Research article

Infimum of the spectrum of Laplace-Beltrami operator on Cartan classical domains of type III and IV

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Abstract: Let \( R_A \) be the Cartan classical domains of type III and IV, and \( \Delta_g \) is assumed to be the Laplace-Beltrami operator associated to the Bergman metric \( g \) on \( R_A \). In this paper, we derive an estimate for \( \lambda_1(\Delta_g) \), which is the bottom of the spectrum of \( \Delta_g \) on \( R_A \).

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1. Introduction

Let \((M, g)\) be a Kähler manifold of complex dimension \( n \) with Kähler metric \( g = \sum_{i,j=1}^{n} u_{ij} dz_i \otimes dz_j \). The Laplace-Beltrami operator with respect to the Kähler metric \( g \) is defined by

\[
\Delta_g = -4 \sum_{i,j=1}^{n} u_{ij} \frac{\partial^2}{\partial z_i \partial \overline{z}_j},
\]

where \([u_{ij}]^\prime = [u_{ij}]^{-1}\). Let

\[
\lambda_1(\Delta_g) := \lambda_1(\Delta_g, M) = \inf \left\{ 4 \int_M u_{ij} \frac{\partial h}{\partial z_i} \frac{\partial h}{\partial \overline{z}_j} dV_g : h \in C_0^\infty(M), \int_M |h|^2 dV_g = 1 \right\}.
\]

Here \( dV_g \) is the volume measure of \( M \) with respect to the Kähler metric \( g \).

Spectral theory or eigenvalue estimates for Laplace-Beltrami operators are important subject for mathematics. When \( M \) is compact and \( \Delta_g \) is uniformly elliptic, \( \lambda_1(\Delta_g) \) is the first eigenvalue of \( \Delta_g \).
Researches on its upper and lower bound estimates have a long history with many results (see [1–5] and the reference therein).

Unlike compact manifolds, for the case when \( M \) is a non-compact manifold, \( \lambda_1(\Delta_g) \) may not be an eigenvalue of \( \Delta_g \), but rather the infimum of the positive spectral of \( \Delta_g \). This naturally leads to an interesting problem: study of estimate for \( \lambda_1(\Delta_g) \) in the complete non-compact case. Quite a bit of research has been done on this problem. For examples, the results on upper bounds and lower bounds estimates obtained by Cheng [6], Li and Wang [7–10], Munteanu [11], Li and Tran [12] are all well known. The rigidity property of manifold may be further obtained when \( \lambda_1 \) achieves its sharp upper or lower bound estimate. For examples, one may see [7–9, 13, 14] and references therein.

We recall the following estimates for \( \lambda_1(\Delta_g) \), in the Riemannian manifolds, a sharp upper bound estimate for \( \lambda_1(\Delta_g) \) is well known from Cheng [6]. For the Kähler manifolds, Li and Wang gave an important sharp upper bound estimate in [9]. They proved \( \lambda_1(\Delta_g) \geq n^2 \) with the assumption that the holomorphic bisectional curvature of \( M \) is bounded below by \(-1\). Later on, Munteanu [11] improved Li and Wang’s result, and derived another sharp estimate in terms of the Ricci curvature is bounded from below by \(-2(n + 1)\).

As a continuation of the work of Li and Wang [9] and the work of Munteanu [11], Li and Tran [12] provided many examples of bounded strongly pseudoconvex domains, on which \( \lambda_1(\Delta_g) \) can be explicitly formulated. However, those domains were exclusive of most of non-smooth domains, like Cartan classical domains. It would be desirable to proceed the study of \( \lambda_1(\Delta_g) \) on the Cartan classical domains.

This paper aims to provide an estimate for \( \lambda_1(\Delta_g) \) on \( \mathcal{R}_\mathcal{A}(\mathcal{A} = \text{III, IV}) \) with \( g \) is the Bergman metric of \( \mathcal{R}_\mathcal{A}(n) \). Regarding the estimate for \( \lambda_1(\Delta_g) \) on the Cartan classical domains of type I and II, it has been studied by the first author in [15]. We continue to use some ideas in [15], but the calculations in the current cases are much more delicate. This is due to the special forms of \( \mathcal{R}_{\text{III}}(n) \) and \( \mathcal{R}_{\text{IV}}(n) \).

Our study is motivated by Li and Tran [12]. We emphasize that the calculation of \( |\partial^2 u|_g^2 \) and the construction of test functions play important roles in our argument, but both are difficult to solve. We use the Bergman metric \( g \) on the Cartan classical domains given by Hua [16] and Lu [17].

Let \( K_{\mathcal{A}}(z) := K_{\mathcal{A}}(z, z) \) be the Bergman kernel function of \( R_{\mathcal{A}}(n) \) and let

\[
u_{\mathcal{A}} = \frac{1}{c_{\mathcal{A}}} \log K_{\mathcal{A}}(z), \quad c_{\text{III}} = n - 1 \quad \text{and} \quad c_{\text{IV}} = 2n.
\]

Our goal is to prove the following results.
**Theorem 1.1.** Let $\Delta_\mathcal{A} = \Delta_g(\mathcal{A} = \text{III, IV})$ be the Laplace-Beltrami operator associated to the Bergman metric $g$ on the Cartan classical domains $\mathcal{R}_\mathcal{A}(n)$ of type $\mathcal{A}$. Let $u = u_\mathcal{A}$ be the strictly plurisubharmonic exhaustion function for $\mathcal{R}_\mathcal{A}(n)$ and $g = g_\mathcal{A}$ be the Kähler metric induced by $u_\mathcal{A}$. Assuming that $|\partial u_\mathcal{A}|^2 \leq \beta$, one has

(i) $\lambda_1(\Delta_\mathcal{A}) \geq N_\mathcal{A}^2 / \beta$ with $N_\text{III} = n(n-1)/2$ and $N_\text{IV} = n$;

(ii) $\lambda_1(\Delta_\mathcal{A}) \leq \beta c_\mathcal{A}^2 (1 - \alpha_\mathcal{A})^2$, where

$$\alpha_\text{III} := \frac{1}{2(n-1)}, \quad \alpha_\text{IV}(n) = \frac{1}{2} \text{ when } n = 1, \quad \alpha_\text{IV}(n) = \frac{1}{n} \text{ when } n > 1;$$

and

$$c_\text{III} = n - 1, \quad c_\text{IV} = 2n.$$

**Corollary 1.1.** Let the notations and assumptions as in Theorem 1.1. Then

$$\lambda_1(\Delta_\text{III}) \in \left\{ \left[ \frac{n(n-1)^2}{4}, \frac{n(2n-3)^2}{4} \right], \quad n = 2k; \right.$$

$$\left. \left[ \frac{n^2(n-1)}{4}, \frac{(n-1)(2n-3)^2}{4} \right], \quad n = 2k - 1. \right.$$

and

$$\lambda_1(\Delta_\text{IV}) \left\{ \begin{array}{ll} 1, & n = 1; \\ \in [n^2, 4(n-1)^2], & n \geq 2. \end{array} \right.$$
Proposition 2.1. [17] Let $A$ and $B$ be two $n \times n$ matrices. Then

$$[A \times A]_{as} [B \times B]_{as} = [AB \times AB]_{as}, \quad ([A \times B]_{as})' = [A' \times B']_{as}$$  \hspace{1cm} (2.5)

and

$$[A \times A]^{-1}_{as} = [A^{-1} \times A^{-1}]_{as}, \quad \det[A \times A]_{as} = (\det A)^{n-1}.$$  \hspace{1cm} (2.6)

A straightforward calculation shows:

Proposition 2.2. Let $C = [c_{pq}]$ be a $s \times s$ matrix where $c_{pq}$ is a function of $z = (z_1, \cdots, z_n) \in C^n$ and $\bar{z}$. Then

$$\frac{\partial \log \det C}{\partial z_k} = \sum_{p,q=1}^s c_{pq} \frac{\partial c_{pq}}{\partial z_k}$$  \hspace{1cm} (2.7)

and

$$\frac{\partial^2 \log \det C}{\partial z_k \partial \bar{z}_\ell} = \sum_{p,q=1}^s c_{pq} \frac{\partial^2 c_{pq}}{\partial z_k \partial \bar{z}_\ell} - \sum_{i,j,p,q=1}^s c_{ij} c_{pq} \frac{\partial c_{pq}}{\partial z_k} \frac{\partial c_{ij}}{\partial \bar{z}_\ell},$$  \hspace{1cm} (2.8)

where

$$\sum_{j=1}^s c_{ij} c_{kj} = \delta_{ik}.$$  

For $Z \in \mathcal{R}_A$, we set

$$z = (z_{12}, \cdots, z_{1m}, z_{23}, \cdots, z_{2n}, \cdots, z_{(n-1)n}) \in C^{n(n-1)}.$$  

Obviously $2||z||^2 = \text{tr}(ZZ^*)$. We know from Hua’s book [16] or Lu’s book [17, Section 3.3] that

$$K_{III}(Z) = \frac{1}{V(\mathcal{R}_{III})} \cdot \frac{1}{\det(I - ZZ^*)^{n-1}},$$  \hspace{1cm} (2.9)

and

$$K_{IV}(z) = \frac{1}{V(\mathcal{R}_{IV})} \cdot \frac{1}{(1 + |\sum_{j=1}^n z_j^2|^2 - 2|z|^2)^n},$$  \hspace{1cm} (2.10)

Here $V(\mathcal{R}_A)$ is the volume of $\mathcal{R}_A$.

Consider the Bergman kernel function of $\mathcal{R}_A$, we construct $u_A$ as (1.5), hence $u_A$ is strictly plurisubharmonic exhaustion function in $\mathcal{R}_A$. Furthermore, we define a complete Kähler metric $g_A$ which is induced by $u_A$ as follows:

$$g_A = \sum_{i,j=1}^N u_{ij} dz_i \otimes d\bar{z}_j,$$  \hspace{1cm} (2.11)

where $N = n(n - 1)/2$ when $A = III$ and $N = n$ when $A = IV$. Consequently,

$$|\partial u_A|_{g_A}^2 = \sum_{i,j=1}^N u_{ij}^2 \frac{\partial u_A}{\partial z_i} \frac{\partial u_A}{\partial \bar{z}_j},$$  \hspace{1cm} (2.12)

where $[u_A]^{-1} = [u_A]^{-1}$.

Let us mention two important consequences about the complex Hessian matrix for $u_A$ on $\mathcal{R}_A$. 

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Proposition 2.3. The complex Hessian matrix for $u_\mathcal{M}$ can be stated by

$$H(u_\mathcal{M})(Z) = 2((I - ZZ^*)^{-1} \times (I - ZZ^*)^{-1})_{zz},$$

(2.13)

$$H(u_{\mathcal{M} M}^{-1})(Z) = \frac{1}{2}((I - ZZ^*)^{-1} \times (I - ZZ^*)_{zz},$$

(2.14)

$$H(u_\mathcal{M})(z) = \frac{1}{r(z)} \left[ r(z)I_n - 2 \left( \frac{z'}{z} \right) \left( \frac{1 - 2|z|^2}{z'z} \right) \right]$$

(2.15)

and

$$H(u_\mathcal{M})(Z) = [r(z)(I - 2z'z) + 2(z' - zz'')(\bar{z} - \bar{z}'z)],$$

(2.16)

where $r(z) = 1 + |\sum_{j=1}^n \bar{z}_j^2| - 2|z|^2$.

Proof. (i) For $Z \in \mathcal{R}_\mathcal{M}$, by applying (2.7) and (2.8), a straightforward calculation shows that

$$\frac{\partial^2 \log K_{\mathcal{M} M}(Z)}{\partial z_{ja} \bar{z}_{k\beta}} \bigg|_{Z=0} = -(n - 1) \frac{\partial^2 \log \det(I_n - Z Z^*)}{\partial z_{ja} \bar{z}_{k\beta}} \bigg|_{Z=0}$$

$$= -(n - 1) \sum_{h,l=1}^n \left( (I_n + Z Z^*)^{-1} \right)_{hl} \frac{\partial^2 (\sum_{j=1}^n \bar{z}_j z_j)}{\partial z_{ja} \bar{z}_{k\beta}} \bigg|_{Z=0}$$

$$= -(n - 1) \sum_{h,l=1}^n \frac{\partial}{\partial z_{ja}} \sum_s (\delta_{jh} \delta_{s,sh} - \delta_{ha} \delta_{j,sh})$$

$$= -(n - 1) \sum_{h,l=1}^n \frac{\partial}{\partial z_{ja}} (\delta_{jh} \bar{z}_{sh} - \delta_{ha} \bar{z}_j)$$

$$= -(n - 1) \sum_{h,l=1}^n \left[ \delta_{jh} (\delta_{ha} \delta_{jhb} - \delta_{h,a} \delta_{ab}) - \delta_{ha} (\delta_{j,k} \delta_{hB} - \delta_{j,k} \delta_{hb}) \right]$$

$$= -2(n - 1)(\delta_{ka} \delta_{jB} - \delta_{jk} \delta_{aB}).$$

(2.17)

Hence, we have

$$H(u_\mathcal{M})(0) = \left[ \frac{\partial^2 u}{\partial z_{ja} \bar{z}_{k\beta}} \right]_{Z=0} = 2[I_m \times I_m]_{zz}.$$

(2.18)

As the proof of equality (3.3.45) in [17], the transformation property of Bergman kernel function and M"{o}bius transform of $\mathcal{R}_\mathcal{M}$, (2.13) follows. By Proposition 2.1, (2.14) obtained.

(ii) For $z = (z_1, z_2, \ldots, z_n) \in \mathcal{R}_\mathcal{M}$, by equality (3.3.57) in [17], we have

$$\frac{\partial^2 \log K_{\mathcal{M} M}(Z)}{\partial z_{ja} \bar{z}_{k\beta}} = \frac{1}{r^2(z)} \left[ r(z)I_n + 4zz'z'z - 2(zz'z'z + zz'z'z) - 2(z'z - z'z) \right]$$

$$= \frac{1}{r^2(z)} \left[ r(z)I_n - 2((z' - 2|z|^2 z' + z'z')z + 2(z'zz - z')z) \right]$$

$$= \frac{1}{r^2(z)} \left[ r(z)I_n - 2 \left( z' - 2|z|^2 z' + z'zz - z'z' - z' \right) \right]$$

$$= \frac{1}{r^2(z)} \left[ r(z)I_n - 2 \left( z' \right) \left( 1 - 2|z|^2 \frac{z'z}{z'} - 1 \right) \right]$$

(2.19)
This implies (2.15). We claim (2.16) holds from discussion of equality (6.1.29) in [17]. □

**Proposition 2.4.** With the notations above, one has

$$\det H(u_{ii})(z) = 2 \frac{e^{\alpha z}}{\det(I - ZZ^*)^{n-1}}$$ 

and

$$\det H(u_{ri})(z) = \frac{1}{r^i(z)}.$$ 

**Proof.** By Proposition 2.1 and Proposition 2.3, it is easy to deduce that

$$\det H(u_{ii})(z) = 2^{\frac{n(n-1)}{2}} \det[(I - ZZ^*)^{-1} \times (I - ZZ^*)^{-1}]_{ii}$$

$$= 2^{\frac{n(n-1)}{2}} \det(I - ZZ^*)^{(n-1)}$$

and

$$\det H(u_{ri})(z) = \det \left( \frac{1}{r^i(z)} \left[ r(z)I_n - 2 \left( \begin{array}{c} z' \\ z \end{array} \right) \left( \begin{array}{cc} 2|z|^2 & \bar{z}z' \\ z' & -1 \end{array} \right) \left( \begin{array}{c} z \\ z' \end{array} \right) \right] \right)$$

$$= \frac{1}{r^i(z)} \det \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) - 2 \left( \begin{array}{cc} 2|z|^2 & \bar{z}z' \\ z' & -1 \end{array} \right) \left( \begin{array}{c} z' \\ z \end{array} \right) \left( \begin{array}{c} z \\ z' \end{array} \right)$$

$$= \frac{1}{r^i(z)} \det \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) - 2 \frac{\bar{z}z'}{r(z)} \left( \begin{array}{cc} |z|^2 & \bar{z}z' \\ z' & -1 \end{array} \right) \left( \begin{array}{c} z' \\ z' \end{array} \right)$$

$$= \frac{1}{r^i(z)} \det \left( \begin{array}{cc} 1 - 2 \frac{z'}{r(z)}(|z|^2 - 2|z|^4 + |z'|^2) & -2 \frac{z'}{r(z)} \bar{z}z' \left( 1 - |z|^2 \right) \\ -2 \frac{z'}{r(z)} \bar{z}z' \left( 1 - |z|^2 \right) & 1 - 2 \frac{z'}{r(z)} \left( |z'|^2 - |z|^2 \right) \end{array} \right)$$

$$= \frac{1}{r^{i+2}(z)} \left( 1 + 2|z'|^2 - 4|z|^2 + 4|z|^4 + |z'|^4 - 4|z|^2 |z'|^2 \right)$$

This completes the proof. □

3. Estimate for $\lambda_1(\Delta_\theta)$

In this section, in order to prove the main theorems, we begin to establish an estimate for $|\partial u_8^2|$. Let

$$\frac{\partial}{\partial z} = \left( \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \right).$$

Since

$$\left( \begin{array}{c} \frac{\partial u}{\partial z} \\ \frac{\partial u}{\partial z} \end{array} \right) = \left( \begin{array}{cccc} \frac{\partial u}{\partial z_1} & \frac{\partial u}{\partial z_2} & \cdots & \frac{\partial u}{\partial z_n} \\ \frac{\partial u}{\partial z_1} & \frac{\partial u}{\partial z_2} & \cdots & \frac{\partial u}{\partial z_n} \end{array} \right),$$

one has

$$|\partial u|^2 = \sum_{i,j=1}^n u_i^* \frac{\partial u_i}{\partial z_j} = \text{tr} \left( \left( \begin{array}{c} \frac{\partial u}{\partial z} \end{array} \right) \left( \begin{array}{c} \frac{\partial u}{\partial z} \end{array} \right)^* \right).$$
Proposition 3.1. With the notations in (1.5), (2.11) and (2.12), one has the following estimates:

\[ |\partial u_m|^2 \leq 2 \left[ \frac{n}{2} \right] \quad \text{and} \quad |\partial u_{\nu'}|^2 \leq 1. \tag{3.3} \]

Proof. (i) For \( z \in \mathcal{R}_m \), according to (3.2), one has

\[
\text{tr} \left( [A \times A]_{\omega m} \frac{\partial u_{\omega m}}{\partial z_i} \frac{\partial u_{\omega m}}{\partial z_j} \right) = \sum_{i,k=1}^{n} \sum_{i<j<k<\ell} a_{(i)(j)(k)} \frac{\partial u_{\omega m}}{\partial z_i} \frac{\partial u_{\omega m}}{\partial z_j} \]

\[
= \sum_{i,k=1}^{n} \sum_{i<j<k<\ell} (a_{ik} a_{j\ell} - a_{i\ell} a_{jk}) \frac{\partial u_{\omega m}}{\partial z_i} \frac{\partial u_{\omega m}}{\partial z_j} \]

\[
= \frac{1}{2} \sum_{i,j,k=1}^{n} \sum_{k<\ell} \left( \sum_{i<j} a_{ik} a_{j\ell} + \sum_{i>j} a_{ji} a_{i\ell} \right) \frac{\partial u_{\omega m}}{\partial z_i} \frac{\partial u_{\omega m}}{\partial z_j} - \left( \sum_{i<j} a_{ij} a_{i\ell} + \sum_{i>j} a_{ji} a_{j\ell} \right) \frac{\partial u_{\omega m}}{\partial z_i} \frac{\partial u_{\omega m}}{\partial z_j} \]

\[
= \frac{1}{2} \sum_{i,j,k=1}^{n} \sum_{k<\ell} \left( \sum_{i<j} a_{ik} a_{j\ell} - \sum_{i>j} a_{ji} a_{i\ell} \right) q_{ij} \frac{\partial u_{\omega m}}{\partial z_i} \frac{\partial u_{\omega m}}{\partial z_j} \]

where

\[ q_{ij} = \begin{cases} 0, & i = j; \\ 1, & i \neq j. \end{cases} \]

Let \( D[\lambda_1, \ldots, \lambda_n] \) be \( n \times n \) diagonal matrix with all diagonal entries are \( \lambda_1, \ldots, \lambda_n \). For \( Z \in \mathcal{R}_m \), since \( ZZ^\ast \) is the Hermitian matrix, there exists \( n \times n \) unitary matrix \( U \) such that

\[
U ZZ^\ast U^\ast = \begin{cases} D[\lambda_1, \ldots, \lambda_n], & n = 2k; \\ D[\lambda_1, \ldots, \lambda_{n-1}, 0], & n = 2k + 1. \end{cases} \tag{3.5} \]

and \( \lambda_j \in [0, 1), \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \). There is no loss of generality in assuming

\[
ZZ^\ast = \begin{cases} D[\lambda_1, \ldots, \lambda_n], & n = 2k; \\ D[\lambda_1, \ldots, \lambda_{n-1}, 0], & n = 2k + 1. \end{cases} \tag{3.6} \]

It follows that

\[
(I_m - ZZ^\ast)^{-1} = \begin{cases} D[\sum_{k=0}^{\infty} \lambda_k^k, \ldots, \sum_{k=0}^{\infty} \lambda_n^k], & n = 2k; \\ D[\sum_{k=0}^{\infty} \lambda_k^k, \ldots, \sum_{k=0}^{\infty} \lambda_n^k, 0], & n = 2k + 1. \end{cases} \tag{3.7} \]

Thus

\[
\frac{\partial u_{\omega m}}{\partial z_{ik}} = z_k \left( \frac{1}{1 - \lambda_i} + \frac{1}{1 - \lambda_k} \right), \quad i < k. \tag{3.8} \]
\[
\frac{\partial u_{ii}}{\partial z_{j\ell}} = z_{j\ell}\left(\frac{1}{1 - \lambda_j} + \frac{1}{1 - \lambda_\ell}\right). \quad j < \ell.
\] (3.9)

Therefore
\[
|\partial u_{ii}|^2_{\text{all}} = \sum_{i,j=1}^n \sum_{k,l,t=1}^n u_{ii}^2 \frac{\partial u_{ii}}{\partial z_{ik}} \frac{\partial u_{ii}}{\partial z_{j\ell}}
\]
\[
= \frac{1}{2} \sum_{i,j,k,t=1}^n (\delta_{ij} - \sum_{s=1}^n z_{is} \bar{z}_{js})(\delta_{kl} - \sum_{t=1}^n z_{kt} \bar{z}_{\ell t})\frac{1}{2} q_{ik} q_{j\ell} \frac{\partial u_{ii}}{\partial z_{ik}} \frac{\partial u_{ii}}{\partial z_{j\ell}}
\]
\[
= \frac{1}{4} \sum_{i=1}^n (1 - \lambda_i)(1 - \lambda_k)(1 - \lambda_j)(1 - \lambda_\ell)\frac{1}{2} q_{ik} q_{j\ell} |q_{ik} z_{ik}|^2
\]
\[
= \frac{1}{2} \left(\text{tr}(Z^T(I - ZZ^T)(I - ZZ^T)^{-1}) + \text{tr}(ZZ^T)\right)
\]
\[= \frac{1}{2} \left(\text{tr}(Z^T(I - Z^T Z)(I - Z^T Z)^{-1}) + \text{tr}(ZZ^T)\right)
\]
\[= \frac{1}{2} \left(\text{tr}(ZZ^T) + \text{tr}(ZZ^T)\right) = \text{tr}(ZZ^T)
\]
\[\leq 2 \left\| \frac{q_i}{2} \right\|.
\] (3.10)

(ii) For \(z \in \mathcal{R}_n\), we have
\[
\frac{\partial u_{iv}}{\partial z_i} = -\frac{1}{2} \frac{\partial \log(r(z))}{\partial z_i} = -\frac{1}{2} \frac{1}{r(z)} \frac{\partial r(z)}{\partial z_i} = -\frac{(\bar{z} z_i - \bar{z}_i)}{r(z)}.
\] (3.11)
\[
\frac{\partial u_{iv}}{\partial z_j} = -\frac{1}{2} \frac{\partial \log(r(z))}{\partial z_j} = -\frac{1}{2} \frac{1}{r(z)} \frac{\partial r(z)}{\partial z_j} = -\frac{(\bar{z} z_j - \bar{z}_j)}{r(z)}.
\] (3.12)

Accordingly,
\[
\frac{\partial u_{iv}}{\partial z_i} \frac{\partial u_{iv}}{\partial z_j} = \frac{1}{r(z)} (|zz'|^2 z_i \bar{z}_j - \bar{z} z_i z_j - \bar{z} z_i z_j + \bar{z}_i z_j).
\] (3.13)

Notice that
\[
A(z) := \sum_{i,j=1}^n (\delta_{ij} - 2z_i \bar{z}_j)(|zz'|^2 z_i \bar{z}_j - \bar{z} z_i z_j - \bar{z} z_i z_j + \bar{z}_i z_j)
\]
\[= (|zz'|^2 |z|^2 - 2|zz'|^2 |z|^2 - 2|zz'|^4 |z|^2 + |z|^4)
\]
\[= 5|zz'|^3 |z|^2 - 2|zz'|^2 - 2|zz'|^4 - 2|z|^4 + |z|^2
\] (3.14)

and
Proposition 3.2. With the notations above, one has,

\[
\int_{\mathcal{R}_{11}} \det(I - ZZ^*)^4 dZ < +\infty \iff \lambda > -\frac{1}{2},
\]

and

\[
\int_{\mathcal{R}_{12}} (\mu(z))^\alpha (v(z))^\beta dz < +\infty \iff \alpha > -1, \beta > -n.
\]

Where \( \mu(z) = (1 - z\bar{z}' - \sqrt{(z\bar{z}')^2 - |z\bar{z}'|^2}), \) \( v(z) = (1 - z\bar{z}' + \sqrt{(z\bar{z}')^2 - |z\bar{z}'|^2}). \)

Now we are ready to prove Theorem 1.1.
Proof of Theorem 1.1. By Proposition 2.1 in [12], it is evident that statement (i) holds. So we only need to prove statement (ii). Let
\[ f_A(Z) = e^{-\tau u_A(z)}. \] (3.20)
By Proposition 2.4, one has
\[ \int_\mathcal{R}_A |f_A(Z)|^2 dV_u(Z) = \int_\mathcal{R}_A K_A(Z, Z) e^{-\tau u_A(z)} dV_u(Z) \]
\[ = C_A \int_\mathcal{R}_A K_A(Z, Z)^{(1-2\tau c_A)} dV(z), \] (3.21)
where \( C_A \) and \( c_A \) are constants which are dependent on \( \mathcal{R}_A \).

By Faraut and Koranyi [18] and Proposition 3.2, there exists \( \alpha_A > 0 \) such that
\[ \int_\mathcal{R}_A K_A(Z, Z) \alpha dV \]
\[ = +\infty \quad \alpha > \alpha_A, \]
\[ < +\infty \quad \alpha < \alpha_A. \] (3.22)
Now we choose \( \tau \) such that
\[ 1 - 2\frac{\tau}{c_A} < \alpha_A \iff \tau > \frac{1}{2} c_A(1 - \alpha_A). \] (3.23)
Applying the argument of the proof of [12, Theorem 2.2] and Proposition 3.1, one has
\[ \lambda_1(\Delta_A) \leq 4 \int_\mathcal{R}_A \sum u^{ik,jf}_{\mathcal{A}} \frac{\partial f_A}{\partial x_k} \frac{\partial f_A}{\partial x_f} dV_u(A) \]
\[ \leq 4 \tau^2 \int_\mathcal{R}_A \sum u^{ik,jf}_{\mathcal{A}} \frac{\partial u_A}{\partial x_k} \frac{\partial u_A}{\partial x_f} dV_u(A) \]
\[ = 4 \tau^2 \int_\mathcal{R}_A \sum u^{ik,jf}_{\mathcal{A}} \partial u_A \frac{\partial u_A}{\partial x_k} dV_u(A) \]
\[ \leq 4 \tau^2 \beta. \] (3.24)
Letting \( \tau \to \frac{1}{2} c_A(1 - \alpha_A) \) we have
\[ \lambda_1(\Delta_A) \leq 4\beta \left[ \frac{1}{2} c_A(1 - \alpha_A) \right]^2 = \beta c_A^2(1 - \alpha_A)^2. \] (3.25)
Which completes the proof. \( \square \)

Remark 3.1. Let
\[ K_{III}(Z, Z) = C_{III}(\det(I - ZZ^*))^{(n-1)} \] (3.26)
and
\[ K_{IV}(Z, Z) = C_{IV}(z)^{-n} v(z)^{-n}. \] (3.27)
We conclude from Proposition 3.2 that
\[ \alpha_{III} = \frac{1}{2(n - 1)}. \] (3.28)
and
Finally, we prove Corollary 1.1 here.

**Proof of Corollary 1.1.** (i) For \( z \in \mathcal{R}_n \) with \( \beta = 2^{\left[ \frac{n}{2} \right]} \), by Theorem 1.1, and Proposition 3.1, (3.28) and (3.29) now lead to

\[
\lambda_1(\Delta_{III}) \geq \left( \frac{n(n-1)}{2} \right)^2 \frac{1}{\beta} = \frac{n^2(n-1)^2}{8} \left[ \frac{n}{2} \right]
\]

and

\[
\lambda_1(\Delta_{III}) \leq \beta c_{\mathcal{A}}^2(1 - \alpha_{\mathcal{A}})^2 = 2\left[ \frac{n}{2} \right](n-1)^2 \left( 1 - \frac{1}{2(n-1)} \right)^2 = 2\left[ \frac{n}{2} \right] \frac{(2n-3)^2}{4}.
\]

Therefore,

\[
\lambda_1(\Delta_{III}) \in \left[ \frac{n(n-1)^2}{4}, \frac{n(2n-3)^2}{4} \right], \quad n = 2k
\]

and

\[
\lambda_1(\Delta_{III}) \in \left[ \frac{n^2(n-1)}{4}, \frac{(n-1)(2n-3)^2}{4} \right], \quad n = 2k - 1.
\]

(ii) For \( z \in \mathcal{R}_n \), when \( n = 1 \), it is evident that

\[
\lambda_1(\Delta_{IV}) = 1.
\]

When \( n \geq 2 \),

\[
\lambda_1(\Delta_{IV}) \geq n^2 \quad \text{and} \quad \lambda_1(\Delta_{IV}) \leq (2n)^2 \left( 1 - \frac{1}{n} \right)^2 = 4(n-1)^2.
\]

The proof of Part (ii) follows. \( \square \)

4. Conclusions

In this paper, we investigate estimate for \( \lambda_1(\Delta_g) \) on the Cartan classical domains of the last two types \( \mathcal{R}_A (A = III, IV) \). Based on theories of harmonic analysis in the Cartan classical domains from Hua [16] and Lu [17]. Firstly, We are dedicated to find the plurisubharmonic exhaustion function \( u_{\mathcal{A}} \) under Bergman kernel function of \( \mathcal{R}_A \). Next, we define a complete Kähler metric \( g_{\mathcal{A}} \) which induced by \( u_{\mathcal{A}} \). Through constructing suitable test function \( f_{\mathcal{A}}(Z) = e^{-u_{\mathcal{A}}(Z)} \), we obtain upper and lower bound estimates for \( \lambda_1(\Delta_g) \) on \( \mathcal{R}_A \) under the assumption that \( |\partial u_{\mathcal{A}}|^2 \leq \beta \). In addition, we provide the value of \( \beta \) by establishing an estimate for \( |\partial u_{\mathcal{A}}|^2 \). This brings us to give an explicit range for \( \lambda_1(\Delta_g) \). Attributed to the special forms of \( \mathcal{R}_{III} \) and \( \mathcal{R}_{IV} \), the approach examined in this present study requires complicated but interesting technical work.

As shown in our study, we actually propose an approach which may be adapted to solve the problem of finding estimates for \( \lambda_1(\Delta_g) \) on other important domains. It is well known that any bounded symmetric domain may be represented as the topological product of irreducible bounded symmetric domains: the class of irreducible bounded symmetric domains consists of four types of Cartan classical domains and two exceptional ones. So we are encouraged to work on the estimates for \( \lambda_1(\Delta_g) \) on bounded symmetric domains. This will be the objective of our future study.
Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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