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Research article

Group codes over symmetric groups

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Abstract: Let $\mathbb{F}_q$ be a finite field of characteristic q and S_n a symmetric group of order $n!$. In this paper, group codes in the symmetric group algebras $\mathbb{F}_qS_n$ with $q > 3$ and $n = 3, 4$ are proposed. We compute the unique (linear and nonlinear) idempotents of $\mathbb{F}_qS_n$ corresponding to the characters of symmetric groups and use the results to characterize the minimum distances and dimensions of group codes. Furthermore, we construct MDS group codes and almost MDS group codes in $\mathbb{F}_qS_3$ and $\mathbb{F}_qS_4$.

Keywords: group codes; group algebra; minimum distance; symmetric group
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1. Introduction

Group codes, a class of important linear codes, play a vital role in error correction coding. A linear code C is called a group code if it is just a one-sided (left or right) ideal in a group algebra $R[G]$, where R is a commutative ring and G is a finite group. If G is abelian, then C is an abelian code.

A brief survey on group codes of some recent results is provided as follows. Polcino Milies et al. [13] calculated the minimum distances and the dimensions of all cyclic codes of length p^n over a finite field $\mathbb{F}_q$, when p is an odd prime and $\mathbb{F}_q$ is a finite field with q elements, assuming that $\bar{q}$ generates the group of invertible elements of the residue ring module p^n, denoted by $\mathbb{Z}_{p^n}$. Jitman et al. [11] gave a characterization and an enumeration of Euclidean self-dual and Euclidean self-orthogonal abelian codes in a principal ideal group algebra. Choosuwan et al. [5] gave the complete enumeration of self-dual abelian codes in nonprincipal ideal group algebras $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ with respect to both the Euclidean and Hermitian inner products, where k and s are positives and A is an abelian group of odd order. In 2017, Boripan et al. [1] studied a family of abelian codes with complementary dual in a group algebra $\mathbb{F}_{p^\nu}[G]$ in the two cases of Euclidean and Hermitian inner products, where p is a prime, ν is a positive integer, and G is an arbitrary finite abelian group. Cao et al. [3,4] proved that any left D_{2n}-code (left ideal of the group algebra $\mathbb{F}_q[D_{2n}]$) is a direct sum of concatenated codes with inner codes A_i and outer codes C_i, where A_i is a minimal self-reciprocal cyclic code over $\mathbb{F}_q$ of length n and C_i is a skew cyclic code of length 2 over an extension field or principal ideal ring of $\mathbb{F}_q$.

Determining the parameters of linear codes is important, but it is difficult to determine the parameters of group codes in group algebras $\mathbb{F}_q[G]$ through the structure of $\mathbb{F}_q[G]$. For instance, Brochero Martínes [2] showed explicitly all central irreducible idempotents and their Wedderburn decomposition of the dihedral group algebra $\mathbb{F}_q[D_{2n}]$, in the case when every divisor of n divides $q-1$. Brochero Martínez et al. [12] determined an explicit expression for the primitive idempotents of $\mathbb{F}_q[G]$, where $\mathbb{F}_q$ is a finite field, G is a finite cyclic group of order p^k and p is an odd prime with $\gcd(q, p) = 1$. Based on the idea, Gao et al. [7] described and counted all linear complementary dual (LCD) codes and self-orthogonal codes in $\mathbb{F}_q[D_{2n,r}]$, where $[D_{2n,r}]$ is a generalized dihedral group. Gao et al. [6] obtained the precise descriptions and enumerations of linear complementary dual (LCD) codes and self-orthogonal codes in the generalized quaternion group algebras $\mathbb{F}_q[Q_{4n}]$. It is a pity that the authors cannot give the parameters of group codes.

To address this issue, we can use the character label of corresponding groups to determine the idempotents of group algebras. Let S_n be a symmetric group of order $n!$. In this paper, we propose group codes in symmetric group algebras $\mathbb{F}_qS_n$ with $q > 3$ and $n = 3, 4$. We compute the unique (linear and nonlinear) idempotents of $\mathbb{F}_qS_n$ corresponding to the characters of symmetric groups and use the results to characterize the minimum distances and dimensions of group codes. Furthermore, we construct MDS group codes and almost MDS group codes in $\mathbb{F}_qS_3$ and $\mathbb{F}_qS_4$.

The paper is organized as follows: Section 2 provides a review of some properties of group algebras and other preliminaries, while Section 3 investigates group codes in symmetric group algebras $\mathbb{F}_qS_n$ with $q > 3$ and $n = 3, 4$, and obtains the parameters of all the above group codes.

2. Preliminaries

Let $\mathbb{F}_q[G]$ be a group algebra, where $\mathbb{F}_q$ is a finite field and G is a finite group. In fact, the group algebra $\mathbb{F}_q[G]$ is a vector space over $\mathbb{F}_q$ with basis G, and it has scalar, additive, and multiplicative operator as follows: for $c, a_g, b_g \in \mathbb{F}_q$ and $g \in G$,

$$c(\sum_{g\in G} a_g g) = \sum_{g\in G} c a_g g,$$

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G}(a_g + b_g)g,$$

$$(\sum_{g\in G} a_g g)(\sum_{h\in G} b_h h) = \sum_{g\in G}(\sum_{uv=g} a_u b_v)g.$$

Then $\mathbb{F}_q[G]$ is an associative $\mathbb{F}_q$-algebra with the identity $1 = 1_{\mathbb{F}_q}1_G$, where $1_{\mathbb{F}_q}$ and 1_G are the identity elements of $\mathbb{F}_q$ and G, respectively. Readers are referred to [14, 16] for more details on group ring or group algebra.

Lemma 2.1. *(Maschke's Theorem [14]) Let R be a ring and G be a group. Then the group ring $R[G]$ is semisimple if and only if the following conditions hold.*

(i) R is a semisimple ring.

(ii) G is finite.

(iii) $|G|$ is invertible in R.

By Lemma 2.1, it is easy to verify that $\mathbb{F}_q[G]$ is semisimple if and only if G is a finite group and $char(\mathbb{F}_q) \nmid |G|$. Let $\mathbb{F}_q[G]$ be a semisimple group algebra. Then $\mathbb{F}_q[G]$ can be decomposed into a direct sum $\mathbb{F}_q[G] = \bigoplus_{i\in\Omega} \mathbb{F}_q[G]e_i$, where $\mathbb{F}_q[G]e_i$ is the minimal ideal generated by the idempotent $e_i, i \in \Omega$ and Ω is the index set (see [9]). If I is any ideal of $\mathbb{F}_q[G]$, then I can be expressed as a direct sum of some minimal ideal $\mathbb{F}_q[G]e_i$ of $\mathbb{F}_q[G]$, i.e., $I = \bigoplus_{i\in\Omega_1\subseteq\Omega} \mathbb{F}_q[G]e_i$. Let I be an ideal generated by a subset Ω_1 and $\Lambda = \Omega \setminus \Omega_1$. Then $I \doteq I_\Lambda = \{\alpha \in \mathbb{F}_q[G] : \alpha e_j = 0, \text{ for all } e_j \in \Lambda\}$.

Suppose that $\alpha = \sum_{g\in G} a_g g \in \mathbb{F}_q[G]$. Then $wt(\alpha) = |\{a_g : a_g \neq 0\}|$ is called the Hamming weight of α ([19]). Let I_Λ be a group code in $\mathbb{F}_q[G]$. Its length n is the order of G in group algebra $\mathbb{F}_q[G]$. Its dimension k is the dimension of I_Λ as a subspace over $\mathbb{F}_q$. Its minimal distance d is defined as $d = \min\{wt(\alpha) : \alpha e_i = 0 \text{ for nonzero element } \alpha \text{ and for all } e_i \in \Lambda\}$. Thus the group code I_Λ is called an $[n, k, d]$ code. Moreover, a linear $[n, k, d]$ group code over $\mathbb{F}_q$ with $d = n - k + 1$ is called a maximum distance separable (MDS) group code. And, a linear $[n, k, d]$ group code over $\mathbb{F}_q$ with $d = n - k$ is called an almost MDS (AMDS) group code. MDS codes and AMDS codes are considered to be an attractive solution for information storage as they operate at the optimal storage versus reliability trade-off (see [8, 15, 17, 18, 20, 21]).

Based on the following lemma, we can give the unique idempotents of $\mathbb{F}_q[G]$.

Lemma 2.2. *[10] If χ is a character of $\mathbb{F}_qG$-module, then the idempotents corresponding to the character χ are given by*

$$e = \frac{1}{|G|}\sum_{g\in G}\chi(g^{-1})g.$$

3. Group codes in $\mathbb{F}_qS_n$

3.1. Group codes in $\mathbb{F}_qS_3$

In the subsection, we will consider the group codes on S_3, where S_3 is a nonabelian group with the smallest order. Let

$$S_3 = \langle a, b : a^3 = b^2 = (ab)^2 = 1\rangle = \{1, a, a^2, b, ab, a^2b\}.$$

It is well-known that S_3 is all the permutations on three elements 1,2,3. In this sense, set $a = (123), b = (12)$, S_3 can be given by $\{(1), (123), (132), (12), (23), (13)\}$. For convenience, we denote g_i $(1 \le i \le 6)$ as the i-th element of S_3 in the above two sets.

In addition, S_3 has three conjugacy classes as follows:

$$C_1 = \{1\}, C_2 = \{b, ab, a^2b\}, C_3 = \{a, a^2\}.$$

Since the commutator group of S_3 is $S_3' = \{1, a, a^2\}$, we have $|S_3/S_3'| = 2$. As a consequence of this, the group S_3 has two linear characters χ_1, χ_2 and one nonlinear character χ_3. The character table of S_3 is as follows (see Table 1).

Table 1. The character table of S_3.

Characters	1	a	b
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

By Lemma 2.2, the unique idempotents of $\mathbb{F}_q S_3$ are given in the following theorem.

Theorem 3.1. *There are three idempotents in $\mathbb{F}_q S_3$ as follows:*

$$\begin{aligned} e_1 &= \frac{1}{6}[1+\overline{C}_2+\overline{C}_3], \\ e_2 &= \frac{1}{6}[1-\overline{C}_2+\overline{C}_3], \\ f_1 &= \frac{1}{6}[2-\overline{C}_3], \end{aligned}$$

where $\overline{C}_i = \sum_{g\in C_i} g, i = 2,3.$

In order to construct the group codes of $\mathbb{F}_q S_3$, we need the product between idempotents e_1, e_2, f_1 and an arbitrary element of $\mathbb{F}_q S_3$. For any element $\alpha = \sum_{i=1}^{6} a_i g_i \in \mathbb{F}_q S_3$, where $a_i \in \mathbb{F}_q, i = 1, 2, \ldots, 6$, we have

$$\alpha e_1 = (\sum_{i=1}^{6} a_i)e_1. \tag{3.1}$$

$$\alpha e_2 = (\sum_{i=1}^{3} a_i - \sum_{i=4}^{6} a_i)e_2. \tag{3.2}$$

$$\begin{aligned} \alpha f_1 = \frac{1}{6}[&(2a_1-a_2-a_3)g_1+(-a_1+2a_2-a_3)g_2+(-a_1-a_2+2a_3)g_3 \\ &+(2a_4-a_5-a_6)g_4+(-a_4+2a_5-a_6)g_5+(-a_4-a_5+2a_6)g_6]. \end{aligned} \tag{3.3}$$

The following theorems give us the parameters of group codes in $\mathbb{F}_q S_3$.

Theorem 3.2. *Let e_1, e_2 and f_1 be idempotents in $\mathbb{F}_q S_3$. Then*

(1) $I_{\{e_1\}}$ is a [6,5,2] group code;

(2) $I_{\{e_2\}}$ is a [6,5,2] group code;

(3) $I_{\{f_1\}}$ is a [6,2,3] group code.

Proof. (1) Clearly, $I_{\{e_1\}} = \{\alpha \in \mathbb{F}_q S_3 | \alpha e_1 = 0\}$. Firstly, we will give the dimension of group code $I_{\{e_1\}}$. For any $\alpha = \sum_{i=1}^{6} a_i g_i \in I_{\{e_1\}}$, we have $\alpha e_1 = 0$. By (3.1), we obtain $\sum_{i=1}^{6} a_i = 0$. Then $\dim(I_{\{e_1\}}) = 5$. Secondly, we will compute the minimal distance of group code $I_{\{e_1\}}$. For some $\alpha = kg_i, 1 \le i \le 6$, if $k \neq 0$, then $\alpha e_1 \neq 0$, i.e., $\alpha = kg_i \notin I_{\{e_1\}}$. So $d(I_{\{e_1\}}) \ge 2$. Set $\alpha = g_1 - g_2$. Since $\alpha e_1 = (g_1 - g_2)e_1 = 0$, we get $\alpha = g_1 - g_2 \in I_{\{e_1\}}$. Hence, $d(I_{\{e_1\}}) = 2$.

(2) Clearly, $I_{\{e_2\}} = \{\alpha \in \mathbb{F}_q S_3 | \alpha e_2 = 0\}$. Firstly, we will give the dimension of group code $I_{\{e_2\}}$. For any $\alpha = \sum_{i=1}^{6} a_i g_i \in I_{\{e_2\}}$, we have $\alpha e_2 = 0$. By (3.2), we obtain $\sum_{i=1}^{3} a_i - \sum_{i=4}^{6} a_i = 0$. Then $\dim(I_{\{e_2\}}) = 5$. Secondly, we will compute the minimal distance of group code $I_{\{e_2\}}$. For some $\alpha = kg_i, 1 \leq i \leq 6$, if $k \neq 0$, then $\alpha e_2 \neq 0$, i.e., $\alpha = kg_i \notin I_{\{e_2\}}$. So $d(I_{\{e_2\}}) \geq 2$. Set $\alpha = g_1 + g_4$. Since $\alpha e_2 = (g_1 + g_4)e_2 = 0$, we get $\alpha = g_1 + g_4 \in I_{\{e_2\}}$. Hence, $d(I_{\{e_2\}}) = 2$.

(3) Clearly, $I_{\{f_1\}} = \{\alpha \in \mathbb{F}_q S_3 | \alpha f_1 = 0\}$. Firstly, we will give the dimension of group code $I_{\{f_1\}}$. For any $\alpha = \sum_{i=1}^{6} a_i g_i \in I_{\{f_1\}}$, we have $\alpha f_1 = 0$. By (3.3), we obtain

$$\begin{cases} 2a_1 - a_2 - a_3 = 0 \\ -a_1 + 2a_2 - a_3 = 0 \\ -a_1 - a_2 + 2a_3 = 0 \\ 2a_4 - a_5 - a_6 = 0 \\ -a_4 + 2a_5 - a_6 = 0 \\ -a_4 - a_5 + 2a_6 = 0 \end{cases}.$$

Since the rank of the coefficient matrix of the above equation is 4, we know that $\dim(I_{\{f_1\}}) = 2$. Secondly, we will compute the minimal distance of group code $I_{\{f_1\}}$. For some $\alpha = k_i g_i + k_j g_j, 1 \leq i, j \leq 6$, if $k_i, k_j \neq 0$, then $\alpha f_1 \neq 0$, i.e., $\alpha = k_i g_i + k_j g_j \notin I_{\{f_1\}}$. So $d(I_{\{f_1\}}) \geq 3$. Set $\alpha = g_1 + g_2 + g_3$. Since $\alpha f_1 = (g_1 + g_2 + g_3) f_1 = 0$, we get $\alpha = g_1 + g_2 + g_3 \in I_{\{f_1\}}$. Hence, $d(I_{\{f_1\}}) = 3$.

This completes the proof. □

Theorem 3.3. *Let e_1, e_2 and f_1 be idempotents in $\mathbb{F}_q S_3$. Then*

(1) $I_{\{e_1,e_2\}}$ is a [6,4,2] group code;

(2) $I_{\{e_1,f_1\}}$ is a [6,1,6] group code;

(3) $I_{\{e_2,f_1\}}$ is a [6,1,6] group code.

Proof. (1) Clearly, $I_{\{e_1,e_2\}} = \{\alpha \in \mathbb{F}_q S_3 | \alpha e_1 = 0, \alpha e_2 = 0\}$. Firstly, we will give the dimension of group code $I_{\{e_1,e_2\}}$. For any $\alpha = \sum_{i=1}^{6} a_i g_i \in I_{\{e_1,e_2\}}$, we have $\alpha e_1 = 0$ and $\alpha e_2 = 0$. From (3.1) and (3.2), we obtain

$$\begin{cases} a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 0 \\ a_1 + a_2 + a_3 - a_4 - a_5 - a_6 = 0 \end{cases}.$$

Since the rank of the coefficient matrix of the above equation is 2, we know that $\dim(I_{\{e_1,e_2\}}) = 4$. Secondly, we will compute the minimal distance of group code $I_{\{e_1,e_2\}}$. For some $\alpha = kg_i, 1 \leq i \leq 6$, if $k \neq 0$, then $\alpha e_1 \neq 0$, i.e., $\alpha = kg_i \notin I_{\{e_1,e_2\}}$. So $d(I_{\{e_1,e_2\}}) \geq 2$. Set $\alpha = g_1 - g_3$. Since $\alpha e_1 = (g_1 - g_3)e_1 = 0$ and $\alpha e_2 = (g_1 - g_3)e_2 = 0$, we get $\alpha = g_1 - g_3 \in I_{\{e_1,e_2\}}$. Hence, $d(I_{\{e_1,e_2\}}) = 2$.

(2) Clearly, $I_{\{e_1,f_1\}} = \{\alpha \in \mathbb{F}_q S_3 | \alpha e_1 = 0, \alpha f_1 = 0\}$. Firstly, we will give the dimension of group code

$I_{\{e_1,f_1\}}$. For any $\alpha = \sum_{i=1}^{6} a_i g_i \in I_{\{e_1,f_1\}}$, we have $\alpha e_1 = 0$ and $\alpha f_1 = 0$. From (3.1) and (3.3), we obtain

$$\begin{cases} a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 0 \\ 2a_1 - a_2 - a_3 = 0 \\ -a_1 + 2a_2 - a_3 = 0 \\ -a_1 - a_2 + 2a_3 = 0 \\ 2a_4 - a_5 - a_6 = 0 \\ -a_4 + 2a_5 - a_6 = 0 \\ -a_4 - a_5 + 2a_6 = 0 \end{cases}.$$

Since the rank of the coefficient matrix of the above equations is 5, we know that $\dim(I_{\{e_1,f_1\}}) = 1$. Secondly, we will compute the minimal distance of group code $I_{\{e_1,f_1\}}$. If we take $\alpha = g_1 + g_2 + g_3$, we have $\alpha f_1 = 0$ and $\alpha e_1 \neq 0$, i.e., $\alpha = g_1+g_2+g_3 \notin I_{\{e_1,f_1\}}$. So $d(I_{\{e_1,e_2\}}) \geq 3$. Set $\alpha = g_1+g_2+g_3-g_4-g_5-g_6$. Since $\alpha e_1 = (g_1 + g_2 + g_3 - g_4 - g_5 - g_6)e_1 = 0$ and $\alpha e_2 = (g_1 + g_2 + g_3 - g_4 - g_5 - g_6) f_1 = 0$, we get $\alpha = g_1 + g_2 + g_3 - g_4 - g_5 - g_6 \in I_{\{e_1,f_1\}}$. Hence, $d(I_{\{e_1,f_1\}}) = 6$.

(3) The result can be obtained by a similar proof of (2).

This completes the proof. □

We can get the following results based on Theorems 3.2 and 3.3.

Remark 3.4. *Let e_1, e_2 and f_1 be idempotents in $\mathbb{F}_q S_3$.*

(1) $I_{\{e_i\}}, I_{\{e_i,f_1\}}, i = 1, 2$ are MDS group codes;

(2) $I_{\{e_1,e_2\}}$ is an AMDS group codes.

3.2. Group codes in $\mathbb{F}_q S_4$

In the subsection, we will consider the group codes on S_4, where S_4 is a nonabelian group of order 24. Let

$$\begin{aligned} S_4 &= \langle a, b : a^4 = b^2 = (ab)^4 = 1\rangle = \{1, a, a^2, a^3, b, ab, a^2b, a^3b, ba, aba, a^2ba, a^3ba, \\ &\quad ba^2, aba^2, a^2ba^2, a^3ba^2, ba^3, aba^3, a^2ba^3, a^3ba^3, ba^2b, aba^2b, a^2ba^2b, a^3ba^2b\}. \end{aligned}$$

It is well-known that S_4 is all the permutations on four elements 1,2,3,4. In this sense, set $a = (1234), b = (12)$, S_4 can be given by {(1), (1234), (13)(24), (1432), (12), (234), (1324), (143), (134), (1243), (142), (23), (1423), (132), (34), (124), (243), (14), (123), (1342), (14)(23), (13), (12)(34), (24)}. For convenience, we denote g_i ($1 \leq i \leq 24$) as the i-th element of S_4 in the above two sets.

In addition, S_4 has five conjugacy classes as follows:

$$\begin{aligned} C_1 &= \{(1)\}, \\ C_2 &= \{(12), (13), (14), (23), (24), (34)\}, \\ C_3 &= \{(12)(34), (13)(24), (14)(23)\}, \\ C_4 &= \{(123), (132), (234), (243), (124), (142), (134), (143)\}, \\ C_5 &= \{(1234), (1432), (1324), (1243), (1423), (1342)\}. \end{aligned}$$

Since the commutator group of S_4 is $S_4' = A_4$, we have $|S_4/S_4'| = 2$. As a consequence of this, the group S_4 has two linear characters χ_1, χ_2 and three nonlinear characters χ_3, χ_4, χ_5. The character table of S_4 is as follows (see Table 2).

Table 2. The character table of S_4.

Characters	(1)	(12)	(12)(34)	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

By Lemma 2.2, the unique idempotents of $\mathbb{F}_q S_4$ are given in the following theorem.

Theorem 3.5. *There are five idempotents in $\mathbb{F}_q S_4$ as follows:*

$$\begin{aligned}
e_1 &= \frac{1}{24}[1 + \overline{C}_2 + \overline{C}_3 + \overline{C}_4 + \overline{C}_5],\\
e_2 &= \frac{1}{24}[1 - \overline{C}_2 + \overline{C}_3 + \overline{C}_4 - \overline{C}_5],\\
f_1 &= \frac{1}{24}[2 + 2\overline{C}_3 - \overline{C}_4],\\
f_2 &= \frac{1}{24}[3 + \overline{C}_2 - \overline{C}_3 - \overline{C}_5],\\
f_3 &= \frac{1}{24}[3 - \overline{C}_2 - \overline{C}_3 + \overline{C}_5],
\end{aligned}$$

where $\overline{C}_i = \sum\limits_{g \in C_i} g, i = 2, 3, 4, 5.$

In order to construct the group codes of $\mathbb{F}_q S_4$, we need the product between idempotents e_1, e_2, f_1, f_2, f_3 and an arbitrary element of $\mathbb{F}_q S_4$. For any element $\alpha = \sum\limits_{i=1}^{24} a_i g_i \in \mathbb{F}_q S_4$, where $a_i \in \mathbb{F}_q, i = 1, 2, \ldots, 24$, we have

$$\alpha e_1 = (\sum_{i=1}^{24} a_i) e_1. \tag{3.4}$$

$$\begin{aligned}
\alpha e_2 &= [(a_1 + a_3 + a_6 + a_8 + a_9 + a_{11} + a_{14} + a_{17} + a_{17} + a_{19} + a_{21} + a_{23})\\
&- (a_2 + a_4 + a_5 + a_7 + a_{10} + a_{12} + a_{13} + a_{15} + a_{18} + a_{20} + a_{22} + a_{24})]e_2. \qquad (3.5)
\end{aligned}$$

$\alpha f_1 = \frac{1}{24}[(2a_1 + 2a_3 + 2a_{21} + 2a_{23} - a_6 - a_8 - a_9 - a_{11} - a_{14} - a_{16} - a_{17} - a_{19})(g_1 + g_3 + g_{21} + g_{23}) + (2a_2 + 2a_4 + 2a_{22} + 2a_{24} - a_5 - a_7 - a_{10} - a_{12} - a_{13} - a_{15} - a_{18} - a_{20})(g_2 + g_4 + g_{22} + g_{24}) + (2a_5 + 2a_7 + 2a_{13} + 2a_{15} - a_2 - a_4 - a_{10} - a_{12} - a_{18} - a_{20} - a_{22} - a_{24})(g_5 + g_7 + g_{13} + g_{15}) + (2a_6 + 2a_8 + 2a_{14} + 2a_{16} - a_1 - a_3 - a_9 - a_{11} - a_{17} -$

$a_{19}-a_{21}-a_{23})(g_6+g_8+g_{14}+g_{16})+(2a_9+2a_{11}+2a_{17}+2a_{19}-a_1-a_3-a_6-a_8-a_{14}-a_{16}-a_{21}-a_{23})(g_9+g_{11}+g_{17}+g_{19})+(2a_{10}+2a_{12}+2a_{18}+2a_{20}-a_2-a_4-a_5-a_7-a_{13}-a_{15}-a_{22}-a_{24})(g_{10}+g_{12}+g_{18}+g_{20})$.

$\alpha f_2=\frac{1}{24}[(3a_1+a_5+a_{12}+a_{15}+a_{18}+a_{22}+a_{24}-a_3-a_{21}-a_{23}-a_2-a_4-a_7-a_{10}-a_{13}-a_{20})g_1+(3a_2+a_6+a_9+a_{16}+a_{19}+a_{21}+a_{23}-a_4-a_{22}-a_{24}-a_1-a_3-a_8-a_{11}-a_{14}-a_{17})g_2+(3a_3+a_7+a_{10}+a_{13}+a_{20}+a_{22}+a_{24}-a_1-a_{21}-a_{23}-a_2-a_4-a_5-a_{12}-a_{15}-a_{18})g_3+(3a_4+a_8+a_{11}+a_{14}+a_{17}+a_{21}+a_{23}-a_2-a_{22}-a_{24}-a_1-a_3-a_6-a_9-a_{16}-a_{19})g_4+(3a_5+a_1+a_{11}+a_{14}+a_{16}+a_{19}+a_{23}-a_7-a_{13}-a_{15}-a_2-a_6-a_8-a_9-a_{17}-a_{21})g_5+(3a_6+a_2+a_{12}+a_{13}+a_{15}+a_{20}+a_{24}-a_8-a_{14}-a_{16}-a_3-a_5-a_7-a_{10}-a_{18}-a_{22})g_6+(3a_7+a_3+a_9+a_{14}+a_{16}+a_{17}+a_{21}-a_5-a_{13}-a_{15}-a_1-a_6-a_8-a_{11}-a_{19}-a_{23})g_7+(3a_8+a_4+a_{10}+a_{13}+a_{15}+a_{18}+a_{22}-a_6-a_{14}-a_{16}-a_2-a_5-a_7-a_{12}-a_{20}-a_{24})g_8+(3a_9+a_2+a_7+a_{15}+a_{18}+a_{20}+a_{22}-a_{11}-a_{17}-a_{19}-a_4-a_5-a_{10}-a_{12}-a_{13}-a_{24})g_9+(3a_{10}+a_3+a_8+a_{16}+a_{17}+a_{19}+a_{23}-a_{12}-a_{18}-a_{20}-a_1-a_6-a_9-a_{11}-a_{14}-a_{21})g_{10}+(3a_{11}+a_4+a_5+a_{13}+a_{18}+a_{20}+a_{24}-a_9-a_{17}-a_{19}-a_2-a_7-a_{10}-a_{12}-a_{15}-a_{22})g_{11}+(3a_{12}+a_1+a_6+a_{14}+a_{17}+a_{19}+a_{21}-a_{10}-a_{18}-a_{20}-a_3-a_8-a_9-a_{11}-a_{16}-a_{23})g_{12}+(3a_{13}+a_3+a_6+a_8+a_{11}+a_{19}+a_{21}-a_5-a_7-a_{15}-a_1-a_9-a_{14}-a_{16}-a_{17}-a_{23})g_{13}+(3a_{14}+a_4+a_5+a_7+a_{12}+a_{20}+a_{22}-a_6-a_8-a_{16}-a_2-a_{10}-a_{13}-a_{15}-a_{18}-a_{24})g_{14}+(3a_{15}+a_1+a_6+a_8+a_9+a_{17}+a_{23}-a_5-a_7-a_{13}-a_3-a_{11}-a_{14}-a_{16}-a_{19}-a_{21})g_{15}+(3a_{16}+a_2+a_5+a_7+a_{10}+a_{18}+a_{24}-a_6-a_8-a_{14}-a_4-a_{12}-a_{13}-a_{15}-a_{20}-a_{22})g_{16}+(3a_{17}+a_4+a_7+a_{10}+a_{12}+a_{15}+a_{24}-a_9-a_{11}-a_{19}-a_2-a_5-a_{13}-a_{18}-a_{20}-a_{22})g_{17}+(3a_{18}+a_1+a_8+a_9+a_{11}+a_{16}+a_{21}-a_{10}-a_{12}-a_{20}-a_3-a_6-a_{14}-a_{17}-a_{19}-a_{23})g_{18}+(3a_{19}+a_2+a_5+a_{10}+a_{12}+a_{13}+a_{22}-a_9-a_{11}-a_{17}-a_4-a_7-a_{15}-a_{18}-a_{20}-a_{24})g_{19}+(3a_{20}+a_3+a_6+a_9+a_{11}+a_{14}+a_{23}-a_{10}-a_{12}-a_{18}-a_1-a_8-a_{16}-a_{17}-a_{19}-a_{21})g_{20}+(3a_{21}+a_2+a_4+a_7+a_{12}+a_{13}+a_{18}-a_1-a_3-a_{23}-a_5-a_{10}-a_{15}-a_{20}-a_{22}-a_{24})g_{21}+(3a_{22}+a_1+a_3+a_8+a_9+a_{14}+a_{19}-a_2-a_4-a_{24}-a_6-a_{11}-a_{16}-a_{17}-a_{21}-a_{23})g_{22}+(3a_{23}+a_2+a_4+a_5+a_{10}+a_{15}+a_{20}-a_1-a_3-a_{21}-a_7-a_{12}-a_{13}-a_{18}-a_{22}-a_{24})g_{23}+(3a_{24}+a_1+a_3+a_6+a_{11}+a_{16}+a_{17}-a_2-a_4-a_{22}-a_8-a_9-a_{14}-a_{19}-a_{21}-a_{23})g_{24}]$.

$\alpha f_3=\frac{1}{24}[(3a_1-a_3-a_5-a_{12}-a_{15}-a_{18}-a_{21}-a_{22}-a_{23}-a_{24}+a_2+a_4+a_7+a_{10}+a_{13}+a_{20})g_1+(3a_2-a_4-a_6-a_9-a_{16}-a_{19}-a_{21}-a_{22}-a_{23}-a_{24}+a_1+a_3+a_8+a_{11}+a_{14}+a_{17})g_2+(3a_3-a_1-a_7-a_{10}-a_{13}-a_{20}-a_{21}-a_{22}-a_{23}-a_{24}+a_2+a_4+a_5+a_{12}+a_{15}+a_{18})g_3+(3a_4-a_2-a_8-a_{11}-a_{14}-a_{17}-a_{21}-a_{22}-a_{23}-a_{24}+a_1+a_3+a_6+a_9+a_{16}+a_{19})g_4+(3a_5-a_1-a_7-a_{11}-a_{13}-a_{14}-a_{15}-a_{16}-a_{19}-a_{23}+a_2+a_6+a_8+a_9+a_{17}+a_{21})g_5+(3a_6-a_2-a_8-a_{12}-a_{13}-a_{14}-a_{15}-a_{16}-a_{20}-a_{24}+a_3+a_5+a_7+a_{10}+a_{18}+a_{22})g_6+(3a_7-a_3-a_5-a_9-a_{13}-a_{14}-a_{15}-a_{16}-a_{17}-a_{21}+a_1+a_6+a_8+a_{11}+a_{19}+a_{23})g_7+(3a_8-a_4-a_6-a_{10}-a_{13}-a_{14}-a_{15}-a_{16}-a_{18}-a_{22}+a_2+a_5+a_7+a_{12}+a_{20}+a_{24})g_8+(3a_9-a_2-a_7-a_{11}-a_{15}-a_{17}-a_{18}-a_{19}-a_{20}-a_{22}+a_4+a_5+a_{10}+a_{12}+a_{13}+a_{24})g_9+(3a_{10}-a_3-a_8-a_{12}-a_{16}-a_{17}-a_{18}-a_{19}-a_{20}-a_{23}+a_1+a_6+a_9+a_{11}+a_{14}+a_{21})g_{10}+(3a_{11}-a_4-a_5-a_9-a_{13}-a_{17}-a_{18}-a_{19}-a_{20}-a_{24}+a_2+a_7+a_{10}+a_{12}+a_{15}+a_{22})g_{11}+(3a_{12}-a_1-a_6-a_{10}-a_{14}-a_{17}-a_{18}-a_{19}-a_{20}-a_{21}+a_3+a_8+a_9+a_{11}+a_{16}+a_{23})g_{12}+(3a_{13}-a_3-a_5-a_6-a_7-a_8-a_{11}-a_{15}-a_{19}-a_{21}+a_1+a_9+a_{14}+a_{16}+a_{17}+a_{23})g_{13}+(3a_{14}-a_4-a_5-a_6-a_7-a_8-a_{12}-a_{16}-a_{20}-a_{22}+a_2+a_{10}+a_{13}+a_{15}+a_{18}+a_{24})g_{14}+(3a_{15}-a_1-a_5-a_6-a_7-a_8-a_9-a_{13}-a_{17}-a_{23}+a_3+a_{11}+a_{14}+a_{16}+a_{19}+a_{21})g_{15}+(3a_{16}-a_2-a_5-a_6-a_7-a_8-a_{10}-a_{14}-a_{18}-a_{24}+a_4+a_{12}+a_{13}+a_{15}+a_{20}+a_{22})g_{16}+(3a_{17}-a_4-a_7-a_9-a_{10}-a_{11}-a_{12}-a_{15}-a_{19}-a_{24}+a_2+a_5+a_{13}+a_{18}+a_{20}+a_{22})g_{17}+(3a_{18}-a_1-a_8-a_9-a_{10}-a_{11}-a_{12}-a_{16}-a_{20}-a_{21}+a_3+a_6+a_{14}+a_{17}+a_{19}+a_{23})g_{18}+(3a_{19}-a_2-a_5-a_9-a_{10}-a_{11}-a_{12}-a_{13}-a_{17}-a_{22}+a_4+a_7+a_{15}+a_{18}+a_{20}+a_{24})g_{19}+(3a_{20}-a_3-a_6-a_9-a_{10}-a_{11}-a_{12}-a_{14}-a_{18}-a_{23}+a_1+a_8+a_{16}+a_{17}+a_{19}+a_{21})g_{20}+(3a_{21}-a_1-a_2-a_3-a_4-a_7-a_{12}-a_{13}-a_{18}-a_{23}+a_5+a_{10}+a_{15}+a_{20}+a_{22}+a_{24})g_{21}+(3a_{22}-a_1-a_2-a_3-a_4-a_8-a_9-a_{14}-a_{19}-a_{24}+a_6+a_{11}+a_{16}+a_{17}+a_{21}+a_{23})g_{22}+(3a_{23}-a_1-a_2-a_3-a_4-a_5-a_{10}-a_{15}-a_{20}-a_{21}+a_7+a_{12}+a_{13}+a_{18}+a_{22}+a_{24})g_{23}+(3a_{24}-a_1-a_2-a_3-a_4-a_6-a_{11}-a_{16}-a_{17}-a_{22}+a_8+a_9+a_{14}+a_{19}+a_{21}+a_{23})g_{24}]$.

The following theorems give us the parameters of group codes in $\mathbb{F}_qS_4$.

Theorem 3.6. *Let $e_i, i=1,2$ and $f_j, j=1,2,3$ be idempotents in $\mathbb{F}_qS_4$. Then*

(1) $I_{\{e_i\}}$ is a [24,23,2] group code, for $i=1,2$;

(2) $I_{\{f_1\}}$ is a [24,20,2] group code;
(3) $I_{\{f_j\}}$ is a [24,15,4] group code, for $j = 2, 3$.

Proof. (1) Clearly, $I_{\{e_i\}} = \{\alpha \in \mathbb{F}_q S_4 | \alpha e_i = 0\}$, for $i = 1, 2$. Firstly, we will give the dimension of group code $I_{\{e_i\}}, i = 1, 2$. For any $\alpha = \sum_{i=1}^{24} a_i g_i \in I_{\{e_i\}}$, we have $\alpha e_i = 0$. By (3.4), we obtain the following equation:

$$\sum_{i=1}^{24} a_i = 0. \tag{3.6}$$

By (3.5), we obtain the following equation:

$$\begin{aligned} & (a_1 + a_3 + a_6 + a_8 + a_9 + a_{11} + a_{14} + a_{17} + a_{17} + a_{19} + a_{21} + a_{23}) \\ - \; & (a_2 + a_4 + a_5 + a_7 + a_{10} + a_{12} + a_{13} + a_{15} + a_{18} + a_{20} + a_{22} + a_{24}) = 0. \end{aligned} \tag{3.7}$$

Then, for $i = 1, 2$, we have $\dim(I_{\{e_i\}}) = 5$. Secondly, we will compute the minimal distance of group code $I_{\{e_i\}}$. For some $\alpha = kg_i, 1 \le i \le 24$, if $k \ne 0$, then $\alpha e_i \ne 0$, i.e., $\alpha = kg_i \notin I_{\{e_i\}}$. So $d(I_{\{e_i\}}) \ge 2$. Set $\alpha = g_1 - g_3$. Since $\alpha e_i = (g_1 - g_3)e_i = 0$, we get $\alpha = g_1 - g_3 \in I_{\{e_i\}}$. Hence, $d(I_{\{e_i\}}) = 2$.

(2) Clearly, $I_{\{f_1\}} = \{\alpha \in \mathbb{F}_q S_4 | \alpha f_1 = 0\}$. Firstly, we will give the dimension of group code $I_{\{f_1\}}$. For any $\alpha = \sum_{i=1}^{24} a_i g_i \in I_{\{f_1\}}$, we have $\alpha f_1 = 0$. Then, we obtain the following equations:

$$2a_1 + 2a_3 + 2a_{21} + 2a_{23} - a_6 - a_8 - a_9 - a_{11} - a_{14} - a_{16} - a_{17} - a_{19} = 0. \tag{3.8}$$

$$2a_2 + 2a_4 + 2a_{22} + 2a_{24} - a_5 - a_7 - a_{10} - a_{12} - a_{13} - a_{15} - a_{18} - a_{20} = 0. \tag{3.9}$$

$$2a_5 + 2a_7 + 2a_{13} + 2a_{15} - a_2 - a_4 - a_{10} - a_{12} - a_{18} - a_{20} - a_{22} - a_{24} = 0. \tag{3.10}$$

$$2a_6 + 2a_8 + 2a_{14} + 2a_{16} - a_1 - a_3 - a_9 - a_{11} - a_{17} - a_{19} - a_{21} - a_{23} = 0. \tag{3.11}$$

$$2a_9 + 2a_{11} + 2a_{17} + 2a_{19} - a_1 - a_3 - a_6 - a_8 - a_{14} - a_{16} - a_{21} - a_{23} = 0. \tag{3.12}$$

$$2a_{10} + 2a_{12} + 2a_{18} + 2a_{20} - a_2 - a_4 - a_5 - a_7 - a_{13} - a_{15} - a_{22} - a_{24} = 0. \tag{3.13}$$

Since the rank of the coefficient matrix of the above equations is 4, we know that $\dim(I_{\{f_1\}}) = 20$. Secondly, we will compute the minimal distance of group code $I_{\{f_1\}}$. For some $\alpha = kg_i, 1 \le i \le 24$, if $k \ne 0$, then $\alpha f_1 \ne 0$, i.e., $\alpha = kg_i \notin I_{\{f_1\}}$. So $d(I_{\{f_1\}}) \ge 2$. Set $\alpha = g_2 - g_4$. Since $\alpha f_1 = (g_2 - g_4)f_1 = 0$, we get $\alpha = g_2 - g_4 \in I_{\{f_1\}}$. Hence, $d(I_{\{f_1\}}) = 2$.

(3) Clearly, $I_{\{f_i\}} = \{\alpha \in \mathbb{F}_q S_4 | \alpha f_i = 0\}, i = 2, 3$. Firstly, we will give the dimension of group code $I_{\{f_2\}}$. For any $\alpha = \sum_{i=1}^{24} a_i g_i \in I_{\{f_2\}}$, we have $\alpha f_2 = 0$. Then, we obtain a corresponding system of equations:

$$\begin{aligned} 3a_1 + a_5 + a_{12} + a_{15} + a_{18} + a_{22} + a_{24} - a_3 - a_{21} \\ -a_{23} - a_2 - a_4 - a_7 - a_{10} - a_{13} - a_{20} = 0. \end{aligned} \tag{3.14}$$

$$3a_2 + a_6 + a_9 + a_{16} + a_{19} + a_{21} + a_{23} - a_4 - a_{22} \\ -a_{24} - a_1 - a_3 - a_8 - a_{11} - a_{14} - a_{17} = 0. \tag{3.15}$$

$$3a_3 + a_7 + a_{10} + a_{13} + a_{20} + a_{22} + a_{24} - a_1 - a_{21} \\ -a_{23} - a_2 - a_4 - a_5 - a_{12} - a_{15} - a_{18} = 0. \tag{3.16}$$

$$3a_4 + a_8 + a_{11} + a_{14} + a_{17} + a_{21} + a_{23} - a_2 - a_{22} \\ -a_{24} - a_1 - a_3 - a_6 - a_9 - a_{16} - a_{19} = 0. \tag{3.17}$$

$$3a_5 + a_1 + a_{11} + a_{14} + a_{16} + a_{19} + a_{23} - a_7 - a_{13} \\ -a_{15} - a_2 - a_6 - a_8 - a_9 - a_{17} - a_{21} = 0. \tag{3.18}$$

$$3a_6 + a_2 + a_{12} + a_{13} + a_{15} + a_{20} + a_{24} - a_8 - a_{14} \\ -a_{16} - a_3 - a_5 - a_7 - a_{10} - a_{18} - a_{22} = 0. \tag{3.19}$$

$$3a_7 + a_3 + a_9 + a_{14} + a_{16} + a_{17} + a_{21} - a_5 - a_{13} \\ -a_{15} - a_1 - a_6 - a_8 - a_{11} - a_{19} - a_{23} = 0. \tag{3.20}$$

$$3a_8 + a_4 + a_{10} + a_{13} + a_{15} + a_{18} + a_{22} - a_6 - a_{14} \\ -a_{16} - a_2 - a_5 - a_7 - a_{12} - a_{20} - a_{24} = 0. \tag{3.21}$$

$$3a_9 - a_2 - a_7 - a_{11} - a_{15} - a_{17} - a_{18} - a_{19} - a_{20} \\ -a_{22} + a_4 + a_5 + a_{10} + a_{12} + a_{13} + a_{24} = 0. \tag{3.22}$$

$$3a_{10} + a_3 + a_8 + a_{16} + a_{17} + a_{19} + a_{23} - a_{12} - a_{18} \\ -a_{20} - a_1 - a_6 - a_9 - a_{11} - a_{14} - a_{21} = 0. \tag{3.23}$$

$$3a_{11} - a_4 - a_5 - a_9 - a_{13} - a_{17} - a_{18} - a_{19} - a_{20} \\ -a_{24} + a_2 + a_7 + a_{10} + a_{12} + a_{15} + a_{22} = 0. \tag{3.24}$$

$$3a_{12} - a_1 - a_6 - a_{10} - a_{14} - a_{17} - a_{18} - a_{19} - a_{20} \\ -a_{21} + a_3 + a_8 + a_9 + a_{11} + a_{16} + a_{23} = 0. \tag{3.25}$$

$$3a_{13} + a_3 + a_6 + a_8 + a_{11} + a_{19} + a_{21} - a_5 - a_7 \\ -a_{15} - a_1 - a_9 - a_{14} - a_{16} - a_{17} - a_{23} = 0. \tag{3.26}$$

$$3a_{14} + a_4 + a_5 + a_7 + a_{12} + a_{20} + a_{22} - a_6 - a_8 \\ -a_{16} - a_2 - a_{10} - a_{13} - a_{15} - a_{18} - a_{24} = 0. \tag{3.27}$$

$$3a_{15} + a_1 + a_6 + a_8 + a_9 + a_{17} + a_{23} - a_5 - a_7 \\ -a_{13} - a_3 - a_{11} - a_{14} - a_{16} - a_{19} - a_{21} = 0. \tag{3.28}$$

$$3a_{16} + a_2 + a_5 + a_7 + a_{10} + a_{18} + a_{24} - a_6 - a_8 \\ -a_{14} - a_4 - a_{12} - a_{13} - a_{15} - a_{20} - a_{22} = 0. \tag{3.29}$$

$$3a_{17} + a_4 + a_7 + a_{10} + a_{12} + a_{15} + a_{24} - a_9 - a_{11} \\ -a_{19} - a_2 - a_5 - a_{13} - a_{18} - a_{20} - a_{22} = 0. \tag{3.30}$$

$$3a_{18} + a_1 + a_8 + a_9 + a_{11} + a_{16} + a_{21} - a_{10} - a_{12} \\ -a_{20} - a_3 - a_6 - a_{14} - a_{17} - a_{19} - a_{23} = 0. \tag{3.31}$$

$$3a_{19} + a_2 + a_5 + a_{10} + a_{12} + a_{13} + a_{22} - a_9 - a_{11} \\ -a_{17} - a_4 - a_7 - a_{15} - a_{18} - a_{20} - a_{24} = 0. \tag{3.32}$$

$$3a_{20} + a_3 + a_6 + a_9 + a_{11} + a_{14} + a_{23} - a_{10} - a_{12} \\ -a_{18} - a_1 - a_8 - a_{16} - a_{17} - a_{19} - a_{21} = 0. \tag{3.33}$$

$$3a_{21} + a_2 + a_4 + a_7 + a_{12} + a_{13} + a_{18} - a_1 - a_3 \\ -a_{23} - a_5 - a_{10} - a_{15} - a_{20} - a_{22} - a_{24} = 0. \tag{3.34}$$

$$3a_{22} + a_1 + a_3 + a_8 + a_9 + a_{14} + a_{19} - a_2 - a_4 \\ -a_{24} - a_6 - a_{11} - a_{16} - a_{17} - a_{21} - a_{23} = 0. \tag{3.35}$$

$$3a_{23} + a_2 + a_4 + a_5 + a_{10} + a_{15} + a_{20} - a_1 - a_3 \\ -a_{21} - a_7 - a_{12} - a_{13} - a_{18} - a_{22} - a_{24} = 0. \tag{3.36}$$

$$3a_{24} + a_1 + a_3 + a_6 + a_{11} + a_{16} + a_{17} - a_2 - a_4 \\ -a_{22} - a_8 - a_9 - a_{14} - a_{19} - a_{21} - a_{23} = 0. \tag{3.37}$$

Since the rank of the coefficient matrix of the above equations is 9, we know that $\dim(I_{\{f_2\}}) = 15$. Secondly, we will compute the minimal distance of group code $I_{\{f_2\}}$. For some $\alpha = kg_i, 1 \leq i \leq 24$, if $k \neq 0$, then $\alpha f_2 \neq 0$, i.e., $\alpha = kg_i \notin I_{\{f_2\}}$. Though the coefficients of f_2, it is not difficult to find that $\alpha f_2 \neq 0$ for $wt(\alpha) \leq 3$. So $d(I_{\{f_2\}}) \geq 4$. Set $\alpha = g_1 + g_2 + g_3 + g_4$. Since $\alpha f_1 = (g_1 + g_2 + g_3 + g_4) f_1 = 0$, we get $\alpha = g_1 + g_2 + g_3 + g_4 \in I_{\{f_2\}}$. Hence, $d(I_{\{f_2\}}) = 4$. We can compute the parameters of group code $I_{\{f_2\}}$ by the similarly method (for $d(I_{\{f_3\}}) = 4$, take $\alpha = -g_1 + g_2 - g_3 + g_4$).

This completes the proof. □

Theorem 3.7. *Let $e_i, i = 1,2$ and $f_j, j = 1,2,3$ be idempotents in $\mathbb{F}_qS_4$. Then*

(1) $I_{\{e_1,e_2\}}$ is a [24,22,2] group code;

(2) $I_{\{e_i,f_1\}}$ is a [24,17,2] group code, for $i = 1,2$;

(3) $I_{\{e_i,f_j\}}$ is a [24,14,8] group code, for $i = 1,2, j = 2,3$;

(4) $I_{\{f_1,f_j\}}$ is a [24,11,8] group code, for $j = 2,3$;

(5) $I_{\{f_2,f_3\}}$ is a [24,6,8] group code.

Proof. (1) Clearly, $I_{\{e_1,e_2\}} = \{\alpha \in \mathbb{F}_qS_4 | \alpha e_1 = 0 \text{ and } \alpha e_2 = 0\}$. Firstly, we will give the dimension of group code $I_{\{e_1,e_2\}}$. For any $\alpha = \sum_{i=1}^{24} a_i g_i \in I_{\{e_1,e_2\}}$, we have $\alpha e_1 = 0$ and $\alpha e_2 = 0$. From (3.4) and (3.5), we obtain the following system of equations:

$$\begin{cases} \sum_{i=1}^{24} a_i = 0 \\ (a_1 + a_3 + a_6 + a_8 + a_9 + a_{11} + a_{14} + a_{17} + a_{17} + a_{19} + a_{21} + a_{23}) \\ -(a_2 + a_4 + a_5 + a_7 + a_{10} + a_{12} + a_{13} + a_{15} + a_{18} + a_{20} + a_{22} + a_{24}) = 0 \end{cases}.$$

Then, we have $\dim(I_{\{e_1,e_2\}}) = 22$. Secondly, we will compute the minimal distance of group code $I_{\{e_1,e_2\}}$. For some $\alpha = kg_i, 1 \le i \le 24$, if $k \neq 0$, then $\alpha e_i \neq 0$, i.e., $\alpha = kg_i \notin I_{\{e_1,e_2\}}$. So $d(I_{\{e_1,e_2\}}) \ge 2$. Set $\alpha = g_1 - g_3$. Since $\alpha e_i = (g_1 - g_3)e_i = 0$, for $i = 1,2$, we get $\alpha = g_1 - g_3 \in I_{\{e_1,e_2\}}$. Hence, $d(I_{\{e_1,e_2\}}) = 2$.

(2) Let $I_{\{e_i,f_1\}}, i = 1,2$ be the set of elements of the form $\alpha = \sum_{i=1}^{24} a_i g_i$ which the coefficients of α satisfy Eqs (3.6) and (3.8)–(3.13) for $i = 1$, while Eqs (3.7)–(3.13) for $i = 2$. Obviously, it does not contain any element with weight 1 of $I_{\{e_i,f_1\}}, i = 1,2$. Also, set $\alpha = g_1 - g_3$, then $\alpha \in I_{\{e_i,f_1\}}, i = 1,2$. Hence, $d(I_{\{e_i,f_1\}}) = 2, i = 1,2$. Moreover, the dimension of $I_{\{e_i,f_2\}}, i = 1,2$ is 17. Therefore, $I_{\{e_i,f_1\}}, i = 1,2$ is a [2,17,24] group code.

(3) Let $I_{\{e_i,f_2\}}, i = 1,2$ be the set of elements of the form $\alpha = \sum_{i=1}^{24} a_i g_i$ which the coefficients of α satisfy Eqs (3.6) and (3.14)–(3.37) for $i = 1$, while Eqs (3.7) and (3.14)–(3.37) for $i = 2$. Obviously, it does not contain any element with weight ≤ 4 of $I_{\{e_i,f_2\}}, i = 1,2$. Also, set $\alpha = g_1 + g_2 + g_3 + g_4 - g_5 - g_6 - g_7 - g_8$, then $\alpha \in I_{\{e_i,f_2\}}, i = 1,2$. Hence, $d(I_{\{e_i,f_2\}}) = 8, i = 1,2$. Moreover, the dimension of $I_{\{e_i,f_2\}}, i = 1,2$ is 14. Therefore, $I_{\{e_i,f_2\}}, i = 1,2$ is a [8,14,24] group code. In addition, we can obtain the parameters of $I_{\{e_i,f_3\}}, i = 1,2$ by a similar proof.

(4) Let $I_{\{f_1,f_2\}}$ be the set of elements of the form $\alpha = \sum_{i=1}^{24} a_i g_i$ which the coefficients of α satisfy Eqs (3.8)–(3.37). Obviously, it does not contain any element with weight ≤ 4 of $I_{\{f_1,f_2\}}$. Also, set $\alpha = g_1 + g_2 + g_3 + g_4 - g_{21} - g_{22} - g_{23} - g_{24}$, then $\alpha \in I_{\{f_1,f_2\}}$. Hence, $d(I_{\{f_1,f_2\}}) = 8$. Moreover, the dimension of $I_{\{f_1,f_2\}}$ is 11. Therefore, $I_{\{f_1,f_2\}}$ is a [24,11,8] group code. In addition, we can obtain the parameters of $I_{\{f_1,f_3\}}$ by a similar proof.

(5) Let $I_{\{f_2,f_3\}}$ be the set of elements of the form $\alpha = \sum_{i=1}^{24} a_i g_i$ which the coefficients of α satisfy the corresponding equations. Obviously, it does not contain any element with weight ≤ 4 of $I_{\{f_2,f_3\}}$. Also, set $\alpha = g_1 + g_2 + g_3 + g_4 - g_{21} - g_{22} - g_{23} - g_{24}$, then $\alpha \in I_{\{f_2,f_3\}}$. Hence, $d(I_{\{f_2,f_3\}}) = 8$. Moreover, the dimension of $I_{\{f_2,f_3\}}$ is 6. Therefore, $I_{\{f_2,f_3\}}$ is a [6,8,24] group code.

This completes the proof. □

We summarize the following results about the group codes of $\mathbb{F}_qS_4$. In fact, these results can be proved by using similar techniques with rigorous derivation that we have used in the previous results in this section.

Theorem 3.8. *Let $e_i, i = 1, 2$ and $f_j, j = 1, 2, 3$ be idempotents in $\mathbb{F}_q S_4$. Then*
(1) $I_{\{e_1,e_2,f_1\}}$ is a [24,18,2] group code;
(2) $I_{\{e_1,e_2,f_j\}}$ is a [24,13,8] group code, for $j = 2, 3$;
(3) $I_{\{e_i,f_1,f_j\}}$ is a [24,10,8] group code, for $i = 1, 2$, $j = 2, 3$;
(4) $I_{\{e_i,f_2,f_3\}}$ is a [24,5,8] group code, for $i = 1, 2$;
(5) $I_{\{f_1,f_2,f_3\}}$ is a [24,2,8] group code, for $i = 1, 2$.

Theorem 3.9. *Let $e_i, i = 1, 2$ and $f_j, j = 1, 2, 3$ be idempotents in $\mathbb{F}_q S_4$. Then*
(1) $I_{\{e_i,f_1,f_2,f_3\}}$ is a [24,1,8] group code, for $i = 1, 2$;
(2) $I_{\{e_1,e_2,f_1,f_j\}}$ is a [24,9,8] group code, for $j = 2, 3$;
(3) $I_{\{e_1,e_2,f_2,f_3\}}$ is a [24,4,8] group code.

We can obtain the following results based on Theorems 3.6–3.9.

Remark 3.10. *Let $e_i, i = 1, 2$ and $f_j, j = 1, 2, 3$ be idempotents in $\mathbb{F}_q S_4$.*
(1) $I_{\{e_i\}}$, $i = 1, 2$, and $I_{\{e_1,e_2\}}$ are MDS group codes.
(2) There are no AMDS group codes in $\mathbb{F}_q S_4$.

4. Conclusions

The main contributions of this paper are the following:

• The unique (linear and nonlinear) idempotents of $\mathbb{F}_q S_3$ and $\mathbb{F}_q S_4$ were described (see Theorems 3.1 and 3.6).

• The minimum distances and dimensions of $\mathbb{F}_q S_3$ and $\mathbb{F}_q S_4$ were characterized (see Theorems 3.2–3.3 and 3.6–3.9).

• The MDS group codes and almost MDS group codes in $\mathbb{F}_q S_3$ and $\mathbb{F}_q S_4$ were constructed (see Remarks 3.4 and 3.10).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

1. A. Boripan, S. Jitman, P. Udomkavanich, Characterization and enumeration of complementary dual abelian codes, *J. Appl. Math. Comput.,* **58** (2018), 527–544. https://doi.org/10.1007/s12190-017-1155-7
2. F. E. B. Martínez, Structure of finite dihedral group algebra, *Finite Fields Appl.,* **35** (2015), 204–214. https://doi.org/10.1016/j.ffa.2015.05.002
3. Y. Cao, Y. Cao, F. Fu, Concatenated structure of left dihedral codes, *Finite Fields Appl.,* **38** (2016), 93–115. https://doi.org/10.1016/j.ffa.2016.01.001
4. Y. Cao, Y. Cao, F. Fu, S. Wang, Left dihedral codes over Galois ring $GR(p^2, m)$, *Discrete Math.,* **341** (2018), 1816–1834. https://doi.org/10.13140/RG.2.2.34951.19365
5. P. Choosuwan, S. Jitman, P. Udomkavanich, Self-dual abelian codes in some nonprincipal ideal group algebras, *Math. Probl. Eng.,* **2016** (2016), 9020173. https://doi.org/10.1155/2016/9020173
6. Y. Gao, Q. Yue, Idempotents of generalized quaternion group algebras and their applications, *Discret. Math.,* **344** (2021), 112342. https://doi.org/10.1016/j.disc.2021.112342
7. Y. Gao, Q. Yue, Y. Wu, LCD codes and self-orthogonal codes in generalized dihedral group algebras, *Des. Codes Cryptogr.,* **88** (2020), 2275–2287. https://doi.org/10.1007/s10623-020-00778-z
8. Z. Heng, C. Ding, The subfield codes of some $[q + 1, 2, q]$ MDS codes, *IEEE Trans. Inf. Theory,* **68** (2022), 3643–3656. https://doi.org/10.1109/TIT.2022.3163813
9. I. M. Isaacs, *Algebra, a graduate course*, Pacific Grove, California: Cole Publishing, 1992. http://dx.doi.org/10.1090/gsm/100
10. G. D. James, M. W. Liebeck, *Representation and characters of groups*, California: Cole Publishing, Pacific Grove, 1992. http://dx.doi.org/10.1063/1.2808641
11. S. Jitman, S. Ling, H. Liu, X. Xie, Abelian codes in principal ideal group algebras, *IEEE Trans. Inf. Theory,* **59** (2013), 3046–3057. https://doi.org/10.1109/TIT.2012.2236383
12. F. E. B. Martínez, C. R. G. Vergara, Explicit idempotents of finite group algebras, *Finite Fields Appl.,* **28** (2014), 123–131. https://doi.org/10.1016/j.ffa.2015.05.002
13. C. P. Milies, F. D. de Melo, On cyclic and abelian codes, *IEEE Trans. Inf. Theory,* **59** (2013), 7314–7319. https://doi.org/10.1109/TIT.2013.2275111
14. C. P. Milies, S. K. Sehgal, *An introduction to group rings*, Pacific Grove, California: Cole Publishing, 1992. http://dx.doi.org/10.1007/978-94-010-0405-3_2
15. Y. Niu, Q. Yue, Y. Wu, L. Hu, Hermitian self-dual, MDS, and generalized Reed-Solomon codes, *IEEE Commun. Lett.,* **23** (2019), 781–784. https://doi.org/10.1109/LCOMM.2019.2908640
16. D. S. Passman, *The algebraic structure of group rings*, New York, Wiley, 1977.
17. J. Sui, Q. Yue, X. Li, D. Huang, MDS, near-MDS or 2-MDS self-dual codes via twisted generalized Reed-Solomon codes, *IEEE Trans. Inf. Theory,* **68** (2022), 7832–7841. https://doi.org/10.1109/TIT.2022.3190676

18. C. Tang, Q. Wang, C. Ding, The subfield codes and subfield subcodes of a family of MDS codes, *IEEE Trans. Inf. Theory,* **68** (2022), 5792–5801. https://doi.org/10.1109/TIT.2022.3163813

19. J. H. van Lint, *Introduction to coding theory*, New York Heidelberg, Berlin: Springer-Verlag, 1982.

20. Y. Wu, J. Y. Hyun, Y. Lee, New LCD MDS codes of non-Reed-Solomon type, *IEEE Trans. Inf. Theory,* **67** (2021), 5069–5078. https://doi.org/10.1109/TIT.2021.3086818

21. G. Xu, X. Cao, L. Qu, Infinite families of 3-designs and 2-designs from almost MDS codes, *IEEE Trans. Inf. Theory,* **68** (2022), 4344–4353. https://doi.org/10.1109/TIT.2022.3157199