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Research article

Positive solutions for a critical quasilinear Schrödinger equation

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Abstract: In our current work we investigate the following critical quasilinear Schrödinger equation

$$-\Delta\Theta + \mathcal{V}(x)\Theta - \Delta(\Theta^2)\Theta = |\Theta|^{22^*-2}\Theta + \lambda\mathcal{K}(x)g(\Theta),\ x\ \in\mathbb{R}^N,$$

where $N \geq 3$, $\lambda > 0$, $\mathcal{V},\ \mathcal{K} \in C(\mathbb{R}^N, \mathbb{R}^+)$ and $g \in C(\mathbb{R}, \mathbb{R})$ has a quasicritical growth condition. We use the dual approach and the mountain pass theorem to show that the considered problem has a positive solution when λ is a large parameter.

Keywords: quasilinear Schrödinger equations; quasicritical growths; dual approaches
Mathematics Subject Classification: 35J20, 35J70, 35P05, 35P30, 34B15, 58E05, 47H04

1. Introduction

In this work we discuss the critical quasilinear Schrödinger equation

$$-\Delta\Theta + \mathcal{V}(x)\Theta - \Delta(\Theta^2)\Theta = |\Theta|^{22^*-2}\Theta + \lambda\mathcal{K}(x)g(\Theta),\ x\ \in\mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $\mathcal{V}, \mathcal{K} \in C(\mathbb{R}^N, \mathbb{R}^+)$, $g \in C(\mathbb{R}, \mathbb{R})$ is a quasilinear growth function, $2^* = 2N/(N-2)$ and 22^* is the critical exponent for (1.1).

Quasilinear equations are often involved in studying standing wave solutions for the quasilinear Schrödinger equation

$$i\psi_t = -\triangle\psi + \mathcal{W}(x)\psi - \kappa\triangle[\rho(|\psi|^2)]\rho'(|\psi|^2)\psi - l(|\psi|^2)\psi, \tag{1.2}$$

where $\mathcal{W}$ is a potential, $\kappa \in \mathbb{R}$, ρ, $l : \mathbb{R} \to \mathbb{R}$. The form of (1.2) has many applications in physics, for example see [1–4]. In [5] the authors used the method of Nehari manifold to discuss the concentration behavior and the exponential decay phenomenon of ground state solutions for the equation

$$-\operatorname{div}\left(\varepsilon^2 g^2(\chi)\nabla\chi\right)+\varepsilon^2 g(\chi)g'(\chi)|\nabla\chi|^2+\mathcal{V}(x)\chi=\mathcal{K}(x)|\chi|^{p-2}\chi, x\in\mathbb{R}^N,$$

where $N\in[3,\infty), \varepsilon\in(0,\infty), p\in(4,22^*), g\in C^1(\mathbb{R},\mathbb{R}^+), \mathcal{V},\mathcal{K}\in C\left(\mathbb{R}^N\right)\cap L^\infty\left(\mathbb{R}^N\right)$. In [6] the authors studied the equation

$$-\Delta\chi+\mathcal{V}(x)\chi-\left[\Delta\left(1+\chi^2\right)^{\alpha/2}\right]\frac{\alpha\chi}{2\left(1+\chi^2\right)^{\frac{2-\alpha}{2}}}=\widetilde{f}(x,\chi),\quad \text{in } \mathbb{R}^N,$$

and obtained that the above problem has infinitely many high energy solutions, where $1\le\alpha<2, \widetilde{f}\in C\left(\mathbb{R}^N\times\mathbb{R},\mathbb{R}\right)$. For some related papers we refer the reader to [7–15] and the references cited therein.

Because of the quasilinear term $\triangle(\Theta^2)\Theta$, we note that the quasilinear case is much more complicated than the semilinear case. Moreover, the main difficulty of (1.1) is there is no suitable space on which the energy functional is well-defined and of the C^1-class except for $N=1$ (see [4]). Also an important problem of (1.1) is the zero mass case, which appears when $\mathcal{V}$ vanishes at infinity, i.e.,

$$\mathcal{V}_\infty:=\lim_{|x|\to\infty}\mathcal{V}(x)=0.$$

In [16], when (1.1) has no critical term and a quasilinear term, the authors studied the zero mass case with

$$g(s)=s^p, 1<p<2^*-1$$

and $\mathcal{V},\mathcal{K}$ satisfy the assumption:

(VK) $\mathcal{V},\mathcal{K}\in C^1(\mathbb{R}^N,\mathbb{R})$, and there exist $\tau,\xi,a_i>0 (i=1,2,3)$ such that

$$\frac{a_1}{1+|x|^\tau}\le\mathcal{V}(x)\le a_2,\ 0<\mathcal{K}(x)\le\frac{a_3}{1+|x|^\xi},\ x\in\mathbb{R}^N,$$

where $\tau,\ \xi$ satisfy

$$\begin{cases}\frac{N+2}{N-2}-\frac{4\xi}{\tau(N-2)}<p, & 0<\xi<\tau,\\ p>1, & \xi\ge\tau.\end{cases}$$

In [17], using **(VK)** the authors established the following result: E is compactly embedded into the Lebesgue space

$$L^{p+1}_{\mathcal{K}}(\mathbb{R}^N)=\left\{\Theta:\mathbb{R}^N\to\mathbb{R}:\Theta\ \text{ is measurable and }\ \int_{\mathbb{R}^N}\mathcal{K}(x)|\Theta|^{p+1}dx<+\infty\right\},$$

where $1<p<2^*-1$ and

$$E:=\{\Theta\in D^{1,2}(\mathbb{R}^N):\int_{\mathbb{R}^N}\mathcal{V}(x)\Theta^2dx<+\infty\},$$

and the norm on E is defined as follows:

$$\|\Theta\|_E^2=\int_{\mathbb{R}^N}(|\nabla\Theta|^2+\mathcal{V}(x)\Theta^2)dx.$$

In [18] the authors also considered the condition **(VK)**, when the inequality of $\mathcal{V}$ is only imposed outside of a ball centered at origin. In [19] the authors introduced some new hypotheses for $\mathcal{K}$, using the Marcinkiewicz spaces $L^{r,\infty}(\mathbb{R}^N)(r > 1)$, which ensures that the embedding $E \hookrightarrow L^q_K(\mathbb{R}^N)$ is continuous and compact for $q > 1$. The space $L^{r,\infty}(\mathbb{R}^N)$ is formed by measurable functions $h : \mathbb{R}^N \to \mathbb{R}$ verifying

$$||h||_{r,\infty} := \sup_{D\subset\mathbb{R}^N} \frac{1}{|D|^{1-\frac{1}{r}}} \int_D |h|dx < +\infty.$$

We will consider the subspace $L_0^{r,\infty}(\mathbb{R}^N)$ of $L^{r,\infty}(\mathbb{R}^N)$, which is the closure of $L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ in $L^{r,\infty}(\mathbb{R}^N)$. In that paper, it was proved that the embedding

$$E \hookrightarrow L^p_K(\mathbb{R}^N)$$

is continuous for $p \in [2, 2^*]$ if $\mathcal{K} \in L^{r,\infty}(\mathbb{R}^N)$. If $\mathcal{K} \in L_0^{r,\infty}(\mathbb{R}^N)$, the above embedding is compact for all $p \in [2, 2^*)$.

In order to study our problem, we first give some assumptions:

We say that $(\mathcal{V}, \mathcal{K}) \in \mathbb{K}$ if the following conditions are satisfied:

($\mathbb{K}$I) $\mathcal{K} \in L^\infty(\mathbb{R}^N)$, $\mathcal{V}(x)$, $\mathcal{K}(x) > 0, x \in \mathbb{R}^N$.

($\mathbb{K}$II) There is a sequence of Borel sets $\{A_n\} \subset \mathbb{R}^N$ such that $|A_n| \le R$ for some $R > 0$, $n \in \mathbb{N}$ and we have

$$\lim_{r\to+\infty} \int_{A_n\cap B_r^c(0)} \mathcal{K}(x)dx = 0 \text{ uniformly for } n \in \mathbb{N}. \tag{K_1}$$

($\mathbb{K}$III) One of the following two conditions occurs:

$$\frac{\mathcal{K}}{\mathcal{V}} \in L^\infty(\mathbb{R}^N) \tag{K_2}$$

or there exists a $p \in (2, 22^*)$ such that

$$\frac{\mathcal{K}}{\mathcal{V}^{\frac{22^*-p}{22^*-2}}} \to 0 \text{ as } |x| \to +\infty. \tag{K_3}$$

Moreover, for the function g, we assume that:

(g_1) $\begin{cases} \lim\limits_{t\to0} \frac{g(t)}{t} = 0 & \text{if } (K_2) \text{ holds,} \\ \lim\limits_{t\to0} \frac{|g(t)|}{|t|^{p-1}} < +\infty & \text{if } (K_3) \text{ holds.} \end{cases}$

(g_2) $\lim\limits_{|t|\to+\infty} \frac{g(t)}{|t|^{22^*-1}} = 0$.

(g_3) $tg(t) - 4G(t) \ge 0, t \in \mathbb{R}$.

(g_4) $g(t)t > 0, t \ne 0$.

Now we give the main theorem:

Theorem 1.1. Let $(\mathcal{V}, \mathcal{K}) \in \mathbb{K}$ and suppose $(g_1) - (g_4)$ are true. Then (1.1) has a positive solution for large λ.

In our paper, C and C_i are utilized in various places to denote different positive constants.

2. Preliminaries

The energy functional of (1.1) is defined as

$$J_\lambda(\Theta) = \frac{1}{2}\int_{\mathbb{R}^N}(1+2\Theta^2)|\nabla\Theta|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}\mathcal{V}(x)\Theta^2dx - \frac{1}{22^*}\int_{\mathbb{R}^N}|\Theta|^{22^*}dx - \lambda\int_{\mathbb{R}^N}\mathcal{K}(x)G(\Theta)dx,$$

where $G(\Theta) := \int_0^\Theta g(s)ds$. Since $J_\lambda(\Theta)$ is not well-defined on E, we cannot adopt directly the variational method to study (1.1). Motivated by [20, 21], let $\Theta = \mathscr{S}(\mathscr{V})$, where $\mathscr{S}$ is defined by

$$\mathscr{S}'(t) = \frac{1}{\sqrt{1+2\mathscr{S}^2(t)}},\ t\in[0,+\infty)$$

and

$$\mathscr{S}(-t) = -\mathscr{S}(t),\ t\in(-\infty,0].$$

By variable transform, we obtain the modified energy functional

$$\begin{aligned} I_\lambda(\mathscr{V}) := J_\lambda(\mathscr{S}(\mathscr{V})) =& \frac{1}{2}\int_{\mathbb{R}^N}|\nabla\mathscr{V}|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}\mathcal{V}(x)\mathscr{S}^2(\mathscr{V})dx - \frac{1}{22^*}\int_{\mathbb{R}^N}|\mathscr{S}(\mathscr{V})|^{22^*}dx \\ &- \lambda\int_{\mathbb{R}^N}\mathcal{K}(x)G(\mathscr{S}(\mathscr{V}))dx. \end{aligned}$$

We easily obtain $I_\lambda \in C^1(E,\mathbb{R})$, and its Gateaux derivative is given by

$$\begin{aligned} \langle I'_\lambda(\mathscr{V}),\varphi\rangle =& \int_{\mathbb{R}^N}\nabla\mathscr{V}\cdot\nabla\varphi dx + \int_{\mathbb{R}^N}\mathcal{V}(x)\mathscr{S}(\mathscr{V})\mathscr{S}'(\mathscr{V})\varphi dx - \int_{\mathbb{R}^N}|\mathscr{S}(\mathscr{V})|^{22^*-2}\mathscr{S}(\mathscr{V})\mathscr{S}'(\mathscr{V})\varphi dx \\ &- \lambda\int_{\mathbb{R}^N}\mathcal{K}(x)g(\mathscr{S}(\mathscr{V}))\mathscr{S}'(\mathscr{V})\varphi dx \end{aligned}$$

for all $\mathscr{V},\varphi\in E$.

For completeness we provide some properties for $\mathscr{S}$.

Lemma 2.1. (see [22–24]) $\mathscr{S}(t)$ has the following properties:

($\mathscr{S}_1$) $\mathscr{S}$ is of class C^∞, and invertible;

($\mathscr{S}_2$) $0 < \mathscr{S}'(t) \le 1,\ t\in\mathbb{R}$;

($\mathscr{S}_3$) $|\mathscr{S}(t)| \le |t|,\ t\in\mathbb{R}$;

($\mathscr{S}_4$) $\lim_{t\to 0}\frac{\mathscr{S}(t)}{t} = 1$;

($\mathscr{S}_5$) $\lim_{t\to+\infty}\frac{\mathscr{S}^2(t)}{t} = \sqrt{2},\ \lim_{t\to-\infty}\frac{\mathscr{S}^2(t)}{t} = -\sqrt{2}$;

($\mathscr{S}_6$) $\frac{\mathscr{S}(t)}{2} \le t\mathscr{S}'(t) \le \mathscr{S}(t),\ \ \forall t\ge 0;\quad \mathscr{S}(t)\le t\mathscr{S}'(t)\le\frac{\mathscr{S}(t)}{2},\ \ t\le 0$;

($\mathscr{S}_7$) $\mathscr{S}^2(t) \le \sqrt{2}|t|,\ t\in\mathbb{R}$;

($\mathscr{S}_8$) $\mathscr{S}^2(t)$ is strictly convex;

($\mathscr{S}_9$) There exists $\theta > 0$ such that

$$|\mathscr{S}(t)| \ge \begin{cases} \theta|t|, & |t|\le 1,\\ \theta|t|^{\frac{1}{2}}, & |t|\ge 1; \end{cases}$$

($\mathscr{S}_{10}$) There exist $C_1, C_2 > 0$ such that

$$|t| \le C_1|\mathscr{S}(t)| + C_2|\mathscr{S}(t)|^2,\ t \in \mathbb{R};$$

($\mathscr{S}_{11}$) $|\mathscr{S}(t)\mathscr{S}'(t)| \le \frac{1}{\sqrt{2}},\ t \in \mathbb{R}$;
($\mathscr{S}_{12}$) $\mathscr{S}(t)$ is odd, $\mathscr{S}^2(t)$ is even;
($\mathscr{S}_{13}$) $\forall \xi > 0$, there exists $C(\xi) > 0$ such that

$$\mathscr{S}^2(\xi t) \le C(\xi)\mathscr{S}^2(t),\ t \in \mathbb{R};$$

($\mathscr{S}_{14}$) $\mathscr{S}(t)\mathscr{S}'(t)t^{-1}$ is strictly decreasing for $t > 0$;
($\mathscr{S}_{15}$) $\mathscr{S}^q(t)\mathscr{S}'(t)t^{-1}$ is strictly increasing for $q \ge 3, t > 0$;
($\mathscr{S}_{16}$) $\mathscr{S}^2(\lambda t) \le \lambda^2\mathscr{S}^2(t), \lambda > 1, t \in \mathbb{R}$;
($\mathscr{S}_{17}$) $\mathscr{S}^2(\frac{1}{\lambda}t) \le \frac{1}{\lambda}\mathscr{S}^2(t), \lambda \ge 1, t \in \mathbb{R}$.

From Lemma 2.1, Proposition 2.1 in [25] or Lemma 2.2 in [26] we can obtain the result:

Lemma 2.2. Let $(\mathcal{V}, \mathcal{K}) \in \mathbb{K}$ and (K_2) or (K_3) be satisfied. Then $\mathscr{V}_n \rightharpoonup \mathscr{V}$ in E implies that

$$\int_{\mathbb{R}^N} \mathcal{K}(x)|\mathscr{S}(\mathscr{V}_n)|^q dx \to \int_{\mathbb{R}^N} \mathcal{K}(x)|\mathscr{S}(\mathscr{V})|^q dx, 2 < q < 22^*.$$

From Lemmas 2.1 and 2.2, Lemma 2.2 in [25] we get the result:

Lemma 2.3. Let $(\mathcal{V}, \mathcal{K}) \in \mathbb{K}$ and $(g_1) - (g_2)$ hold. If $\mathscr{V}_n \rightharpoonup \mathscr{V}$ in E, then we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \mathcal{K}(x)G(\mathscr{S}(\mathscr{V}_n))dx = \int_{\mathbb{R}^N} \mathcal{K}(x)G(\mathscr{S}(\mathscr{V}))dx,$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \mathcal{K}(x)g(\mathscr{S}(\mathscr{V}_n))\mathscr{S}(\mathscr{V}_n)dx = \int_{\mathbb{R}^N} \mathcal{K}(x)g(\mathscr{S}(\mathscr{V}))\mathscr{S}(\mathscr{V})dx$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \mathcal{K}(x)g(\mathscr{S}(\mathscr{V}_n))\mathscr{S}(\mathscr{V})dx = \int_{\mathbb{R}^N} \mathcal{K}(x)g(\mathscr{S}(\mathscr{V}))\mathscr{S}(\mathscr{V})dx.$$

Proof. (i) If (K_2) holds, then Lemma 2.2 implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \mathcal{K}(x)|\mathscr{S}(\mathscr{V}_n)|^q dx = \int_{\mathbb{R}^N} \mathcal{K}(x)|\mathscr{S}(\mathscr{V})|^q dx,\ 2 < q < 22^*.$$

Therefore, for all $\varepsilon > 0$, there exists $r > 0$ such that $\int_{B_r^c(0)} \mathcal{K}(x)|\mathscr{S}(\mathscr{V}_n)|^q dx < \varepsilon$ for large n. Moreover, (g_1) and (g_2) imply that

$$|\mathcal{K}(x)G(s)| \le \varepsilon C[\mathcal{V}(x)s^2 + |s|^{22^*}] + C\mathcal{K}(x)|s|^q,\ s \in \mathbb{R}.$$

Hence, from ($\mathscr{S}_3$) and ($\mathscr{S}_7$) we have

$$\int_{B_r^c(0)} \mathcal{K}(x)G(\mathscr{S}(\mathscr{V}_n))dx < C\varepsilon \tag{2.1}$$

for large n. By the compactness lemma of Strauss (see [27]) we have

$$\lim_{n\to\infty} \int_{B_r(0)} \mathcal{K}(x)G(\mathscr{S}(\mathscr{V}_n))dx = \int_{B_r(0)} \mathcal{K}(x)G(\mathscr{S}(\mathscr{V}))dx. \tag{2.2}$$

Consequently, from (2.1) and (2.2) we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\mathcal{K}(x)G(\mathscr{S}(\mathscr{V}_n))dx=\int_{\mathbb{R}^N}\mathcal{K}(x)G(\mathscr{S}(\mathscr{V}))dx.$$

(ii) If (K_3) holds, then for any $\varepsilon>0$, there exists a sufficient large r such that

$$\mathcal{K}(x)\le\varepsilon[\mathcal{V}(x)|s|^{2-p}+|s|^{22^*-p}],\ s\in\mathbb{R},\ |x|\ge r.$$

Hence, we have

$$\mathcal{K}(x)|G(s)|\le\varepsilon[\mathcal{V}(x)|G(s)||s|^{2-p}+|G(s)||s|^{22^*-p}],\ s\in\mathbb{R},\ |x|\ge r. \tag{2.3}$$

Combining (2.3) with (g_1) and (g_2), there exist $0<s_0<s_1$ such that

$$\mathcal{K}(x)|G(s)|\le\varepsilon C[\mathcal{V}(x)s^2+|s|^{22^*}],\ s\in I,\ |x|\ge r,$$

where $I:=\{s\in\mathbb{R}:|s|<s_0\ \text{ or }\ |s|>s_1\}$. Consequently, we obtain

$$\mathcal{K}(x)|G(s)|\le\varepsilon C[\mathcal{V}(x)s^2+|s|^{22^*}]+C\mathcal{K}(x)\chi_{[s_0,s_1]}(|s|),\ s\in\mathbb{R},\ |x|\ge r. \tag{2.4}$$

Moreover, there exists a $M_1>0$ such that

$$\|\mathscr{V}_n\|_E\le M_1\ \text{ and }\ \int_{\mathbb{R}^N}|\mathscr{V}_n|^{2^*}dx\le M_1,\ n\in\mathbb{N}.$$

Let

$$A_n:=\{x\in\mathbb{R}^N:s_0\le|\mathscr{V}_n(x)|\le s_1\}.$$

Then $s_0^{2^*}|A_n|\le\int_{A_n}|\mathscr{V}_n|^{2^*}dx\le M_1,\ n\in\mathbb{N}$. This implies that $\sup_{n\in\mathbb{N}}|A_n|<+\infty$. Thus (K_1) implies that

$$\int_{A_n\cap B_r^c(0)}\mathcal{K}(x)dx<\varepsilon,\ n\in\mathbb{N},$$

because r is big enough. Hence, $(\mathscr{S}_3)$, $(\mathscr{S}_7)$ and (2.4) imply that

$$\int_{B_r^c(0)}\mathcal{K}(x)|G(\mathscr{S}(\mathscr{V}_n))|dx\le\varepsilon C\int_{\mathbb{R}^N}[\mathcal{V}(x)\mathscr{V}_n^2+|\mathscr{V}_n|^{2^*}]dx+C\int_{A_n\cap B_r^c(0)}\mathcal{K}(x)dx\le C\varepsilon.$$

Similarly, by the compactness lemma of Strauss we have

$$\lim_{n\to\infty}\int_{B_r(0)}\mathcal{K}(x)G(\mathscr{S}(\mathscr{V}_n))dx=\int_{B_r(0)}\mathcal{K}(x)G(\mathscr{S}(\mathscr{V}))dx.$$

Consequently,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\mathcal{K}(x)G(\mathscr{S}(\mathscr{V}_n))dx=\int_{\mathbb{R}^N}\mathcal{K}(x)G(\mathscr{S}(\mathscr{V}))dx.$$

Similarly, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\mathcal{K}(x)g(\mathscr{S}(\mathscr{V}_n))\mathscr{S}(\mathscr{V}_n)dx=\int_{\mathbb{R}^N}\mathcal{K}(x)g(\mathscr{S}(\mathscr{V}))\mathscr{S}(\mathscr{V})dx$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\mathcal{K}(x)g(\mathscr{S}(\mathscr{V}_n))\mathscr{S}(\mathscr{V})dx=\int_{\mathbb{R}^N}\mathcal{K}(x)g(\mathscr{S}(\mathscr{V}))\mathscr{S}(\mathscr{V})dx.$$

The proof is completed. □

Lemma 2.4. ([28, 29]) Let X be a real Banach space and $I\in C^1(X,\mathbb{R})$. Let Σ be a closed subset of X which disconnects (arcwise) X into distinct connected components X_1 and X_2. Suppose that $I(0)=0$ and

(I_1) $0\in X_1$, and there exists $\alpha>0$ such that $I|_\Sigma\geq\alpha>0$.
(I_2) there exists a $e\in X_2$ such that $I(e)<0$.

Then I possesses a Cerami sequence with $c\geq\alpha>0$ given by

$$c:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t)),$$

where

$$\Gamma=\{\gamma\in C([0,1],X):\gamma(0)=0\text{ and }\ I(\gamma(1))<0\}.$$

Now, we prove that I_λ has the mountain pass geometry.

Lemma 2.5. Suppose that $(V,K)\in\mathbb{K}$ and g satisfies (g_1), (g_2) and (g_4). Then I_λ satisfies the conditions in Lemma 2.4 (I_1) and (I_2).

Proof. For any $\rho>0$, let $S_\rho=\{\mathscr{V}\in E:Q(\mathscr{V})=\rho^2\}$, where $Q:E\to\mathbb{R}$ is given by

$$Q(\mathscr{V}):=\int_{\mathbb{R}^N}[|\nabla\mathscr{V}|^2+\mathcal{V}(x)\mathscr{S}^2(\mathscr{V})]dx.$$

Since $Q(\mathscr{V})$ is continuous on E, S_ρ is a closed subset of E and it disconnects E into distinct connected components E_1 and E_2.

If either (K_2) or (K_3) hold, (g_1) and (g_2) imply that for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$|g(s)|\leq\varepsilon|s|+C_\varepsilon|s|^{22^*-1}$$

for all $s\in\mathbb{R}$. Hence, by an inequality (see [30, (4.5)]) we have

$$\begin{aligned}\lambda\int_{\mathbb{R}^N}\mathcal{K}(x)G(\mathscr{S}(\mathscr{V}))dx&\leq\lambda\int_{\mathbb{R}^N}\mathcal{K}(x)[\varepsilon\mathscr{S}^2(\mathscr{V})+C_\varepsilon|\mathscr{S}(\mathscr{V})|^{22^*}]dx\\&\leq\lambda\varepsilon C_1\int_{\mathbb{R}^N}\mathscr{S}^2(\mathscr{V})dx+\lambda CC_\varepsilon\int_{\mathbb{R}^N}|\mathscr{S}(\mathscr{V})|^{22^*}dx\\&\leq\lambda\varepsilon C_1\int_{\mathbb{R}^N}[|\nabla\mathscr{S}(\mathscr{V})|^2+\mathcal{V}(x)\mathscr{S}^2(\mathscr{V})]dx+\lambda CC_\varepsilon\int_{\mathbb{R}^N}|\mathscr{S}(\mathscr{V})|^{22^*}dx\\&\leq\lambda\varepsilon C_1\int_{\mathbb{R}^N}[|\nabla\mathscr{V}|^2+\mathcal{V}(x)\mathscr{S}^2(\mathscr{V})]dx+\lambda CC_\varepsilon\int_{\mathbb{R}^N}|\mathscr{S}(\mathscr{V})|^{22^*}dx.\end{aligned}$$

Moreover, by $(\mathscr{S}_7)$ we have

$$\int_{\mathbb{R}^N}|\mathscr{S}(\mathscr{V})|^{22^*}dx\leq2^{\frac{2^*}{2}}\int_{\mathbb{R}^N}|\mathscr{V}|^{2^*}dx\leq C[\int_{\mathbb{R}^N}|\nabla\mathscr{V}|^2dx]^{\frac{2^*}{2}}\leq C\rho^{2^*},\forall\mathscr{V}\in S_\rho.$$

Consequently, for $\mathscr{V} \in S_\rho$, we obtain

$$I_\lambda(\mathscr{V}) \geq \frac{1}{2}\rho^2 - \lambda\varepsilon C_1\rho^2 - \lambda CC_\varepsilon\rho^{2^*} \geq \frac{1}{4}\rho^2 - \lambda CC_\varepsilon\rho^{2^*} := \alpha > 0$$

for $\rho > 0$ and $\varepsilon > 0$ small enough.

In what follows, we take a function $\varphi \in C_0^\infty(\mathbb{R}^N)$ with supp$\varphi = \overline{B_1}$ and $\varphi \in [0,1]$, $x \in B_1$. For any $t > 0$, since $\frac{\mathscr{S}(t)}{t}$ is decreasing about $t \geq 0$, we get $\mathscr{S}(t)\varphi(x) \leq \mathscr{S}(t\varphi(x))$, for $x \in B_1$ and $t > 0$. Hence by $(\mathscr{S}_3)$, $(\mathscr{S}_5)$ and (g_4) we have

$$\begin{aligned}
I_\lambda(t\varphi) =& \frac{1}{2}t^2\int_{\mathbb{R}^N}|\nabla\varphi|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}\mathcal{V}(x)\mathscr{S}^2(t\varphi)dx - \frac{1}{22^*}\int_{\mathbb{R}^N}|\mathscr{S}(t\varphi)|^{22^*}dx - \lambda\int_{\mathbb{R}^N}\mathcal{K}(x)G(\mathscr{S}(t\varphi))dx \\
\leq& \frac{1}{2}t^2\int_{\mathbb{R}^N}[|\nabla\varphi|^2 + \mathcal{V}(x)\varphi^2]dx - \frac{1}{22^*}\int_{B_1}|\mathscr{S}(t\varphi)|^{22^*}dx \\
\leq& \frac{1}{2}t^2\int_{\mathbb{R}^N}[|\nabla\varphi|^2 + \mathcal{V}(x)\varphi^2]dx - \frac{1}{22^*}\int_{B_1}\mathscr{S}^{22^*}(t)|\varphi|^{22^*}dx \\
=& t^2\left[\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla\varphi|^2 + \mathcal{V}(x)\varphi^2)dx - \frac{1}{22^*}\frac{\mathscr{S}^{22^*}(t)}{t^2}\int_{B_1}|\varphi|^{22^*}dx\right] \\
\to& -\infty
\end{aligned}$$

as $t \to +\infty$, i.e., $I_\lambda(t\varphi) \to -\infty$ as $t \to +\infty$. Consequently, let $e := t^*\varphi$ be such that $I_\lambda(e) < 0$(t^* large enough). The proof is completed. □

Lemma 2.6. Let $(V, K) \in \mathbb{K}$ and $(g_1) - (g_3)$ hold. Then there is a bounded Cerami sequence $\{\mathscr{V}_n\} \subset E$ with $I_\lambda(\mathscr{V}_n) \to c_\lambda \geq \alpha > 0$, where

$$c_\lambda := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_\lambda(\gamma(t)), \Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0 \ \text{ and } \ I_\lambda(\gamma(1)) < 0\},$$

and α is found in Lemma 2.5.

Proof. Step 1: We prove that the sequence $\{Q(\mathscr{V}_n)\}$ is bounded. Let $\varphi_n = \frac{\mathscr{S}(\mathscr{V}_n)}{\mathscr{S}'(\mathscr{V}_n)}$. Then $\|\varphi_n\|_E \leq 2\|\mathscr{V}_n\|_E$. Consequently, we get

$$\begin{aligned}
c_\lambda + o_n(1) =& I_\lambda(\mathscr{V}_n) - \frac{1}{4}\langle I'_\lambda(\mathscr{V}_n), \frac{\mathscr{S}(\mathscr{V}_n)}{\mathscr{S}'(\mathscr{V}_n)}\rangle \\
=& \int_{\mathbb{R}^N}[\frac{1}{2} - \frac{1}{4}(1 + \frac{2\mathscr{S}^2(\mathscr{V}_n)}{1 + 2\mathscr{S}^2(\mathscr{V}_n)})]|\nabla\mathscr{V}_n|^2dx + \frac{1}{4}\int_{\mathbb{R}^N}\mathcal{V}(x)\mathscr{S}^2(\mathscr{V}_n)dx \\
&+ (\frac{1}{4} - \frac{1}{22^*})\int_{\mathbb{R}^N}|\mathscr{S}(\mathscr{V}_n)|^{22^*}dx + \lambda\int_{\mathbb{R}^N}\mathcal{K}(x)[\frac{1}{4}g(\mathscr{S}(\mathscr{V}_n))\mathscr{S}(\mathscr{V}_n) - G(\mathscr{S}(\mathscr{V}_n))]dx \\
=& \frac{1}{4}\int_{\mathbb{R}^N}\mathscr{S}'^2(\mathscr{V}_n)|\nabla\mathscr{V}_n|^2dx + \frac{1}{4}\int_{\mathbb{R}^N}\mathcal{V}(x)\mathscr{S}^2(\mathscr{V}_n)dx \\
&+ (\frac{1}{4} - \frac{1}{22^*})\int_{\mathbb{R}^N}|\mathscr{S}(\mathscr{V}_n)|^{22^*}dx + \lambda\int_{\mathbb{R}^N}\mathcal{K}(x)[\frac{1}{4}g(\mathscr{S}(\mathscr{V}_n))\mathscr{S}(\mathscr{V}_n) - G(\mathscr{S}(\mathscr{V}_n))]dx.
\end{aligned}$$

Hence, from (g_3) we have $\{\|\mathscr{S}(\mathscr{V}_n)\|_E\}$ and $\{\|\mathscr{S}(\mathscr{V}_n)\|_{22^*}\}$ are bounded, and

$$0 \leq \lambda\int_{\mathbb{R}^N}\mathcal{K}(x)[\frac{1}{4}g(\mathscr{S}(\mathscr{V}_n))\mathscr{S}(\mathscr{V}_n) - G(\mathscr{S}(\mathscr{V}_n))]dx \leq C.$$

Note that (g_1) and (g_2) imply that

$$|G(t)| \le C(t^2 + |t|^{22^*}),\ t \in \mathbb{R}.$$

Therefore, we have

$$c_\lambda + o_n(1) = I_\lambda(\mathscr{V}_n) = \frac{1}{2}Q(\mathscr{V}_n) - \frac{1}{22^*}\|\mathscr{S}(\mathscr{V}_n)\|_{22^*}^{22^*} - \lambda \int_{\mathbb{R}^N} \mathcal{K}(x)G(\mathscr{S}(\mathscr{V}_n))dx$$

and thus $\{Q(\mathscr{V}_n)\}$ is bounded.

Step 2: We verify that there exists $C > 0$ such that $Q(\mathscr{V}_n) + [Q(\mathscr{V}_n)]^{\frac{2^*}{2}} \ge C\|\mathscr{V}_n\|_E^2$. In view of $(\mathscr{S}_9)$, the Sobolev embedding theorem implies that

$$\int_{|\mathscr{V}_n|\le 1} \mathcal{V}(x)\mathscr{V}_n^2 dx \le \frac{1}{\theta^2}\int_{|\mathscr{V}_n|\le 1} \mathcal{V}(x)\mathscr{S}^2(\mathscr{V}_n)dx \le \frac{1}{\theta^2}Q(\mathscr{V}_n)$$

and

$$\int_{|\mathscr{V}_n|>1} \mathcal{V}(x)\mathscr{V}_n^2 dx \le C\int_{|\mathscr{V}_n|>1} |\mathscr{V}_n|^{2^*} dx \le C[Q(\mathscr{V}_n)]^{\frac{2^*}{2}}.$$

Hence, we obtain

$$\int_{\mathbb{R}^N} \mathcal{V}(x)\mathscr{V}_n^2 dx \le \frac{1}{\theta^2}Q(\mathscr{V}_n) + C[Q(\mathscr{V}_n)]^{\frac{2^*}{2}}.$$

Consequently,

$$\|\mathscr{V}_n\|_E^2 \le Q(\mathscr{V}_n) + \frac{1}{\theta^2}Q(\mathscr{V}_n) + C[Q(\mathscr{V}_n)]^{\frac{2^*}{2}},$$

i.e., $C\|\mathscr{V}_n\|_E^2 \le Q(\mathscr{V}_n) + [Q(\mathscr{V}_n)]^{\frac{2^*}{2}}$.

Combining with the two steps we have $\{\|\mathscr{V}_n\|_E\}$ is bounded. The proof is completed. □

Lemma 2.7. Let $(V, K) \in \mathbb{K}$ and (g_1), (g_2), (g_4) hold. Then there exists $\lambda^* > 0$ such that $0 < c_\lambda < \frac{1}{2N}S^{\frac{N}{2}}$ for all $\lambda > \lambda^*$.

Proof. Suppose the contrary. Then there exists a sequence $\{\lambda_n\}$ with $\lambda_n \to +\infty$ such that $c_{\lambda_n} \ge \frac{1}{2N}S^{\frac{N}{2}}$. Choosing $\mathscr{V} \in E\backslash\{0\}$, then from (g_1), (g_2) and (g_4) there exists a unique $t_{\lambda_n} > 0$ such that $\max_{t\ge 0} I_{\lambda_n}(t\mathscr{V}) = I_{\lambda_n}(t_{\lambda_n}\mathscr{V})$. From $(\mathscr{S}_3)$ and $(\mathscr{S}_6)$ we have

$$\begin{aligned} t_{\lambda_n}^2\|\mathscr{V}\|_E^2 &\ge \int_{\mathbb{R}^N} |\mathscr{S}(t_{\lambda_n}\mathscr{V})|^{22^*-2}\mathscr{S}(t_{\lambda_n}\mathscr{V})\mathscr{S}'(t_{\lambda_n}\mathscr{V})t_{\lambda_n}\mathscr{V}dx + \lambda_n\int_{\mathbb{R}^N} \mathcal{K}(x)g(\mathscr{S}(t_{\lambda_n}\mathscr{V}))\mathscr{S}'(t_{\lambda_n}\mathscr{V})t_{\lambda_n}\mathscr{V}dx \\ &\ge \int_{\mathbb{R}^N} |\mathscr{S}(t_{\lambda_n}\mathscr{V})|^{22^*-2}\mathscr{S}(t_{\lambda_n}\mathscr{V})\mathscr{S}'(t_{\lambda_n}\mathscr{V})t_{\lambda_n}\mathscr{V}dx. \end{aligned}$$

From $(\mathscr{S}_5)$ and $(\mathscr{S}_6)$ we have $\{t_{\lambda_n}\}$ is bounded. Hence, there exists $t_0 \ge 0$ such that $t_{\lambda_n} \to t_0$ as $n \to \infty$. If $t_0 > 0$, then by (g_4) and Fatou's Lemma we have

$$\lim_{n\to\infty}[\int_{\mathbb{R}^N} |\mathscr{S}(t_{\lambda_n}\mathscr{V})|^{22^*-2}\mathscr{S}(t_{\lambda_n}\mathscr{V})\mathscr{S}'(t_{\lambda_n}\mathscr{V})t_{\lambda_n}\mathscr{V}dx + \lambda_n\int_{\mathbb{R}^N} \mathcal{K}(x)g(\mathscr{S}(t_{\lambda_n}\mathscr{V}))\mathscr{S}'(t_{\lambda_n}\mathscr{V})t_{\lambda_n}\mathscr{V}dx] = +\infty. \tag{2.5}$$

However we note that

$$\begin{aligned} &\int_{\mathbb{R}^N} |\mathscr{S}(t_{\lambda_n}\mathscr{V})|^{22^*-2}\mathscr{S}(t_{\lambda_n}\mathscr{V})\mathscr{S}'(t_{\lambda_n}\mathscr{V})t_{\lambda_n}\mathscr{V}dx + \lambda_n\int_{\mathbb{R}^N} \mathcal{K}(x)g(\mathscr{S}(t_{\lambda_n}\mathscr{V}))\mathscr{S}'(t_{\lambda_n}\mathscr{V})t_{\lambda_n}\mathscr{V}dx \\ &\quad \le t_{\lambda_n}^2\|\mathscr{V}\|_E^2 \to t_0^2\|\mathscr{V}\|_E^2, \end{aligned}$$

and this contradicts (2.5). Hence $t_0 = 0$. Let $\mathscr{W} = t_{\lambda_n}\mathscr{V}$. Then we have

$$I_{\lambda_n}(\mathscr{W}) = I_{\lambda_n}(t_{\lambda_n}\mathscr{V}) = \max_{t\geq 0} I_{\lambda_n}(t\mathscr{W}).$$

Consequently, by $(\mathscr{S}_3)$ and (g_4) we have

$$\max_{t\geq 0} I_{\lambda_n}(t\mathscr{W}) = I_{\lambda_n}(t_{\lambda_n}\mathscr{V}) \leq \frac{1}{2}t_{\lambda_n}^2\|\mathscr{V}\|_E^2 \to 0$$

as $n \to \infty$. Hence

$$0 < \frac{1}{2N}S^{\frac{N}{2}} \leq c_{\lambda_n} \leq \inf_{\mathscr{V}\in E\setminus\{0\}} \sup_{t\geq 0} I_{\lambda_n}(t\mathscr{V}) \leq \sup_{t\geq 0} I_{\lambda_n}(t\mathscr{W}) \to 0,$$

and this is a contradiction. Therefore, we have our conclusion in this lemma. The proof is completed. □

3. Proof of Theorem 1.1

Lemma 2.7 indicates that there is a $\lambda^* > 0$ such that $0 < c_\lambda < \frac{1}{2N}S^{\frac{N}{2}}$ for all $\lambda > \lambda^*$. For a fixed $\lambda > \lambda^*$, by Lemma 2.6 there exists a bounded Cerami sequence $\{\mathscr{V}_n\} \subset E$ with $I_\lambda(\mathscr{V}_n) \to c_\lambda \geq \alpha > 0$, where

$$c_\lambda := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I_\lambda(\gamma(t)), \Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0 \text{ and } I_\lambda(\gamma(1)) < 0\},$$

and α is found in Lemma 2.5. Hence, there is a $\mathscr{V} \in E$ such that

$$\begin{aligned}
&\mathscr{V}_n \rightharpoonup \mathscr{V} \text{ in } E,\\
&\mathscr{V}_n \to \mathscr{V} \text{ in } L^s_{loc}(\mathbb{R}^N) \text{ for } 2 \leq s < 2^*,\\
&\mathscr{V}_n(x) \to \mathscr{V}(x) \text{ a.e. on } \mathbb{R}^N.
\end{aligned} \tag{3.1}$$

By a standard argument we obtain $I'_\lambda(\mathscr{V}) = 0$, i.e., $\mathscr{V}$ is a weak solution of (1.1). Indeed, for any $\psi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}
o_n(1) =& \langle I'_\lambda(\mathscr{V}_n), \psi\rangle\\
=& \int_{\mathbb{R}^N} \nabla\mathscr{V}_n \cdot \nabla\psi dx + \int_{\mathbb{R}^N} \mathcal{V}(x)\mathscr{S}(\mathscr{V}_n)\mathscr{S}'(\mathscr{V}_n)\psi dx - \int_{\mathbb{R}^N} |\mathscr{S}(\mathscr{V}_n)|^{22^*-2}\mathscr{S}(\mathscr{V}_n)\mathscr{S}'(\mathscr{V}_n)\psi dx\\
&- \lambda\int_{\mathbb{R}^N} \mathcal{K}(x)g(\mathscr{S}(\mathscr{V}_n))\mathscr{S}'(\mathscr{V}_n)\psi dx.
\end{aligned}$$

From (3.1) we have

$$\int_{\mathbb{R}^N} \nabla\mathscr{V}_n \cdot \nabla\psi dx \to \int_{\mathbb{R}^N} \nabla\mathscr{V} \cdot \nabla\psi dx,$$

$$\int_{\mathbb{R}^N} \mathcal{V}(x)\mathscr{S}(\mathscr{V}_n)\mathscr{S}'(\mathscr{V}_n)\psi dx \to \int_{\mathbb{R}^N} \mathcal{V}(x)\mathscr{S}(\mathscr{V})\mathscr{S}'(\mathscr{V})\psi dx,$$

$$\int_{\mathbb{R}^N} |\mathscr{S}(\mathscr{V}_n)|^{22^*-2}\mathscr{S}(\mathscr{V}_n)\mathscr{S}'(\mathscr{V}_n)\psi dx \to \int_{\mathbb{R}^N} |\mathscr{S}(\mathscr{V})|^{22^*-2}\mathscr{S}(\mathscr{V})\mathscr{S}'(\mathscr{V})\psi dx$$

and

$$\int_{\mathbb{R}^N} \mathcal{K}(x)g(\mathscr{S}(\mathscr{V}_n))\mathscr{S}'(\mathscr{V}_n)\psi dx \to \int_{\mathbb{R}^N} \mathcal{K}(x)g(\mathscr{S}(\mathscr{V}))\mathscr{S}'(\mathscr{V})\psi dx.$$

Consequently, we obtain

$$\begin{aligned}0 = &\int_{\mathbb{R}^N} \nabla\mathscr{V}\cdot\nabla\psi dx + \int_{\mathbb{R}^N} \mathcal{V}(x)\mathscr{S}(\mathscr{V})\mathscr{S}'(\mathscr{V})\psi dx - \int_{\mathbb{R}^N} |\mathscr{S}(\mathscr{V})|^{22^*-2}\mathscr{S}(\mathscr{V})\mathscr{S}'(\mathscr{V})\psi dx\\ &- \lambda\int_{\mathbb{R}^N} \mathcal{K}(x)g(\mathscr{S}(\mathscr{V}))\mathscr{S}'(\mathscr{V})\psi dx, \forall\psi\in C_0^\infty(\mathbb{R}^N).\end{aligned}$$

For any $\varphi\in E$, there exists a sequence $\{\psi_n\}\subset C_0^\infty(\mathbb{R}^N)$ such that $\psi_n\to\varphi$ in E. Hence

$$\begin{aligned}0 = &\int_{\mathbb{R}^N} \nabla\mathscr{V}\cdot\nabla\psi_n dx + \int_{\mathbb{R}^N} \mathcal{V}(x)\mathscr{S}(\mathscr{V})\mathscr{S}'(\mathscr{V})\psi_n dx - \int_{\mathbb{R}^N} |\mathscr{S}(\mathscr{V})|^{22^*-2}\mathscr{S}(\mathscr{V})\mathscr{S}'(\mathscr{V})\psi_n dx\\ &- \lambda\int_{\mathbb{R}^N} \mathcal{K}(x)g(\mathscr{S}(\mathscr{V}))\mathscr{S}'(\mathscr{V})\psi_n dx.\end{aligned}$$

Let $n\to\infty$ and we get

$$\begin{aligned}0 = &\int_{\mathbb{R}^N} \nabla\mathscr{V}\cdot\nabla\varphi dx + \int_{\mathbb{R}^N} \mathcal{V}(x)\mathscr{S}(\mathscr{V})\mathscr{S}'(\mathscr{V})\varphi dx - \int_{\mathbb{R}^N} |\mathscr{S}(\mathscr{V})|^{22^*-2}\mathscr{S}(\mathscr{V})\mathscr{S}'(\mathscr{V})\varphi dx\\ &- \lambda\int_{\mathbb{R}^N} \mathcal{K}(x)g(\mathscr{S}(\mathscr{V}))\mathscr{S}'(\mathscr{V})\varphi dx,\end{aligned}$$

i.e., $\langle I_\lambda'(\mathscr{V}),\varphi\rangle = 0$ for all $\varphi\in E$. Hence $I_\lambda'(\mathscr{V}) = 0$. Now, let $\mathscr{V}^+ := \max\{\mathscr{V},0\}$ and $\mathscr{V}^- := \min\{\mathscr{V},0\}$. Then we replace I_λ with the functional

$$I_\lambda^+(\mathscr{V}) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\mathscr{V}|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \mathcal{V}(x)\mathscr{S}^2(\mathscr{V})dx - \frac{1}{22^*}\int_{\mathbb{R}^N} |\mathscr{S}(\mathscr{V}^+)|^{22^*} dx - \lambda\int_{\mathbb{R}^N} \mathcal{K}(x)G(\mathscr{S}(\mathscr{V}^+))dx.$$

Consequently, we obtain that $\mathscr{V}$ is a solution for the equation

$$-\Delta\mathscr{V} = -\mathcal{V}(x)\mathscr{S}(\mathscr{V})\mathscr{S}'(\mathscr{V}) + |\mathscr{S}(\mathscr{V}^+)|^{22^*-2}\mathscr{S}'(\mathscr{V}^+) - \lambda\mathcal{K}(x)g(\mathscr{S}(\mathscr{V}^+))\mathscr{S}'(\mathscr{V}^+),\ x\in\mathbb{R}^N.$$

Let $\mathscr{V}^-$ be a test function, and we get

$$0 \le \int_{\mathbb{R}^N} |\nabla\mathscr{V}^-|^2 dx = -\int_{\mathbb{R}^N} \mathcal{V}(x)\mathscr{S}(\mathscr{V})\mathscr{S}'(\mathscr{V})\mathscr{V}^- dx \le 0.$$

Consequently, $\int_{\mathbb{R}^N} \mathcal{V}(x)\mathscr{S}(\mathscr{V})\mathscr{S}'(\mathscr{V})\mathscr{V}^- dx = 0$, i.e., $\mathscr{V}\ge 0$. By Lemma 2.3 we find

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \mathcal{K}(x)g(\mathscr{S}(\mathscr{V}_n))\mathscr{S}(\mathscr{V}_n)dx = \int_{\mathbb{R}^N} \mathcal{K}(x)g(\mathscr{S}(\mathscr{V}))\mathscr{S}(\mathscr{V})dx$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \mathcal{K}(x)G(\mathscr{S}(\mathscr{V}_n))dx = \int_{\mathbb{R}^N} \mathcal{K}(x)G(\mathscr{S}(\mathscr{V}))dx.$$

From $\|\frac{\mathscr{S}(\mathscr{V}_n)}{\mathscr{S}'(\mathscr{V}_n)}\|_E \leq 2\|\mathscr{V}_n\|_E$ we obtain

$$\begin{aligned}c_\lambda + o_n(1) =& I_\lambda(\mathscr{V}_n)\\ \geq& \frac{1}{4}\int_{\mathbb{R}^N}[1+\frac{2\mathscr{S}^2(\mathscr{V}_n)}{1+2\mathscr{S}^2(\mathscr{V}_n)}]|\nabla\mathscr{V}_n|^2dx + \frac{1}{4}\int_{\mathbb{R}^N}\mathcal{V}(x)\mathscr{S}^2(\mathscr{V}_n)dx\\ &-\frac{1}{22^*}\int_{\mathbb{R}^N}|\mathscr{S}(\mathscr{V}_n)|^{22^*}dx - \lambda\int_{\mathbb{R}^N}\mathcal{K}(x)G(\mathscr{S}(\mathscr{V}_n))dx\end{aligned}$$

and

$$\begin{aligned}o_n(1) =& \langle I'_\lambda(\mathscr{V}_n), \frac{\mathscr{S}(\mathscr{V}_n)}{\mathscr{S}'(\mathscr{V}_n)}\rangle\\ =& \int_{\mathbb{R}^N}\Big[1+\frac{2\mathscr{S}^2(\mathscr{V}_n)}{1+2\mathscr{S}^2(\mathscr{V}_n)}\Big]|\nabla\mathscr{V}_n|^2dx + \int_{\mathbb{R}^N}\mathcal{V}(x)\mathscr{S}^2(\mathscr{V}_n)dx\\ &-\int_{\mathbb{R}^N}|\mathscr{S}(\mathscr{V}_n)|^{22^*}dx - \lambda\int_{\mathbb{R}^N}\mathcal{K}(x)g(\mathscr{S}(\mathscr{V}_n))\mathscr{S}(\mathscr{V}_n)dx.\end{aligned} \tag{3.2}$$

Let

$$\Lambda(\mathscr{V}_n) = \int_{\mathbb{R}^N}\Big[1+\frac{2\mathscr{S}^2(\mathscr{V}_n)}{1+2\mathscr{S}^2(\mathscr{V}_n)}\Big]|\nabla\mathscr{V}_n|^2dx + \int_{\mathbb{R}^N}\mathcal{V}(x)\mathscr{S}^2(\mathscr{V}_n)dx.$$

Then $c_\lambda > 0$ implies that $\Lambda(\mathscr{V}_n)$ has a positive lower bound. Otherwise, $\Lambda(\mathscr{V}_n) \to 0 (n \to \infty)$, and we have

$$\begin{aligned}c_\lambda + o_n(1) = I_\lambda(\mathscr{V}_n) =& \frac{1}{2}\int_{\mathbb{R}^N}|\nabla\mathscr{V}_n|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}\mathcal{V}(x)\mathscr{S}^2(\mathscr{V}_n)dx - \frac{1}{22^*}\int_{\mathbb{R}^N}|\mathscr{S}(\mathscr{V}_n)|^{22^*}dx\\ &-\lambda\int_{\mathbb{R}^N}\mathcal{K}(x)G(\mathscr{S}(\mathscr{V}_n))dx\\ \leq& \frac{1}{2}\int_{\mathbb{R}^N}\Big[1+\frac{2\mathscr{S}^2(\mathscr{V}_n)}{1+2\mathscr{S}^2(\mathscr{V}_n)}\Big]|\nabla\mathscr{V}_n|^2dx + \frac{1}{2}\int_{\mathbb{R}^N}\mathcal{V}(x)\mathscr{S}^2(\mathscr{V}_n)dx\\ =& \frac{1}{2}\Lambda(\mathscr{V}_n) \to 0 \text{ as } n \to \infty.\end{aligned}$$

This contradicts $c_\lambda > 0$. Therefore, we have

$$\frac{o_n(1)}{\Lambda(\mathscr{V}_n)} \to 0.$$

If $\mathscr{V} \equiv 0$, then

$$\begin{aligned}c_\lambda + o_n(1) =& I_\lambda(\mathscr{V}_n) - \frac{1}{22^*}\langle I'_\lambda(\mathscr{V}_n), \frac{\mathscr{S}(\mathscr{V}_n)}{\mathscr{S}'(\mathscr{V}_n)}\rangle\\ \geq& \left(\frac{1}{4}-\frac{1}{22^*}\right)\int_{\mathbb{R}^N}\Big[(1+\frac{2\mathscr{S}^2(\mathscr{V}_n)}{1+2\mathscr{S}^2(\mathscr{V}_n)})|\nabla\mathscr{V}_n|^2 + \mathcal{V}(x)\mathscr{S}^2(\mathscr{V}_n)\Big]dx.\end{aligned}$$

Moreover, Lemma 2.7 and (3.2) imply that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\Big[1+\frac{2\mathscr{S}^2(\mathscr{V}_n)}{1+2\mathscr{S}^2(\mathscr{V}_n)}\Big]|\nabla\mathscr{V}_n|^2dx+\int_{\mathbb{R}^N}\mathcal{V}(x)\mathscr{S}^2(\mathscr{V}_n)dx\\
\geq&\int_{\mathbb{R}^N}|\nabla\mathscr{S}^2(\mathscr{V}_n)|^2dx\\
\geq& S\left\{\int_{\mathbb{R}^N}|\mathscr{S}(\mathscr{V}_n)|^{22^*}dx\right\}^{\frac{2}{2^*}}\\
=& S\left\{\int_{\mathbb{R}^N}\Big[(1+\frac{2\mathscr{S}^2(\mathscr{V}_n)}{1+2\mathscr{S}^2(\mathscr{V}_n)})|\nabla\mathscr{V}_n|^2+\mathcal{V}(x)\mathscr{S}^2(\mathscr{V}_n)\Big]dx+o_n(1)\right\}^{\frac{2}{2^*}}\\
=& S\left\{\int_{\mathbb{R}^N}\Big[(1+\frac{2\mathscr{S}^2(\mathscr{V}_n)}{1+2\mathscr{S}^2(\mathscr{V}_n)})|\nabla\mathscr{V}_n|^2+\mathcal{V}(x)\mathscr{S}^2(\mathscr{V}_n)\Big]dx\right\}^{\frac{2}{2^*}}+o_n(1).
\end{aligned}$$

From $c_\lambda > 0$ we have

$$\left\{\int_{\mathbb{R}^N}\Big[(1+\frac{2\mathscr{S}^2(\mathscr{V}_n)}{1+2\mathscr{S}^2(\mathscr{V}_n)})|\nabla\mathscr{V}_n|^2+\mathcal{V}(x)\mathscr{S}^2(\mathscr{V}_n)\Big]dx\right\}^{1-\frac{2}{2^*}}\geq S+o_n(1),$$

i.e.,

$$\int_{\mathbb{R}^N}\Big[(1+\frac{2\mathscr{S}^2(\mathscr{V}_n)}{1+2\mathscr{S}^2(\mathscr{V}_n)})|\nabla\mathscr{V}_n|^2+\mathcal{V}(x)\mathscr{S}^2(\mathscr{V}_n)\Big]dx\geq[S+o_n(1)]^{\frac{N}{2}}=S^{\frac{N}{2}}+o_n(1).$$

Consequently,

$$c_\lambda\geq\left(\frac{1}{4}-\frac{1}{22^*}\right)\lim_{n\to\infty}\int_{\mathbb{R}^N}\Big[(1+\frac{2\mathscr{S}^2(\mathscr{V}_n)}{1+2\mathscr{S}^2(\mathscr{V}_n)})|\nabla\mathscr{V}_n|^2+\mathcal{V}(x)\mathscr{S}^2(\mathscr{V}_n)\Big]dx\geq\frac{1}{2N}S^{\frac{N}{2}},$$

and this has a contradiction. Hence, $\mathscr{V}\neq 0$, and by the maximum principle we have $\mathscr{V}>0$. The proof is completed. □

4. Conclusions

In this paper we use the dual approach and the mountain pass theorem to investigate the existence of positive solutions for the critical quasilinear Schrödinger equation (1.1) considering suitable conditions about nonlinearity g and the potential V. It is interesting to notice that our work gives some weaker conditions than those in the cited works, and generalizes the corresponding ones in the literature.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

References

1. A. D. Bouard, N. Hayashi, J. Saut, Global existence of small solutions to a relativistic nonlinear Schrödinger equation, *Commun. Math. Phys.*, **189** (1997), 73–105. https://doi.org/10.1007/s002200050191
2. X. L. Chen, R. N. Sudan, Necessary and sufficient conditions for self-focusing of short ultraintense laser pulse in underdense plasma, *Phys. Rev. Lett.*, **70** (1993), 2082–2085. https://doi.org/10.1103/PhysRevLett.70.2082
3. H. Lange, M. Poppenberg, H. Teismann, Nash-Moser methods for the solution of quasilinear Schrödinger equations, *Commun. Part. Diff. Eq.*, **24** (1999), 1399–1418. https://doi.org/10.1080/03605309908821469
4. M. Poppenberg, K. Schmitt, Z. Q. Wang, On the existence of soliton solutions to quasilinear Schrödinger equations, *Calc. Var. Partial Dif.*, **14** (2002), 329–344. https://doi.org/10.1007/s005260100105
5. J. Chen, X. Huang, B. Cheng, X. Tang, Existence and concentration behavior of ground state solutions for a class of generalized quasilinear Schrödinger equations in $\mathbb{R}^N$, *Acta Math. Sci.*, **40B** (2020), 1495–1524. https://doi.org/10.1007/s10473-020-0519-5
6. X. Zhang, L. Liu, Y. Wu, Y. Cui, Existence of infinitely solutions for a modified nonlinear Schrödinger equation via dual approach, *Electron. J. Differ. Equ.*, **147** (2018), 1–15.
7. S. Chen, X. Wu, Existence of positive solutions for a class of quasilinear Schrödinger equations of Choquard type, *J. Math. Anal. Appl.*, **475** (2019), 1754–1777. https://doi.org/10.1016/j.jmaa.2019.03.051
8. W. Zhu, C. Chen, Ground state sign-changing solutions for a class of quasilinear Schrödinger equations, *Open Math.*, **19** (2021), 1746–1754. https://doi.org/10.1515/math-2021-0134
9. K. Tu, Y. Cheng, On a class of quasilinear Schrödinger equations with the supercritical growth, *J. Math. Phys.*, **62** (2021), 121508. https://doi.org/10.1063/5.0072312
10. Y. Xue, L. Yu, J. Han, Existence of ground state solutions for generalized quasilinear Schrödinger equations with asymptotically periodic potential, *Qual. Theor. Dyn. Syst.*, **21** (2022), 67. https://doi.org/10.1007/s12346-022-00590-1
11. Y. Wei, C. Chen, H. Yang, Z. Xiu, Existence and nonexistence of entire large solutions to a class of generalized quasilinear Schrödinger equations, *Appl. Math. Lett.*, **133** (2002), 108296. https://doi.org/10.1016/j.aml.2022.108296

12. S. Zhang, Positive ground state solutions for asymptotically periodic generalized quasilinear Schrödinger equations, *AIMS Math.*, **7** (2021), 1015–1034. https://doi.org/10.3934/math.2022061
13. Q. Jin, Standing wave solutions for a generalized quasilinear Schrödinger equation with indefinite potential, *Appl. Anal.*, 2022. https://doi.org/10.1080/00036811.2022.2107907
14. W. Wang, Y. Zhang, Positive solutions for a relativistic nonlinear Schrödinger equation with critical exponent and Hardy potential, *Complex Var. Elliptic*, **67** (2022), 2924–2943. https://doi.org/10.1080/17476933.2021.1958798
15. J. Zhang, Multiple solutions for a quasilinear Schrödinger-Poisson system, *Bound. Value Probl.*, **2021** (2021), 78. https://doi.org/10.1186/s13661-021-01553-2
16. A. Ambrosetti, V. Felli, A. Malchiodi, Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity, *J. Eur. Math. Soc. (JEMS)*, **7** (2005), 117–144. https://doi.org/10.4171/JEMS/24
17. B. Opic, A. Kufner, *Hardy-type inequalities, Pitman research notes in mathematics series*, Longman Scientific & Technical, Harlow, 1990.
18. A. Ambrosetti, Z. Q. Wang, Nonlinear Schrödinger equations with vanishing and decaying potentials, *Differ. Integral Equ.*, **18** (2005), 1321–1332.
19. D. Bonheure, J. Van Schaftingen, Ground states for nonlinear Schrödinger equation with potential vanishing at infinity, *Ann. Mat. Pur. Appl.*, **189** (2010), 273–301. Available from: `https://link.springer.com/article/10.1007/s10231-009-0109-6.`
20. M. Colin, L. Jeanjean, Solutions for a quasilinear Schrödinger equation: A dual approach, *Nonlinear Anal.*, **56** (2004), 213–226. https://doi.org/10.1016/j.na.2003.09.008
21. J. Liu, Y. Wang, Z. Q. Wang, Soliton solutions for quasilinear Schrödinger equations, II, *J. Differ. Equ.*, **187** (2003), 473–493. https://doi.org/10.1016/S0022-0396(02)00064-5
22. J. M. Bezerra do Ó, O. H. Miyagaki, S. H. M. Soares, Soliton solutions for quasilinear Schrödinger equations with critical growth, *J. Differ. Equ.*, **248** (2010), 722–744. https://doi.org/10.1016/j.jde.2009.11.030
23. X. He, A. Qian, W. Zou, Existence and concentration of positive solutions for quasilinear Schrödinger equations with critical growth, *Nonlinearity*, **26** (2013), 3137–3168. https://doi.org/10.1088/0951-7715/26/12/3137
24. E. Gloss, Existence and concentration of positive solutions for a quasilinear equation in $\mathbb{R}^N$, *J. Math. Anal. Appl.*, **371** (2010), 465–484. https://doi.org/10.1016/j.jmaa.2010.05.033
25. C. O. Alves, M. A. S. Souto, Existence of solutions for a class of nonlinear Schrödinger equations with potential vanishing at infinity, *J. Differ. Equ.*, **254** (2013), 1977–1991. https://doi.org/10.1016/j.jde.2012.11.013
26. Q. Li, K. Teng, X. Wu, Existence of positive solutions for a class of critical fractional Schrödinger equations with potential vanishing at infinity, *Mediterr. J. Math.*, **14** (2017), 80. https://doi.org/10.1007/s00009-017-0846-5
27. H. Berestycki, P. L. Lions, Nonlinear scalar field equations, I existence of a ground state, *Arch. Ration. Mech. Anal.*, **82** (1983), 313–346. Available from: `https://link.springer.com/article/10.1007/BF00250555.`

28. E. A. B. Silva, G. F. Vieira, Quasilinear asymptotically periodic Schrödinger equations with subcritical growth, *Nonlinear Anal.*, **72** (2010), 2935–2949. https://doi.org/10.1016/j.na.2009.11.037
29. E. A. B. Silva, G. F. Vieira, Quasilinear asymptotically periodic Schrödinger equations with critical growth, *Calc. Var. Part. Dif.*, **39** (2010), 1–33. https://doi.org/10.1007/s00526-009-0299-1
30. X. Liu, J. Liu, Z. Q. Wang, Ground states for quasilinear Schrödinger equations with critical growth, *Calc. Var. Part. Dif.*, **46** (2013), 641–669. https://doi.org/10.1007/s00526-012-0497-0

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