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Research article

Infimum of the spectrum of Laplace-Beltrami operator on Cartan classical domains of type III and IV

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Abstract: Let $R_{\mathcal{A}}$ be the Cartan classical domains of type III and IV, and Δ_g is assumed to be the Laplace-Beltrami operator associated to the Bergman metric g on $R_{\mathcal{A}}$. In this paper, we derive an estimate for $\lambda_1(\Delta_g)$, which is the bottom of the spectrum of Δ_g on $R_{\mathcal{A}}$.

Keywords: Laplace-Beltrami operator; Cartan classical domains; Bergman metric
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1. Introduction

Let (M, g) be a Kähler manifold of complex dimension n with Kähler metric $g = \sum_{i,j=1}^n u_{i\bar{j}} dz_i \otimes d\bar{z}_j$. The Laplace-Beltrami operator with respect to the Kähler metric g is defined by

$$\Delta_g = -4 \sum_{i,j=1}^n u^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}, \tag{1.1}$$

where $[u^{i\bar{j}}]^t = [u_{i\bar{j}}]^{-1}$. Let

$$\lambda_1(\Delta_g) := \lambda_1(\Delta_g, M) = \inf\Big\{4 \int_M u^{i\bar{j}} \frac{\partial h}{\partial z_i} \frac{\partial h}{\partial \bar{z}_j} dV_g : h \in C_0^\infty(M), \int_M |h|^2 dV_g = 1\Big\}. \tag{1.2}$$

Here dV_g is the volume measure of M with respect to the Kähler metric g.

Spectral theory or eigenvalue estimates for Laplace-Beltrami operators are important subject for mathematics. When M is compact and Δ_g is uniformly elliptic, $\lambda_1(\Delta_g)$ is the first eigenvalue of Δ_g.

Researches on its upper and lower bound estimates have a long history with many results (see [1–5] and the reference therein).

Unlike compact manifolds, for the case when M is a non-compact manifold, $\lambda_1(\Delta_g)$ may not be an eigenvalue of Δ_g, but rather the infimum of the positive spectral of Δ_g. This naturally leads to an interesting problem: study of estimate for $\lambda_1(\Delta_g)$ in the complete non-compact case. Quite a bit of research has been done on this problem. For examples, the results on upper bounds and lower bounds estimates obtained by Cheng [6], Li and Wang [7–10], Munteanu [11], Li and Tran [12] are all well known. The rigidity property of manifold may be further obtained when λ_1 achieves its sharp upper or lower bound estimate. For examples, one may see [7–9, 13, 14] and references therein.

We recall the following estimates for $\lambda_1(\Delta_g)$, in the Riemannian manifolds, a sharp upper bound estimate for $\lambda_1(\Delta_g)$ is well known from Cheng [6]. For the Kähler manifolds, Li and Wang gave an important sharp upper bound estimate in [9]. They proved $\lambda_1(\Delta_g) \geq n^2$ with the assumption that the holomorphic bisectional curvature of M is bounded below by -1. Later on, Munteanu [11] improved Li and Wang's result, and derived another sharp estimate in terms of the Ricci curvature is bounded from below by $-2(n+1)$.

As a continuation of the work of Li and Wang [9] and the work of Munteanu [11], Li and Tran [12] provided many examples of bounded strongly pseudoconvex domains, on which $\lambda_1(\Delta_g)$ can be explicitly formulated. However, those domains were exclusive of most of non-smooth domains, like Cartan classical domains. It would be desirable to proceed the study of $\lambda_1(\Delta_g)$ on the Cartan classical domains.

Suppose $\mathcal{R}_{\mathcal{A}}(n)$ be the Cartan classical domains of type $\mathcal{A}(\mathcal{A} = III, IV)$. Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $u(z) \in C^\infty(D)$ be a strictly plurisubharmonic exhaustion function for D. Let $g = \sum_{i,j=1}^n \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j$ be the Kähler metric induced by u. We set

$$|\partial u|_g^2 = \sum_{i,j=1}^n g^{i\bar{j}} \frac{\partial u}{\partial z_i} \frac{\partial u}{\partial \bar{z}_j} \tag{1.3}$$

and

$$\alpha_g = \sup\{\alpha : \int_D (\det[g_{i\bar{j}}])^\alpha dv < \infty\}. \tag{1.4}$$

This paper aims to provide an estimate for $\lambda_1(\Delta_g)$ on $\mathcal{R}_{\mathcal{A}}(n)(\mathcal{A} = III, IV)$ with g is the Bergman metric of $\mathcal{R}_{\mathcal{A}}(n)$. Regarding the estimate for $\lambda_1(\Delta_g)$ on the Cartan classical domains of type I and II, it has been studied by the first author in [15]. We continue to use some ideas in [15], but the calculations in the current cases are much more delicate. This is due to the special forms of $\mathcal{R}_{III}(n)$ and $\mathcal{R}_{IV}(n)$.

Our study is motivated by Li and Tran [12]. We emphasize that the calculation of $|\partial u|_g^2$ and the construction of test functions play important roles in our argument, but both are difficult to solve. We use the Bergman metric g on the Cartan classical domains given by Hua [16] and Lu [17].

Let $K_{\mathcal{A}}(z) := K_{\mathcal{A}}(z, z)$ be the Bergman kernel function of $R_{\mathcal{A}}(n)$ and let

$$u_{\mathcal{A}} = \frac{1}{c_{\mathcal{A}}} \log K_{\mathcal{A}}(z), \quad c_{III} = n - 1 \quad \text{and} \quad c_{IV} = 2n. \tag{1.5}$$

Our goal is to prove the following results.

Theorem 1.1. *Let $\Delta_{\mathcal{A}} = \Delta_g$ ($\mathcal{A} = III, IV$) be the Laplace-Beltrami operator associated to the Bergman metric g on the Cartan classical domains $R_{\mathcal{A}}(n)$ of type $\mathcal{A}$. Let $u = u_{\mathcal{A}}$ be the strictly plurisubharmonic exhaustion function for $\mathcal{R}_{\mathcal{A}}(n)$ and $g = g_{\mathcal{A}}$ be the Kähler metric induced by $u_{\mathcal{A}}$. Assuming that $|\partial u|_g^2 \le \beta$, one has*

(i) $\lambda_1(\Delta_{\mathcal{A}}) \ge N_{\mathcal{A}}^2/\beta$ *with* $N_{III} = n(n-1)/2$ *and* $N_{IV} = n$*;*
(ii) $\lambda_1(\Delta_{\mathcal{A}}) \le \beta c_{\mathcal{A}}^2 (1-\alpha_{\mathcal{A}})^2$*, where*

$$\alpha_{III} := \frac{1}{2(n-1)}, \quad \alpha_{IV}(n) = \frac{1}{2} \text{ when } n = 1, \quad \alpha_{IV}(n) = \frac{1}{n} \text{ when } n > 1;$$

and

$$c_{III} = n-1, \quad c_{IV} = 2n.$$

Corollary 1.1. *Let the notations and assumptions as in Theorem 1.1. Then*

$$\lambda_1(\Delta_{III}) \in \begin{cases} [\frac{n(n-1)^2}{4}, \frac{n(2n-3)^2}{4}], & n = 2k; \\ [\frac{n^2(n-1)}{4}, \frac{(n-1)(2n-3)^2}{4}], & n = 2k-1. \end{cases}$$

and

$$\lambda_1(\Delta_{IV}) \begin{cases} = 1, & n = 1; \\ \in [n^2, 4(n-1)^2], & n \ge 2. \end{cases}$$

The remainder of the paper is organized as follows. In Section 2, we introduce the notion and provide some preliminary results concerning $\mathcal{R}_{\mathcal{A}}(n)$. In Section 3, our main results are stated and proved. Section 4 contains a brief summary of our study.

2. Cartan classical domains of type III and IV

Let $M^{(n)}$ be the set of all $n \times n$ matrices with entries in $\mathbb{C}$. For any $A = [a_{ij}] \in M^{(n)}$, let

$$A^* = \overline{A}' = [\overline{a_{ji}}].$$

We denote by I_n the $n \times n$ identity matrix. The Cartan classical domains of the type III and IV can be represented as follows:

$$\mathcal{R}_{III} := \mathcal{R}_{III}(n) = \{Z \in M^{(n)} : Z = -Z', I_n - ZZ^* > 0\}. \tag{2.1}$$

$$\mathcal{R}_{IV} := \mathcal{R}_{IV}(n) = \{z = (z_1, \cdots, z_n) \in \mathbb{C}^n, 1 + |\sum_{j=1}^n z_j^2|^2 - 2|z|^2 > 0, |\sum_{j=1}^n z_j^2| < 1\}. \tag{2.2}$$

Suppose A and B are two $n \times n$ matrix. Then we define an $\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$ matrix $[A \dot{\times} B]_{as}$, which consists of the entries $(A \dot{\times} B)_{(ij)(k\ell)}$ with $i < j$ and $k < \ell$ as follows:

$$(A \dot{\times} B)_{(ij)(k\ell)} = a_{ik} b_{j\ell} - a_{i\ell} b_{jk}, \quad 1 \le i < j \le n, 1 \le k < \ell \le n. \tag{2.3}$$

In particular,

$$(A \dot{\times} A)_{(ij)(k\ell)} = a_{(ij)(k\ell)} = a_{ik} a_{j\ell} - a_{i\ell} a_{jk}, \quad 1 \le i < j \le n, 1 \le k < \ell \le n. \tag{2.4}$$

The following result can be found in [17, p317–318].

Proposition 2.1. *[17] Let A and B be two $n \times n$ matrices. Then*

$$[A\dot{\times}A]_{as}[B\dot{\times}B]_{as} = [AB\dot{\times}AB]_{as}, \quad ([A\dot{\times}B]_{as})' = [A'\dot{\times}B']_{as} \tag{2.5}$$

and

$$[A\dot{\times}A]_{as}^{-1} = [A^{-1}\dot{\times}A^{-1}]_{as}, \quad \det[A\dot{\times}A]_{as} = (\det A)^{n-1}. \tag{2.6}$$

A straightforward calculation shows:

Proposition 2.2. *Let $C = [c_{pq}]$ be a $s \times s$ matrix where c_{pq} is a function of $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ and $\bar{z}$. Then*

$$\frac{\partial \log \det C}{\partial z_k} = \sum_{p,q=1}^{s} c^{pq} \frac{\partial c_{pq}}{\partial z_k} \tag{2.7}$$

and

$$\frac{\partial^2 \log \det C}{\partial z_k \partial \bar{z}_\ell} = \sum_{p,q=1}^{s} c^{pq} \frac{\partial^2 c_{pq}}{\partial z_k \partial \bar{z}_\ell} - \sum_{ij,pq=1}^{s} c^{iq} c^{pj} \frac{\partial c_{pq}}{\partial z_k} \frac{\partial c_{ij}}{\partial \bar{z}_\ell}, \tag{2.8}$$

where

$$\sum_{j=1}^{s} c^{ij} c_{kj} = \delta_{ik}.$$

For $Z \in \mathcal{R}_{III}$, we set

$$z = (z_{12}, \cdots, z_{1m}, z_{23}, \cdots, z_{2n}, \cdots, z_{(n-1)n}) \in \mathbb{C}^{\frac{n(n-1)}{2}}.$$

Obviously $2\|z\|^2 = \mathrm{tr}(ZZ^*)$. We know from Hua's book [16] or Lu's book [17, Section 3.3] that

$$K_{III}(Z) = \frac{1}{V(\mathcal{R}_{III})} \cdot \frac{1}{\det(I - ZZ^*)^{n-1}}, \tag{2.9}$$

and

$$K_{IV}(z) = \frac{1}{V(\mathcal{R}_{IV})} \cdot \frac{1}{(1 + |\sum_{j=1}^{n} z_j^2|^2 - 2|z|^2)^n}. \tag{2.10}$$

Here $V(\mathcal{R}_\mathcal{A})$ is the volume of $\mathcal{R}_\mathcal{A}$.

Consider the Bergman kernel function of $\mathcal{R}_\mathcal{A}$, we construct $u_\mathcal{A}$ as (1.5), hence $u_\mathcal{A}$ is strictly plurisubharmonic exhaustion function in $\mathcal{R}_\mathcal{A}$. Furthermore, we define a complete Kähler metric $g_\mathcal{A}$ which is induced by $u_\mathcal{A}$ as follows:

$$g_\mathcal{A} = \sum_{i,j=1}^{N} u_{\mathcal{A}i\bar{j}} \mathrm{d}z_i \otimes \mathrm{d}\bar{z}_j, \tag{2.11}$$

where $N = n(n-1)/2$ when $\mathcal{A} = III$ and $N = n$ when $\mathcal{A} = IV$. Consequently,

$$|\partial u_\mathcal{A}|^2_{g_\mathcal{A}} = \sum_{i,j=1}^{N} u_\mathcal{A}{}^{i\bar{j}} \frac{\partial u_\mathcal{A}}{\partial z_i} \frac{\partial u_\mathcal{A}}{\partial \bar{z}_j}, \tag{2.12}$$

where $[u_\mathcal{A}{}^{i\bar{j}}]^t = [u_{\mathcal{A}i\bar{j}}]^{-1}$.

Let us mention two important consequences about the complex Hessian matrix for $u_\mathcal{A}$ on $\mathcal{R}_\mathcal{A}$.

Proposition 2.3. *The complex Hessian matrix for $u_{\mathcal{A}}$ can be stated by*

$$H(u_{III})(Z) = 2[(I - \overline{ZZ}^*)^{-1} \dot{\times} (I - \overline{ZZ}^*)^{-1}]_{as}, \tag{2.13}$$

$$H(u_{III})^{-1}(Z) = \frac{1}{2}[(I - \overline{ZZ}^*) \dot{\times} (I - \overline{ZZ}^*)]_{as}, \tag{2.14}$$

$$H(u_{IV})(z) = \frac{1}{r^2(z)} \left[r(z) I_n - 2 \begin{pmatrix} z \\ \overline{z} \end{pmatrix}' \begin{pmatrix} 1 - 2|z|^2 & \overline{zz'} \\ zz' & -1 \end{pmatrix} \overline{\begin{pmatrix} z \\ \overline{z} \end{pmatrix}} \right] \tag{2.15}$$

and

$$H(u_{IV})^{-1}(z) = [r(z)(I - 2\overline{z}'z) + 2(z' - zz'\overline{z}')(\overline{z} - \overline{zz}'z)], \tag{2.16}$$

where $r(z) = 1 + |\sum_{j=1}^n z_j^2|^2 - 2|z|^2$.

Proof. (i) For $Z \in \mathcal{R}_{III}$, by applying (2.7) and (2.8), a straightforward calculation shows that

$$\begin{aligned}
\frac{\partial^2 \log K_{III}(Z)}{\partial z_{j\alpha} \partial \overline{z}_{k\beta}}\Big|_{Z=0} &= -(n-1) \frac{\partial^2 \log \det(I_n - ZZ^*)}{\partial z_{j\alpha} \partial \overline{z}_{k\beta}}\Big|_{Z=0} \\
&= -(n-1) \sum_{h,\ell=1}^{n} \left((I_n + ZZ^*)^{-1} \right)_{h\ell} \frac{\partial^2 (\sum_{s=1}^n z_{hs} \overline{z}_{s\ell})}{\partial z_{j\alpha} \partial \overline{z}_{k\beta}}\Big|_{Z=0} \\
&= -(n-1) \sum_{h=1}^{n} \frac{\partial}{\partial \overline{z}_{k\beta}} \sum_s (\delta_{jh} \delta_{s\alpha} \overline{z}_{sh} - \delta_{h\alpha} \delta_{js} \overline{z}_{sh}) \\
&= -(n-1) \sum_{h=1}^{n} \frac{\partial}{\partial \overline{z}_{k\beta}} (\delta_{jh} \overline{z}_{\alpha h} - \delta_{h\alpha} \overline{z}_{jh}) \\
&= -(n-1) \sum_{h=1}^{n} \left[\delta_{jh} (\delta_{k\alpha} \delta_{h\beta} - \delta_{hk} \delta_{\alpha\beta}) - \delta_{h\alpha} (\delta_{jk} \delta_{h\beta} - \delta_{j\beta} \delta_{hk}) \right] \\
&= -2(n-1)(\delta_{k\alpha} \delta_{j\beta} - \delta_{jk} \delta_{\alpha\beta}).
\end{aligned} \tag{2.17}$$

Hence, we have

$$H(u_{III})(0) = \left[\frac{\partial^2 u}{\partial z_{j\alpha} \partial \overline{z}_{k\beta}} \right]\Big|_{Z=0} = 2[I_m \dot{\times} I_m]_{as}. \tag{2.18}$$

As the proof of equality (3.3.45) in [17], the transformation property of Bergman kernel function and Möbius transform of $R_{\mathcal{A}}$, (2.13) follows. By Proposition 2.1, (2.14) obtained.

(ii) For $z = (z_1, z_2 \cdots, z_n) \in \mathcal{R}_{IV}$, by equality (3.3.57) in [17], we have

$$\begin{aligned}
\frac{\partial^2 \log K_{IV}(z)}{\partial z_i \partial \overline{z}_j} &= \frac{1}{r^2(z)} [r(z) I_n + 4zz^* z' \overline{z} - 2(\overline{z} z^* z' z + zz' z^* \overline{z}) - 2(z' \overline{z} - z^* z)] \\
&= \frac{1}{r^2(z)} \left\{ r(z) I_n - 2[(z' - 2|z|^2 z' + z^* zz') \overline{z} + 2(z' \overline{z} z^* - z^*) z] \right\} \\
&= \frac{1}{r^2(z)} \left[r(z) I_n - 2 \begin{pmatrix} z' - 2|z|^2 z' + z^* zz' & z' \overline{z} z^* - z^* \end{pmatrix} \overline{\begin{pmatrix} z \\ \overline{z} \end{pmatrix}} \right] \\
&= \frac{1}{r^2(z)} \left[r(z) I_n - 2 \begin{pmatrix} z' & z^* \end{pmatrix} \begin{pmatrix} 1 - 2|z|^2 & \overline{zz'} \\ zz' & -1 \end{pmatrix} \overline{\begin{pmatrix} z \\ \overline{z} \end{pmatrix}} \right]
\end{aligned} \tag{2.19}$$

This implies (2.15). We claim (2.16) holds from discussion of equality (6.1.29) in [17]. □

Proposition 2.4. *With the notations above, one has*

$$\det H(u_{III})(Z) = 2^{\frac{n(n-1)}{2}} \frac{1}{\det(I - ZZ^*)^{(n-1)}} \tag{2.20}$$

and

$$\det H(u_{IV})(z) = \frac{1}{r^n(z)}. \tag{2.21}$$

Proof. By Proposition 2.1 and Proposition 2.3, it is easy to deduce that

$$\begin{aligned}\det H(u_{III})(z) =& 2^{\frac{n(n-1)}{2}} \det[(I - \overline{ZZ}^*)^{-1} \dot{\times} (I - \overline{ZZ}^*)^{-1}]_{as}\\ =& 2^{\frac{n(n-1)}{2}} \det(I - \overline{ZZ}^*)^{-(n-1)}\\ =& 2^{\frac{n(n-1)}{2}} \det(I - ZZ^*)^{-(n-1)}\end{aligned}$$

and

$$\begin{aligned}
\det H(u_{IV})(z) &= \det\left\{\frac{1}{r^2(z)}\left[r(z)I_n - 2\begin{pmatrix} z \\ \bar{z}\end{pmatrix}' \begin{pmatrix} 1-2|z|^2 & \overline{zz'} \\ zz' & -1\end{pmatrix}\overline{\begin{pmatrix} z \\ \bar{z}\end{pmatrix}}\right]\right\}\\
&= \frac{1}{r^n(z)} \det\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{r(z)}\begin{pmatrix} 1-2|z|^2 & \overline{zz'} \\ zz' & -1\end{pmatrix}\begin{pmatrix} \bar{z} \\ z\end{pmatrix}\begin{pmatrix} z' & \bar{z}'\end{pmatrix}\right]\\
&= \frac{1}{r^n(z)} \det\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{r(z)}\begin{pmatrix} 1-2|z|^2 & \overline{zz'} \\ zz' & -1\end{pmatrix}\begin{pmatrix} |z|^2 & \overline{zz'} \\ zz' & |z|^2\end{pmatrix}\right]\\
&= \frac{1}{r^n(z)} \det\begin{pmatrix} 1 - \frac{2}{r(z)}(|z|^2 - 2|z|^4 + |zz'|^2) & -\frac{2}{r(z)}\overline{zz'}(1-|z|^2) \\ -\frac{2}{r(z)}zz'(|z|^2-1) & 1 - \frac{2}{r(z)}(|zz'|^2 - |z|^2)\end{pmatrix}\\
&= \frac{1}{r^{n+2}(z)}(1 + 2|zz'|^2 - 4|z|^2 + 4|z|^4 + |zz'|^4 - 4|z|^2|zz'|^2)\\
&= \frac{1}{r^n(z)}.
\end{aligned}$$

This completes the proof. □

3. Estimate for $\lambda_1(\Delta_g)$

In this section, in order to prove the main theorems, we begin to establish an estimate for $|\partial u|_g^2$. Let

$$\frac{\partial}{\partial z} = (\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}).$$

Since

$$(\frac{\partial u}{\partial z})' \frac{\partial u}{\partial \bar{z}} = \begin{pmatrix} \frac{\partial u}{\partial z_1}\frac{\partial u}{\partial \bar{z}_1} & \cdots & \frac{\partial u}{\partial z_1}\frac{\partial u}{\partial \bar{z}_n} \\ \vdots & & \vdots \\ \frac{\partial u}{\partial z_n}\frac{\partial u}{\partial \bar{z}_1} & \cdots & \frac{\partial u}{\partial z_n}\frac{\partial u}{\partial \bar{z}_n}\end{pmatrix}, \tag{3.1}$$

one has

$$|\partial u|_g^2 = \sum_{i,j=1}^n u^{i\bar{j}} \frac{\partial u}{\partial z_i}\frac{\partial u}{\partial \bar{z}_j} = \mathrm{tr}\Big([u^{i\bar{j}}](\frac{\partial u}{\partial z})' \frac{\partial u}{\partial \bar{z}}\Big). \tag{3.2}$$

Proposition 3.1. *With the notations in (1.5), (2.11) and (2.12), one has the following estimates:*

$$|\partial u_{III}|^2_{g_{III}} \le 2[\frac{n}{2}] \qquad \text{and} \quad |\partial u_{IV}|^2_{g_{IV}} \le 1. \tag{3.3}$$

Proof. (i) For $z \in \mathcal{R}_{III}$, according to (3.2), one has

$$\begin{aligned}
&\operatorname{tr}\left([A\dot{\times}A]_{as}\frac{\partial u_{III}}{\partial z'}\frac{\partial u_{III}}{\partial \bar{z}}\right)\\
&= \sum_{i,k=1}^{n}\sum_{i<j,k<\ell} a_{(ij)(k\ell)}\frac{\partial u_{III}}{\partial z_{ij}}\frac{\partial u_{III}}{\partial \bar{z}_{k\ell}}\\
&= \sum_{i,k=1}^{n}\sum_{i<j,k<\ell} (a_{ik}a_{j\ell} - a_{i\ell}a_{jk})\frac{\partial u_{III}}{\partial z_{ij}}\frac{\partial u_{III}}{\partial \bar{z}_{k\ell}}\\
&= \frac{1}{2}\sum_{i,k=1}^{n}\sum_{k<\ell}\Big[\Big(\sum_{i<j} a_{ik}a_{j\ell}\frac{\partial u_{III}}{\partial z_{ij}} + \sum_{i>j} a_{j\ell}a_{ik}\frac{\partial u_{III}}{\partial z_{ij}}\Big) - \Big(\sum_{i>j} a_{jk}a_{i\ell}\frac{\partial u}{\partial z_{ij}} + \sum_{i<j} a_{i\ell}a_{jk}\frac{\partial u_{III}}{\partial z_{ij}}\Big)\Big]\frac{\partial u_{III}}{\partial \bar{z}_{k\ell}}\\
&= \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{n}\sum_{k<\ell}(a_{ik}a_{j\ell} - a_{i\ell}a_{jk})q_{ij}\frac{\partial u_{III}}{\partial z_{ij}}\frac{\partial u_{III}}{\partial \bar{z}_{k\ell}}\\
&= \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{n}\Big(\sum_{k<\ell} a_{ik}a_{j\ell}\frac{\partial u_{III}}{\partial \bar{z}_{k\ell}} - \sum_{k>\ell} a_{ik}a_{j\ell}\frac{\partial u_{III}}{\partial \bar{z}_{k\ell}}\Big)q_{ij}\frac{\partial u_{III}}{\partial z_{ij}}\\
&= \frac{1}{2}\sum_{i,k=1}^{n}\sum_{k,\ell=1}^{n} a_{ik}a_{j\ell}q_{ij}q_{k\ell}\frac{\partial u_{III}}{\partial z_{ij}}\frac{\partial u_{III}}{\partial \bar{z}_{k\ell}},
\end{aligned} \tag{3.4}$$

where

$$q_{ij} = \begin{cases} 0, & i = j;\\ 1, & i \ne j.\end{cases}$$

Let $D[\lambda_1, \dots, \lambda_n]$ be $n \times n$ diagonal matrix with all diagonal entries are $\lambda_1, \dots, \lambda_n$. For $Z \in \mathcal{R}_{III}$, since ZZ^* is the Hermitian matrix, there exists $n \times n$ unitary matrix U such that

$$UZZ^*U^* = \begin{cases} D[\lambda_1, \cdots, \lambda_n], & n = 2k;\\ D[\lambda_1, \cdots, \lambda_{n-1}, 0], & n = 2k+1.\end{cases} \tag{3.5}$$

and $\lambda_j \in [0, 1), \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$. There is no loss of generality in assuming

$$ZZ^* = \begin{cases} D[\lambda_1, \cdots, \lambda_n], & n = 2k;\\ D[\lambda_1, \cdots, \lambda_{n-1}, 0], & n = 2k+1.\end{cases} \tag{3.6}$$

It follows that

$$(I_m - ZZ^*)^{-1} = \begin{cases} D[\sum_{k=0}^{\infty}\lambda_1^k, \dots, \sum_{k=0}^{\infty}\lambda_n^k], & n = 2k;\\ D[\sum_{k=0}^{\infty}\lambda_1^k, \dots, \sum_{k=0}^{\infty}\lambda_n^k, 0], & n = 2k+1.\end{cases} \tag{3.7}$$

Thus

$$\frac{\partial u_{III}}{\partial z_{ik}} = \bar{z}_{ik}\Big(\frac{1}{1-\lambda_i} + \frac{1}{1-\lambda_k}\Big), \quad i < k; \tag{3.8}$$

$$\frac{\partial u_{III}}{\partial \bar{z}_{j\ell}} = z_{j\ell}\Big(\frac{1}{1-\lambda_j} + \frac{1}{1-\lambda_\ell}\Big). \quad j < \ell. \tag{3.9}$$

Therefore

$$\begin{aligned}
|\partial u_{III}|^2_{g_{III}} &= \sum_{i,j=1}^{n} \sum_{k>i,\ell>j} u_{III}^{i\bar{j},k\bar{\ell}} \frac{\partial u_{III}}{\partial z_{ik}} \frac{\partial u_{III}}{\partial \bar{z}_{j\ell}} \\
&= \frac{1}{2} \sum_{i,j,k,\ell=1}^{n} (\delta_{ij} - \sum_{s=1}^{n} z_{is}\bar{z}_{js})(\delta_{k\ell} - \sum_{t=1}^{n} z_{kt}\bar{z}_{\ell t}) \frac{1}{2} q_{ik} q_{j\ell} \frac{\partial u_{III}}{\partial z_{ik}} \frac{\partial u_{III}}{\partial \bar{z}_{j\ell}} \\
&= \frac{1}{4} \sum_{i=1}^{n} \sum_{k=1}^{n} (1-\lambda_i)(1-\lambda_k)\Big(\frac{1}{1-\lambda_i} + \frac{1}{1-\lambda_k}\Big)^2 |q_{ik} z_{ik}|^2 \\
&= \frac{1}{2} \sum_{i,k=1}^{n} \Big(\frac{1-\lambda_k}{1-\lambda_i} + 1\Big) |q_{ik} z_{ik}|^2 \\
&= \frac{1}{2}\Big(tr(-Z^*(I-ZZ^*)Z(I-ZZ^*)^{-1}) + tr(ZZ^*)\Big) \\
&= \frac{1}{2}\Big(tr(-Z^*Z(I-Z^*Z)(I-ZZ^*)^{-1}) + tr(ZZ^*)\Big) \\
&= \frac{1}{2}\Big(tr(ZZ^*) + tr(ZZ^*)\Big) = tr(ZZ^*) \\
&\le 2\Big[\frac{n}{2}\Big].
\end{aligned} \tag{3.10}$$

(ii) For $z \in \mathcal{R}_{IV}$, we have

$$\frac{\partial u_{IV}}{\partial z_i} = -\frac{1}{2} \frac{\partial \log(r(z))}{\partial z_i} = -\frac{1}{2} \frac{1}{r(z)} \frac{\partial r(z)}{\partial z_i} = -\frac{(\overline{zz'} z_i - \bar{z}_i)}{r(z)}, \tag{3.11}$$

$$\frac{\partial u_{IV}}{\partial \bar{z}_j} = -\frac{1}{2} \frac{\partial \log(r(z))}{\partial \bar{z}_j} = -\frac{1}{2} \frac{1}{r(z)} \frac{\partial r(z)}{\partial \bar{z}_j} = -\frac{(zz'\bar{z}_j - z_j)}{r(z)}. \tag{3.12}$$

Accordingly,

$$\frac{\partial u_{IV}}{\partial z_i} \frac{\partial u_{IV}}{\partial \bar{z}_j} = \frac{1}{r^2(z)}(|zz'|^2 z_i\bar{z}_j - \overline{zz'} z_i z_j - zz'\bar{z}_i\bar{z}_j + \bar{z}_i z_j). \tag{3.13}$$

Notice that

$$\begin{aligned}
A(z) :=& \sum_{i,j=1}^{n} (\delta_{ij} - 2z_i\bar{z}_j)(|zz'|^2 z_i\bar{z}_j - \overline{zz'} z_i z_j - zz'\bar{z}_i\bar{z}_j + \bar{z}_i z_j) \\
=& (|zz'|^2|z|^2 - 2|zz'|^2 + |z|^2) - 2(|zz'|^4 - 2|zz'|^2|z|^2 + |z|^4) \\
=& 5|zz'|^2|z|^2 - 2|zz'|^2 - 2|zz'|^4 - 2|z|^4 + |z|^2
\end{aligned} \tag{3.14}$$

and

$$
\begin{aligned}
B(z) :=& \sum_{i,j=1}^{n} (\bar{z}_i - \overline{zz'} z_i)(z_j - zz'\bar{z}_j)(\overline{zz'} z_i - \bar{z}_i)(zz'\bar{z}_j - z_j) \\
=& \sum_{i=1}^{n} (\bar{z}_i - \overline{zz'} z_i)(\overline{zz'} z_i - \bar{z}_i) \sum_{j=1}^{n} (z_j - zz'\bar{z}_j)(zz'\bar{z}_j - z_j) \\
=&(2\overline{zz'}|z|^2 - \overline{zz'} - \overline{zz'}|zz'|^2)(2zz'|z|^2 - zz' - zz'|zz'|^2) \\
=&|zz'|^2(2|z|^2 - 1 - |zz'|^2)^2 \\
=&r^2(z)|zz'|^2. \qquad (3.15)
\end{aligned}
$$

Then (3.14) and (3.15) yield

$$
r(z)A(z) + 2B(z) = r(z)(5|zz'|^2|z|^2 - 2|zz'|^2 - 2|zz'|^4 - 2|z|^4 + |z|^2) + 2r^2(z)|zz'|^2 = r^2(z)|z|^2. \qquad (3.16)
$$

Consequently,

$$
\begin{aligned}
|\partial u_{IV}|^2_{g_{IV}} =& \sum_{i,j=1}^{n} u_{IV}^{i\bar{j}} \frac{\partial u_{IV}}{\partial z_i} \frac{\partial u_{IV}}{\partial \bar{z}_j} \\
=& \sum_{i,j=1}^{n} [r(z)(\delta_{ij} - 2z_i\bar{z}_j) + 2(\bar{z}_i - \overline{zz'} z_i)(z_j - zz'\bar{z}_j)] \frac{\partial u_{IV}}{\partial z_i} \frac{\partial u_{IV}}{\partial \bar{z}_j} \\
=& \frac{1}{r^2(z)} \sum_{i,j=1}^{n} [r(z)(\delta_{ij} - 2z_i\bar{z}_j) + 2(\bar{z}_i - \overline{zz'} z_i)(z_j - zz'\bar{z}_j)](\overline{zz'} z_i - \bar{z}_i)(zz'\bar{z}_j - z_j) \\
=& \frac{1}{r^2(z)} [r(z)A(z) + 2B(z)] \\
=&|z|^2 \\
\leq&1, \qquad (3.17)
\end{aligned}
$$

which implies the desired conclusion. The proof of the proposition is complete. □

To prove Theorem 1.1 and for the convenience of the reader, we recall the following proposition from [16].

Proposition 3.2. *With the notations above, one has,*

$$
\int_{\mathcal{R}_{III}} \det(I - ZZ^*)^{\lambda} dZ < +\infty \iff \lambda > -\frac{1}{2}, \qquad (3.18)
$$

and

$$
\int_{\mathcal{R}_{IV}} (\mu(z))^{\alpha} (\nu(z))^{\beta} dz < +\infty \iff \alpha > -1, \alpha + \beta > -n. \qquad (3.19)
$$

Where $\mu(z) = (1 - z\bar{z}' - \sqrt{(z\bar{z}')^2 - |zz'|^2})$, $\nu(z) = (1 - z\bar{z}' + \sqrt{(z\bar{z}')^2 - |zz'|^2})$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.1 in [12], it is evident that statement (i) holds. So we only need to prove statement (ii). Let

$$f_{\mathcal{A}}(Z) = e^{-\tau u_{\mathcal{A}}(z)}. \tag{3.20}$$

By Proposition 2.4, one has

$$\begin{aligned}\int_{\mathcal{R}_{\mathcal{A}}} |f_{\mathcal{A}}(Z)|^2 dV_u(Z) = &\int_{\mathcal{R}_{\mathcal{A}}} K_{\mathcal{A}}(Z,Z)^{\frac{-2\tau}{c_{\mathcal{A}}}} dV_{u_{\mathcal{A}}}(Z)\\ =&C_{\mathcal{A}} \int_{\mathcal{R}_{\mathcal{A}}} K_{\mathcal{A}}(Z,Z)^{(1-2\frac{\tau}{c_{\mathcal{A}}})} dV(z),\end{aligned} \tag{3.21}$$

where $C_{\mathcal{A}}$ and $c_{\mathcal{A}}$ are constants which are dependent on $\mathcal{R}_{\mathcal{A}}$.

By Faraut and Koranyi [18] and Proposition 3.2, there exists $\alpha_{\mathcal{A}} > 0$ such that

$$\int_{\mathcal{R}_{\mathcal{A}}} K_{\mathcal{A}}(Z,Z)^{\alpha} dv \begin{cases} = +\infty & \alpha > \alpha_{\mathcal{A}}, \\ < +\infty & \alpha < \alpha_{\mathcal{A}}. \end{cases} \tag{3.22}$$

Now we choose τ such that

$$1 - 2\frac{\tau}{c_{\mathcal{A}}} < \alpha_{\mathcal{A}} \iff \tau > \frac{1}{2} c_{\mathcal{A}}(1 - \alpha_{\mathcal{A}}). \tag{3.23}$$

Applying the argument of the proof of [12, Theorem 2.2] and Proposition 3.1, one has

$$\begin{aligned}\lambda_1(\Delta_{\mathcal{A}}) \leq& 4\frac{\int_{\mathcal{R}_{\mathcal{A}}} \sum u_{\mathcal{A}}^{ik,j\ell} \frac{\partial f_{\mathcal{A}}}{\partial z_{ik}} \frac{\partial f_{\mathcal{A}}}{\partial \bar{z}_{j\ell}} dV_{u_{\mathcal{A}}}}{\int_{\mathcal{R}_{\mathcal{A}}} |f_{\mathcal{A}}|^2 dV_{u_{\mathcal{A}}}}\\ =&4\tau^2 \frac{\int_{\mathcal{R}_{\mathcal{A}}} |f_{\mathcal{A}}|^2 \sum u_{\mathcal{A}}^{ik,j\ell} \frac{\partial u_{\mathcal{A}}}{\partial z_{ik}} \frac{\partial u_{\mathcal{A}}}{\partial \bar{z}_{j\ell}} dV_{u_{\mathcal{A}}}}{\int_{\mathcal{R}_{\mathcal{A}}} |f_{\mathcal{A}}|^2 dV_{u_{\mathcal{A}}}}\\ =&4\tau^2 \frac{\int_{\mathcal{R}_{\mathcal{A}}} |f_{\mathcal{A}}|^2 |\partial u_{\mathcal{A}}|^2_{g_{\mathcal{A}}} dV_{u_{\mathcal{A}}}}{\int_{R_{\mathcal{A}}} |f_{\mathcal{A}}|^2 dV_{u_{\mathcal{A}}}}\\ \leq& 4\tau^2 \beta.\end{aligned} \tag{3.24}$$

Letting $\tau \to \frac{1}{2} c_{\mathcal{A}}(1 - \alpha_{\mathcal{A}})$ we have

$$\lambda_1(\Delta_{\mathcal{A}}) \leq 4\beta[\frac{1}{2} c_{\mathcal{A}}(1 - \alpha_{\mathcal{A}})]^2 = \beta c_{\mathcal{A}}^2 (1 - \alpha_{\mathcal{A}})^2. \tag{3.25}$$

Which completes the proof. □

Remark 3.1. *Let*

$$K_{III}(Z,Z) = C_{III}(\det(I - ZZ^*))^{-(n-1)} \tag{3.26}$$

and

$$K_{IV}(Z,Z) = C_{IV} \mu(z)^{-n} \nu(z)^{-n}. \tag{3.27}$$

We conclude from Proposition 3.2 that

$$\alpha_{III} = \frac{1}{2(n-1)}, \tag{3.28}$$

and

$$\alpha_{IV} = \begin{cases} \frac{1}{2} & n = 1, \\ \frac{1}{n} & n \geq 2. \end{cases} \tag{3.29}$$

Finally, we prove Corollary 1.1 here.

Proof of Corollary 1.1. (i) For $Z \in \mathcal{R}_{III}$ with $\beta = 2[\frac{n}{2}]$, by Theorem 1.1, and Proposition 3.1, (3.28) and (3.29) now lead to

$$\lambda_1(\Delta_{III}) \geq \Big(\frac{n(n-1)}{2}\Big)^2 \frac{1}{\beta} = \frac{n^2(n-1)^2}{8} \frac{1}{[\frac{n}{2}]}$$

and

$$\lambda_1(\Delta_{III}) \leq \beta c_{\mathcal{A}}^2 (1-\alpha_{\mathcal{A}})^2 = 2[\frac{n}{2}](n-1)^2(1-\frac{1}{2(n-1)})^2 = 2[\frac{n}{2}]\frac{(2n-3)^2}{4}.$$

Therefore,

$$\lambda_1(\Delta_{III}) \in [\frac{n(n-1)^2}{4}, \frac{n(2n-3)^2}{4}], \quad n = 2k$$

and

$$\lambda_1(\Delta_{III}) \in [\frac{n^2(n-1)}{4}, \frac{(n-1)(2n-3)^2}{4}], \quad n = 2k-1.$$

(ii) For $z \in \mathcal{R}_{IV}$, when $n = 1$, it is evident that

$$\lambda_1(\Delta_{IV}) = 1.$$

When $n \geq 2$,

$$\lambda_1(\Delta_{IV}) \geq n^2 \quad \text{and} \quad \lambda_1(\Delta_{IV}) \leq (2n)^2(1-\frac{1}{n})^2 = 4(n-1)^2.$$

The proof of Part (ii) follows. □

4. Conclusions

In this paper, we investigate estimate for $\lambda_1(\Delta_g)$ on the Cartan classical domains of the last two types $\mathcal{R}_{\mathcal{A}}(\mathcal{A} = III, IV)$. Based on theories of harmonic analysis in the Cartan classical domains from Hua [16] and Lu [17]. Firstly, We are dedicated to find the plurisubharmonic exhaustion function $u_{\mathcal{A}}$ under Bergman kernel function of $\mathcal{R}_{\mathcal{A}}$. Next, we define a complete Kähler metric $g_{\mathcal{A}}$ which induced by $u_{\mathcal{A}}$. Through constructing suitable test function $f_{\mathcal{A}}(Z) = e^{-\tau u_{\mathcal{A}}(z)}$, we obtain upper and lower bound estimates for $\lambda_1(\Delta_g)$ on $\mathcal{R}_{\mathcal{A}}$ under the assumption that $|\partial u|_g^2 < \beta$. In addition, we provide the value of β by establishing an estimate for $|\partial u|_g^2$. This brings us to give an explicit range for $\lambda_1(\Delta_g)$. Attributed to the special forms of $\mathcal{R}_{III}$ and $\mathcal{R}_{IV}$, the approach examined in this present study requires complicated but interesting technical work.

As shown in our study, we actually propose an approach which may be adapted to solve the problem of finding estimates for $\lambda_1(\Delta_g)$ on other important domains. It is well known that any bounded symmetric domain may be represented as the topological product of irreducible bounded symmetric domains: the class of irreducible bounded symmetric domains consists of four types of Cartan classical domains and two exceptional ones. So we are encouraged to work on the estimates for $\lambda_1(\Delta_g)$ on bounded symmetric domains. This will be the objective of our future study.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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