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Research article

Bi-slant lightlike submanifolds of golden semi-Riemannian manifolds

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Abstract: In this paper, we introduce the notion of a bi-slant lightlike submanifold of a golden semi-Riemannian manifold. We provide two examples. We give some characterizations about the geometry of such submanifolds.

Keywords: degenerate metric; indefinite metrics; bi-slant lightlike submanifold; golden structure; golden semi-Riemannian manifold
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1. Introduction

The notion of lightlike submanifolds was introduced by K. L. Duggal and A. Bejancu in [1]. The study of lightlike submanifolds is remarkably more interesting due to non-trivial intersection of normal vector bundle and tangent bundle. Many geometers enriched the study of lightlike submanifolds (see [2, 3]). B. Y. Chen [4] defined and studied slant submanifolds of an almost Hermitian manifold as generalization of complex submanifolds and totally real submanifolds for which the angle Ψ between $\overline{\varrho}\zeta$ and the tangent space is constant for any tangent vector field ζ. A. Lotta [5] investigated the concept of slant submanifolds in contact geometry. B. Sahin [6] studied slant submanifolds of an almost product Riemannian manifold. J. L. Cabrerizo et al. [7] analysed slant submanifolds of a Sasakian manifold. M. A. Khan et al. [8] introduced slant submanifolds in LP-contact manifolds. N. Papaghuic [9] introduced the notion of semi-slant submanifolds of Kaehler manifold. P. Alegre and A. Carriazo [10] introduced and studied bi-slant submanifolds of a para Hermitian manifold. On the other hand, Crasmareanu and Hretcanu [11, 12] define golden structure $\overline{\varrho}$ as a (1,1) tensor-field satisfying $\overline{\varrho}^2 = \overline{\varrho} + I$. A Riemannian manifold $\overline{O}$ with a golden structure $\overline{\varrho}$ is called almost golden Riemannian manifold. M. A. Qayyoom and M. Ahmad (see [13–15]) studied hypersurfaces, warped product skew semi-invariant submanifolds, skew semi-invariant submanifolds of a golden Riemannian manifold. They also studied submanifolds of locally metallic Riemannian manifolds [16]. B. Sahin [17] studied slant lightlike submanifolds of indefinite Hermitian manifolds. R. S. Gupta and A. Sharfuddin [18] studied slant lightlike submanifolds of indefinite Kenmotsu manifolds. J. W. Lee and D. H. Jin [19] studied slant lightlike submanifolds of an indefinite Sasakian manifold. Semi-slant submanifolds were defined and studied by V. Khan et al. [20]. Thereafter, many geometers studied semi-slant submanifolds in various spaces (see [21–23]). Moreover, many authors studied geometry of bi-slant lightlike submanifolds [24, 25]. N. Poyraz and E. Yasar [3] studied lightlike submanifolds of golden semi-Riemannian manifolds. S. Kumar and A. Yadav [26] studied semi-slant lightlike submanifolds of golden semi-Riemannian manifolds. Johnson and Whitt studied foliations that has great importance in diffrential geometry, they considered each leaf of the foliation to be a totally geodesic submanifold of the ambient space [27].

In this paper, we introduce bi-slant lightlike submanifolds of golden semi-Riemannian manifolds. The paper is organized as follows: In Section 2, we define golden semi-Riemannian manifolds and present some basic concepts of lightlike submanifolds. In Section 3, we define and investigate some results on bi-slant lightlike submanifolds of golden semi-Riemannian manifolds and give two examples. We also discuss the integrability conditions of distributions on bi-slant lightlike submanifolds. In the last section, we obtain necessary and sufficient conditions for foliations determined by distributions on bi-slant lightlike submanifolds of golden semi-Riemannian manifolds to be geodesic.

2. Basic concepts

Let $(\overline{O}, g)$ be a semi-Riemannian manifold. A golden structure on $(\overline{O}, g)$ is a non-null tensor $\overline{\varrho}$ of type (1,1) which satisfies the equation

$$\overline{\varrho}^2 = \overline{\varrho} + I, \tag{2.1}$$

where I is the identity transformation. We say that the metric g is $\overline{\varrho}$-compatible if

$$g(\overline{\varrho}\zeta, \eta) = g(\zeta, \overline{\varrho}\eta) \tag{2.2}$$

for all ζ,η vector fields on $\overline{O}$. If we substitute $\overline{\varrho}\zeta$ into ζ in (2.2), then we have

$$g(\overline{\varrho}\zeta, \overline{\varrho}\eta) = g(\overline{\varrho}\zeta, \eta) + g(\zeta, \eta). \tag{2.3}$$

The semi-Riemannian metric is called $\overline{\varrho}$-compatible and $(\overline{O}, \overline{\varrho}, g)$ is called a golden semi-Riemannian manifold. Also, if

$$\overline{\nabla}_\zeta \overline{\varrho}\eta = \overline{\varrho}\,\overline{\nabla}_\zeta \eta \tag{2.4}$$

for all $\zeta, \eta \in \Gamma(TO)$, where $\overline{\nabla}$ is the Levi-Civita connection with respect to g, then $(\overline{O}, \overline{\varrho}, g)$ is called a locally golden semi-Riemannian manifold.

A submanifold (O^m, g) immersed in a semi-Riemannian manifold $(\overline{O}^{m+n}, \overline{g})$ is called a lightlike submanifold if the metric g induced from $\overline{g}$ is degenerate and the radical distribution $Rad(TO)$ is of rank r, where $1 \leq r \leq m$. Let $S(TO)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TO)$ in TO, i.e., $TO = Rad(TO) \perp S(TO)$.

Consider a screen transversal vector bundle $S(TO^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad(TO)$ in $TO^{\perp}$. Since, for any local basis $\{\xi_i\}$ of $Rad(TO)$, there exists a local null frame $\{\varpi_i\}$ of sections with values in the orthogonal complement of $S(TO^{\perp})$ in $[S(TO)]^{\perp}$ such that $\overline{g}(\xi_i, \varpi_j) = \delta_{ij}$, it follows that there exists a lightlike transversal vector bundle $ltr(TO)$ locally spanned by $\{\varpi_i\}$. Let $tr(TO)$ be complementary (but not orthogonal) vector bundle to TO in $T\overline{O}|_O$. Then

$$tr(TO) = ltr(TO) \perp S(TO^{\perp}),$$

$$T\overline{O}|_O = S(TO) \perp [Rad(TO) \oplus ltr(TO)] \perp S(TO^{\perp}).$$

A submanifold $(O, g, S(TO), S(TO^{\perp}))$ of $\overline{O}$ is said to be
(i) r-lightlike if $r < min\{m, n\}$;
(ii) Coisotropic if $r = n < m, S(TO^{\perp}) = \{0\}$;
(iii) Isotropic if $r = m < n, S(TO) = \{0\}$;
(iv) Totally lightlike if $r = m = n, S(TO) = \{0\} = S(TO^{\perp})$.
Let $\overline{\nabla}$, ∇ and ∇^t denote the linear connections on $\overline{O}$, O and vector bundle $tr(TO)$, respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_{\zeta}\eta = \nabla_{\zeta}\eta + h(\zeta, \eta), \quad \forall \ \zeta, \eta \in \Gamma(TO), \tag{2.5}$$

$$\overline{\nabla}_{\zeta}\eta = -A_{\eta}\zeta + \nabla^t_{\zeta}\eta, \quad \forall \ \zeta \in \Gamma(TO), \eta \in \Gamma tr(TO),$$

where $\{\nabla_{\zeta}\eta, A_{\eta}\zeta\}$ and $\{h(\zeta, \eta), \nabla^t_{\zeta}\eta\}$ belong to $\Gamma(TO)$ and $\Gamma(ltr(TO))$, respectively. ∇ and ∇^t_{ζ} are linear connections on O and on the vector bundle $ltr(TO)$, respectively. The second fundamental form h is a symmetric $F(O)$-bilinear form on $\Gamma(TO)$ with values in $\Gamma(tr(TO))$ and the shape operator A_{η} is a linear endomorphism of $\Gamma(TO)$. Then we have

$$\overline{\nabla}_{\zeta}\eta = \nabla_{\zeta}\eta + h^l(\zeta, \eta) + h^s(\zeta, \eta), \tag{2.6}$$

$$\overline{\nabla}_{\zeta}\varpi = -A_{\varpi}\zeta + \nabla^l_{\zeta}(\varpi) + D^s(\zeta, \varpi), \tag{2.7}$$

$$\overline{\nabla}_{\zeta}\omega = -A_{\omega}\zeta + \nabla^s_{\zeta}(\omega) + D^l(\zeta, \omega), \quad \forall \ \zeta, \eta \in \Gamma(TO), \varpi \in \Gamma(ltr(TO)) \tag{2.8}$$

and $\omega \in \Gamma(S(TO^{\perp}))$. Denote the projection of TO on $S(TO)$ by $\overline{P}$. Then, by using (2.1), (2.3)–(2.5) and having the fact that $\overline{\nabla}$ is a metric connection we obtain

$$\overline{g}(h^s(\zeta, \eta), \omega) + \overline{g}(\eta, D^l(\zeta, \omega)) = g(A_{\omega}\zeta, \eta),$$

$$\overline{g}(D^s(\zeta, \varpi), \omega) = \overline{g}(\varpi, A_{\omega}\zeta).$$

We set

$$\nabla_{\zeta}\overline{P}\eta = \nabla^*_{\zeta}\overline{P}\eta + h^*(\zeta, \overline{P}\eta), \tag{2.9}$$

$$\nabla_{\zeta}\xi = -A^*_{\xi}\zeta + \nabla^{*t}_{\zeta}\xi \tag{2.10}$$

for $\zeta, \eta \in \Gamma(TO)$ and $\xi \in \Gamma(RadTO)$. By using above equations, we obtain

$$\overline{g}(h^l(\zeta, \overline{P}\eta), \xi) = g(A^*_{\xi}\zeta, \overline{P}\eta),$$

$$\overline{g}(h^*(\zeta, \overline{P}\eta), \varpi) = g(A_{\varpi}\zeta, \overline{P}\eta),$$

$$\overline{g}(h^l(\zeta,\xi),\xi) = 0, A^*_{\xi}\xi = 0.$$

In general, the induced connection ∇ on O is not metric connection. Since $\overline{\nabla}$ is a metric connection, by using (2.3) we get

$$(\nabla_{\zeta} g)(\eta,\varsigma) = \overline{g}(h^l(\zeta,\eta),\varsigma) + \overline{g}(h^l(\zeta,\varsigma),\eta).$$

However, it is important to note that ∇^* is a metric connection on $S(TO)$. From now on, we briefly denote $(O, g, S(TO), S(TO^{\perp}))$ by O in this paper.

3. Bi-slant lightlike submanifolds

To define bi-slant lightlike submanifolds, we require a lemma:

Lemma 3.1. [28] *Let O be a q-lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$ of index $2q$. Suppose there exists a screen distribution $S(TO)$ such that $\overline{\varrho}Rad(TO) \subset S(TO)$ and $\overline{\varrho}ltr(TO) \subset S(TO)$. Then, $\overline{\varrho}Rad(TO) \bigcap \overline{\varrho}ltr(TO) = \{0\}$ and any complementary distribution to $\overline{\varrho}Rad(TO) \oplus \overline{\varrho}ltr(TO)$ in $S(TO)$ is Riemannian.*

Definition 3.1. Let O be a q-lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$ of index $2q$ such that $2q < dim(O)$. Then, we say that O is a bi-slant lightlike submanifold of $\overline{O}$ if the following conditions are satisfied:
(i) $\overline{\varrho}Rad(TO)$ is a distribution on O such that $Rad(TO) \bigcap \overline{\varrho}RadTO = \{0\}$;
(ii) there exists non-degenerate orthogonal distributions D, D_1 and D_2 on O such that

$$S(TO) = (\overline{\varrho}Rad(TO) \oplus \overline{\varrho}ltr(TO)) \perp D \perp D_1 \perp D_2;$$

(iii) the distribution D is an invariant distribution, i.e., $\overline{\varrho}D = D$;
(iv) the distribution D_1 is slant with angle $\Psi_1(\neq 0)$, i.e., for each $\zeta \in O$ and each non-zero vector $\zeta \in (D_1)_{\zeta}$, the angle Ψ_1 between $\overline{\varrho}\zeta$ and the vector space $(D_1)_{\zeta}$ is a non-zero constant, which is independent of the choice of $\zeta \in O$ and $\zeta \in (D_1)_{\zeta}$;
(v) the distribution D_2 is slant with angle $\Psi_2(\neq 0)$, i.e., for each $\zeta \in O$ and each non-zero vector $\zeta \in (D_2)_{\zeta}$, the angle Ψ_2 between $\overline{\varrho}\zeta$ and the vector space $(D_2)_{\zeta}$ is a non-zero constant, which is independent of the choice of $\zeta \in O$ and $\zeta \in (D_2)_{\zeta}$.

These constant angle Ψ_1 and Ψ_2 are called the slant angles of distributions D_1 and D_2 respectively. A bi-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}, D_2 \neq \{0\}$ and $\Psi_1 \neq \frac{\pi}{2}, \Psi_2 \neq \frac{\pi}{2}$.

From the above definition, we have the following decomposition

$$TO = Rad(TO) \perp (\overline{\varrho}Rad(TO) \oplus \overline{\varrho}ltr(TO)) \perp D \perp D_1 \perp D_2.$$

In particular, we have
(i) If $D = 0$, and any one of D_1 and D_2 is zero, then O is a slant lightlike submanifold;
(ii) If $D \neq 0$, and any one of D_1 and D_2 is zero, then O is a semi-slant lightlike submanifold;
(iii) If $D \neq 0$ and $\Psi_1 = \frac{\pi}{2}, \Psi_2 = \frac{\pi}{2}$, then O is CR-lightlike submanifold.
Thus, the above new class of lightlike submanifolds of a golden semi-Riemannian manifold includes slant, semi-slant and Cauchy-Riemann lightlike submanifolds as its subcases.

Example 3.1. Let $(R_2^{16}, \overline{g})$ be a semi-Euclidean space of signature $(-,-,+,+,+,+,+,+,+,+,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial\zeta_1, \partial\zeta_2, \partial\zeta_3, \partial\zeta_4, \partial\zeta_5, \partial\zeta_6, \partial\zeta_7, \partial\zeta_8, \partial\zeta_9, \partial\zeta_{10}, \partial\zeta_{11}, \partial\zeta_{12}, \partial\zeta_{13}, \partial\zeta_{14}, \partial\zeta_{15}, \partial\zeta_{16}\}$ with the golden structure $\overline{\varrho}$ defined by

$$\begin{aligned}&\overline{\varrho}(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7, \zeta_8, \zeta_9, \zeta_{10}, \zeta_{11}, \zeta_{12}, \zeta_{13}, \zeta_{14}, \zeta_{15}, \zeta_{16})\\ =&(\tau\zeta_1, \overline{\tau}\zeta_2, \tau\zeta_3, \overline{\tau}\zeta_4, \overline{\tau}\zeta_5, \overline{\tau}\zeta_6, \overline{\tau}\zeta_7, \overline{\tau}\zeta_8, \overline{\tau}\zeta_9, \overline{\tau}\zeta_{10}, \tau\zeta_{11}, \tau\zeta_{12}, \tau\zeta_{13}, \tau\zeta_{14}, \tau\zeta_{15}, \tau\zeta_{16}),\end{aligned}$$

where $\tau = \frac{1+\sqrt{5}}{2}$ and $\overline{\tau} = \frac{1-\sqrt{5}}{2}$ are the roots of $\zeta^2 - \zeta - 1 = 0$. Thus, $\overline{\varrho}^2 = \overline{\varrho} + I$ and $\overline{\varrho}$ is a golden structure on R_2^{16}.

Let O be a 9-dimensional submanifold of $(R_2^{16}, \overline{g})$ given by

$$\begin{aligned}&\zeta_1 = \eta_1 + \tau\eta_2 - \tau\eta_3, && \zeta_2 = \tau\eta_1 - \eta_2 + \eta_3, && \zeta_3 = \eta_1 + \tau\eta_2 + \tau\eta_3, && \zeta_4 = -\tau\eta_1 + \eta_2 + \eta_3,\\ &\zeta_5 = \overline{\tau}\eta_4, && \zeta_6 = \overline{\tau}\eta_5, && \zeta_7 = \tau\eta_4, && \zeta_8 = \tau\eta_5,\\ &\zeta_9 = \overline{\tau}\eta_6, && \zeta_{10} = \overline{\tau}\eta_7, && \zeta_{11} = \tau\eta_6, && \zeta_{12} = \tau\eta_7,\\ &\zeta_{13} = \overline{\tau}\eta_8, && \zeta_{14} = \overline{\tau}\eta_9, && \zeta_{15} = \tau\eta_8, && \zeta_{16} = \tau\eta_9.\end{aligned}$$

Then, TO is spanned by $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6, \varsigma_7, \varsigma_8, \varsigma_9$, where

$$\begin{aligned}&\varsigma_1 = \partial\zeta_1 + \tau\partial\zeta_2 + \partial\zeta_3 - \tau\partial\zeta_4, && \varsigma_2 = \tau\partial\zeta_1 - \partial\zeta_2 + \tau\partial\zeta_3 + \partial\zeta_4, && \varsigma_3 = -\tau\partial\zeta_1 + \partial\zeta_2 + \tau\partial\zeta_3 + \partial\zeta_4,\\ &\varsigma_4 = \overline{\tau}\partial\zeta_5 + \tau\partial\zeta_7, && \varsigma_5 = \overline{\tau}\partial\zeta_6 + \tau\partial\zeta_8, && \varsigma_6 = \overline{\tau}\partial\zeta_9 + \tau\partial\zeta_{11},\\ &\varsigma_7 = \overline{\tau}\partial\zeta_{10} + \tau\partial\zeta_{12}, && \varsigma_8 = \overline{\tau}\partial\zeta_{13} + \tau\partial\zeta_{15}, && \varsigma_9 = \overline{\tau}\partial\zeta_{14} + \tau\partial\zeta_{16}.\end{aligned}$$

This implies that $Rad(TO) = span\{\varsigma_1\}$ and $S(TO) = span\{\varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5, \varsigma_6, \varsigma_7, \varsigma_8, \varsigma_9\}$.

Now, $ltr(TO)$ is spannned by

$$\varpi = \frac{1}{2(2+\tau)}\{-\partial\zeta_1 - \tau\partial\zeta_2 + \partial\zeta_3 - \tau\partial\zeta_4\}$$

and $S(TO^\perp)$ is spanned by

$$\begin{aligned}&\omega_1 = \tau\partial\zeta_5 - \overline{\tau}\partial\zeta_7, && \omega_2 = \tau\partial\zeta_6 - \overline{\tau}\partial\zeta_8, && \omega_3 = \tau\partial\zeta_9 - \overline{\tau}\partial\zeta_{11},\\ &\omega_4 = \tau\partial\zeta_{10} - \overline{\tau}\partial\zeta_{12}, && \omega_5 = \tau\partial\zeta_{13} - \overline{\tau}\partial\zeta_{15}, && \omega_6 = \tau\partial\zeta_{14} - \overline{\tau}\partial\zeta_{16}.\end{aligned}$$

Now, $\overline{\varrho}\varsigma_1 = \varsigma_2, \overline{\varrho}\varpi = \varsigma_3$ and $\overline{\varrho}\varsigma_4 = \overline{\tau}\varsigma_4, \overline{\varrho}\varsigma_5 = \overline{\tau}\varsigma_5$, which means D is invariant, i.e., $\overline{\varrho}D = D$ and $D = span\{\varsigma_4, \varsigma_5\}$ and $D_1 = span\{\varsigma_6, \varsigma_7\}, D_2 = span\{\varsigma_8, \varsigma_9\}$ are slant distributions with slant angles $\Psi_1 = arccos(\frac{4}{\sqrt{21}})$ and $\Psi_2 = arccos(\frac{1+\sqrt{5}}{\sqrt{2(3+\sqrt{5})}})$ respectively. Hence, O is a bi-slant 1-lightlike submanifold of R_2^{16}.

Example 3.2. Let $(R_4^{16}, \overline{g})$ be a semi-Euclidean space of signature $(+,+,-,-,+,+,+,+,+,+,+,-,+,+,+,-)$ with respect to the canonical basis $\{\partial\zeta_1, \partial\zeta_2, \partial\zeta_3, \partial\zeta_4, \partial\zeta_5, \partial\zeta_6, \partial\zeta_7, \partial\zeta_8, \partial\zeta_9, \partial\zeta_{10}, \partial\zeta_{11}, \partial\zeta_{12}, \partial\zeta_{13}, \partial\zeta_{14}, \partial\zeta_{15}, \partial\zeta_{16}\}$ with the golden structure $\overline{\varrho}$ defined by

$$\overline{\varrho}(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5,\zeta_6,\zeta_7,\zeta_8,\zeta_9,\zeta_{10},\zeta_{11},\zeta_{12},\zeta_{13},\zeta_{14},\zeta_{15},\zeta_{16})$$
$$=(\overline{\tau}\zeta_1,\tau\zeta_2,\overline{\tau}\zeta_3,\tau\zeta_4,\tau\zeta_5,\tau\zeta_6,\overline{\tau}\zeta_7,\overline{\tau}\zeta_8,\overline{\tau}\zeta_9,\tau\zeta_{10},\overline{\tau}\zeta_{11},\overline{\tau}\zeta_{12},\overline{\tau}\zeta_{13},\tau\zeta_{14},\overline{\tau}\zeta_{15},\overline{\tau}\zeta_{16}),$$

where $\tau = \frac{1+\sqrt{5}}{2}$ and $\overline{\tau} = \frac{1-\sqrt{5}}{2}$ are the roots of $\zeta^2-\zeta-1 = 0$. Thus, $\overline{\varrho}^2 = \overline{\varrho}+I$ and $\overline{\varrho}$ is a golden structure on R^{16}_4.

Let O be a 9-dimensional submanifold of $(R^{16}_4,\overline{g})$ given by

$$\zeta_1 = \tau\eta_1-\eta_2-\eta_3,\quad \zeta_2 = \eta_1+\tau\eta_2+\tau\eta_3,\quad \zeta_3 = \tau\eta_1-\eta_2+\eta_3,\quad \zeta_4 = \eta_1+\tau\eta_2-\tau\eta_3,$$
$$\zeta_5 = \tau\eta_4,\quad \zeta_6 = \overline{\tau}\eta_4,\quad \zeta_7 = \tau\eta_5,\quad \zeta_8 = \overline{\tau}\eta_5,$$
$$\zeta_9 = \tau\eta_6,\quad \zeta_{10} = \overline{\tau}\eta_6,\quad \zeta_{11} = \tau\eta_7,\quad \zeta_{12} = \overline{\tau}\eta_7,$$
$$\zeta_{13} = \tau\eta_8,\quad \zeta_{14} = \overline{\tau}\eta_8,\quad \zeta_{15} = \tau\eta_9,\quad \zeta_{16} = \overline{\tau}\eta_9.$$

Then, TO is spanned by $\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4,\varsigma_5,\varsigma_6,\varsigma_7,\varsigma_8,\varsigma_9$, where

$$\varsigma_1 = \tau\partial\zeta_1+\partial\zeta_2+\tau\partial\zeta_3+\partial\zeta_4,\quad \varsigma_2 = -\partial\zeta_1+\tau\partial\zeta_2-\partial\zeta_3+\tau\partial\zeta_4,\quad \varsigma_3 = -\partial\zeta_1+\tau\partial\zeta_2+\partial\zeta_3-\tau\partial\zeta_4,$$
$$\varsigma_4 = \tau\partial\zeta_5+\overline{\tau}\partial\zeta_6,\quad \varsigma_5 = \tau\partial\zeta_7+\overline{\tau}\partial\zeta_8,\quad \varsigma_6 = \tau\partial\zeta_9+\overline{\tau}\partial\zeta_{10},$$
$$\varsigma_7 = \tau\partial\zeta_{11}+\overline{\tau}\partial\zeta_{12},\quad \varsigma_8 = \tau\partial\zeta_{13}+\overline{\tau}\partial\zeta_{14},\quad \varsigma_9 = \tau\partial\zeta_{15}+\overline{\tau}\partial\zeta_{16}.$$

This implies that $Rad(TO) = span\{\varsigma_1\}$ and $S(TO) = span\{\varsigma_2,\varsigma_3,\varsigma_4,\varsigma_5,\varsigma_6,\varsigma_7,\varsigma_8,\varsigma_9\}$.

Now, $ltr(TO)$ is spannned by

$$\varpi = \frac{1}{2(2+\tau)}\{\tau\partial\zeta_1+\partial\zeta_2-\tau\partial\zeta_3-\partial\zeta_4\},$$

and $S(TO^{\perp})$ is spanned by

$$\omega_1 = \overline{\tau}\partial\zeta_5-\tau\partial\zeta_6,\quad \omega_2 = \overline{\tau}\partial\zeta_7-\tau\partial\zeta_8,\quad \omega_3 = \overline{\tau}\partial\zeta_9-\tau\partial\zeta_{10},$$
$$\omega_4 = \overline{\tau}\partial\zeta_{11}+\tau\partial\zeta_{12},\quad \omega_5 = \overline{\tau}\partial\zeta_{13}-\tau\partial\zeta_{14},\quad \omega_6 = \overline{\tau}\partial\zeta_{15}+\tau\partial\zeta_{16}.$$

Now, $\overline{\varrho}\varsigma_1 = \varsigma_2, \overline{\varrho}\varpi = \varsigma_3$ and $\overline{\varrho}\varsigma_4 = \tau\varsigma_4, \overline{\varrho}\varsigma_5 = \overline{\tau}\varsigma_5$, which means D is invariant, i.e., $\overline{\varrho}D = D$ and $D = span\{\varsigma_4,\varsigma_5\}$ and $D_1 = span\{\varsigma_6,\varsigma_8\}, D_2 = span\{\varsigma_7,\varsigma_9\}$ are slant distributions with slant angles $\Psi_1 = arccos(-\frac{1}{\sqrt{6}})$ and $\Psi_2 = arccos(\frac{1-\sqrt{5}}{\sqrt{2(3-\sqrt{5})}})$ respectively. Hence, O is a bi-slant 1-lightlike submanifold of R^{16}_4.

Further, for any vector field ζ tangent to O, we put

$$\overline{\varrho}\zeta = f\zeta+F\zeta, \tag{3.1}$$

where $f\zeta$ and $F\zeta$ are tangential and transversal parts of $\overline{\varrho}\zeta$ respectively. We denote the projections on $Rad(TO), \overline{\varrho}Rad(TO), \overline{\varrho}ltr(TO), D, D_1$ and D_2 in TO by P_1, P_2, P_3, P_4, P_5 and P_6 respectively. Similarly, we denote the projections of $tr(TO)$ on $ltr(TO)$ and $S(TO^{\perp})$ by Q_1 and Q_2 respectively. Thus, for any $\zeta \in \Gamma(TO)$, we get

$$\zeta = P_1\zeta+P_2\zeta+P_3\zeta+P_4\zeta+P_5\zeta+P_6\zeta.$$

Now, applying $\overline{\varrho}$ to above, we get

$$\overline{\varrho}\zeta = \overline{\varrho}P_1\zeta + \overline{\varrho}P_2\zeta + \overline{\varrho}P_3\zeta + \overline{\varrho}P_4\zeta + \overline{\varrho}P_5\zeta + \overline{\varrho}P_6\zeta,$$

which gives

$$\overline{\varrho}\zeta = \overline{\varrho}P_1\zeta + \overline{\varrho}P_2\zeta + \overline{\varrho}P_3\zeta + \overline{\varrho}P_4\zeta + fP_5\zeta + FP_5\zeta + fP_6\zeta + FP_6\zeta, \tag{3.2}$$

where $\overline{\varrho}P_2\zeta = K_1\overline{\varrho}P_2\zeta + K_2\overline{\varrho}P_2\zeta, \overline{\varrho}P_3\zeta = L_1\overline{\varrho}P_3\zeta + L_2\overline{\varrho}P_3\zeta$ and $fP_5\zeta, fP_6\zeta$ (resp. $FP_5\zeta, FP_6\zeta$) denotes the tangential (resp. transversal) component of $\overline{\varrho}P_5\zeta$ and $\overline{\varrho}P_6\zeta$. Thus, we get $\overline{\varrho}P_1\zeta \in \Gamma(\overline{\varrho}Rad(TO)), K_1\overline{\varrho}P_2\zeta \in \Gamma(RadTO), K_2\overline{\varrho}P_2\zeta \in \Gamma(\overline{\varrho}Rad(TO)), L_1\overline{\varrho}P_3\zeta \in \Gamma(ltr(TO)), L_2\overline{\varrho}P_3\zeta \in \Gamma(\overline{\varrho}ltr(TO)), \overline{\varrho}P_4\zeta \in \Gamma(\overline{\varrho}D), fP_5\zeta \in \Gamma(D_1), fP_6\zeta \in \Gamma(D_2)$ and $FP_5\zeta, FP_6\zeta \in \Gamma(S(TO^{\perp}))$. Also, for any $\omega \in \Gamma(tr(TO))$, we have

$$\omega = Q_1\omega + Q_2\omega.$$

Applying $\overline{\varrho}$ to above, we obtain

$$\overline{\varrho}\omega = \overline{\varrho}Q_1\omega + \overline{\varrho}Q_2\omega,$$

which gives

$$\overline{\varrho}\omega = \overline{\varrho}Q_1\omega + BQ_2^{'}\omega + CQ_2^{'}\omega + BQ_2^{''}\omega + CQ_2^{''}\omega, \tag{3.3}$$

where $BQ_2^{'}\omega, BQ_2^{''}\omega$(resp. $CQ_2^{'}\omega, CQ_2^{''}\omega$) denote the tangential(resp. transversal) component of $\overline{\varrho}Q_2\omega$. Thus, we get $\overline{\varrho}Q_1\omega \in \Gamma(\overline{\varrho}ltr(TO))$, $BQ_2^{'}\omega \in \Gamma(D_1)$, $BQ_2^{''}\omega \in \Gamma(D_2)$ and $CQ_2^{'}\omega, CQ_2^{''}\omega \in \Gamma(S(TO^{\perp}))$.

Theorem 3.2. *Let O be a q-lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$ of index $2q$. Then, O is a bi-slant lightlike submanifold iff*
(i) there exist a distribution $\overline{\varrho}Rad(TO)$ on O such that $Rad(TO) \bigcap \overline{\varrho}Rad(TO) = \{0\}$;
(ii) there exist a screen distribution $S(TO)$ which can be splitted as

$$S(TO) = (\overline{\varrho}Rad(TO) \oplus \overline{\varrho}ltr(TO)) \perp D \perp D_1 \perp D_2$$

such that D is an invariant distribution on O, i.e., $\overline{\varrho}D = D$;
(iii) there exist a constant $\kappa_1 \in [0, 1)$ such that $f^2\zeta = \kappa_1(\overline{\varrho} + I)\zeta$, $\forall\ \zeta \in \Gamma(D_1)$;
(iv) there exist a constant $\kappa_2 \in [0, 1)$ such that $f^2\zeta = \kappa_2(\overline{\varrho} + I)\zeta$, $\forall\ \zeta \in \Gamma(D_2)$.
In that case, $\kappa_1 = cos^2\Psi_1$ and $\kappa_2 = cos^2\Psi_2$, where Ψ_1 and Ψ_2 represents the slant angles of D_1 and D_2 respectively.

Proof. Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. Then, the distribution D is invariant with respect to $\overline{\varrho}$ and $\overline{\varrho}Rad(TO)$ is a distribution on O such that $Rad(TO) \bigcap \overline{\varrho}Rad(TO) = \{0\}$.

For any $\zeta \in \Gamma(D_1)$, we have

$$|f\zeta| = |\overline{\varrho}\zeta| cos\Psi_1,$$

$$cos\Psi_1 = \frac{|f\zeta|}{|\overline{\varrho}\zeta|}. \tag{3.4}$$

$$cos^2\Psi_1 = \frac{|f\zeta|^2}{|\overline{\varrho}\zeta|^2} = \frac{g(f\zeta, f\zeta)}{g(\overline{\varrho}\zeta, \overline{\varrho}\zeta)} = \frac{g(\zeta, f^2\zeta)}{g(\zeta, \overline{\varrho}^2\zeta)},$$

$$g(\zeta, f^2\zeta) = cos^2\Psi_1 g(\zeta, \overline{\varrho}^2\zeta). \tag{3.5}$$

Since, O is a bi-slant lightlike submanifold, $cos^2\Psi_1 = \kappa_1$ (constant)$\in [0, 1)$ and therefore from (3.5), we have

$$g(\zeta, f^2\zeta) = \kappa_1 g(\zeta, \overline{\varrho}^2\zeta) = g(\zeta, \kappa_1\overline{\varrho}^2\zeta).$$

For all $\zeta \in \Gamma(D_1)$,

$$g(\zeta, (f^2 - \kappa_1\overline{\varrho}^2)\zeta) = 0. \tag{3.6}$$

Since, $(f^2 - \kappa_1\overline{\varrho}^2)\zeta \in \Gamma(D_1)$ and the induced metric $g = g|_{D_1\times D_1}$ is non-degenerate (positive definite), from (3.6), we have

$$(f^2 - \kappa_1\overline{\varrho}^2)\zeta = 0,$$

$$f^2\zeta = \kappa_1\overline{\varrho}^2\zeta = \kappa_1(\overline{\varrho} + I)\zeta, \ \forall \ \zeta \in \Gamma(D_1). \tag{3.7}$$

This proves (iii).

Suppose for any $\zeta \in \Gamma(D_2)$, we have

$$|f\zeta| = |\overline{\varrho}\zeta| cos\Psi_2,$$

$$cos\Psi_2 = \frac{|f\zeta|}{|\overline{\varrho}\zeta|}. \tag{3.8}$$

Now, by using similar steps as in proof of (iii), we obtain

$$f^2\zeta = \kappa_2(\overline{\varrho} + I)\zeta, \ \forall \ \zeta \in \Gamma(D_2), \tag{3.9}$$

which proves (iv).

Conversely, suppose that conditions (i)–(iv) are satisfied. From (iii), we have $f^2\zeta = \kappa_1\overline{\varrho}^2\zeta$ for all $\zeta \in \Gamma(D_1)$, where κ_1(constant)$\in [0, 1)$.

Now,

$$cos\Psi_1 = \frac{g(\overline{\varrho}\zeta, f\zeta)}{|\overline{\varrho}\zeta||f\zeta|} = \frac{g(\zeta, \overline{\varrho}f\zeta)}{|\overline{\varrho}\zeta||f\zeta|} = \frac{g(\zeta, f(f\zeta) + F(f\zeta))}{|\overline{\varrho}\zeta||f\zeta|}.$$

Using (3.7), we have

$$\begin{aligned} cos\Psi &= \frac{g(\zeta, f^2\zeta)}{|\overline{\varrho}\zeta||f\zeta|} = \frac{g(\zeta, \kappa_1\overline{\varrho}^2\zeta)}{|\overline{\varrho}\zeta||f\zeta|} \\ &= \frac{\kappa_1 g(\zeta, \overline{\varrho}^2\zeta)}{|\overline{\varrho}\zeta||f\zeta|} = \frac{\kappa_1 g(\zeta, \overline{\varrho}(\overline{\varrho}\zeta))}{|\overline{\varrho}\zeta||f\zeta|}, \end{aligned}$$

$$cos\Psi_1 = \frac{\kappa_1 g(\overline{\varrho}\zeta, \overline{\varrho}\zeta)}{|\overline{\varrho}\zeta||f\zeta|} = \frac{\kappa_1|\overline{\varrho}\zeta|^2}{|\overline{\varrho}\zeta||f\zeta|},$$

$$cos\Psi_1 = \kappa_1\frac{|\overline{\varrho}\zeta|}{|f\zeta|}.$$

Using (3.4), we have

$$cos\Psi_1 = \kappa_1\frac{1}{cos\Psi_1},$$

$$cos^2\Psi_1 = \kappa_1(constant).$$

Further, from (iv) we have $f^2\zeta = \kappa_2\overline{\varrho}^2\zeta$ for all $\zeta \in \Gamma(D_2)$, where κ_2(constant)$\in [0, 1)$. Now, by proceeding in the same way as above, we get $cos^2\Psi_2 = \kappa_2(constant)$. This completes the proof. Hence, O is a bi-slant lightlike submanifold. □

Theorem 3.3. *Let O be a q-lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$ of index $2q$. Then, O is a bi-slant lightlike submanifold iff*
(i) there exist a distribution $\overline{\varrho}Rad(TO)$ on O such that $Rad(TO)\bigcap\overline{\varrho}Rad(TO)=\{0\}$;

$$S(TO)=(\overline{\varrho}Rad(TO)\oplus\overline{\varrho}ltr(TO))\perp D\perp D_1\perp D_2$$

such that D is an invariant distribution on O, i.e., $\overline{\varrho}D=D$;
(ii) there exist a constant $\nu_1\in[0,1)$ such that $BF\zeta=\nu_1(\overline{\varrho}+I)\zeta$, $\forall\ \zeta\in\Gamma(D_1)$;
(iii) there exist a constant $\nu_2\in[0,1)$ such that $BF\zeta=\nu_2(\overline{\varrho}+I)\zeta$, $\forall\ \zeta\in\Gamma(D_2)$.

In that case, $\nu_1=\sin^2\Psi_1$ and $\nu_2=\sin^2\Psi_2$, where Ψ_1 and Ψ_2 represents the slant angles of D_1 and D_2, respectively.

Proof. Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. Then, the distribution D is invariant with respect to $\overline{\varrho}$ and $\overline{\varrho}Rad(TO)$ is a distribution on O such that $Rad(TO)\bigcap\overline{\varrho}Rad(TO)=\{0\}$.

Now, $\forall$ vector field $\zeta\in\Gamma(D_1)$, we have

$$\overline{\varrho}\zeta=f\zeta+F\zeta, \tag{3.10}$$

where $f\zeta$ and $F\zeta$ represents the tangential and transversal parts of $\overline{\varrho}\zeta$ respectively. Applying $\overline{\varrho}$ to (3.10), we get

$$\overline{\varrho}^2\zeta=\overline{\varrho}f\zeta+\overline{\varrho}F\zeta,$$
$$\overline{\varrho}^2\zeta=f(f\zeta)+F(f\zeta)+B(F\zeta)+C(F\zeta).$$

Now, comparing the tangential components, we get

$$\overline{\varrho}^2\zeta=f^2\zeta+BF\zeta,\ \ \forall\ \ \zeta\in\Gamma(D_2). \tag{3.11}$$

Since, O is a bi-slant lightlike submanifold, $f^2\zeta=\kappa_1\overline{\varrho}^2\zeta$, $\forall\ \ \zeta\in\Gamma(D_1)$, where κ_1(constant)$\in[0,1)$. From (3.11), we get

$$\overline{\varrho}^2\zeta=\kappa_1\overline{\varrho}^2\zeta+BF\zeta,$$
$$(1-\kappa_1)\overline{\varrho}^2\zeta=BF\zeta,$$
$$BF\zeta=\nu_1\overline{\varrho}^2\zeta=\nu_1(\overline{\varrho}+I)\zeta,\ \ \forall\ \ \zeta\in\Gamma(D_1), \tag{3.12}$$

where $1-\kappa_1=\nu_1$(constant)$\in[0,1)$.

This proves (*iii*).

Suppose for any $\zeta\in\Gamma(D_2)$, we have

$$\overline{\varrho}\zeta=f\zeta+F\zeta, \tag{3.13}$$

where $f\zeta$ and $F\zeta$ are the tangential and transversal parts of $\overline{\varrho}\zeta$ respectively.

Now, by using similar steps as in proof of (iii), we obtain

$$BF\zeta=\nu_2\overline{\varrho}^2\zeta=\nu_2(\overline{\varrho}+I)\zeta,\ \ \forall\ \ \zeta\in\Gamma(D_2), \tag{3.14}$$

where $1-\kappa_2=\nu_2$(constant)$\in[0,1)$.

This proves (iv).

Conversely, suppose that conditions (i)–(iv) are satisfied. From (3.11), we have

$$\overline{\varrho}^2\zeta = f^2\zeta + \nu_1\overline{\varrho}^2\zeta, \ \ \forall\ \zeta \in \Gamma(D_1),$$

$$f^2\zeta = (1-\nu_1)\overline{\varrho}^2\zeta,$$

$$f^2\zeta = \kappa_1\overline{\varrho}^2\zeta = \kappa_1(\overline{\varrho}+I)\zeta, \ \ \forall\ \zeta \in \Gamma(D_1), \tag{3.15}$$

where $1-\nu_1 = \kappa_1$(constant)$\in [0,1)$.

Furthermore, from (iv) we have $BF\zeta = \nu_2\overline{\varrho}^2\zeta$, for all $\zeta \in \Gamma(D_2)$, where ν_2(constant)$\in [0,1)$. Now, by proceeding in the same way as above, we get

$$f^2\zeta = \kappa_2\overline{\varrho}^2\zeta = \kappa_2(\overline{\varrho}+I)\zeta, \ \ \forall\ \zeta \in \Gamma(D_2), \tag{3.16}$$

where $1-\nu_2 = \kappa_2$(constant)$\in [0,1)$. Now, the proof follows from Theorem 3.2.

Hence, O is a bi-slant lightlike submanifold. □

Corollary 3.1. *Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. Then for any slant distribution $D^{'}$ of O with slant angle Ψ, we have*

$$g(f\zeta, f\eta) = cos^2\Psi[g(\overline{\varrho}\zeta,\eta) + g(\zeta,\eta)],$$

$$g(F\zeta, F\eta) = sin^2\Psi[g(\overline{\varrho}\zeta,\eta) + g(\zeta,\eta)],$$

$\forall\ \zeta,\eta \in \Gamma(D^{'})$.

The proof of the above corollary follows by using similar steps as in the proof of Corollary 3.1 of [17].

Theorem 3.4. *Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. Then, the integrability of $Rad(TO)$ holds iff*
(i) $\overline{g}(h^l(\zeta,\overline{\varrho}\eta),\xi) = \overline{g}(h^l(\eta,\overline{\varrho}\zeta),\xi)$;
(ii) $\overline{g}(h^*(\zeta,\overline{\varrho}\eta),\varpi) = \overline{g}(h^*(\eta,\overline{\varrho}\zeta),\varpi)$;
(iii) $\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta - \nabla^*_\eta\overline{\varrho}\zeta, \overline{J}\varsigma) = \overline{g}(\nabla^*_\zeta\overline{\varrho}\eta - \nabla^*_\eta\overline{\varrho}\zeta, \varsigma)$;
(iv) $\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta - \nabla^*_\eta\overline{\varrho}\zeta, f\varsigma_1) + \overline{g}(h^s(\zeta,\overline{\varrho}\eta) - h^s(\eta,\overline{\varrho}\zeta), F\varsigma_1) = \overline{g}(\nabla^*_\zeta\overline{\varrho}\eta - \nabla^*_\eta\overline{\varrho}\zeta, \varsigma_1)$;
(v) $\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta - \nabla^*_\eta\overline{\varrho}\zeta, f\varsigma_2) + \overline{g}(h^s(\zeta,\overline{\varrho}\eta) - h^s(\eta,\overline{\varrho}\zeta), F\varsigma_2) = \overline{g}(\nabla^*_\zeta\overline{\varrho}\eta - \nabla^*_\eta\overline{\varrho}\zeta, \varsigma_2)$
$\forall\ \zeta,\eta,\xi \in \Gamma(RadTO), \varsigma \in \Gamma(D), \varsigma_1 \in \Gamma(D_1), \varsigma_2 \in \Gamma(D_2)$ and $\varpi \in \Gamma(ltr(TO))$.

Proof. Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. $Rad(TO)$ is integrable iff

$$\overline{g}([\zeta,\eta],\overline{\varrho}\xi) = \overline{g}([\zeta,\eta],\overline{\varrho}\varpi) = \overline{g}([\zeta,\eta],\varsigma) = \overline{g}([\zeta,\eta],\varsigma_1) = \overline{g}([\zeta,\eta],\varsigma_2) = 0,$$

$\forall\ \zeta,\eta,\xi \in \Gamma(RadTO), \varsigma \in \Gamma(D), \varsigma_1 \in \Gamma(D_1), \varsigma_2 \in \Gamma(D_2)$ and $\varpi \in \Gamma(ltr(TO))$. $\overline{\nabla}$ being a metric connection and using (2.3), (2.6), (2.9) and (3.1), we obtain

$$\begin{aligned}\overline{g}([\zeta,\eta],\overline{\varrho}\xi) &= \overline{g}(\overline{\nabla}_\zeta\eta - \overline{\nabla}_\eta\zeta, \overline{\varrho}\xi) = \overline{g}(\overline{\varrho}(\overline{\nabla}_\zeta\eta - \overline{\nabla}_\eta\zeta),\xi)\\ &= \overline{g}(\overline{\varrho}\ \overline{\nabla}_\zeta\eta - \overline{\varrho}\ \overline{\nabla}_\eta\zeta),\xi) = \overline{g}(\overline{\nabla}_\zeta\ \overline{\varrho}\eta - \overline{\nabla}_\eta\ \overline{\varrho}\zeta,\xi)\\ &= \overline{g}(h^l(\zeta,\overline{\varrho}\eta) - h^l(\eta,\overline{\varrho}\zeta),\xi),\end{aligned} \tag{3.17}$$

$$\begin{aligned}\overline{g}([\zeta,\eta],\overline{\varrho}\varpi) =&\overline{g}(\overline{\nabla}_\zeta\eta-\overline{\nabla}_\eta\zeta,\overline{\varrho}\varpi)=\overline{g}(\overline{\varrho}(\overline{\nabla}_\zeta\eta-\overline{\nabla}_\eta\zeta),\varpi)\\ =&\overline{g}(\overline{\varrho}\,\overline{\nabla}_\zeta\eta-\overline{\varrho}\,\overline{\nabla}_\eta\zeta),\varpi)=\overline{g}(\overline{\nabla}_\zeta\,\overline{\varrho}\eta-\overline{\nabla}_\eta\,\overline{\varrho}\zeta,\varpi)\\ =&\overline{g}(h^*(\zeta,\overline{\varrho}\eta)-h^*(\eta,\overline{\varrho}\zeta),\varpi),\end{aligned}\tag{3.18}$$

$$\begin{aligned}\overline{g}([\zeta,\eta],\varsigma) =&\overline{g}(\overline{\varrho}[\zeta,\eta],\overline{\varrho}\varsigma)-\overline{g}(\overline{\varrho}[\zeta,\eta],\varsigma)\\ =&\overline{g}(\overline{\varrho}(\overline{\nabla}_\zeta\eta-\overline{\nabla}_\eta\zeta),\overline{\varrho}\varsigma)-\overline{g}(\overline{\varrho}(\overline{\nabla}_\zeta\eta-\overline{\nabla}_\eta\zeta),\varsigma)\\ =&\overline{g}(\overline{\nabla}_\zeta\overline{\varrho}\eta-\overline{\nabla}_\eta\overline{\varrho}\zeta,\overline{\varrho}\varsigma)-\overline{g}(\overline{\nabla}_\zeta\overline{\varrho}\eta-\overline{\nabla}_\eta\overline{\varrho}\zeta,\varsigma)\\ =&\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta-\nabla^*_\eta\overline{\varrho}\zeta,\overline{\varrho}\varsigma)-\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta-\nabla^*_\eta\overline{\varrho}\zeta,\varsigma),\end{aligned}\tag{3.19}$$

$$\begin{aligned}\overline{g}([\zeta,\eta],\varsigma_1) =&\overline{g}(\overline{\varrho}[\zeta,\eta],\overline{\varrho}\varsigma_1)-\overline{g}(\overline{\varrho}[\zeta,\eta],\varsigma_1)\\ =&\overline{g}(\overline{\nabla}_\zeta\overline{\varrho}\eta-\overline{\nabla}_\eta\overline{\varrho}\zeta,f\varsigma_1+F\varsigma_1)-\overline{g}(\overline{\nabla}_\zeta\overline{\varrho}\eta-\overline{\nabla}_\eta\overline{\varrho}\zeta,\varsigma_1)\\ =&\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta-\nabla^*_\eta\overline{\varrho}\zeta,f\varsigma_1)+\overline{g}(h^s(\zeta,\overline{\varrho}\eta)-h^s(\eta,\overline{\varrho}\zeta),F\varsigma_1)-\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta-\nabla^*_\eta\overline{\varrho}\zeta,\varsigma_1).\end{aligned}\tag{3.20}$$

$$\begin{aligned}\overline{g}([\zeta,\eta],\varsigma_2) =&\overline{g}(\overline{\varrho}[\zeta,\eta],\overline{\varrho}\varsigma_2)-\overline{g}(\overline{\varrho}[\zeta,\eta],\varsigma_2)\\ =&\overline{g}(\overline{\nabla}_\zeta\overline{\varrho}\eta-\overline{\nabla}_\eta\overline{\varrho}\zeta,f\varsigma_2+F\varsigma_2)-\overline{g}(\overline{\nabla}_\zeta\overline{\varrho}\eta-\overline{\nabla}_\eta\overline{\varrho}\zeta,\varsigma_2)\\ =&\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta-\nabla^*_\eta\overline{\varrho}\zeta,f\varsigma_2)+\overline{g}(h^s(\zeta,\overline{\varrho}\eta)-h^s(\eta,\overline{\varrho}\zeta),F\varsigma_2)-\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta-\nabla^*_\eta\overline{\varrho}\zeta,\varsigma_2).\end{aligned}\tag{3.21}$$

Now, the proof follows from (3.17)–(3.21). □

Theorem 3.5. *Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. Then, the integrability of $\overline{\varrho}Rad(TO)$ holds iff*
(i) $\overline{g}(h^l(\overline{\varrho}\zeta,\eta)-h^l(\overline{\varrho}\eta,\zeta),\overline{\varrho}\xi)=-\overline{g}(h^l(\overline{\varrho}\zeta,\eta)-h^l(\overline{\varrho}\eta,\zeta),\xi)$;
(ii) $\overline{g}(A^*_\zeta\overline{\varrho}\eta,\overline{\varrho}\varsigma)=\overline{g}(A^*_\eta\overline{\varrho}\zeta,\overline{\varrho}\varsigma)$;
(iii) $\overline{g}(A^*_\zeta\overline{\varrho}\eta-A^*_\eta\overline{\varrho}\zeta,f\varsigma_1)=\overline{g}(h^s(\overline{\varrho}\eta,\zeta)-h^s(\overline{\varrho}\zeta,\eta),F\varsigma_1)$;
(iv) $\overline{g}(A^*_\zeta\overline{\varrho}\eta-A^*_\eta\overline{\varrho}\zeta,f\varsigma_2)=\overline{g}(h^s(\overline{\varrho}\eta,\zeta)-h^s(\overline{\varrho}\zeta,\eta),F\varsigma_2)$;
(v) $\overline{g}(A_\varpi\overline{\varrho}\zeta,\overline{\varrho}\eta)=\overline{g}(A_\varpi\overline{\varrho}\eta,\overline{\varrho}\zeta)$
$\forall\ \zeta,\eta,\xi\in\Gamma(RadTO),\varsigma\in\Gamma(D),\varsigma_1\in\Gamma(D_1),\varsigma_2\in\Gamma(D_2)$ *and* $\varpi\in\Gamma(ltr(TO))$.

Proof. Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. $\overline{\varrho}Rad(TO)$ is integrable iff

$$\overline{g}([\overline{\varrho}\zeta,\overline{\varrho}\eta],\overline{\varrho}\xi)=\overline{g}([\overline{\varrho}\zeta,\overline{\varrho}\eta],\varsigma)=\overline{g}([\overline{\varrho}\zeta,\overline{\varrho}\eta],\varsigma_1)=\overline{g}([\overline{\varrho}\zeta,\overline{\varrho}\eta],\varsigma_2)=\overline{g}([\overline{\varrho}\zeta,\overline{\varrho}\eta],\varpi)=0,$$

$\forall\ \zeta,\eta,\xi\in\Gamma Rad(TO),\varsigma\in\Gamma(D),\varsigma_1\in\Gamma(D_1),\varsigma_2\in\Gamma(D_2)$ and $\varpi\in\Gamma(ltr(TO))$. $\overline{\nabla}$ being a metric connection and using (2.3), (2.6), (2.7), (2.10) and (3.1), we obtain

$$\begin{aligned}\overline{g}([\overline{\varrho}\zeta,\overline{\varrho}\eta],\overline{\varrho}\xi) =&\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\overline{\varrho}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\overline{\varrho}\zeta,\overline{\varrho}\xi)=\overline{g}(\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\eta}\zeta,\overline{\varrho}\xi)\\ =&\overline{g}(\overline{\varrho}(\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\zeta),\overline{\varrho}\xi)=\overline{g}(\overline{\varrho}(\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\zeta),\xi)+\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\zeta,\xi)\\ =&\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\zeta,\overline{\varrho}\xi)+\overline{g}(h^l(\overline{\varrho}\zeta,\eta)-h^l(\overline{\varrho}\eta,\zeta),\xi)\\ =&\overline{g}(h^l(\overline{\varrho}\zeta,\eta)-h^l(\overline{\varrho}\eta,\zeta),\overline{\varrho}\xi)+\overline{g}(h^l(\overline{\varrho}\zeta,\eta)-h^l(\overline{\varrho}\eta,\zeta),\xi),\end{aligned}\tag{3.22}$$

$$\begin{aligned}\overline{g}([\overline{\varrho}\zeta,\overline{\varrho}\eta],\varsigma) =&\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\overline{\varrho}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\overline{\varrho}\zeta,\varsigma)=\overline{g}(\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\eta}\zeta,\varsigma)\\ =&\overline{g}(\overline{\varrho}(\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\zeta),\varsigma)=\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\zeta,\overline{\varrho}\varsigma)=\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\eta,\overline{\varrho}\varsigma)-\overline{g}(\overline{\nabla}_{\overline{\varrho}\eta}\zeta,\overline{\varrho}\varsigma)\\ =&-\overline{g}(A^*_\eta\overline{\varrho}\zeta,\overline{\varrho}\varsigma)+\overline{g}(A^*_\zeta\overline{\varrho}\eta,\overline{\varrho}\varsigma)\\ =&\overline{g}(A^*_\zeta\overline{\varrho}\eta,\overline{\varrho}\varsigma)-\overline{g}(A^*_\eta\overline{\varrho}\zeta,\overline{\varrho}\varsigma),\end{aligned}\tag{3.23}$$

$$\begin{aligned}\overline{g}([\overline{\varrho}\zeta,\overline{\varrho}\eta],\varsigma_1)=&\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\overline{\varrho}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\overline{\varrho}\zeta,\varsigma_1)=\overline{g}(\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\eta}\zeta,\varsigma_1)\\=&\overline{g}(\overline{\varrho}(\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\zeta),\varsigma_1)=\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\zeta,\overline{\varrho}\varsigma_1)=\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\zeta,f\varsigma_1+F\varsigma_1)\\=&\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\eta,f\varsigma_1+F\varsigma_1)-\overline{g}(\overline{\nabla}_{\overline{\varrho}\eta}\zeta,f\varsigma_1+F\varsigma_1)\\=&-\overline{g}(A^*_\eta\overline{\varrho}\zeta,f\varsigma_1)+\overline{g}(h^s(\overline{\varrho}\zeta,\eta),F\varsigma_1)+\overline{g}(A^*_\zeta\overline{\varrho}\eta,f\varsigma_1)-\overline{g}(h^s(\overline{\varrho}\eta,\zeta),F\varsigma_1)\\=&\overline{g}(A^*_\zeta\overline{\varrho}\eta-A^*_\eta\overline{\varrho}\zeta,f\varsigma_1)-\overline{g}(h^s(\overline{\varrho}\eta,\zeta)-h^s(\overline{\varrho}\zeta,\eta),F\varsigma_1),\end{aligned}\tag{3.24}$$

$$\begin{aligned}\overline{g}([\overline{\varrho}\zeta,\overline{\varrho}\eta],\varsigma_2)=&\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\overline{\varrho}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\overline{\varrho}\zeta,\varsigma_2)=\overline{g}(\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\eta}\zeta,\varsigma_2)\\=&\overline{g}(\overline{\varrho}(\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\zeta),\varsigma_2)=\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\zeta,\overline{\varrho}\varsigma_2)=\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\zeta,f\varsigma_2+F\varsigma_2)\\=&\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\eta,f\varsigma_2+F\varsigma_2)-\overline{g}(\overline{\nabla}_{\overline{\varrho}\eta}\zeta,f\varsigma_2+F\varsigma_2)\\=&-\overline{g}(A^*_\eta\overline{\varrho}\zeta,f\varsigma_2)+\overline{g}(h^s(\overline{\varrho}\zeta,\eta),F\varsigma_2)+\overline{g}(A^*_\zeta\overline{\varrho}\eta,f\varsigma_2)-\overline{g}(h^s(\overline{\varrho}\eta,\zeta),F\varsigma_2)\\=&\overline{g}(A^*_\zeta\overline{\varrho}\eta-A^*_\eta\overline{\varrho}\zeta,f\varsigma_2)-\overline{g}(h^s(\overline{\varrho}\eta,\zeta)-h^s(\overline{\varrho}\zeta,\eta),F\varsigma_2),\end{aligned}\tag{3.25}$$

$$\begin{aligned}\overline{g}([\overline{\varrho}\zeta,\overline{\varrho}\eta],\varpi)=&\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\overline{\varrho}\eta-\overline{\nabla}_{\overline{\varrho}\eta}\overline{\varrho}\zeta,\varpi)=\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\overline{\varrho}\eta,\varpi)-\overline{g}(\overline{\nabla}_{\overline{\varrho}\eta}\overline{\varrho}\zeta,\varpi)\\=&-\overline{g}(\overline{\varrho}\eta,\overline{\nabla}_{\overline{\varrho}\zeta}\varpi)+\overline{g}(\overline{\varrho}\zeta,\overline{\nabla}_{\overline{\varrho}\eta}\varpi)=-\overline{g}(\overline{\nabla}_{\overline{\varrho}\zeta}\varpi,\overline{\varrho}\eta)+\overline{g}(\overline{\nabla}_{\overline{\varrho}\eta}\varpi,\overline{\varrho}\zeta)\\=&\overline{g}(A_\varpi\overline{\varrho}\zeta,\overline{\varrho}\eta)-\overline{g}(A_\varpi\overline{\varrho}\eta,\overline{\varrho}\zeta).\end{aligned}\tag{3.26}$$

Now, the proof follows from (3.22)–(3.26). □

Theorem 3.6. *Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. Then, the integrability of $\overline{\varrho}ltr(TO)$ holds iff*
(i) $\overline{g}(A_{\varpi_1}\overline{\varrho}\varpi_2-A_{\varpi_2}\overline{\varrho}\varpi_1,\overline{\varrho}\varpi)=-\overline{g}(A_{\varpi_1}\overline{\varrho}\varpi_2-A_{\varpi_2}\overline{\varrho}\varpi_1,\varpi)$;
(ii) $\overline{g}(A_{\varpi_1}\overline{\varrho}\varpi_2,\overline{\varrho}\varsigma)=\overline{g}(A_{\varpi_2}\overline{\varrho}\varpi_1,\overline{\varrho}\varsigma)$;
(iii) $\overline{g}(A_{\varpi_1}\overline{\varrho}\varpi_2-A_{\varpi_2}\overline{\varrho}\varpi_1,f\varsigma_1)=\overline{g}(D^s(\overline{\varrho}\varpi_2,\varpi_1)-D^s(\overline{\varrho}\varpi_1,\varpi_2),F\varsigma_1)$;
(iv)$\overline{g}(A_{\varpi_1}\overline{\varrho}\varpi_2-A_{\varpi_2}\overline{\varrho}\varpi_1,f\varsigma_2)=\overline{g}(D^s(\overline{\varrho}\varpi_2,\varpi_1)-D^s(\overline{\varrho}\varpi_1,\varpi_2),F\varsigma_2)$;
(v) $\overline{g}(A_\varpi\overline{\varrho}\varpi_1,\overline{\varrho}\varpi_2)=\overline{g}(A_\varpi\overline{\varrho}\varpi_2,\overline{\varrho}\varpi_1)$
$\forall\ \varpi_1,\varpi_2,\varpi\in\Gamma(ltr(TO)),\varsigma\in\Gamma(D),\varsigma_1\in\Gamma(D_1)$ *and* $\varsigma_2\in\Gamma(D_2)$.

Proof. Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. $\overline{\varrho}ltr(TO)$ is integrable iff

$$\begin{aligned}\overline{g}([\overline{\varrho}\varpi_1,\overline{\varrho}\varpi_2],\overline{\varrho}\varpi)=&\overline{g}([\overline{\varrho}\varpi_1,\overline{\varrho}\varpi_2],\varsigma)=\overline{g}([\overline{\varrho}\varpi_1,\overline{\varrho}\varpi_2],\varsigma_1)\\=&\overline{g}([\overline{\varrho}\varpi_1,\overline{\varrho}\varpi_2],\varsigma_2)=\overline{g}([\overline{\varrho}\varpi_1,\overline{\varrho}\varpi_2],\varpi)=0,\end{aligned}$$

$\forall\ \varpi_1,\varpi_2,\varpi\in\Gamma(ltr(TO)),\varsigma\in\Gamma(D),\varsigma_1\in\Gamma(D_1)$ and $\varsigma_2\in\Gamma(D_2)$. $\overline{\nabla}$ being a metric connection and using (2.3), (2.6), (2.7), (2.9) and (3.1), we obtain

$$\begin{aligned}\overline{g}([\overline{\varrho}\varpi_1,\overline{\varrho}\varpi_2],\overline{\varrho}\varpi)=&\overline{g}(\overline{\nabla}_{\overline{\varrho}\varpi_1}\overline{\varrho}\varpi_2-\overline{\nabla}_{\overline{\varrho}\varpi_2}\overline{\varrho}\varpi_1,\overline{\varrho}\varpi)=\overline{g}(\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\varpi_1}\varpi_2-\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\varpi_2}\varpi_1,\overline{\varrho}\varpi)\\=&\overline{g}(\overline{\varrho}(\overline{\nabla}_{\overline{\varrho}\varpi_1}\varpi_2-\overline{\nabla}_{\overline{\varrho}\varpi_2}\varpi_1),\overline{\varrho}\varpi)\\=&\overline{g}(\overline{\varrho}(\overline{\nabla}_{\overline{\varrho}\varpi_1}\varpi_2-\overline{\nabla}_{\overline{\varrho}\varpi_2}\varpi_1),\varpi)+\overline{g}(\overline{\nabla}_{\overline{\varrho}\varpi_1}\varpi_2-\overline{\nabla}_{\overline{\varrho}\varpi_2}\varpi_1,\varpi)\\=&\overline{g}(\overline{\nabla}_{\overline{\varrho}\varpi_1}\varpi_2-\overline{\nabla}_{\overline{\varrho}\varpi_2}\varpi_1,\overline{\varrho}\varpi)+\overline{g}(\overline{\nabla}_{\overline{\varrho}\varpi_1}\varpi_2-\overline{\nabla}_{\overline{\varrho}\varpi_2}\varpi_1,\varpi)\\=&\overline{g}(-A_{\varpi_2}\overline{\varrho}\varpi_1+A_{\varpi_1}\overline{\varrho}\varpi_2,\overline{\varrho}\varpi)+\overline{g}(-A_{\varpi_2}\overline{\varrho}\varpi_1+A_{\varpi_1}\overline{\varrho}\varpi_2,\varpi)\\=&\overline{g}(A_{\varpi_1}\overline{\varrho}\varpi_2-A_{\varpi_2}\overline{\varrho}\varpi_1,\overline{\varrho}\varpi)+\overline{g}(A_{\varpi_1}\overline{\varrho}\varpi_2-A_{\varpi_2}\overline{\varrho}\varpi_1,\varpi),\end{aligned}\tag{3.27}$$

$$\begin{aligned}\overline{g}([\overline{\varrho}\varpi_1,\overline{\varrho}\varpi_2],\varsigma)=&\overline{g}(\overline{\nabla}_{\overline{\varrho}\varpi_1}\overline{\varrho}\varpi_2-\overline{\nabla}_{\overline{\varrho}\varpi_2}\overline{\varrho}\varpi_1,\varsigma)=\overline{g}(\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\varpi_1}\varpi_2-\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\varpi_2}\varpi_1,\varsigma)\\=&\overline{g}(\overline{\varrho}(\overline{\nabla}_{\overline{\varrho}\varpi_1}\varpi_2-\overline{\nabla}_{\overline{\varrho}\varpi_2}\varpi_1),\varsigma)=\overline{g}(\overline{\nabla}_{\overline{\varrho}\varpi_1}\varpi_2-\overline{\nabla}_{\overline{\varrho}\varpi_2}\varpi_1,\overline{\varrho}\varsigma)\\=&\overline{g}(\overline{\nabla}_{\overline{\varrho}\varpi_1}\varpi_2,\overline{\varrho}\varsigma)-\overline{g}(\overline{\nabla}_{\overline{\varrho}\varpi_2}\varpi_1,\overline{\varrho}\varsigma)=-\overline{g}(A_{\varpi_2}\overline{\varrho}\varpi_1,\overline{\varrho}\varsigma)+\overline{g}(A_{\varpi_1}\overline{\varrho}\varpi_2,\overline{\varrho}\varsigma)\\=&\overline{g}(A_{\varpi_1}\overline{\varrho}\varpi_2,\overline{\varrho}\varsigma)-\overline{g}(A_{\varpi_2}\overline{\varrho}\varpi_1,\overline{\varrho}\varsigma),\end{aligned}\tag{3.28}$$

$$\begin{aligned}\overline{g}([\overline{\varrho}\varpi_1,\overline{\varrho}\varpi_2],\varsigma_1)=&\overline{g}(\overline{\nabla}_{\overline{\varrho}\varpi_1}\overline{\varrho}\varpi_2-\overline{\nabla}_{\overline{\varrho}\varpi_2}\overline{\varrho}\varpi_1,\varsigma_1)=\overline{g}(\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\varpi_1}\varpi_2-\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\varpi_2}\varpi_1,\varsigma_1)\\=&-\overline{g}(A_{\varpi_2}\overline{\varrho}\varpi_1,f\varsigma_1)+\overline{g}(A_{\varpi_1}\overline{\varrho}\varpi_2,f\varsigma_1)+\overline{g}(D^s(\overline{\varrho}\varpi_1,\varpi_2),F\varsigma_1)-\overline{g}(D^s(\overline{\varrho}\varpi_2,\varpi_1),F\varsigma_1)\\=&\overline{g}(A_{\varpi_1}\overline{\varrho}\varpi_2-A_{\varpi_2}\overline{\varrho}\varpi_1,f\varsigma_1)-\overline{g}(D^s(\overline{\varrho}\varpi_2,\varpi_1)-D^s(\overline{\varrho}\varpi_1,\varpi_2),F\varsigma_1),\end{aligned}\tag{3.29}$$

$$\begin{aligned}\overline{g}([\overline{\varrho}\varpi_1,\overline{\varrho}\varpi_2],\varsigma_2)=&\overline{g}(\overline{\nabla}_{\overline{\varrho}\varpi_1}\overline{\varrho}\varpi_2-\overline{\nabla}_{\overline{\varrho}\varpi_2}\overline{\varrho}\varpi_1,\varsigma_2)=\overline{g}(\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\varpi_1}\varpi_2-\overline{\varrho}\,\overline{\nabla}_{\overline{\varrho}\varpi_2}\varpi_1,\varsigma_2)\\=&-\overline{g}(A_{\varpi_2}\overline{\varrho}\varpi_1,f\varsigma_2)+\overline{g}(A_{\varpi_1}\overline{\varrho}\varpi_2,f\varsigma_2)+\overline{g}(D^s(\overline{\varrho}\varpi_1,\varpi_2),F\varsigma_2)-\overline{g}(D^s(\overline{\varrho}\varpi_2,\varpi_1),F\varsigma_2)\\=&\overline{g}(A_{\varpi_1}\overline{\varrho}\varpi_2-A_{\varpi_2}\overline{\varrho}\varpi_1,f\varsigma_2)-\overline{g}(D^s(\overline{\varrho}\varpi_2,\varpi_1)-D^s(\overline{\varrho}\varpi_1,\varpi_2),F\varsigma_2),\end{aligned}\tag{3.30}$$

$$\begin{aligned}\overline{g}([\overline{\varrho}\varpi_1,\overline{\varrho}\varpi_2],\varpi)=&\overline{g}(\overline{\nabla}_{\overline{\varrho}\varpi_1}\overline{\varrho}\varpi_2-\overline{\nabla}_{\overline{\varrho}\varpi_2}\overline{\varrho}\varpi_1,\varpi)=\overline{g}(\overline{\nabla}_{\overline{\varrho}\varpi_1}\overline{\varrho}\varpi_2,\varpi)-\overline{g}(\overline{\nabla}_{\overline{\varrho}\varpi_2}\overline{\varrho}\varpi_1,\varpi)\\=&-\overline{g}(\overline{\varrho}\varpi_2,\overline{\nabla}_{\overline{\varrho}\varpi_1}\varpi)+\overline{g}(\overline{\varrho}\varpi_1,\overline{\nabla}_{\overline{\varrho}\varpi_2}\varpi)=-\overline{g}(\overline{\nabla}_{\overline{\varrho}\varpi_1}\varpi,\overline{\varrho}\varpi_2)+\overline{g}(\overline{\nabla}_{\overline{\varrho}\varpi_2}\varpi,\overline{\varrho}\varpi_1)\\=&\overline{g}(A_{\varpi}\overline{\varrho}\varpi_1,\overline{\varrho}\varpi_2)-\overline{g}(A_{\varpi}\overline{\varrho}\varpi_2,\overline{\varrho}\varpi_1).\end{aligned}\tag{3.31}$$

The proof follows from (3.27)–(3.31). □

Theorem 3.7. *Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. Then, the integrability of D holds iff*
(i) $\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta-\nabla^*_\eta\overline{\varrho}\zeta,f\varsigma_1)+\overline{g}(h^s(\zeta,\overline{\varrho}\eta)-h^s(\eta,\overline{\varrho}\zeta),F\varsigma_1)=\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta-\nabla^*_\eta\overline{\varrho}\zeta,\varsigma_1)$;
(ii) $\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta-\nabla^*_\eta\overline{\varrho}\zeta,f\varsigma_2)+\overline{g}(h^s(\zeta,\overline{\varrho}\eta)-h^s(\eta,\overline{\varrho}\zeta),F\varsigma_2)=\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta-\nabla^*_\eta\overline{\varrho}\zeta,\varsigma_2)$;
(iii) $\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta-\nabla^*_\eta\overline{\varrho}\zeta,\overline{\varrho}\varpi)=\overline{g}(h^*(\zeta,\overline{\varrho}\eta)-h^*(\eta,\overline{\varrho}\zeta),\varpi)$;
(iii) $\overline{g}(A_\varpi\zeta,\overline{\varrho}\eta)=\overline{g}(A_\varpi\eta,\overline{\varrho}\zeta)$; $\forall\ \zeta,\eta\in\Gamma(D),\varsigma_1\in\Gamma(D_1),\varsigma_2\in\Gamma(D_2)$ *and* $\varpi\in\Gamma(ltr(TO))$.

Proof. Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. D is integrable iff

$$\overline{g}([\zeta,\eta],\varsigma_1)=\overline{g}([\zeta,\eta],\varsigma_2)=\overline{g}([\zeta,\eta],\varpi)=\overline{g}([\zeta,\eta],\overline{\varrho}\varpi)=0,$$

$\forall\ \zeta,\eta\in\Gamma(D),\varsigma_1\in\Gamma(D_1),\varsigma_2\in\Gamma(D_2)$ and $\varpi\in\Gamma(ltr(TO))$. $\overline{\nabla}$ being a metric connection and using (2.3), (2.6), (2.7), (2.9) and (3.1), we obtain

$$\begin{aligned}\overline{g}([\zeta,\eta],\varsigma_1)=&\overline{g}(\overline{\varrho}[\zeta,\eta],\overline{\varrho}\varsigma_1)-\overline{g}(\overline{\varrho}[\zeta,\eta],\varsigma_1)\\=&\overline{g}(\overline{\varrho}(\overline{\nabla}_\zeta\eta-\overline{\nabla}_\eta\zeta),f\varsigma_1+F\varsigma_1)-\overline{g}(\overline{\varrho}(\overline{\nabla}_\zeta\eta-\overline{\nabla}_\eta\zeta),\varsigma_1)\\=&\overline{g}(\overline{\nabla}_\zeta\overline{\varrho}\eta-\overline{\nabla}_\eta\overline{\varrho}\zeta,f\varsigma_1+F\varsigma_1)-\overline{g}(\overline{\nabla}_\zeta\overline{\varrho}\eta-\overline{\nabla}_\eta\overline{\varrho}\zeta,\varsigma_1)\\=&\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta-\nabla^*_\eta\overline{\varrho}\zeta,f\varsigma_1)+\overline{g}(h^s(\zeta,\overline{\varrho}\eta)-h^s(\eta,\overline{\varrho}\zeta),F\varsigma_1)-\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta-\nabla^*_\eta\overline{\varrho}\zeta,\varsigma_1),\end{aligned}\tag{3.32}$$

$$\begin{aligned}\overline{g}([\zeta,\eta],\varsigma_2)=&\overline{g}(\overline{\varrho}[\zeta,\eta],\overline{\varrho}\varsigma_2)-\overline{g}(\overline{\varrho}[\zeta,\eta],\varsigma_2)\\=&\overline{g}(\overline{\varrho}(\overline{\nabla}_\zeta\eta-\overline{\nabla}_\eta\zeta),f\varsigma_2+F\varsigma_2)-\overline{g}(\overline{\varrho}(\overline{\nabla}_\zeta\eta-\overline{\nabla}_\eta\zeta),\varsigma_2)\\=&\overline{g}(\overline{\nabla}_\zeta\overline{\varrho}\eta-\overline{\nabla}_\eta\overline{\varrho}\zeta,f\varsigma_2+F\varsigma_2)-\overline{g}(\overline{\nabla}_\zeta\overline{\varrho}\eta-\overline{\nabla}_\eta\overline{\varrho}\zeta,\varsigma_2)\\=&\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta-\nabla^*_\eta\overline{\varrho}\zeta,f\varsigma_2)+\overline{g}(h^s(\zeta,\overline{\varrho}\eta)-h^s(\eta,\overline{\varrho}\zeta),F\varsigma_2)-\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta-\nabla^*_\eta\overline{\varrho}\zeta,\varsigma_2),\end{aligned}\tag{3.33}$$

$$\begin{aligned}\overline{g}([\zeta,\eta],\varpi) =&\overline{g}(\overline{\varrho}[\zeta,\eta],\overline{\varrho}\varpi)-\overline{g}(\overline{\varrho}[\zeta,\eta],\varpi)\\ =&\overline{g}(\overline{\varrho}(\overline{\nabla}_{\zeta}\eta-\overline{\nabla}_{\eta}\zeta),\overline{\varrho}\varpi)-\overline{g}(\overline{\varrho}(\overline{\nabla}_{\zeta}\eta-\overline{\nabla}_{\eta}\zeta),\varpi)\\ =&\overline{g}(\overline{\nabla}_{\zeta}\overline{\varrho}\eta-\overline{\nabla}_{\eta}\overline{\varrho}\zeta,\overline{\varrho}\varpi)-\overline{g}(\overline{\nabla}_{\zeta}\overline{\varrho}\eta-\overline{\nabla}_{\eta}\overline{\varrho}\zeta,\varpi)\\ =&\overline{g}(\nabla^{*}_{\zeta}\overline{\varrho}\eta-\nabla^{*}_{\eta}\overline{\varrho}\zeta,\overline{\varrho}\varpi)-\overline{g}(h^{*}(\zeta,\overline{\varrho}\eta)-h^{*}(\eta,\overline{\varrho}\zeta),\varpi),\end{aligned}\tag{3.34}$$

$$\begin{aligned}\overline{g}([\zeta,\eta],\overline{\varrho}\varpi) =&\overline{g}(\overline{\nabla}_{\zeta}\eta-\overline{\nabla}_{\eta}\zeta,\overline{\varrho}\varpi)=\overline{g}(\overline{\nabla}_{\zeta}\eta,\overline{\varrho}\varpi)-\overline{g}(\overline{\nabla}_{\eta}\zeta,\overline{\varrho}\varpi))\\ =&-\overline{g}(\overline{\varrho}\eta,\overline{\nabla}_{\zeta}\varpi)+\overline{g}(\overline{\varrho}\zeta,\overline{\nabla}_{\eta}\varpi)=-\overline{g}(\overline{\nabla}_{\zeta}\varpi,\overline{\varrho}\eta)+\overline{g}(\overline{\nabla}_{\eta}\varpi,\overline{\varrho}\zeta)\\ =&\overline{g}(A_{\varpi}\zeta,\overline{\varrho}\eta)-\overline{g}(A_{\varpi}\eta,\overline{\varrho}\zeta).\end{aligned}\tag{3.35}$$

The proof follows from (3.32)–(3.35). □

Theorem 3.8. *Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. Then, any slant distribution $D^{'}$ (in particular D_1, D_2) is integrable iff*
(i) $\overline{g}(\nabla_{\zeta}f\eta-A_{F\eta}\zeta,\overline{\varrho}\varsigma)+\overline{g}(\nabla_{\eta}f\zeta-A_{F\zeta}\eta,\varsigma)=\overline{g}(\nabla_{\zeta}f\eta-A_{F\eta}\zeta,\varsigma)+\overline{g}(\nabla_{\eta}f\zeta-A_{F\zeta}\eta,\overline{\varrho}\varsigma)$;
(ii) $\overline{g}(\nabla_{\zeta}f\eta-A_{F\eta}\zeta,\overline{\varrho}\varpi)+\overline{g}(\nabla_{\eta}f\zeta-A_{F\zeta}\eta,\varpi)=\overline{g}(\nabla_{\zeta}f\eta-A_{F\eta}\zeta,\varpi)+\overline{g}(\nabla_{\eta}f\zeta-A_{F\zeta}\eta,\overline{\varrho}\varpi)$;
(iii) $\overline{g}(\nabla_{\zeta}f\eta-A_{F\eta}\zeta,\varpi)=\overline{g}(\nabla_{\eta}f\zeta-A_{F\zeta}\eta,\varpi)$;
$\forall\ \zeta,\eta\in\Gamma(D^{'})$ *(in particular* $\Gamma(D_1),\Gamma(D_2)$*),* $\varsigma\in\Gamma(D)$ *and* $\varpi\in\Gamma(ltr(TO))$*.*

Proof. Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. $D^{'}$ is integrable iff

$$\overline{g}([\zeta,\eta],\varsigma)=\overline{g}([\zeta,\eta],\varpi)=\overline{g}([\zeta,\eta],\overline{\varrho}\varpi)=0,$$

$\forall\ \zeta,\eta\in\Gamma(D^{'})$ (in particular $\Gamma(D_1),\Gamma(D_2)$), $\varsigma\in\Gamma(D)$ and $\varpi\in\Gamma(ltr(TO))$. Then, from (2.3), (2.6), (2.8) and (3.1), we obtain

$$\begin{aligned}\overline{g}([\zeta,\eta],\varsigma) =&\overline{g}(\overline{\varrho}[\zeta,\eta],\overline{\varrho}\varsigma)-\overline{g}(\overline{\varrho}[\zeta,\eta],\varsigma)\\ =&\overline{g}(\overline{\varrho}(\overline{\nabla}_{\zeta}\eta-\overline{\nabla}_{\eta}\zeta),\overline{\varrho}\varsigma)-\overline{g}(\overline{\varrho}(\overline{\nabla}_{\zeta}\eta-\overline{\nabla}_{\eta}\zeta),\varsigma)\\ =&\overline{g}(\overline{\nabla}_{\zeta}\overline{\varrho}\eta-\overline{\nabla}_{\eta}\overline{\varrho}\zeta,\overline{\varrho}\varsigma)-\overline{g}(\overline{\nabla}_{\zeta}\overline{\varrho}\eta-\overline{\nabla}_{\eta}\overline{\varrho}\zeta,\varsigma)\\ =&\overline{g}(\nabla_{\zeta}f\eta-A_{F\eta}\zeta,\overline{\varrho}\varsigma)-\overline{g}(\nabla_{\eta}f\zeta-A_{F\zeta}\eta,\overline{\varrho}\varsigma)\\ &-\overline{g}(\nabla_{\zeta}f\eta-A_{F\eta}\zeta,\varsigma)+\overline{g}(\nabla_{\eta}f\zeta-A_{F\zeta}\eta,\varsigma),\end{aligned}\tag{3.36}$$

$$\begin{aligned}\overline{g}([\zeta,\eta],\varpi) =&\overline{g}(\overline{\varrho}[\zeta,\eta],\overline{\varrho}\varpi)-\overline{g}(\overline{\varrho}[\zeta,\eta],\varpi)\\ =&\overline{g}(\overline{\varrho}(\overline{\nabla}_{\zeta}\eta-\overline{\nabla}_{\eta}\zeta),\overline{\varrho}\varpi)-\overline{g}(\overline{\varrho}(\overline{\nabla}_{\zeta}\eta-\overline{\nabla}_{\eta}\zeta),\varpi)\\ =&\overline{g}(\overline{\nabla}_{\zeta}\overline{\varrho}\eta-\overline{\nabla}_{\eta}\overline{\varrho}\zeta,\overline{\varrho}\varpi)-\overline{g}(\overline{\nabla}_{\zeta}\overline{\varrho}\eta-\overline{\nabla}_{\eta}\overline{\varrho}\zeta,\varpi)\\ =&\overline{g}(\nabla_{\zeta}f\eta-A_{F\eta}\zeta,\overline{\varrho}\varpi)-\overline{g}(\nabla_{\eta}f\zeta-A_{F\zeta}\eta,\overline{\varrho}\varpi)\\ &-\overline{g}(\nabla_{\zeta}f\eta-A_{F\eta}\zeta,\varpi)+\overline{g}(\nabla_{\eta}f\zeta-A_{F\zeta}\eta,\varpi),\end{aligned}\tag{3.37}$$

$$\begin{aligned}\overline{g}([\zeta,\eta],\overline{\varrho}\varpi) =&\overline{g}(\overline{\nabla}_{\zeta}\overline{\varrho}\eta-\overline{\nabla}_{\eta}\overline{\varrho}\zeta,\varpi)\\ =&\overline{g}(\nabla_{\zeta}f\eta-A_{F\eta}\zeta,\varpi)-\overline{g}(\nabla_{\eta}f\zeta-A_{F\zeta}\eta,\varpi).\end{aligned}\tag{3.38}$$

The proof follows from (3.36)–(3.38). □

4. Foliations determined by distributions

This section includes the necessary and sufficient conditions for foliations determined by distributions on a bi-slant lightlike submanifold of a golden semi-Riemannian manifold to be totally geodesic.

Definition 4.1. [2] A bi-slant lightlike submanifold O of a golden semi-Riemannian manifold $\overline{O}$ is said to be mixed geodesic if its second fundamental form h satisfies $h(\zeta,\eta)=0$ for all $\zeta\in\Gamma(D_1)$ and $\eta\in\Gamma(D_2)$. Thus, O is a mixed geodesic bi-slant lightlike submanifold if $h^l(\zeta,\eta)=0$ and $h^s(\zeta,\eta)=0$ for all $\zeta\in\Gamma(D_1)$ and $\eta\in\Gamma(D_2)$.

Theorem 4.1. *Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. Then, $Rad(TO)$ defines a totally geodesic foliation iff*

$$\begin{aligned}
&\overline{g}(\nabla_\zeta K_1\overline{\varrho}P_2\varsigma+\nabla_\zeta K_2\overline{\varrho}P_2\varsigma-A_{L_1\overline{\varrho}P_3\varsigma}\zeta+\nabla_\zeta L_2\overline{\varrho}P_3\varsigma+\nabla_\zeta\overline{\varrho}P_4\varsigma\\
&\quad+\nabla_\zeta fP_5\varsigma-A_{FP_5\varsigma}\zeta+\nabla_\zeta fP_6\varsigma-A_{FP_6\varsigma}\zeta,\overline{\varrho}\eta)\\
=&\overline{g}(h^l(\zeta,K_1\overline{\varrho}P_2\varsigma)+h^l(\zeta,K_2\overline{\varrho}P_2\varsigma)+\nabla^l_\zeta L_1\overline{\varrho}P_3\varsigma+h^l(\zeta,L_2\overline{\varrho}P_3\varsigma)+h^l(\zeta,\overline{\varrho}P_4\varsigma)\\
&\quad+h^l(\zeta,fP_5\varsigma)+D^l(\zeta,FP_5\varsigma)+h^l(\zeta,fP_6\varsigma)+D^l(\zeta,FP_6\varsigma),\eta)
\end{aligned}$$

$\forall\ \zeta,\eta\in\Gamma(Rad(TO))$ and $\varsigma\in\Gamma(S(TO))$.

Proof. Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. The distribution $Rad(TO)$ defines a totally geodesic foliation iff $\nabla_\zeta\eta\in\Gamma(Rad(TO))$ $\forall\ \zeta,\eta\in\Gamma(Rad(TO)$. $\overline{\nabla}$ being a metric connection and using (2.3),(2.6),(2.7),(2.8) and (3.1) $\forall\ \zeta,\eta\in\Gamma(Rad(TO)$ and $\varsigma\in\Gamma(S(TO))$, we get

$$\begin{aligned}
\overline{g}(\nabla_\zeta\eta,\varsigma)&=-\overline{g}(\eta,\overline{\nabla}_\zeta\varsigma)=-\overline{g}(\overline{\varrho}\eta,\overline{\varrho}\,\overline{\nabla}_\zeta\varsigma)+\overline{g}(\overline{\varrho}\eta,\overline{\nabla}_\zeta\varsigma)\\
&=-\overline{g}(\overline{\varrho}\eta,\overline{\varrho}\,\overline{\nabla}_\zeta\varsigma)-\overline{g}(\eta,\overline{\varrho}\,\overline{\nabla}_\zeta\varsigma)\\
&=-\overline{g}(\overline{\varrho}\eta,\overline{\nabla}_\zeta\overline{\varrho}P_2\varsigma+\overline{\nabla}_\zeta\overline{\varrho}P_3\varsigma+\overline{\nabla}_\zeta\overline{\varrho}P_4\varsigma+\overline{\nabla}_\zeta fP_5\varsigma+\overline{\nabla}_\zeta FP_5\varsigma+\overline{\nabla}_\zeta fP_6\varsigma+\overline{\nabla}_\zeta FP_6\varsigma)\\
&\quad+\overline{g}(\eta,\overline{\nabla}_\zeta\overline{\varrho}P_2\varsigma+\overline{\nabla}_\zeta\overline{\varrho}P_3\varsigma+\overline{\nabla}_\zeta\overline{\varrho}P_4\varsigma+\overline{\nabla}_\zeta fP_5\varsigma+\overline{\nabla}_\zeta FP_5\varsigma+\overline{\nabla}_\zeta fP_6\varsigma+\overline{\nabla}_\zeta FP_6\varsigma),
\end{aligned}$$

$$\begin{aligned}
\overline{g}(\nabla_\zeta\eta,\varsigma)=&-\overline{g}(\nabla_\zeta K_1\overline{\varrho}P_2\varsigma+\nabla_\zeta K_2\overline{\varrho}P_2\varsigma-A_{L_1\overline{\varrho}P_3\varsigma}\zeta+\nabla_\zeta L_2\overline{\varrho}P_3\varsigma\\
&+\nabla_\zeta\overline{\varrho}P_4\varsigma+\nabla_\zeta fP_5\varsigma-A_{FP_5\varsigma}\zeta+\nabla_\zeta fP_6\varsigma-A_{FP_6\varsigma}\zeta,\overline{\varrho}\eta)\\
&+\overline{g}(h^l(\zeta,K_1\overline{\varrho}P_2\varsigma)+h^l(\zeta,K_2\overline{\varrho}P_2\varsigma)+\nabla^l_\zeta L_1\overline{\varrho}P_3\varsigma+h^l(\zeta,L_2\overline{\varrho}P_3\varsigma)+h^l(\zeta,\overline{\varrho}P_4\varsigma)\\
&+h^l(\zeta,fP_5\varsigma)+D^l(\zeta,FP_5\varsigma)+h^l(\zeta,fP_6\varsigma)+D^l(\zeta,FP_6\varsigma),\eta).
\end{aligned}$$

Thus, the theorem is completed. □

Theorem 4.2. *Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. Then, D defines a totally geodesic foliation iff*
(i) $\overline{g}(\nabla_\zeta f\varsigma-A_{F\varsigma}\zeta,\overline{\varrho}\eta)=\overline{g}(\nabla_\zeta f\varsigma-A_{F\varsigma}\zeta,\eta)$;
(ii) $\overline{g}(\nabla^*_\zeta\overline{\varrho}\eta,\overline{\varrho}\varpi)=\overline{g}(h^*(\zeta,\overline{\varrho}\eta),\varpi)$;
(iii) $h^*(\zeta,\overline{\varrho}\eta)$ has no components in $\Gamma(Rad(TO))$
$\forall\ \zeta,\eta\in\Gamma(D)$, $\varsigma\in\Gamma(D')$(in particular $\Gamma(D_1),\Gamma(D_2)$) where D' is any slant distribution and $\varpi\in\Gamma(ltr(TO))$.

Proof. Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. The distribution D defines a totally geodesic foliation iff $\nabla_\zeta \eta \in \Gamma(D)$, $\forall\ \zeta, \eta \in \Gamma(D)$. $\overline{\nabla}$ being a metric connection and using (2.3), (2.6), (2.8) and (3.1), $\forall\ \zeta, \eta \in \Gamma(D)$ and $\varsigma \in \Gamma(D^{'})$, we obtain

$$\begin{aligned}\overline{g}(\nabla_\zeta \eta, \varsigma) &= -\,\overline{g}(\eta, \overline{\nabla}_\zeta \varsigma) = \overline{g}(\overline{\varrho}\eta, \overline{\nabla}_\zeta \varsigma) - \overline{g}(\overline{\varrho}\eta, \overline{\varrho}(\overline{\nabla}_\zeta \varsigma))\\ &= \overline{g}(\eta, \overline{\varrho}(\overline{\nabla}_\zeta \varsigma)) - \overline{g}(\overline{\varrho}\eta, \overline{\varrho}(\overline{\nabla}_\zeta \varsigma))\\ &= \overline{g}(\eta, \overline{\nabla}_\zeta f\varsigma + \overline{\nabla}_\zeta F\varsigma) - \overline{g}(\overline{\varrho}\eta, \overline{\nabla}_\zeta f\varsigma + \overline{\nabla}_\zeta F\varsigma),\end{aligned}$$

$$\overline{g}(\nabla_\zeta \eta, \varsigma) = \overline{g}(\eta, \nabla_\zeta f\varsigma - A_{F\varsigma}\zeta) - \overline{g}(\overline{\varrho}\eta, \nabla_\zeta f\varsigma - A_{F\varsigma}\zeta).$$

Now, from (2.3), (2.6) and (2.9), $\forall\ \zeta, \eta \in \Gamma(D)$ and $\varpi \in \Gamma(ltr(TO))$, we obtain

$$\begin{aligned}\overline{g}(\nabla_\zeta \eta, \varpi) &= \overline{g}(\overline{\nabla}_\zeta \eta, \varpi) = \overline{g}(\overline{\varrho}(\overline{\nabla}_\zeta \eta), \overline{\varrho}\varpi) - \overline{g}(\overline{\varrho}(\overline{\nabla}_\zeta \eta), \varpi)\\ &= \overline{g}(\overline{\nabla}_\zeta \overline{\varrho}\eta, \overline{\varrho}\varpi) - \overline{g}(\overline{\nabla}_\zeta \overline{\varrho}\eta, \varpi),\end{aligned}$$

$$\overline{g}(\nabla_\zeta \eta, \varpi) = \overline{g}(\nabla^*_\zeta \overline{\varrho}\eta, \overline{\varrho}\varpi) - \overline{g}(h^*(\zeta, \overline{\varrho}\eta), \varpi).$$

Also, from (2.2), (2.6) and (2.9), $\forall\ \zeta, \eta \in \Gamma(D)$ and $\varpi \in \Gamma(ltr(TO))$, we obtain

$$\overline{g}(\nabla_\zeta \eta, \overline{\varrho}\varpi) = \overline{g}(\overline{\nabla}_\zeta \overline{\varrho}\eta, \varpi),$$

which implies

$$\overline{g}(\nabla_\zeta \eta, \overline{\varrho}\varpi) = \overline{g}(h^*(\zeta, \overline{\varrho}\eta), \varpi),$$

which completes the proof. □

Theorem 4.3. *Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. Then, the slant distribution $D^{'}$ (in particular D_1, D_2) defines a totally geodesic foliation iff*
(i) $\overline{g}(f\eta, \nabla_\zeta \overline{\varrho}\varsigma) + \overline{g}(F\eta, h^s(\zeta, \overline{\varrho}\varsigma)) = \overline{g}(\nabla_\zeta \overline{\varrho}\varsigma, \eta)$;
(ii) $\overline{g}(\nabla_\zeta f\eta - A_{F\eta}\zeta, \overline{\varrho}\varpi) = \overline{g}(\nabla_\zeta f\eta - A_{F\eta}\zeta, \varpi)$;
(iii) $\nabla_\zeta f\eta - A_{F\eta}\zeta$ *has no components in* $\Gamma(Rad(TO))$
$\forall\ \zeta, \eta \in \Gamma(D^{'})$ *(in particular* $\Gamma(D_1), \Gamma(D_2)$*),* $\varsigma \in \Gamma(D)$ *and* $\varpi \in \Gamma(ltr(TO))$.

Proof. Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. The distribution $D^{'}$ defines a totally geodesic foliation iff $\nabla_\zeta \eta \in \Gamma(D^{'})$ $\forall\ \zeta, \eta \in \Gamma(D^{'})$. $\overline{\nabla}$ being a metric connection and using (2.3), (2.6) and (3.1), $\forall\ \zeta, \eta \in \Gamma(D^{'})$ and $\varsigma \in \Gamma(D)$, we obtain

$$\begin{aligned}\overline{g}(\nabla_\zeta \eta, \varsigma) &= -\,\overline{g}(\eta, \overline{\nabla}_\zeta \varsigma) = \overline{g}(\overline{\varrho}\eta, \overline{\nabla}_\zeta \varsigma) - \overline{g}(\overline{\varrho}\eta, \overline{\varrho}(\overline{\nabla}_\zeta \varsigma))\\ &= \overline{g}(\eta, \overline{\nabla}_\zeta \overline{\varrho}\varsigma) - \overline{g}(f\eta + F\eta, \overline{\nabla}_\zeta(\overline{\varrho}\varsigma)),\end{aligned}$$

which implies

$$\overline{g}(\nabla_\zeta \eta, \varsigma) = \overline{g}(\eta, \nabla_\zeta \overline{\varrho}\varsigma) - \overline{g}(f\eta, \nabla_\zeta \overline{\varrho}\varsigma) - \overline{g}(F\eta, h^s(\zeta, \overline{\varrho}\varsigma)).$$

Now, from (2.3), (2.6), (2.8) and (3.1), $\forall\ \zeta, \eta \in \Gamma(D^{'})$ and $\varpi \in \Gamma(ltr(TO))$, we obtain

$$\begin{aligned}\overline{g}(\nabla_\zeta \eta, \varpi) &= \overline{g}(\overline{\nabla}_\zeta \eta, \varpi) = \overline{g}(\overline{\varrho}(\overline{\nabla}_\zeta \eta), \overline{\varrho}\varpi) - \overline{g}(\overline{\varrho}(\overline{\nabla}_\zeta \eta), \varpi)\\ &= \overline{g}(\overline{\nabla}_\zeta \overline{\varrho}\eta, \overline{\varrho}\varpi) - \overline{g}(\overline{\nabla}_\zeta \overline{\varrho}\eta, \varpi),\end{aligned}$$

which implies

$$\overline{g}(\nabla_\zeta \eta, \varpi) = \overline{g}(\nabla_\zeta f\eta - A_{F\eta}\zeta, \overline{\varrho}\varpi) - \overline{g}(\nabla_\zeta f\eta - A_{F\eta}\zeta, \varpi).$$

Now, from (2.2), (2.6), (2.8) and (3.1), $\forall\ \zeta, \eta \in \Gamma(D')$ and $\varpi \in \Gamma(ltr(TO))$, we obtain

$$\overline{g}(\nabla_\zeta \eta, \overline{\varrho}\varpi) = \overline{g}(\overline{\nabla}_\zeta \eta, \overline{\varrho}\varpi) = \overline{g}(\overline{\varrho}(\overline{\nabla}_\zeta \eta), \varpi) = \overline{g}(\overline{\nabla}_\zeta \overline{\varrho}\eta, \varpi) = \overline{g}(\overline{\nabla}_\zeta f\eta + \overline{\nabla}_\zeta F\eta, \varpi),$$

which gives

$$\overline{g}(\nabla_\zeta \eta, \overline{\varrho}\varpi) = \overline{g}(\nabla_\zeta f\eta - A_{F\eta}\zeta, \varpi).$$

This completes the proof. □

Theorem 4.4. *Let O be a mixed geodesic bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. Then, the slant distribution D' (in particular D_1, D_2) defines a totally geodesic foliation iff*
(i) $\overline{g}(f\eta, \nabla_\zeta \overline{\varrho}\varsigma) = \overline{g}(\eta, \nabla_\zeta \overline{\varrho}\varsigma)$;
(ii) $\overline{g}(\nabla_\zeta \overline{\varrho}\varpi, f\eta) + \overline{g}(h^s(\zeta, \overline{\varrho}\varpi), F\eta) = \overline{g}(\nabla_\zeta \overline{\varrho}\varpi, \eta)$;
(iii) $\nabla_\zeta f\eta - A_{F\eta}\zeta$ has no components in $\Gamma(Rad(TO))$
$\forall\ \zeta, \eta \in \Gamma(D')$ *(in particular $\Gamma(D_1), \Gamma(D_2)$), $\varsigma \in \Gamma(D)$ and $\varpi \in \Gamma(ltr(TO))$.*

Proof. Let O be a mixed geodesic bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$, we have $h^s(\zeta, \overline{\varrho}\varsigma) = 0$, $\forall\ \zeta \in \Gamma(D')$ and $\varsigma \in \Gamma(D)$. The distribution D' defines a totally geodesic foliation iff $\nabla_\zeta \eta \in \Gamma(D')$. $\overline{\nabla}$ being a metric connection and using (2.3), (2.6) and (3.1), $\forall$ $\zeta, \eta \in \Gamma(D')$ and $\varsigma \in \Gamma(D)$, we obtain

$$\begin{aligned}\overline{g}(\nabla_\zeta \eta, \varsigma) &= -\,\overline{g}(\eta, \overline{\nabla}_\zeta \varsigma) = \overline{g}(\overline{\varrho}\eta, \overline{\nabla}_\zeta \varsigma) - \overline{g}(\overline{\varrho}\eta, \overline{\varrho}(\overline{\nabla}_\zeta \varsigma)) \\ &= \overline{g}(\eta, \overline{\nabla}_\zeta \overline{\varrho}\varsigma) - \overline{g}(f\eta + F\eta, \overline{\nabla}_\zeta(\overline{\varrho}\varsigma)) \\ &= \overline{g}(\eta, \nabla_\zeta \overline{\varrho}\varsigma) - \overline{g}(f\eta, \nabla_\zeta \overline{\varrho}\varsigma) - \overline{g}(F\eta, h^s(\zeta, \overline{\varrho}\varsigma)),\end{aligned}$$

which implies

$$\overline{g}(\nabla_\zeta \eta, \varsigma) = \overline{g}(\eta, \nabla_\zeta \overline{\varrho}\varsigma) - \overline{g}(f\eta, \nabla_\zeta \overline{\varrho}\varsigma).$$

From (2.3), (2.6) and (3.1), $\forall\ \zeta, \eta \in \Gamma(D')$ and $\varpi \in \Gamma(ltr(TO))$, we obtain

$$\begin{aligned}\overline{g}(\nabla_\zeta \eta, \varpi) &= -\,\overline{g}(\eta, \overline{\nabla}_\zeta \varpi) = \overline{g}(\overline{\varrho}\eta, \overline{\nabla}_\zeta \varpi) - \overline{g}(\overline{\varrho}\eta, \overline{\varrho}(\overline{\nabla}_\zeta \varpi)) \\ &= \overline{g}(\eta, \overline{\nabla}_\zeta \overline{\varrho}\varpi) - \overline{g}(f\eta + F\eta, \overline{\nabla}_\zeta(\overline{\varrho}\varpi)) \\ &= \overline{g}(\eta, \nabla_\zeta \overline{\varrho}\varpi) - \overline{g}(f\eta, \nabla_\zeta \overline{\varrho}\varpi) - \overline{g}(F\eta, h^s(\zeta, \overline{\varrho}\varpi)),\end{aligned}$$

which implies

$$\overline{g}(\nabla_\zeta \eta, \varpi) = \overline{g}(\nabla_\zeta \overline{\varrho}\varpi, \eta) - \overline{g}(\nabla_\zeta \overline{\varrho}\varpi, f\eta) - \overline{g}(h^s(\zeta, \overline{\varrho}\varpi), F\eta).$$

Now, from (2.2), (2.6), (2.8) and (3.1), $\forall\ \zeta, \eta \in \Gamma(D')$ and $\varpi \in \Gamma(ltr(TO))$, we obtain

$$\overline{g}(\nabla_\zeta \eta, \overline{\varrho}\varpi) = \overline{g}(\overline{\nabla}_\zeta \eta, \overline{\varrho}\varpi) = \overline{g}(\overline{\varrho}(\overline{\nabla}_\zeta \eta), \varpi) = \overline{g}(\overline{\nabla}_\zeta \overline{\varrho}\eta, \varpi) = \overline{g}(\overline{\nabla}_\zeta f\eta + \overline{\nabla}_\zeta F\eta, \varpi),$$

which gives

$$\overline{g}(\nabla_\zeta \eta, \overline{\varrho}\varpi) = \overline{g}(\nabla_\zeta f\eta - A_{F\eta}\zeta, \varpi).$$

This completes the proof. □

Theorem 4.5. *Let O be a bi-slant lightlike submanifold of a golden semi-Riemannian manifold $\overline{O}$. Then, O is mixed geodesic iff the following holds*
(i) $F(\nabla_\zeta f\varsigma - A_{F\varsigma}\zeta) = -C(h^s(\zeta, f\varsigma) + \nabla^s_\zeta F\varsigma)$*;*
(ii) $h^l(\zeta, f\varsigma) + D^l(\zeta, F\varsigma) = h^s(\zeta, f\varsigma) + \nabla^s_\zeta F\varsigma$
$\forall\ \zeta \in \Gamma(D)$ *and* $\varsigma \in \Gamma(D')$ *(in particular* $\Gamma(D_1), \Gamma(D_2)$*).*

Proof. From (2.5), (2.6), (2.8), (3.1), (3.2) and (3.3), we obtain

$$\begin{aligned} h(\zeta,\varsigma) =& \overline{\nabla}_\zeta\varsigma - \nabla_\zeta\varsigma \\ =& \overline{\varrho}(\overline{\varrho}\,\overline{\nabla}_\zeta\varsigma) - \overline{\varrho}(\overline{\nabla}_\zeta\varsigma) - \nabla_\zeta\varsigma \\ =& \overline{\varrho}(\overline{\nabla}_\zeta\overline{\varrho}\varsigma) - (\overline{\nabla}_\zeta\overline{\varrho}\varsigma) - \nabla_\zeta\varsigma \\ =& \overline{\varrho}(\overline{\nabla}_\zeta f\varsigma + \overline{\nabla}_\zeta F\varsigma) - (\overline{\nabla}_\zeta f\varsigma + \overline{\nabla}_\zeta F\varsigma) - \nabla_\zeta\varsigma \\ =& \overline{\varrho}(\nabla_\zeta f\varsigma + h^l(\zeta, f\varsigma) + h^s(\zeta, f\varsigma) - A_{F\varsigma}\zeta + \nabla^s_\zeta F\varsigma + D^l(\zeta, F\varsigma)) \\ & - (\nabla_\zeta f\varsigma + h^l(\zeta, f\varsigma) + h^s(\zeta, f\varsigma) - A_{F\varsigma}\zeta + \nabla^s_\zeta F\varsigma + D^l(\zeta, F\varsigma)) - \nabla_\zeta\varsigma. \end{aligned}$$

Taking transversal part of above equation, we get

$$\begin{aligned} h(\zeta,\varsigma) =& F(\nabla_\zeta f\varsigma - A_{F\varsigma}\zeta) + C(h^s(\zeta, f\varsigma) + \nabla^s_\zeta F\varsigma) \\ & - h^l(\zeta, f\varsigma) - h^s(\zeta, f\varsigma) - \nabla^s_\zeta F\varsigma - D^l(\zeta, F\varsigma). \end{aligned}$$

Hence, $h(\zeta, \varsigma) = 0$, iff (i) and (ii) hold, which completes the proof. □

5. Conclusions

We investigate some interesting results on bi-slant lightlike submanifolds of golden semi-Riemannian manifolds and give two examples on such submanifolds. We also discuss the integrability of distributions on bi-slant lightlike submanifolds. Certain conditions on foliations determined by distributions on bi-slant lightlike submanifolds of golden semi-Riemannian manifolds are derived.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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