$ imply $(a \wedge e, z) = (a \wedge e, z^{\circ\circ} \wedge (z \vee d_L)) \in \theta$. Consequently, we deduce that $(x, a \wedge e) \in \Phi$ and $(a \wedge e, z) \in \theta$, $a \wedge e \in L$. Then $(x, z) \in \Phi \circ \theta$, and hence θ, Φ are 2-permutable. □

Now, we construct two examples to clarify the above theorem.

Example 4.1. *Consider the principal MS -algebra L in Example 2.1 (Figure 1). From Table 1, we show that L has* 2*-permutable congruences. Also, Tables 2 and 3 show that* $L^{\circ\circ}$ *and* $D(L)$ *have* 2*-permutable congruences, respectively. Where* $\alpha_{L^{\circ\circ}} = \beta_{L^{\circ\circ}}$ *and* $\delta_{L^{\circ\circ}} = \triangle_{L^{\circ\circ}}$.

Table 1. $(Con(L); \circ)$.

$\circ$	$\triangle_L$	α	β	γ	δ	∇_L
$\triangle_L$	$\triangle_L$	α	β	γ	δ	∇_L
α	α	α	β	∇_L	β	∇_L
β	β	β	β	∇_L	β	∇_L
γ	γ	∇_L	∇_L	γ	∇_L	∇_L
δ	δ	β	β	∇_L	δ	∇_L
∇_L	∇_L	∇_L	∇_L	∇_L	∇_L	∇_L

Table 2. $(Con(L^{\circ\circ}); \circ)$.

$\circ$	$\triangle_{L^{\circ\circ}}$	$\alpha_{L^{\circ\circ}}$	$\gamma_{L^{\circ\circ}}$	$\nabla_{L^{\circ\circ}}$
$\triangle_{L^{\circ\circ}}$	$\triangle_{L^{\circ\circ}}$	$\alpha_{L^{\circ\circ}}$	$\gamma_{L^{\circ\circ}}$	$\nabla_{L^{\circ\circ}}$
$\alpha_{L^{\circ\circ}}$	$\alpha_{L^{\circ\circ}}$	$\alpha_{L^{\circ\circ}}$	$\nabla_{L^{\circ\circ}}$	$\nabla_{L^{\circ\circ}}$
$\gamma_{L^{\circ\circ}}$	$\gamma_{L^{\circ\circ}}$	$\nabla_{L^{\circ\circ}}$	$\gamma_{L^{\circ\circ}}$	$\nabla_{L^{\circ\circ}}$
$\nabla_{L^{\circ\circ}}$	$\nabla_{L^{\circ\circ}}$	$\nabla_{L^{\circ\circ}}$	$\nabla_{L^{\circ\circ}}$	$\nabla_{L^{\circ\circ}}$

Table 3. $(Con(D(L)); \circ)$.

$\circ$	$\triangle_{D(L)}$	$\nabla_{D(L)}$
$\triangle_{D(L)}$	$\triangle_{D(L)}$	$\nabla_{D(L)}$
$\nabla_{D(L)}$	$\nabla_{D(L)}$	$\nabla_{D(L)}$

In the following example, we give a principal MS-algebra L which has not 2-permutable congruences as well as $D(L)$ has not also 2-permutable congruences.

Example 4.2. *Consider the principal MS -algebra L in Figure* 5.

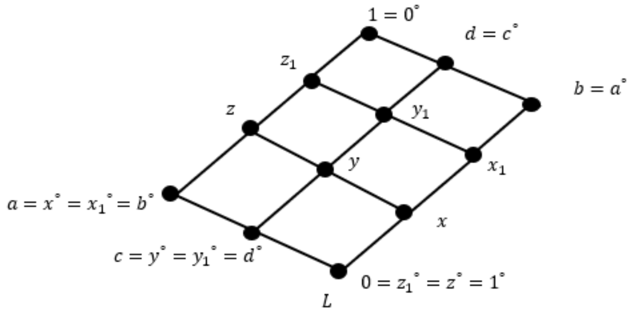

Figure 5. L is a principal MS-algebra with $D(L) = [z)$.

It is clear that $D(L) = \{z, z_1, 1\}$ *and* $L^{\circ\circ} = \{0, c, a, b, d, 1\}$. *Consider* $\theta, \Phi \in Con(L)$ *as follows:*

$$\theta = \triangle_L \cup \{\{x_1, b\}, \{y_1, d\}, \{z_1, 1\}\},$$
$$\Phi = \triangle_L \cup \{\{x, x_1\}, \{y, y_1\}, \{z, z_1\}\}.$$

We observe that $\theta \circ \Phi \neq \Phi \circ \theta$ *as* $(z, 1) \in \Phi \circ \theta$ *but* $(z, 1) \notin \theta \circ \Phi$.

Then L has not 2*-permutable congruences and* $D(L)$ *has not* 2*-permutable congruences as* $\theta_{D(L)} \circ \Phi_{D(L)} \neq \Phi_{D(L)} \circ \theta_{D(L)}$.

5. n-permutability of principal MS-algebras

We generalize the concept of 2-permutability to the concept of n-permutability of congruences of principal MS-algebras.

Definition 5.1. *A principal MS-algebra L is said to have n-permutable congrunces if every two congruences* θ, Φ *of L are n-permutable, that is,*

$$\theta \circ \Phi \circ \theta \circ ... = \Phi \circ \theta \circ \Phi \circ ...(n\ times).$$

At first, we need the following two lemmas.

Lemma 5.1. *Let L be a principal MS- algebra with the smallest dense element* d_L. *Let* θ, Φ *be two congruences on L. Then we have*

(i) $(\theta \circ \Phi \circ \theta \circ ...)_{L^{\circ\circ}} = \theta_{L^{\circ\circ}} \circ \Phi_{L^{\circ\circ}} \circ ...(n\ times)$,

(ii) $(\theta \circ \Phi \circ \theta \circ ...)_{D(L)} = \theta_{D(L)} \circ \Phi_{D(L)} \circ ...(n\ times)$.

Proof.

(i) Recall that $\theta_{L^{\circ\circ}}, \Phi_{L^{\circ\circ}}$ are the restrictions of θ, Φ on $L^{\circ\circ}$, respectively. Let $a, b \in L^{\circ\circ}$ and $(a, b) \in (\theta_{L^{\circ\circ}} \circ \Phi_{L^{\circ\circ}} \circ ...)$. Then there exist $c_1, c_2, ..., c_{n-1} \in L^{\circ\circ}$ such that

$$(a, c_1) \in \theta_{L^{\circ\circ}}, (c_1, c_2) \in \Phi_{L^{\circ\circ}}, ..., (c_{n-1}, b) \in \Psi, \text{where } \Psi = \begin{cases} \theta_{L^{\circ\circ}}, & \text{if } n \text{ odd}, \\ \Phi_{L^{\circ\circ}}, & \text{if } n \text{ even}. \end{cases}$$

Then

$$(a, c_1) \in \theta, (c_1, c_2) \in \Phi, ..., (c_{n-1}, b) \in \Psi, \text{where } \Psi = \begin{cases} \theta, & \text{if } n \text{ odd}, \\ \Phi, & \text{if } n \text{ even}. \end{cases}$$

Then $(a, b) \in (\theta \circ \Phi \circ \theta \circ ...)$ and hence $(a, b) \in (\theta \circ \Phi \circ \theta \circ ...)_{L^{\circ\circ}}$. Therefore

$$\theta_{L^{\circ\circ}} \circ \Phi_{L^{\circ\circ}} \circ \theta_{L^{\circ\circ}} ... \subseteq (\theta \circ \Phi \circ \theta \circ ...)_{L^{\circ\circ}}.$$

Conversely, let $a, b \in L^{\circ\circ}$ such that $(a, b) \in (\theta \circ \Phi \circ \theta \circ ...)_{L^{\circ\circ}}$. Then $(a, b) \in (\theta \circ \Phi \circ \theta \circ ...)$. Then there exist $c_1, c_2, ..., c_{n-1} \in L$ such that

$$(a, c_1) \in \theta_{L^{\circ\circ}}, (c_1, c_2) \in \Phi_{L^{\circ\circ}}, ..., (c_{n-1}, b) \in \Psi, \text{where } \Psi = \begin{cases} \theta_{L^{\circ\circ}}, & \text{if } n \text{ odd}, \\ \Phi_{L^{\circ\circ}}, & \text{if } n \text{ even}. \end{cases}$$

Then we get

$$(a, {c_1}^{\circ\circ}) \in \theta, ({c_1}^{\circ\circ}, {c_2}^{\circ\circ}) \in \Phi, ..., ({c_{n-1}}^{\circ\circ}, b) \in \Psi, \text{where } \Psi = \begin{cases} \theta, & \text{if } n \text{ odd}, \\ \Phi, & \text{if } n \text{ even}. \end{cases}$$

Since ${c_1}^{\circ\circ}, {c_2}^{\circ\circ}, ..., {c_{n-1}}^{\circ\circ} \in L^{\circ\circ}$, we have

$$(a, {c_1}^{\circ\circ}) \in \theta_{L^{\circ\circ}}, ({c_1}^{\circ\circ}, {c_2}^{\circ\circ}) \in \Phi_{L^{\circ\circ}}, ..., ({c_{n-1}}^{\circ\circ}, b) \in \Psi, \text{where } \Psi = \begin{cases} \theta_{L^{\circ\circ}}, & \text{if } n \text{ odd}, \\ \Phi_{L^{\circ\circ}}, & \text{if } n \text{ even}. \end{cases}$$

Therefore

$$(a, b) \in (\theta_{L^{\circ\circ}} \circ \Phi_{L^{\circ\circ}} \circ \theta_{L^{\circ\circ}} \circ ...)$$

and hence

$$(\theta \circ \Phi \circ \theta \circ ...)_{L^{\circ\circ}} \subseteq \theta_{L^{\circ\circ}} \circ \Phi_{L^{\circ\circ}} \circ \theta_{L^{\circ\circ}} \circ ...(\text{n times}).$$

Then we get

$$(\theta \circ \Phi \circ \theta \circ ...)_{L^{\circ\circ}} = \theta_{L^{\circ\circ}} \circ \Phi_{L^{\circ\circ}} \circ \theta_{L^{\circ\circ}} \circ ...(\text{n times}).$$

(ii) Take $x, y \in D(L)$, let $(x, y) \in (\theta_{D(L)} \circ \Phi_{D(L)} \circ \Phi_{D(L)} \circ \theta_{D(L)} \circ ...)$. Then $(x, y) \in (\theta \circ \Phi \circ \theta \circ ...)$ and hence $(a, b) \in (\theta \circ \Phi \circ \theta \circ ...)_{D(L)}$. Then $(\theta \circ \Phi \circ \theta \circ ...) \subseteq (\theta \circ \Phi \circ \theta \circ ...)_{D(L)}$.

Conversely, let $(x, y) \in (\theta \circ \Phi \circ \theta \circ ...)_{D(L)}$, then $(x, y) \in (\theta \circ \Phi \circ ...)$. There exist $d_1, d_2, ...d_{n-1} \in L$, such that

$$(x, d_1) \in \theta, (d_1, d_2) \in \Phi, ..., (d_{n-1}, y) \in \psi, \text{where } \psi = \begin{cases} \theta, & \text{if } n \text{ odd}, \\ \Phi, & \text{if } n \text{ even}. \end{cases}$$

Since $x, y \geq d_L$, we get,

$$(x, d_1) = (x \vee d_L, d_1 \vee d_L) \in \theta_{D(L)}, (d_1 \vee d_L, d_2 \vee d_L) \in \Phi_{D(L)}, ...,$$
$$(d_{n-1}, b \vee d_{L_1}) = (d_{n-1}, y) \in \psi, \text{where } \psi = \begin{cases} \theta_{D(L)}, & \text{if } n \text{ odd}, \\ \Phi_{D(L)}, & \text{if } n \text{ even}. \end{cases}$$

Hence $(x, y) \in (\theta_{D(L)} \circ \Phi_{D(L)} \circ \theta_{D(L)} \circ ...)$. Then the required equality is proved. □

Lemma 5.2. *Let θ and Φ be congruences on a principal MS-algebra L. Let ψ denoted to the relation $\theta \circ \Phi \circ \theta \circ ...$(n times). Then $(a, b) \in \psi$ and $(c, d) \in \psi$ imply $(a \wedge c, b \wedge d) \in \psi$.*

Proof. Let $(a,b) \in \psi$ and $(c,d) \in \psi$. Then there exist $a_1, a_2, ..., a_{n-1}$, $c_1, c_2, ..., c_{n-1} \in L$ such that

$$(a, a_1) \in \theta, (a_1, a_2) \in \Phi, ..., (a_{n-1}, b) \in \Psi, \text{where } \Psi = \begin{cases} \theta, & \text{if } \ n \text{ odd,} \\ \Phi, & \text{if } \ n \text{ even.} \end{cases}$$

And

$$(c, c_1) \in \theta, (c_1, c_2) \in \Phi, ..., (c_{n-1}, d) \in \Psi, \text{where } \Psi = \begin{cases} \theta, & \text{if } \ n \text{ odd,} \\ \Phi, & \text{if } \ n \text{ even.} \end{cases}$$

Since θ, Φ are congruence on L then we get,

$$(a \wedge c, a_1 \wedge c_1) \in \theta, (a_1 \wedge c_1, a_2 \wedge c_2) \in \Phi, ..., (a_{n-1} \wedge c_{n-1}, b \wedge d) \in \Psi, \text{where } \Psi = \begin{cases} \theta, & \text{if } \ n \text{ odd,} \\ \Phi, & \text{if } \ n \text{ even.} \end{cases}$$

Thus $(a \wedge c, b \wedge d) \in \psi$. □

Now, a characterization of n-permutability of principal MS-algebras is given.

Theorem 5.1. *Two congruences θ, Φ on a principal MS-algebra L are n-permutable if and only if their restrictions $\theta_{L^{\circ\circ}}, \Phi_{L^{\circ\circ}}$ and $\theta_{D(L)}, \Phi_{D(L)}$ with respect to $L^{\circ\circ}$ and $D(L)$ respectively are n-permutable.*

Proof. Let L has n-permutable congruences. Let θ, Φ be any two congruences on L. Then by (i) and (ii) of Lemma 5.1, respectively, we have

$$\theta_{L^{\circ\circ}} \circ \Phi_{L^{\circ\circ}} \circ \theta_{L^{\circ\circ}} \circ ... = (\theta \circ \Phi \circ \theta \circ ...)_{L^{\circ\circ}} = (\Phi \circ \theta \circ \Phi \circ ...)_{L^{\circ\circ}} = \Phi_{L^{\circ\circ}} \circ \theta_{L^{\circ\circ}} \circ \Phi_{L^{\circ\circ}} \circ ...,$$

and

$$\theta_{D(L)} \circ \Phi_{D(L)} \circ \theta_{D(L)} \circ ... = (\theta \circ \Phi \circ \theta \circ ...)_{D(L)} = (\Phi \circ \theta \circ \Phi \circ ...)_{D(L)} = \Phi_{D(L)} \circ \theta_{D(L)} \circ \Phi_{D(L)} \circ ....$$

Therefore $L^{\circ\circ}$ and $D(L)$ have n-permutable congruences.

Conversely, let $L^{\circ\circ}$ and $D(L)$ have n-permutable congruences. If $(a,b) \in (\theta \circ \Phi \circ \theta \circ ...)$, using Theorem 2.3, we get

$$(a^{\circ\circ}, b^{\circ\circ}) \in (\theta \circ \Phi \circ \theta \circ ...)_{L^{\circ\circ}},$$

and

$$(a \vee d_L, b \vee d_L) \in (\theta \circ \Phi \circ \theta \circ ...)_{D_L}.$$

By lemma 5.1, we get

$$(a^{\circ\circ}, b^{\circ\circ}) \in (\theta_{L^{\circ\circ}} \circ \Phi_{L^{\circ\circ}} \circ \theta_{L^{\circ\circ}} \circ ...),$$

and

$$(a \vee d_L, b \vee d_L) \in (\theta_{D(L)} \circ \Phi_{D(L)} \circ \Phi_{D(L)} \circ ...).$$

Since $\theta_{L^{\circ\circ}}, \Phi_{L^{\circ\circ}}$ and $\theta_{D(L)}, \Phi_{D(L)}$ are n-permutable on $L^{\circ\circ}, D(L)$, respectively, we have

$$(a^{\circ\circ}, b^{\circ\circ}) \in (\Phi_{L^{\circ\circ}} \circ \theta_{L^{\circ\circ}} \circ \Phi_{L^{\circ\circ}} \circ ...),$$

and

$$(a \vee d_L \equiv b \vee d_L) \in \ (\Phi_{D(L)} \circ \theta_{D(L)} \circ \Phi_{D(L)} \circ ...).$$

It follows that, $(a^{\circ\circ}, b^{\circ\circ}) \in (\Phi \circ \theta \circ \Phi \circ ...)$ and $(a \vee d_L, b \vee d_L) \in (\Phi \circ \theta \circ \Phi \circ ...)$. Since L is a principal MS-algebra, then $a = a^{\circ\circ} \wedge (a \vee d_L)$ and $b = b^{\circ\circ} \wedge (b \vee d_L)$. By Lemma 5.2, we get

$$(a^{\circ\circ} \wedge (a \vee d_L), b) = (a, b) \in (\Phi \circ \theta \circ \Phi \circ ...).$$

Thus $(\theta \circ \Phi \circ \theta \circ ...) \subseteq (\Phi \circ \theta \circ \Phi \circ ...)$. Similarly we can show that $(\Phi \circ \theta \circ \Phi \circ ...) \subseteq (\theta \circ \Phi \circ \theta \circ ...)$. Then L has n-permutable congruences. □

6. Conclusions

With the aid of the technique of MS-congruence pairs, many properties were investigated for principal MS-algebras, deal with a characterization of congruence pairs of a principal MS-algebra (a Stone algebra), a description of the lattice $A(L)$ of all MS-congruence pairs of L via Boolean elements of L, characterizations of 2-permutability and n-permutability of congruences of a principal MS-algebra. This work leads us in the future to study many aspects of principal MS-algebras and related structures, for instance, it can be applied to triple construction of principal MS-algebra, affine and locally complete of principal MS-algebras.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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