AIMS Mathematics, 8(8): 19442–19460.
DOI:10.3934/math.2023992
Received: 10 April 2023
Revised: 17 May 2023
Accepted: 28 May 2023
Published: 09 June 2023

Mathematics

http://www.aimspress.com/journal/Math

Research article

Complete convergence and complete moment convergence for maximal weighted sums of extended negatively dependent random variables under sub-linear expectations

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Abstract: In this article, we study the complete convergence and complete moment convergence for maximal weighted sums of extended negatively dependent random variables under sub-linear expectations. We also give some sufficient assumptions for the convergence. Moreover, we get the Marcinkiewicz-Zygmund type strong law of large numbers for weighted sums of extended negatively dependent random variables. The results obtained in this paper generalize the relevant ones in probability space.

Keywords: extended negatively dependent; complete convergence; complete moment convergence; maximal weighted sums; sub-linear expectations
Mathematics Subject Classification: 60F05, 60F15

1. Introduction

Peng [1, 2] introduced important concepts of the sub-linear expectations space to describe the uncertainty in probability. Stimulated by the works of Peng [1, 2], many scholars tried to discover the results under sub-linear expectations space, similar to those in classic probability space. Zhang [3, 4] proved exponential inequalities and Rosenthal's inequality under sub-linear expectations. Xu et al. [5], Xu and Kong [6] obtained complete convergence and complete moment convergence of weighted sums of negatively dependent random variables under sub-linear expectations. For more limit theorems under sub-linear expectations, the readers could refer to Zhang [7], Xu and Zhang [8, 9], Wu and Jiang [10], Zhang and Lin [11], Zhong and Wu [12], Hu et al. [13], Gao and Xu [14], Kuczmaszewska [15], Xu and Cheng [16, 17], Zhang [18], Chen [19], Zhang [20], Chen and Wu [21], Xu et al. [5], Xu and Kong [6], and references therein.

In classic probability space, Yan [22] established complete convergence and complete moment convergence for maximal weighted sums of extended negatively dependent random variables. For references on complete moment convergence and complete convergence in probability space, the reader could refer to Hsu and Robbins [23], Chow [24], Hosseini and Nezakati [25], Meng et al. [26] and refercences therein. Stimulated by the works of Yan [22], Xu et al. [5], Xu and Kong [6], we try to prove complete convergence and complete moment convergence for maximal weighted sums of extended negatively dependent random variables under sub-linear expectations, and the corresponding Marcinkiewicz-Zygmund strong law of large number, which extends the corresponding results in Yan [22]. Another main innovation point here is Rosenthal-type inequality for extended negatively dependent random variables, provided by Lemma 2.3.

We organize the remainders of this article as follows. We present relevant basic notions, concepts and properties, and give relevant lemmas under sub-linear expectations in Section 2. In Section 3, we give our main results, Theorems 3.1–3.3, the proofs of which are presented in Section 4.

2. Preliminary

Hereafter, we use notions similar to that in the works by Peng [2] and Zhang [4]. Assume that $(\Omega,\mathcal{F})$ is a given measurable space. Suppose that $\mathcal{H}$ is a set of all random variables on $(\Omega,\mathcal{F})$ fulfilling $\varphi(X_1,\cdots,X_n)\in\mathcal{H}$ for $X_1,\cdots,X_n\in\mathcal{H}$, and each $\varphi\in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ is the set of φ fulfilling

$$|\varphi(\mathbf{x})-\varphi(\mathbf{y})|\le C(1+|\mathbf{x}|^m+|\mathbf{y}|^m)(|\mathbf{x}-\mathbf{y}|),\forall\mathbf{x},\mathbf{y}\in\mathbb{R}^n$$

for $C>0$, $m\in\mathbb{N}$ relying on φ.

Definition 2.1. *A sub-linear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E}:\mathcal{H}\mapsto\bar{\mathbb{R}}:=[-\infty,\infty]$ fulfilling the following: for every $X,Y\in\mathcal{H}$,*

(a) $X\ge Y$ *implies* $\mathbb{E}[X]\ge\mathbb{E}[Y]$;

(b) $\mathbb{E}[c]=c$, $\forall c\in\mathbb{R}$;

(c) $\mathbb{E}[\lambda X]=\lambda\mathbb{E}[X]$, $\forall\lambda\ge 0$;

(d) $\mathbb{E}[X+Y]\le\mathbb{E}[X]+\mathbb{E}[Y]$ *whenever* $\mathbb{E}[X]+\mathbb{E}[Y]$ *is not of the form* $\infty-\infty$ *or* $-\infty+\infty$.

Definition 2.2. *We say that $\{X_n;n\ge 1\}$ is stochastically dominated by a random variable X under $(\Omega,\mathcal{H},\mathbb{E})$, if there exist a constant C such that $\forall n\ge 1$, for all non-negative $h\in C_{l,Lip}(\mathbb{R})$, $\mathbb{E}(h(X_n))\le C\mathbb{E}(h(X))$.*

$V:\mathcal{F}\mapsto[0,1]$ is named to be a capacity if

(a) $V(\emptyset)=0$, $V(\Omega)=1$.

(b) $V(A)\le V(B)$, $A\subset B$, $A,B\in\mathcal{F}$.

Furthermore, if V is continuous, then V obeys

(c) $A_n\uparrow A$ yields $V(A_n)\uparrow V(A)$.

(d) $A_n\downarrow A$ yields $V(A_n)\downarrow V(A)$.

V is said to be sub-additive when $V(A \bigcup B) \leq V(A) + V(B)$, $A, B \in \mathcal{F}$.

Under $(\Omega, \mathcal{H}, \mathbb{E})$, set $\mathbb{V}(A) := \inf\{\mathbb{E}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}$, $\forall A \in \mathcal{F}$ (cf. Zhang [3]). $\mathbb{V}$ is a sub-additive capacity. Write

$$C_{\mathbb{V}}(X) := \int_0^{\infty} \mathbb{V}(X > x)\mathrm{d}x + \int_{-\infty}^{0} (\mathbb{V}(X > x) - 1)\mathrm{d}x.$$

Under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, $\{X_n; n \geq 1\}$ are called to be upper (resp. lower) extended negatively dependent if there is a constant $K \geq 1$ fulfilling

$$\mathbb{E}\left[\prod_{i=1}^{n} g_i(X_i)\right] \leq K \prod_{i=1}^{n} \mathbb{E}\left[g_i(X_i)\right], \quad n \geq 1,$$

whenever the non-negative functions $g_i \in C_{b,Lip}(\mathbb{R})$, $i = 1, 2, \ldots$ are all non-decreasing (resp. all non-increasing) (cf. Definition 2.4 of Zhang [18]). They are named extended negatively dependent (END) if they are both upper extended negatively dependent and lower extended negatively dependent.

Suppose $\mathbf{X}_1$ and $\mathbf{X}_2$ are two n-dimensional random vectors under $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$ respectively. They are said to be identically distributed if for every $\psi \in C_{l,Lip}(\mathbb{R}^n)$,

$$\mathbb{E}_1[\psi(\mathbf{X}_1)] = \mathbb{E}_2[\psi(\mathbf{X}_2)].$$

$\{X_n; n \geq 1\}$ is called to be identically distributed if for every $i \geq 1$, X_i and X_1 are identically distributed.

Throughout this paper, we suppose that $\mathbb{E}$ is countably sub-additive, i.e., $\mathbb{E}(X) \leq \sum_{n=1}^{\infty} \mathbb{E}(X_n)$ could be implied by $X \leq \sum_{n=1}^{\infty} X_n$, $X, X_n \in \mathcal{H}$, and $X \geq 0$, $X_n \geq 0$, $n = 1, 2, \ldots$. Write $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$. Let C denote a positive constant which may change from line to line. $I(A)$ or I_A is the indicator function of A. The symbol $a_x \approx b_x$ means that there exists two positive constants C_1, C_2 fulfilling $C_1|b_x| \leq |a_x| \leq C_2|b_x|$, x^+ stands for $\max\{x, 0\}$, $x^- = (-x)^+$, for $x \in \mathbb{R}$.

As in Zhang [18], if $X_1, X_2, \ldots, X_n$ are extended negatively dependent random variables and $f_1(x), f_2(x), \ldots, f_n(x) \in C_{l,Lip}(\mathbb{R})$ are all non increasing (or non decreasing) functions, then $f_1(X_1)$, $f_2(X_2), \ldots, f_n(X_n)$ are extended negatively dependent random variables.

We cite the following under sub-linear expectations.

Lemma 2.1. *(Cf. Lemma 4.5 (iii) of Zhang [3]) If $\mathbb{E}$ is countably sub-additive under $(\Omega, \mathcal{H}, \mathbb{E})$, then for $X \in \mathcal{H}$,*

$$\mathbb{E}|X| \leq C_{\mathbb{V}}(|X|).$$

Lemma 2.2. *Assume that $p > 1$ and $\{X_n; n \geq 1\}$ is a sequence of upper extended negatively dependent random varables with $\mathbb{E}[X_k] \leq 0$, $k \geq 0$, under $(\Omega, \mathcal{H}, \mathbb{E})$. Then for every $n \geq 1$, there exists a positive constant $C = C(p)$ relying on p such that for $p \geq 2$,*

$$\mathbb{E}\left[\left(\left(\sum_{j=1}^{n} X_j\right)^+\right)^p\right] \leq C_{\mathbb{V}}\left[\left(\left(\sum_{j=1}^{n} X_j\right)^+\right)^p\right] \leq C\left\{\sum_{i=1}^{n} C_{\mathbb{V}}\left[(X_i^+)^p\right] + \left(\sum_{i=1}^{n} \mathbb{E}[X_i^2]\right)^{p/2}\right\}. \tag{2.1}$$

By (2.1) of Lemma 2.2 and similar proof of Lemma 2.4 of Xu et al. [5], we could get the following.

Lemma 2.3. *Assume that $p > 1$ and $\{X_n; n \geq 1\}$ is a sequence of upper extended negatively dependent random varables with $\mathbb{E}[X_k] \leq 0$, $k \geq 0$, under $(\Omega, \mathcal{H}, \mathbb{E})$. Then for every $n \geq 1$, there exists a positive constant $C = C(p)$ relying on p such that for $p \geq 2$,*

$$\mathbb{E}\left[\max_{1 \leq i \leq n}\left(\left(\sum_{j=1}^{i} X_j\right)^+\right)^p\right] \leq C(\log n)^p\left\{\sum_{i=1}^{n} C_{\mathbb{V}}\left[(X_i^+)^p\right] + \left(\sum_{i=1}^{n} \mathbb{E}[X_i^2]\right)^{p/2}\right\}. \tag{2.2}$$

We first give two lemmas.

Lemma 2.4. *Suppose that* $0 < \alpha < 2$, $\gamma > 0$, *and* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *is an array of real numbers fulfilling*

$$\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n), \text{ for some } \alpha > 0. \tag{2.3}$$

Assume that $\{X_{ni}, i \ge 1, n \ge 1\}$ *is stochastically dominated by a random variable* X *with* $C_{\mathbb{V}}\{|X|^{\alpha}\} < \infty$. *Moreover, suppose that* $\mathbb{E}(a_{ni}X_{ni}) = 0$ *for* $1 < \alpha < 2$ *and* $b_n = n^{1/\alpha}(\log n)^{3/\gamma}$ *for some* $\gamma > 0$. *Then*

$$\frac{1}{b_n} \max_{1\le j\le n} \sum_{i=1}^{j} \mathbb{E}(Y_{ni}) \le C(\log n)^{-3\alpha/\gamma} C_{\mathbb{V}}\{|X|^{\alpha}\} \to 0, \text{ as } n \to \infty, \tag{2.4}$$

where $Y_{ni} = a_{ni}X_{ni}I\left(|a_{ni}X_{ni}| \le b_n\right) + b_n I\left(a_{ni}X_{ni} > b_n\right) - b_n I\left(a_{ni}X_{ni} < b_n\right)$ *for each* $1 \le i \le n$, $n \ge 1$.

Lemma 2.5. *Assume that* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *is an array of real numbers fuffilling (2.3) and* $X \in \mathcal{H}$ *under sub-linear expectation space* $(\Omega, \mathcal{H}, \mathbb{E})$. *Suppose* b_n *is as in Lemma 2.4.*

(i) *If* $p > \max\{\alpha, \gamma(\beta+1)/3\}$ *for some* β, *then*

$$\sum_{n=2}^{\infty} \frac{(\log n)^{\beta}}{n b_n^{p}} \sum_{i=1}^{n} \int_0^{b_n^{p}} \mathbb{V}\{|a_{ni}X|^{p} > x\} \mathrm{d}x \le \begin{cases} CC_{\mathbb{V}}\{|X|^{\alpha}\}, & \text{for } \alpha > \gamma(\beta+1)/3, \\ CC_{\mathbb{V}}\{|X|^{\alpha}\log(|X|+1)\}, & \text{for } \alpha = \gamma(\beta+1)/3, \\ CC_{\mathbb{V}}\{|X|^{\gamma(\beta+1)/3}\}, & \text{for } \alpha < \gamma(\beta+1)/3. \end{cases}$$

(ii) *If* $p = \alpha$, $\beta = 2$, *then*

$$\sum_{n=2}^{\infty} \frac{(\log n)^{2}}{n b_n^{\alpha}} \sum_{i=1}^{n} \int_{b_n^{\alpha}}^{\infty} \mathbb{V}\{|a_{ni}X|^{\alpha} > x\} \mathrm{d}x \le \begin{cases} CC_{\mathbb{V}}\{|X|^{\alpha}\}, & \text{for } \alpha > \gamma, \\ CC_{\mathbb{V}}\{|X|^{\alpha}\log(1+|X|)\}, & \text{for } \alpha = \gamma, \\ CC_{\mathbb{V}}\{|X|^{\gamma}\}, & \text{for } \alpha < \gamma. \end{cases}$$

3. Main results

Below are our main results.

Theorem 3.1. *Suppose* $\{X_{ni}, i \ge 1, n \ge 1\}$ *is an array of rowwise END random variables, which is stochastic dominated by* X *under* $(\Omega, \mathcal{H}, \mathbb{E})$. *Assume that for some* $0 < \alpha < 2$, $0 < \gamma < 2$, $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *is an array of real numbers, being all non-negative or all non-positive, fulfilling (2.3) and* b_n *is as in Lemma 2.5. Moreover, suppose that* $\mathbb{E}(a_{ni}X_{ni}) = 0$ *for* $1 < \alpha < 2$. *If*

$$\begin{cases} C_{\mathbb{V}}\{|X|^{\alpha}\} < \infty, & \text{for } \alpha > \gamma, \\ C_{\mathbb{V}}\{|X|^{\alpha}\log(1+|X|)\} < \infty, & \text{for } \alpha = \gamma, \\ C_{\mathbb{V}}\{|X|^{\gamma}\} < \infty, & \text{for } \alpha < \gamma, \end{cases} \tag{3.1}$$

then for all $\varepsilon > 0$,

$$\sum_{n=2}^{\infty} \frac{1}{n} \mathbb{V}\left(\max_{1\le j\le n} \left(\sum_{i=1}^{j} a_{ni}X_{ni} \right) > \varepsilon b_n \right) < \infty. \tag{3.2}$$

Similarly, moreover, with the condition that $\mathbb{E}(-a_{ni}X_{ni}) = 0$ *for* $1 < \alpha < 2$ *in place of that* $\mathbb{E}(a_{ni}X_{ni}) = 0$ *for* $1 < \alpha < 2$*, we have for all* $\varepsilon > 0$*,*

$$\sum_{n=2}^{\infty} \frac{1}{n} \mathbb{V}\left(\max_{1\le j\le n} \left(\sum_{i=1}^{j} (-a_{ni}X_{ni}) \right) > \varepsilon b_n \right) < \infty. \tag{3.3}$$

Moreover, suppose that $\mathbb{E}(X_{ni}) = \mathbb{E}(-X_{ni}) = 0$ *for* $1 < \alpha < 2$*. Then (3.1) implies*

$$\sum_{n=2}^{\infty} \frac{1}{n} \mathbb{V}\left(\max_{1\le j\le n} \left| \sum_{i=1}^{j} a_{ni}X_{ni} \right| > \varepsilon b_n \right) < \infty, \text{ for all } \varepsilon > 0. \tag{3.4}$$

Theorem 3.2. *Suppose* $\{X_{ni}, i \ge 1, n \ge 1\}$ *is an array of rowwise END random variables, which is stochastic dominated by* X *under* $(\Omega, \mathcal{H}, \mathbb{E})$*. Assume that for some* $0 < \alpha < 2$*,* $0 < \gamma < 2$*,* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *is an array of real numbers, being all non-negative or all non-positive, fulfilling (2.3) and* b_n *is as in Lemma 2.5. Suppose (3.1) holds. Moreover, suppose that* $\mathbb{E}(a_{ni}X_{ni}) = 0$ *for* $1 < \alpha < 2$*. Then for* $0 < \tau < \alpha$*,*

$$\sum_{n=2}^{\infty} \frac{1}{n} C_{\mathbb{V}} \left\{ \left(\frac{1}{b_n} \max_{1\le j\le n} \sum_{i=1}^{j} a_{ni}X_{ni} - \varepsilon \right)_+^{\tau} \right\} < \infty, \text{ for all } \varepsilon > 0. \tag{3.5}$$

Similarly, moreover, with the condition that $\mathbb{E}(-a_{ni}X_{ni}) = 0$ *for* $1 < \alpha < 2$ *in place of that* $\mathbb{E}(a_{ni}X_{ni}) = 0$ *for* $1 < \alpha < 2$*, we have for* $0 < \tau < \alpha$*,*

$$\sum_{n=2}^{\infty} \frac{1}{n} C_{\mathbb{V}} \left\{ \left(\frac{1}{b_n} \max_{1\le j\le n} \sum_{i=1}^{j} (-a_{ni}X_{ni}) - \varepsilon \right)_+^{\tau} \right\} < \infty, \text{ for all } \varepsilon > 0. \tag{3.6}$$

Moreover, if $\mathbb{E}(X_{ni}) = \mathbb{E}(-X_{ni}) = 0$ *for* $1 < \alpha < 2$*, then*

$$\sum_{n=2}^{\infty} \frac{1}{n} C_{\mathbb{V}} \left\{ \left(\frac{1}{b_n} \max_{1\le j\le n} \left| \sum_{i=1}^{j} a_{ni}X_{ni} \right| - \varepsilon \right)_+^{\tau} \right\} < \infty, \text{ for all } \varepsilon > 0. \tag{3.7}$$

Remark 3.1. *From (3.7) follows that (3.4) holds. Hence we know that the complete moment convergence implies the complete convergence.*

Theorem 3.3. *Under the same conditions of Theorem 3.1, and assume that* $\mathbb{V}$ *induced by* $\mathbb{E}$ *is countably sub-additive, we have*

$$\mathbb{V}\left(\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} a_{ni}X_{ni}}{b_n} \le 0 \right) = 1,$$
$$\mathbb{V}\left(\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} (-a_{ni}X_{ni})}{b_n} \le 0 \right) = 1.$$

Moreover, if $\mathbb{E}(X_{ni}) = \mathbb{E}(-X_{ni}) = 0$ *for* $1 < \alpha < 2$*, then*

$$\mathbb{V}\left(\limsup_{n\to\infty} \left| \frac{\sum_{i=1}^{n} a_{ni}X_{ni}}{b_n} \right| = 0 \right) = 1.$$

4. Proofs of main results

Proof of Lemma 2.4. We will investigate (2.4) in two cases.
Case I. $0 < \alpha \le 1$.

By Markov's inequality under sub-linear expectations and $C_{\mathbb{V}}\{|X|^\alpha\} < \infty$, we see that

$$\begin{aligned}
\frac{1}{b_n}\max_{1\le j\le n}\sum_{i=1}^{j}\mathbb{E}(Y_{ni}) &\le \frac{1}{b_n}\max_{1\le j\le n}\sum_{i=1}^{j}\mathbb{E}(Y_{ni}^+) \le \frac{1}{b_n}\sum_{i=1}^{n}\mathbb{E}(Y_{ni}^+) \le C\frac{1}{b_n}\sum_{i=1}^{n}\mathbb{E}((Y'_{ni})^+) \\
&\le \frac{C}{b_n}\sum_{i=1}^{n}C_{\mathbb{V}}\left\{(Y'_{ni})^+\right\} \le \frac{C}{b_n}\sum_{i=1}^{n}\int_0^{b_n}\mathbb{V}\{|a_{ni}X| > x\}\mathrm{d}x \\
&\le \frac{C}{b_n}\sum_{i=1}^{n}\int_0^{b_n}\frac{\mathbb{E}|a_{ni}X|^\alpha}{x^\alpha}\mathrm{d}x \\
&= \frac{C}{b_n}b_n^{1-\alpha}\sum_{i=1}^{n}|a_{ni}|^\alpha C_{\mathbb{V}}\{|X|^\alpha\} \le C(\log n)^{-3\alpha/\gamma}C_{\mathbb{V}}\{|X|^\alpha\} \to 0, \qquad (4.1)
\end{aligned}$$

where $Y'_{ni} \equiv a_{ni}XI\{|a_{ni}X| \le b_n\} + b_nI\{a_{ni}X > b_n\} - b_nI\,(a_{ni}X_< b_n)$.
Case II. $1 < \alpha < 2$.

For all $1 \le i \le n$, $n \ge 1$, write

$$Z_{ni} = a_{ni}X_{ni} - Y_{ni} = (a_{ni}X_{ni} - b_n)I\{a_{ni}X_{ni} > b_n\} + (a_{ni}X_{ni} + b_n)I\{a_{ni}X_{ni} < -b_n\}.$$

Then $0 < Z_{ni} = a_{ni}X_{ni} - b_n < a_{ni}X_{ni}$ for $a_{ni}X_{ni} > b_n$, $a_{ni}X_{ni} < Z_{ni} = a_{ni}X_{ni} + b_n < 0$ for $a_{ni}X_{ni} < -b_n$. Therefore, $|Z_{ni}| \le |a_{ni}X_{ni}|I\{|a_{ni}X_{ni}| > b_n\}$.

From $\mathbb{E}(a_{ni}X_{ni}) = 0$ for $1 < \alpha < 2$ and $C_{\mathbb{V}}\{|X|^\alpha\} < \infty$ follows that

$$\begin{aligned}
\frac{1}{b_n}\max_{1\le j\le n}\sum_{i=1}^{j}\mathbb{E}(Y_{ni}) &\le \frac{1}{b_n}\max_{1\le j\le n}\left|\sum_{i=1}^{j}\mathbb{E}(Y_{ni})\right| \le \frac{1}{b_n}\max_{1\le j\le n}\sum_{i=1}^{j}|\mathbb{E}(Y_{ni})| \\
&\le \frac{1}{b_n}\max_{1\le j\le n}\sum_{i=1}^{j}|\mathbb{E}(Y_{ni}) - \mathbb{E}(a_{ni}X_{ni})| \\
&\le \frac{1}{b_n}\sum_{i=1}^{n}\mathbb{E}|Z_{ni}| \le \frac{1}{b_n}\sum_{i=1}^{n}\mathbb{E}|Z'_{ni}| \le \frac{1}{b_n}\sum_{i=1}^{n}C_{\mathbb{V}}\left\{|Z'_{ni}|\right\} \\
&\le \frac{1}{b_n}\sum_{i=1}^{n}\int_0^{\infty}\mathbb{V}\left\{|a_{ni}X|I\{|a_{ni}X| > b_n\} > x\right\}\mathrm{d}x \\
&\le \frac{1}{b_n}\sum_{i=1}^{n}\int_0^{\infty}\mathbb{V}\left\{|a_{ni}X|^\alpha I\{|a_{ni}X| > b_n\} > xb_n^{\alpha-1}\right\}\mathrm{d}x \\
&\le \frac{1}{b_n}\sum_{i=1}^{n}\int_0^{\infty}\mathbb{V}\left\{|a_{ni}X|^\alpha I\{|a_{ni}X| > b_n\} > x\right\}\mathrm{d}x/b_n^{\alpha-1} \\
&\le \frac{1}{b_n^\alpha}\sum_{i=1}^{n}|a_{ni}|^\alpha C_{\mathbb{V}}\left\{|X|^\alpha\right\}
\end{aligned}$$

$$\leq \quad C(\log n)^{-3\alpha/\gamma} C_{\mathbb{V}}\left\{|X|^{\alpha}\right\} \to 0, \text{ as } n \to \infty, \tag{4.2}$$

where $Z'_{ni} \equiv (a_{ni}X - b_n)I\{a_{ni}X > b_n\} + (a_{ni}X + b_n)I\{a_{ni}X < -b_n\}$. Combining (4.1) and (4.2) results in (2.4). □

Proof of Lemma 2.5. Without loss of generality, we suppose that

$$\sum_{i=1}^{n} |a_{ni}|^{\alpha} \leq n, \text{ for some } \alpha > 0.$$

Write

$$I_{nj} := \{1 \leq i \leq n : n^{1/\alpha}(j+1)^{-1/\alpha} < |a_{ni}| \leq n^{1/\alpha} j^{-1/\alpha}\}.$$

Then by Lemma 2.4 of Yan [22], we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{(\log n)^{\beta}}{n b_n^p} \sum_{i=1}^{n} \int_0^{b_n^p} \mathbb{V}\{|a_{ni}X|^p > x\} \mathrm{d}x \\
&= \sum_{n=2}^{\infty} n^{-1-p/\alpha} (\log n)^{\beta - 3p/\gamma} \sum_{i=1}^{n} |a_{ni}|^p \int_0^{b_n^p/|a_{ni}|^p} \mathbb{V}\{|X|^p > x\} \mathrm{d}x \\
&= \sum_{n=2}^{\infty} n^{-1-p/\alpha} (\log n)^{\beta - 3p/\gamma} \sum_{j=1}^{\infty} \sum_{i \in I_{nj}} |a_{ni}|^p \int_0^{b_n^p/|a_{ni}|^p} \mathbb{V}\{|X|^p > x\} \mathrm{d}x \\
&\leq \sum_{n=2}^{\infty} n^{-1} (\log n)^{\beta - 3p/\gamma} \sum_{j=1}^{\infty} \# I_{nj} j^{-p/\alpha} \int_0^{(j+1)^{p/\alpha} (\log n)^{3p/\gamma}} \mathbb{V}\left\{|X|^p > x\right\} \mathrm{d}x \\
&= \sum_{n=2}^{\infty} n^{-1} (\log n)^{\beta - 3p/\gamma} \sum_{j=1}^{\infty} \# I_{nj} j^{-p/\alpha} \sum_{k=0}^{j} \int_{k^{p/\alpha} (\log n)^{3p/\gamma}}^{(k+1)^{p/\alpha} (\log n)^{3p/\gamma}} \mathbb{V}\left\{|X|^p > x\right\} \mathrm{d}x \\
&\leq \sum_{n=2}^{\infty} n^{-1} (\log n)^{\beta - 3p/\gamma} \sum_{j=1}^{\infty} \# I_{nj} j^{-p/\alpha} \int_0^{(\log n)^{3p/\gamma}} \mathbb{V}\left\{|X|^p > x\right\} \mathrm{d}x \\
&\quad + \sum_{n=2}^{\infty} n^{-1} (\log n)^{\beta - 3p/\gamma} \sum_{j=1}^{\infty} \# I_{nj} j^{-p/\alpha} \sum_{k=1}^{j} \int_{k^{p/\alpha} (\log n)^{3p/\gamma}}^{(k+1)^{p/\alpha} (\log n)^{3p/\gamma}} \mathbb{V}\left\{|X|^p > x\right\} \mathrm{d}x \\
&=: I_1 + I_2.
\end{aligned}$$

First, for I_1, when $\alpha > \frac{\gamma(\beta+1)}{3}$, from Lemma 2.4 of Yan [22] and $p > \alpha$ follows that

$$\begin{aligned}
I_1 &\leq C \sum_{n=2}^{\infty} n^{-1} (\log n)^{\beta - 3p/\gamma} \int_0^{(\log n)^{3/\gamma}} \mathbb{V}\left\{|X| > x\right\} x^{p-1} \mathrm{d}x \\
&\leq C \sum_{n=2}^{\infty} n^{-1} (\log n)^{\beta - 3p/\gamma} \int_0^{(\log n)^{3/\gamma}} \mathbb{V}\left\{|X| > x\right\} x^{\alpha - 1} (\log n)^{3(p-\alpha)/\gamma} \mathrm{d}x \\
&\leq C \sum_{n=2}^{\infty} n^{-1} (\log n)^{\beta - 3\alpha/\gamma} C_{\mathbb{V}}\{|X|^{\alpha}\} \leq C C_{\mathbb{V}}\{|X|^{\alpha}\}.
\end{aligned}$$

When $\alpha \le \frac{\gamma(\beta+1)}{3}$, from Lemma 2.4 of Yan [22], and $p > \frac{\gamma(\beta+1)}{3}$ follows that

$$\begin{aligned}
I_1 &\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-3p/\gamma}\int_0^{(\log n)^{3/\gamma}} \mathbb{V}\{|X|>x\}\,x^{p-1}\mathrm{d}x \\
&= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-3p/\gamma}\sum_{m=2}^{n}\int_{(\log(m-1))^{3/\gamma}}^{(\log m)^{3/\gamma}} \mathbb{V}\{|X|>x\}\,x^{p-1}\mathrm{d}x \\
&= C\sum_{m=2}^{\infty}\int_{(\log(m-1))^{3/\gamma}}^{(\log m)^{3/\gamma}} \mathbb{V}\{|X|>x\}\,x^{p-1}\mathrm{d}x\sum_{n=m}^{\infty} n^{-1}(\log n)^{\beta-3p/\gamma} \\
&\le C\sum_{m=2}^{\infty}\int_{(\log(m-1))^{3/\gamma}}^{(\log m)^{3/\gamma}} \mathbb{V}\{|X|>x\}\,x^{p-1}(\log m)^{\beta-3p/\gamma+1}\mathrm{d}x \\
&\le C\sum_{m=2}^{\infty}\int_{(\log(m-1))^{3/\gamma}}^{(\log m)^{3/\gamma}} \mathbb{V}\{|X|>x\}\,x^{\frac{\gamma(\beta+1)}{3}-1}(\log m)^{\beta-3p/\gamma+1}(\log m)^{\frac{3}{\gamma}\left(p-\frac{\gamma(\beta+1)}{3}\right)}\mathrm{d}x \\
&\le CC_{\mathbb{V}}\left\{|X|^{\frac{\gamma(\beta+1)}{3}}\right\}.
\end{aligned}$$

Next for I_2, from Lemma 2.4 of Yan [22] and $p > \alpha$ follows that

$$\begin{aligned}
I_2 &\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-3p/\gamma}\sum_{k=1}^{\infty}\int_{k^{p/\alpha}(\log n)^{3p/\gamma}}^{(k+1)^{p/\alpha}(\log n)^{3p/\gamma}} \mathbb{V}\{|X|^p>x\}\,\mathrm{d}x\sum_{j=k}^{\infty}\#I_{nj}\,j^{-p/\alpha} \\
&\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-3p/\gamma}\sum_{k=1}^{\infty}(k+1)^{1-p/\alpha}\int_{k^{p/\alpha}(\log n)^{3p/\gamma}}^{(k+1)^{p/\alpha}(\log n)^{3p/\gamma}} \mathbb{V}\{|X|^p>x\}\,\mathrm{d}x \\
&\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-3p/\gamma}\sum_{k=1}^{\infty}(k+1)^{1-p/\alpha}\int_{k^{1/\alpha}(\log n)^{3/\gamma}}^{(k+1)^{1/\alpha}(\log n)^{3/\gamma}} \mathbb{V}\{|X|>x\}\,x^{\alpha-1}x^{p-\alpha}\mathrm{d}x \\
&\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-3\alpha/\gamma}\sum_{k=1}^{\infty}\int_{k^{1/\alpha}(\log n)^{3/\gamma}}^{(k+1)^{1/\alpha}(\log n)^{3/\gamma}} \mathbb{V}\{|X|>x\}\,x^{\alpha-1}\mathrm{d}x \\
&\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-3\alpha/\gamma}\sum_{k=1}^{\infty}\int_{k^{1/\alpha}(\log n)^{3/\gamma}}^{(k+1)^{1/\alpha}(\log n)^{3/\gamma}} \mathbb{V}\{|X|>x\}\,x^{\alpha-1}\mathrm{d}x \\
&= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-3\alpha/\gamma}\int_{(\log n)^{3/\gamma}}^{\infty} \mathbb{V}\{|X|>x\}\,x^{\alpha-1}\mathrm{d}x \\
&= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{\beta-3\alpha/\gamma}\sum_{m=n}^{\infty}\int_{(\log m)^{3/\gamma}}^{(\log(m+1))^{3/\gamma}} \mathbb{V}\{|X|>x\}\,x^{\alpha-1}\mathrm{d}x \\
&\le C\sum_{m=2}^{\infty}\int_{(\log m)^{3/\gamma}}^{(\log(m+1))^{3/\gamma}} \mathbb{V}\{|X|>x\}\,x^{\alpha-1}\mathrm{d}x\sum_{n=2}^{m} n^{-1}(\log n)^{\beta-3\alpha/\gamma}.
\end{aligned}$$

Noting that

$$\sum_{n=2}^{m} n^{-1}(\log n)^{\beta-3\alpha/\gamma} \le \begin{cases} C, & \text{for } \alpha > \frac{\gamma(\beta+1)}{3}, \\ C\log\log m, & \text{for } \alpha = \frac{\gamma(\beta+1)}{3}, \\ C(\log m)^{\beta-\frac{3\alpha}{\gamma}+1}, & \text{for } \alpha < \frac{\gamma(\beta+1)}{3}, \end{cases}$$

we obtain

$$I_2 \leq \begin{cases} CC_{\mathbb{V}}\{|X|^{\alpha}\}, & \text{for } \alpha > \frac{\gamma(\beta+1)}{3}, \\ CC_{\mathbb{V}}\{|X|^{\alpha}\log(1+|X|)\}, & \text{for } \alpha = \frac{\gamma(\beta+1)}{3}, \\ CC_{\mathbb{V}}\left\{|X|^{\frac{\gamma(\beta+1)}{3}}\right\}, & \text{for } \alpha < \frac{\gamma(\beta+1)}{3}. \end{cases}$$

Hence,

$$I = I_1 + I_2 \leq \begin{cases} CC_{\mathbb{V}}\{|X|^{\alpha}\}, & \text{for } \alpha > \frac{\gamma(\beta+1)}{3}, \\ CC_{\mathbb{V}}\{|X|^{\alpha}\log(1+|X|)\}, & \text{for } \alpha = \frac{\gamma(\beta+1)}{3}, \\ CC_{\mathbb{V}}\left\{|X|^{\frac{\gamma(\beta+1)}{3}}\right\}, & \text{for } \alpha < \frac{\gamma(\beta+1)}{3}. \end{cases}$$

The proof is completed. □

Proof of Theorem 3.1. By (2.3) and $a_{ni} = a_{ni}^{+} - a_{ni}^{-}$, without loss of generality, we suppose that $a_{ni} \geq 0$ and $\sum_{i=1}^{n} a_{ni}^{\alpha} \leq n$. We need only to prove (3.2).

For fixed $n \geq 1$, write Z_{ni} as in the proof of Lemma 2.4, and

$$A = \bigcap_{i=1}^{n} \{Y_{ni} = a_{ni}X_i\},$$

$$B = \bar{A} = \bigcup_{i=1}^{n} \{Y_{ni} \neq a_{ni}X_{ni}\} = \bigcup_{i=1}^{n} \{|a_{ni}X_{ni}| > b_n\},$$

$$E_n = \left\{ \max_{1\leq j\leq n} \sum_{i=1}^{j} a_{ni}X_{ni} > \varepsilon b_n \right\}.$$

We easily see that for all $\varepsilon > 0$,

$$E_n = E_nA \bigcup E_nB \subset \left\{ \max_{1\leq j\leq n} \sum_{i=1}^{j} Y_{ni} > \varepsilon b_n \right\} \bigcup \left\{ \bigcup_{i=1}^{n} \{|a_{ni}X_{ni}| > b_n\} \right\}.$$

Then from Lemma 2.4, for n sufficiently large follows that

$$\begin{aligned} \mathbb{V}(E_n) &\leq \mathbb{V}\left(\max_{1\leq j\leq n} \sum_{i=1}^{j} Y_{ni} > \varepsilon b_n \right) + \mathbb{V}\left(\bigcup_{i=1}^{n} \{|a_{ni}X_{ni}| > b_n\} \right) \\ &\leq \mathbb{V}\left(\max_{1\leq j\leq n} \sum_{i=1}^{j} (Y_{ni} - \mathbb{E}Y_{ni}) > \varepsilon b_n - \max_{1\leq j\leq n} \sum_{i=1}^{j} \mathbb{E}Y_{ni} \right) + \mathbb{V}\left(\bigcup_{i=1}^{n} \{|a_{ni}X_{ni}| > b_n\} \right) \\ &\leq \mathbb{V}\left(\max_{1\leq j\leq n} \sum_{i=1}^{j} (Y_{ni} - \mathbb{E}Y_{ni}) > \varepsilon b_n/2 \right) + \mathbb{V}\left(\bigcup_{i=1}^{n} \{|a_{ni}X_{ni}| > b_n\} \right). \end{aligned} \tag{4.3}$$

To establish (4.3), we only need to prove that

$$J_1 := \sum_{n=2}^{\infty} \frac{1}{n} \mathbb{V}\left(\max_{1\leq j\leq n} \sum_{i=1}^{j} (Y_{ni} - \mathbb{E}Y_{ni}) > \varepsilon b_n/2 \right) < \infty, \tag{4.4}$$

$$J_2 := \sum_{n=2}^{\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{V}\left(|a_{ni}X_{ni}| > b_n\right) < \infty. \tag{4.5}$$

For J_1, we see that $\{Y_{ni} - \mathbb{E}(Y_{ni}), 1 \le i \le n, n \ge 1\}$ is an array of rowwise END random variables under sub-linear expectations. Therefore, by Markov's inequality under sub-linear expectations, Lemmas 2.1, 2.3, and similar proof of (2.8) of Zhang [20], we obtain

$$\begin{aligned}
J_1 &\le \sum_{n=2}^{\infty} \frac{1}{n} \mathbb{V}\left(\max_{1\le j\le n} \left(\sum_{i=1}^{j} (Y_{ni} - \mathbb{E}Y_{ni}) \right)^{+} > \varepsilon b_n/2 \right) \le C \sum_{n=1}^{\infty} \frac{1}{nb_n^2} \mathbb{E}\left(\max_{1\le j\le n} \left(\left(\sum_{i=1}^{j} (Y_{ni} - \mathbb{E}Y_{ni}) \right)^{+} \right)^2 \right) \\
&\le C \sum_{n=2}^{\infty} \frac{(\log n)^2}{nb_n^2} \left(\sum_{i=1}^{n} C_{\mathbb{V}} \left\{ ((Y_{ni} - \mathbb{E}(Y_{ni}))^{+})^2 \right\} + \sum_{i=1}^{n} \mathbb{E}\left((Y_{ni} - \mathbb{E}(Y_{ni}))^2 \right) \right) \\
&\le C \sum_{n=2}^{\infty} \frac{(\log n)^2}{nb_n^2} \sum_{i=1}^{n} C_{\mathbb{V}}\{|Y_{ni}|^2\} + C \sum_{n=2}^{\infty} \frac{(\log n)^2}{nb_n^2} \left(\sum_{i=1}^{n} |\mathbb{E}(Y_{ni})| \right)^2 \\
&\le C \sum_{n=2}^{\infty} \frac{(\log n)^2}{nb_n^2} \sum_{i=1}^{n} C_{\mathbb{V}}\{|Y'_{ni}|^2\} + C \sum_{n=2}^{\infty} \frac{(\log n)^2}{nb_n^2} \left(\sum_{i=1}^{n} |\mathbb{E}(Y_{ni})| \right)^2 \\
&\le C \sum_{n=2}^{\infty} \frac{(\log n)^2}{nb_n^2} \sum_{i=1}^{n} \int_0^{b_n^2} \mathbb{V}\{|a_{ni}X|^2 > x\} \mathrm{d}x + C \sum_{n=2}^{\infty} \frac{(\log n)^2}{nb_n^2} \left(\sum_{i=1}^{n} |\mathbb{E}(Y_{ni})| \right)^2 \\
&=: J_{11} + J_{12}.
\end{aligned}$$

By Lemma 2.5(i) (for $p = \beta = 2$) and its proof, we have $J_{11} < \infty$.

For J_{12}, when $0 < \alpha \le 1$, by Lemma 2.4 of Yan [22] and its proof, we see that

$$\begin{aligned}
J_{12} &\le C \sum_{n=2}^{\infty} \frac{(\log n)^2}{nb_n^2} \left(\sum_{i=1}^{n} \mathbb{E}(|Y_{ni}|) \right)^2 \le C \sum_{n=2}^{\infty} \frac{(\log n)^2}{nb_n^2} \left(\sum_{i=1}^{n} \mathbb{E}(|Y'_{ni}|) \right)^2 \\
&\le C \sum_{n=2}^{\infty} \frac{(\log n)^2}{nb_n^2} \left(\sum_{i=1}^{n} C_{\mathbb{V}}\{|Y'_{ni}|\} \right)^2 \\
&\le C \sum_{n=2}^{\infty} n^{-1-2/\alpha} (\log n)^{2-6/\gamma} \left(\sum_{i=1}^{n} \int_0^{b_n} \mathbb{V}\{|a_{ni}X| > x\} \mathrm{d}x \right)^2 \\
&\le C \sum_{n=2}^{\infty} n^{-1-2/\alpha} (\log n)^{2-6/\gamma} \left(\sum_{j=1}^{\infty} \sum_{i\in I_{nj}} \int_0^{(\log n)^{3/\gamma} n^{1/\alpha}} \mathbb{V}\{|a_{ni}X| > x\} \mathrm{d}x \right)^2 \\
&\le C \sum_{n=2}^{\infty} n^{-1-2/\alpha} (\log n)^{2-6/\gamma} \left(\sum_{j=1}^{\infty} \# I_{nj} \int_0^{(\log n)^{3/\gamma} n^{1/\alpha}} \mathbb{V}\{|X| > x j^{1/\alpha}/n^{1/\alpha}\} \mathrm{d}x \right)^2 \\
&\le C \sum_{n=2}^{\infty} n^{-1-2/\alpha} (\log n)^{2-6/\gamma} \left(\sum_{j=1}^{\infty} \# I_{nj} \int_0^{(\log n)^{3/\gamma} j^{1/\alpha}} \mathbb{V}\{|X| > x\} n^{1/\alpha} j^{-1/\alpha} \mathrm{d}x \right)^2 \\
&\le C \sum_{n=2}^{\infty} n^{-1} (\log n)^{2-6/\gamma} \left(\sum_{j=1}^{\infty} \# I_{nj} j^{-1/\alpha} \sum_{k=0}^{j-1} \int_{(\log n)^{3/\gamma} k^{1/\alpha}}^{(\log n)^{3/\gamma} (k+1)^{1/\alpha}} \mathbb{V}\{|X| > x\} \mathrm{d}x \right)^2
\end{aligned}$$

$$\le C\sum_{n=2}^{\infty}n^{-1}(\log n)^{2-6/\gamma}\left(\sum_{k=0}^{\infty}\int_{(\log n)^{3/\gamma}k^{1/\alpha}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}}\mathbb{V}\{|X|>x\}\mathrm{d}x\sum_{j=k+1}^{\infty}\#I_{nj}j^{-1/\alpha}\right)^2$$

$$\le C\sum_{n=2}^{\infty}n^{-1}(\log n)^{2-6/\gamma}\left(\sum_{k=0}^{\infty}\int_{(\log n)^{3/\gamma}k^{1/\alpha}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}}(k+1)^{1-1/\alpha}\mathbb{V}\{|X|>x\}\mathrm{d}x\right)^2$$

$$\le J_{121}+C\sum_{n=2}^{\infty}n^{-1}(\log n)^{2-6/\gamma}$$

$$\times\begin{cases}\left(\sum_{k=1}^{\infty}\int_{(\log n)^{3/\gamma}k^{1/\alpha}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}}x^{\alpha-1}\mathbb{V}\{|X|>x\}\mathrm{d}x/(\log n)^{3(\alpha-1)/\gamma}\right)^2, & \text{for } \alpha\ge\gamma,\\ \left(\sum_{k=1}^{\infty}\int_{(\log n)^{3/\gamma}k^{1/\alpha}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}}x^{\gamma-1}\mathbb{V}\{|X|>x\}\mathrm{d}x\cdot k^{1-\gamma/\alpha}/(\log n)^{3(\gamma-1)/\gamma}\right)^2, & \text{for } \alpha<\gamma,\end{cases}$$

$$\le J_{121}+\begin{cases}C\sum_{n=2}^{\infty}n^{-1}(\log n)^{2-6\alpha/\gamma}(C_{\mathbb{V}}\{|X|^{\alpha}\})^2<\infty, & \text{for } \alpha\ge\gamma,\\ C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-4}(C_{\mathbb{V}}\{|X|^{\gamma}\})^2<\infty, & \text{for } \alpha<\gamma,\end{cases}$$

where for $0<\gamma<2$,

$$\begin{aligned}J_{121} &= C\sum_{n=2}^{\infty}n^{-1}(\log n)^{2-6/\gamma}\left(\int_0^{(\log n)^{3/\gamma}}\mathbb{V}\{|X|>x\}\mathrm{d}x\right)^2\\ &\le C\int_2^{\infty}y^{-1}(\log y)^{2-6/\gamma}\mathrm{d}y\int_0^{(\log y)^{3/\gamma}}\mathbb{V}\{|X|>z\}\mathrm{d}z\int_0^{z}\mathbb{V}\{|X|>x\}\mathrm{d}x\\ &\le C\int_2^{\infty}y^{-1}(\log y)^{2-6/\gamma}\mathrm{d}y\int_0^{(\log y)^{3/\gamma}}z\mathbb{V}\{|X|>z\}\mathrm{d}z\\ &\le C\int_0^{\infty}z\mathbb{V}\{|X|>z\}\mathrm{d}z\int_{\max\{2,e^{z^{\gamma/3}}\}}^{\infty}y^{-1}(\log y)^{2-6/\gamma}\mathrm{d}y\\ &\le C+C\int_1^{\infty}z^{\gamma-1}\mathbb{V}\{|X|>z\}\mathrm{d}z\\ &\le C+CC_{\mathbb{V}}\{|X|^{\gamma}\}<\infty.\end{aligned}$$

When $1<\alpha<2$, by $\mathbb{E}(a_{ni}X_{ni})=0$, we have

$$\begin{aligned}J_{12} &\le C\sum_{n=2}^{\infty}\frac{(\log n)^2}{nb_n^2}\left(\sum_{i=1}^{n}|\mathbb{E}(Y_{ni})-\mathbb{E}(a_{ni}X_{ni})|\right)^2\le C\sum_{n=2}^{\infty}\frac{(\log n)^2}{nb_n^2}\left(\sum_{i=1}^{n}\mathbb{E}(|Y_{ni}-a_{ni}X_{ni}|)\right)^2\\ &\le C\sum_{n=2}^{\infty}\frac{(\log n)^2}{nb_n^2}\left(\sum_{i=1}^{n}\mathbb{E}(|Y'_{ni}-a_{ni}X|)\right)^2\le C\sum_{n=2}^{\infty}\frac{(\log n)^2}{nb_n^2}\left(\sum_{i=1}^{n}C_{\mathbb{V}}(|Y'_{ni}-a_{ni}X|)\right)^2\\ &\le C\sum_{n=2}^{\infty}\frac{(\log n)^2}{nb_n^2}\left(\sum_{i=1}^{n}C_{\mathbb{V}}\{|a_{ni}X|I\{|a_{ni}X|>b_n\}\}\right)^2\\ &\le C\sum_{n=2}^{\infty}(\log n)^{2-6/\gamma}n^{-1-2/\alpha}\left(\sum_{i=1}^{n}|a_{ni}|C_{\mathbb{V}}\{|X|I\{|a_{ni}X|>b_n\}\}\right)^2\\ &\le C\sum_{n=2}^{\infty}(\log n)^{2-6/\gamma}n^{-1-2/\alpha}\left(\sum_{j=1}^{\infty}\sum_{i\in I_{nj}}n^{1/\alpha}j^{-1/\alpha}\int_0^{\infty}\mathbb{V}\{|X|I\{|X|>(\log n)^{3/\gamma}j^{1/\alpha}\}>x\}\mathrm{d}x\right)^2\end{aligned}$$

$$\begin{aligned}
&\leq C\sum_{n=2}^{\infty}(\log n)^{2-6/\gamma}n^{-1-2/\alpha}\left(\sum_{j=1}^{\infty}\sum_{i\in I_{nj}}n^{1/\alpha}j^{-1/\alpha}\int_0^{(\log n)^{3/\gamma}j^{1/\alpha}}\mathbb{V}\{|X|>(\log n)^{3/\gamma}j^{1/\alpha}\}\mathrm{d}x\right)^2\\
&\quad+\sum_{n=2}^{\infty}(\log n)^{2-6/\gamma}n^{-1-2/\alpha}\left(\sum_{j=1}^{\infty}\sum_{i\in I_{nj}}n^{1/\alpha}j^{-1/\alpha}\sum_{k=j}^{\infty}\int_{(\log n)^{3/\gamma}k^{1/\alpha}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}}\mathbb{V}\{|X|>x\}\mathrm{d}x\right)^2\\
&=:J_{121}+J_{122}.
\end{aligned}$$

By $\#I_{nj}\leq(j+1)$, we see that

$$\begin{aligned}
J_{121}&\leq C\sum_{n=2}^{\infty}(\log n)^{2-6/\gamma}n^{-1-2/\alpha}\left(\sum_{j=1}^{\infty}\#I_{nj}n^{1/\alpha}(\log n)^{3/\gamma}\mathbb{V}\{|X|>(\log n)^{3/\gamma}j^{1/\alpha}\}\right)^2\\
&\leq C\sum_{n=2}^{\infty}(\log n)^{2-6/\gamma}n^{-1-2/\alpha}\left(\sum_{j=1}^{\infty}j\cdot n^{1/\alpha}(\log n)^{3/\gamma}\mathbb{V}\{|X|>(\log n)^{3/\gamma}j^{1/\alpha}\}\right)^2\\
&\leq\begin{cases}C\sum_{n=2}^{\infty}(\log n)^{2-6\alpha/\gamma}n^{-1}\left(\sum_{j=1}^{\infty}j\cdot(\log n)^{3\alpha/\gamma}\mathbb{V}\{|X|^{\alpha}>(\log n)^{3\alpha/\gamma}j\}\right)^2, & \text{for } \alpha>\gamma,\\ C\sum_{n=2}^{\infty}(\log n)^{2-6}n^{-1}\left(\sum_{j=1}^{\infty}j^{\gamma/\alpha-1}\cdot(\log n)^{3}\mathbb{V}\{|X|^{\gamma}>(\log n)^{3}j^{\gamma/\alpha}\}j^{2-2\gamma/\alpha}\right)^2, & \text{for } \alpha\leq\gamma,\end{cases}\\
&\leq\begin{cases}C\sum_{n=2}^{\infty}(\log n)^{2-6\alpha/\gamma}n^{-1}\left(C_{\mathbb{V}}\{|X|^{\alpha}\}\right)^2<\infty, & \text{for } \alpha>\gamma,\\ C\sum_{n=2}^{\infty}(\log n)^{-4}n^{-1}\left(C_{\mathbb{V}}\{|X|^{\gamma}\}\right)^2<\infty, & \text{for } \alpha\leq\gamma.\end{cases}
\end{aligned}$$

By $\sum_{j=1}^{k}\#I_{nj}j^{-1}\leq C$, $\sum_{j=1}^{k}\#I_{nj}j^{-1/\alpha}\leq\sum_{j=1}^{k}\#I_{nj}j^{-1/\alpha}j^{1-1/\alpha}\leq Ck^{1-1/\alpha}$, we see that

$$\begin{aligned}
J_{122}&\leq C\sum_{n=2}^{\infty}(\log n)^{2-6/\gamma}n^{-1-2/\alpha}\left(\sum_{j=1}^{\infty}\sum_{i\in I_{nj}}n^{1/\alpha}j^{-1/\alpha}\sum_{k=j}^{\infty}\int_{(\log n)^{3/\gamma}k^{1/\alpha}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}}\mathbb{V}\{|X|>x\}\mathrm{d}x\right)^2\\
&\leq C\sum_{n=2}^{\infty}(\log n)^{2-6/\gamma}n^{-1}\left(\sum_{k=1}^{\infty}\int_{(\log n)^{3/\gamma}k^{1/\alpha}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}}\mathbb{V}\{|X|>x\}\mathrm{d}x\sum_{j=1}^{k}\#I_{nj}j^{-1/\alpha}\right)^2\\
&\leq C\sum_{n=2}^{\infty}(\log n)^{2-6/\gamma}n^{-1}\left(\sum_{k=1}^{\infty}\int_{(\log n)^{3/\gamma}k^{1/\alpha}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}}\mathbb{V}\{|X|>x\}\mathrm{d}x\cdot k^{1-1/\alpha}\right)^2\\
&\leq\begin{cases}C\sum_{n=2}^{\infty}(\log n)^{2-6\alpha/\gamma}n^{-1}\left(\sum_{k=1}^{\infty}\int_{(\log n)^{3/\gamma}k^{1/\alpha}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}}\mathbb{V}\{|X|>x\}x^{\alpha-1}\mathrm{d}x\right)^2, & \text{for } \alpha>\gamma,\\ C\sum_{n=2}^{\infty}(\log n)^{2-6}n^{-1}\left(\sum_{k=1}^{\infty}\int_{(\log n)^{3/\gamma}k^{1/\alpha}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}}\mathbb{V}\{|X|>x\}x^{\gamma-1}\mathrm{d}x\cdot k^{1-1/\alpha-(\gamma-1)/\alpha}\right)^2, & \text{for } \alpha\leq\gamma,\end{cases}\\
&\leq\begin{cases}C\sum_{n=2}^{\infty}(\log n)^{2-6\alpha/\gamma}n^{-1}\left(C_{\mathbb{V}}\{|X|^{\alpha}\}\right)^2, & \text{for } \alpha>\gamma,\\ C\sum_{n=2}^{\infty}(\log n)^{-4}n^{-1}\left(C_{\mathbb{V}}\{|X|^{\gamma}\}\right)^2, & \text{for } \alpha\leq\gamma.\end{cases}
\end{aligned}$$

Hence, we prove that $J_1<\infty$.

Define $g_{\mu}(x)\in C_{l,Lip}(\mathbb{R})$ such that $I\{|x|\leq\mu\}\leq g_{\mu}(|x|)<I\{|x|\leq 1\}$, for some $0<\mu<1$. Then $I\{|x|>\mu\}\geq 1-g_{\mu}(|x|)\geq I\{|x|>1\}$. For J_2, by $\sum_{j=1}^{\infty}\frac{\#I_{nj}}{j+1}\leq 1$, we see that

$$J_2\leq\sum_{n=2}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{V}\left(|a_{ni}X_{ni}|>n^{1/\alpha}(\log n)^{3/\gamma}\right)$$

$$
\begin{aligned}
&\le C\sum_{n=2}^{\infty}\frac{1}{n}\sum_{j=1}^{\infty}\sum_{i\in I_{nj}}\mathbb{E}\left(1-g_\mu\left(\left|\frac{a_{ni}X_{ni}}{n^{1/\alpha}(\log n)^{3/\gamma}}\right|\right)\right)\\
&\le C\sum_{n=2}^{\infty}\frac{1}{n}\sum_{j=1}^{\infty}\sum_{i\in I_{nj}}\mathbb{E}\left(1-g_\mu\left(\left|\frac{a_{ni}X}{n^{1/\alpha}(\log n)^{3/\gamma}}\right|\right)\right)\\
&\le C\sum_{n=2}^{\infty}\frac{1}{n}\sum_{j=1}^{\infty}\#I_{nj}\mathbb{V}\left\{|X|>\mu j^{1/\alpha}(\log n)^{3/\gamma}\right\}\\
&\le C\sum_{n=2}^{\infty}\frac{1}{n}\sum_{j=1}^{\infty}\frac{\#I_{nj}}{j+1}(j+1)\mathbb{V}\left\{|X|>\mu j^{1/\alpha}(\log n)^{3/\gamma}\right\}\\
&\le C\sum_{n=2}^{\infty}\frac{1}{n}\max_{y\ge1}y\cdot\mathbb{V}\left\{|X|>\mu y^{1/\alpha}(\log n)^{3/\gamma}\right\}\\
&\le \begin{cases} C\sum_{n=2}^{\infty}\frac{1}{n\cdot(\log n)^{3\alpha/\gamma}}\max_{y\ge\mu^\alpha}y(\log n)^{3\alpha/\gamma}\cdot\mathbb{V}\left\{|X|^\alpha>y(\log n)^{3\alpha/\gamma}\right\}, & \text{for } \alpha>\gamma,\\ C\sum_{n=2}^{\infty}\frac{1}{n(\log n)^3}\max_{y\ge\mu^\alpha}y^{\gamma/\alpha}\cdot\mathbb{V}\left\{|X|^\gamma>y^{\gamma/\alpha}(\log n)^3\right\}\cdot y^{1-\gamma/\alpha}, & \text{for } \alpha\le\gamma,\end{cases}\\
&\le \begin{cases} C\sum_{n=2}^{\infty}\frac{1}{n\cdot(\log n)^{3\alpha/\gamma}}<\infty, & \text{for } \alpha>\gamma,\\ C\sum_{n=2}^{\infty}\frac{1}{n(\log n)^3}<\infty, & \text{for } \alpha\le\gamma.\end{cases}
\end{aligned}
$$

The proof of Theorem 3.1 is finished. □

Proof of Theorem 3.2. We only prove (3.5). For all $\varepsilon>0$, we see that

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\frac{1}{n}C_{\mathbb{V}}\left\{\left(\frac{1}{b_n}\max_{1\le j\le n}\sum_{i=1}^{j}a_{ni}X_{ni}-\varepsilon\right)_+^{\tau}\right\}\\
&\quad=\sum_{n=2}^{\infty}\frac{1}{n}\int_0^{\infty}\mathbb{V}\left\{\frac{1}{b_n}\max_{1\le j\le n}\sum_{i=1}^{j}a_{ni}X_{ni}-\varepsilon>t^{1/\tau}\right\}\mathrm{d}t\\
&\quad=\sum_{n=2}^{\infty}\frac{1}{n}\int_0^{1}\mathbb{V}\left\{\frac{1}{b_n}\max_{1\le j\le n}\sum_{i=1}^{j}a_{ni}X_{ni}-\varepsilon>t^{1/\tau}\right\}\mathrm{d}t\\
&\qquad+\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\mathbb{V}\left\{\frac{1}{b_n}\max_{1\le j\le n}\sum_{i=1}^{j}a_{ni}X_{ni}-\varepsilon>t^{1/\tau}\right\}\mathrm{d}t\\
&\quad\le\sum_{n=2}^{\infty}\frac{1}{n}\mathbb{V}\left\{\max_{1\le j\le n}\sum_{i=1}^{j}a_{ni}X_{ni}>\varepsilon b_n\right\}+\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\mathbb{V}\left\{\max_{1\le j\le n}\sum_{i=1}^{j}a_{ni}X_{ni}>b_nt^{1/\tau}\right\}\mathrm{d}t\\
&\quad=:K_1+K_2. \qquad (4.6)
\end{aligned}
$$

To establish (3.5), it is enough to prove that $K_1<\infty$ and $K_2<\infty$. By Theorem 3.1, we know $K_1<\infty$. For K_2, for each $1\le i\le n$, $n\ge1$, and $t\ge1$, write

$$Y'_{ni}=a_{ni}X_{ni}I\{|a_{ni}X_{ni}|\le b_nt^{1/\tau}\}+b_nt^{1/\tau}I\{a_{ni}X_{ni}>b_nt^{1/\tau}\}-b_nt^{1/\tau}I\{a_{ni}X_{ni}<-b_nt^{1/\tau}\},$$

$$Z'_{ni}=a_{ni}X_{ni}-Y'_{ni},\quad A'=\bigcap_{i=1}^{n}\{Y'_{ni}=a_{ni}X_{ni}\},$$

$$B' = \bar{A}' = \bigcup_{i=1}^{n}\{Y'_{ni} \neq a_{ni}X_{ni}\} = \bigcup_{i=1}^{n}\{|a_{ni}X_{ni}| > b_n t^{1/\tau}\},$$

$$E'_n = \left\{\max_{1\le j\le n}\sum_{i=1}^{j} a_{ni}X_{ni} > b_n t^{1/\tau}\right\}.$$

By Lemma 2.4, for all $t \geq 1$ and n large sufficiently, we obtain

$$\begin{aligned}
\mathbb{V}\{E'_n\} &\leq \mathbb{V}\left\{\max_{1\le j\le n}\sum_{i=1}^{j} Y'_{ni} > b_n t^{1/\tau}\right\} + \mathbb{V}\left\{\bigcup_{i=1}^{n}\left\{|a_{ni}X_{ni}| > b_n t^{1/\tau}\right\}\right\} \\
&\leq \mathbb{V}\left\{\max_{1\le j\le n}\sum_{i=1}^{j}(Y'_{ni} - \mathbb{E}(Y'_{ni})) > b_n t^{1/\tau} - \max_{1\le j\le n}\sum_{i=1}^{j}\mathbb{E}(Y'_{ni})\right\} + \sum_{i=1}^{n}\mathbb{V}\left\{|a_{ni}X_{ni}| > b_n t^{1/\tau}\right\} \\
&\leq \mathbb{V}\left\{\max_{1\le j\le n}\left(\sum_{i=1}^{j}(Y'_{ni} - \mathbb{E}(Y'_{ni}))\right)^{+} > b_n t^{1/\tau}/2\right\} + \sum_{i=1}^{n}\mathbb{V}\left\{|a_{ni}X_{ni}| > b_n t^{1/\tau}\right\}. \qquad (4.7)
\end{aligned}$$

To establish $K_2 < \infty$, we only need to prove that

$$K_{21} := \sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\mathbb{V}\left\{\max_{1\le j\le n}\left(\sum_{i=1}^{j}(Y'_{ni} - \mathbb{E}(Y'_{ni}))\right)^{+} > b_n t^{1/\tau}/2\right\}\mathrm{d}t < \infty, \qquad (4.8)$$

$$K_{22} := \sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}\mathbb{V}\left\{|a_{ni}X_{ni}| > b_n t^{1/\tau}\right\}\mathrm{d}t < \infty. \qquad (4.9)$$

We know that $\{Y'_{ni} - \mathbb{E}(Y'_{ni}), 1 \leq i \leq n, n \geq 1\}$ is an array of rowwise END random variables under sub-linear expectations. Therefore, by Markov's inequality under sub-linear expectations, Lemmas 2.1, 2.3, and the similar proof of (2.8) of Zhang [20], we obtain

$$\begin{aligned}
K_{21} &\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^2 t^{2/\tau}}\mathbb{E}\left\{\max_{1\le j\le n}\left(\left(\sum_{i=1}^{j}(Y'_{ni} - \mathbb{E}(Y'_{ni}))\right)^{+}\right)^{2}\right\}\mathrm{d}t \\
&\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2 t^{2/\tau}}\left\{\sum_{i=1}^{n}C_{\mathbb{V}}\{((Y'_{ni} - \mathbb{E}(Y'_{ni}))^{+})^{2}\} + \sum_{i=1}^{n}\mathbb{E}[(Y'_{ni} - \mathbb{E}(Y'_{ni}))^{2}]\right\}\mathrm{d}t \\
&\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2 t^{2/\tau}}\sum_{i=1}^{n}C_{\mathbb{V}}\{(Y'_{ni})^{2}\}\mathrm{d}t + C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2 t^{2/\tau}}\left(\sum_{i=1}^{n}|\mathbb{E}[Y'_{ni}]|\right)^{2}\mathrm{d}t \\
&\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2 t^{2/\tau}}\sum_{i=1}^{n}\int_0^{b_n}\mathbb{V}\left(|a_{ni}X| > x\right)x\mathrm{d}x\mathrm{d}t \\
&\quad + C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2 t^{2/\tau}}\sum_{i=1}^{n}\int_{b_n}^{b_n t^{1/\tau}}\mathbb{V}\left(|a_{ni}X| > x\right)x\mathrm{d}x\mathrm{d}t \\
&\quad + C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2 t^{2/\tau}}\left(\sum_{i=1}^{n}|\mathbb{E}[Y'_{ni}]|\right)^{2}\mathrm{d}t =: K_{211} + K_{212} + K_{213}. \qquad (4.10)
\end{aligned}$$

By $0 < \tau < \alpha < 2$ and Lemma 2.5 and its proof, we have

$$K_{211} = C\sum_{n=2}^{\infty}\frac{(\log n)^2}{nb_n^2}\sum_{i=1}^{n}\int_0^{b_n}\mathbb{V}\left(|a_{ni}X| > x\right)x\mathrm{d}x < \infty. \tag{4.11}$$

By using $t = x^\tau$, Markov's inequality under sub-linear expectations, Lemmas 2.1 and 2.5, we see that

$$\begin{aligned}
K_{212} &\leq C\sum_{n=2}^{\infty}\frac{(\log n)^2}{nb_n^2}\int_1^{\infty}x^{\tau-3}\sum_{i=1}^{n}\int_{b_n}^{b_n x}\mathbb{V}\left(|a_{ni}X| > y\right)y\mathrm{d}y\mathrm{d}x \\
&\leq C\sum_{n=2}^{\infty}\frac{(\log n)^2}{nb_n^2}\sum_{m=1}^{\infty}\int_m^{m+1}x^{\tau-3}\sum_{i=1}^{n}\int_{b_n}^{b_n x}\mathbb{V}\left(|a_{ni}X| > y\right)y\mathrm{d}y\mathrm{d}x \\
&\leq C\sum_{n=2}^{\infty}\frac{(\log n)^2}{nb_n^2}\sum_{m=1}^{\infty}m^{\tau-3}\sum_{i=1}^{n}\int_{b_n}^{b_n(m+1)}\mathbb{V}\left(|a_{ni}X| > y\right)y\mathrm{d}y \\
&\leq C\sum_{n=2}^{\infty}\frac{(\log n)^2}{nb_n^2}\sum_{i=1}^{n}\sum_{m=1}^{\infty}m^{\tau-3}\sum_{s=1}^{m}\int_{b_n s}^{b_n(s+1)}\mathbb{V}\left(|a_{ni}X| > y\right)y\mathrm{d}y \\
&\leq C\sum_{n=2}^{\infty}\frac{(\log n)^2}{nb_n^2}\sum_{i=1}^{n}\sum_{s=1}^{\infty}\int_{b_n s}^{b_n(s+1)}\mathbb{V}\left(|a_{ni}X| > y\right)y\mathrm{d}y\sum_{m=s}^{\infty}m^{\tau-3} \\
&\leq C\sum_{n=2}^{\infty}\frac{(\log n)^2}{nb_n^2}\sum_{i=1}^{n}\sum_{s=1}^{\infty}s^{\tau-2}\int_{b_n s}^{b_n(s+1)}\mathbb{V}\left(|a_{ni}X| > y\right)y\mathrm{d}y \\
&\leq C\sum_{n=2}^{\infty}\frac{(\log n)^2}{nb_n^2}\sum_{i=1}^{n}\sum_{s=1}^{\infty}s^{\tau-2}\int_{b_n s}^{b_n(s+1)}\mathbb{V}\left(|a_{ni}X| > y\right)y^{\tau-1}y^{2-\tau}\mathrm{d}y \\
&\leq C\sum_{n=2}^{\infty}\frac{(\log n)^2}{nb_n^\tau}\sum_{i=1}^{n}\int_{b_n}^{\infty}\mathbb{V}\left(|a_{ni}X| > y\right)y^{\tau-1}\mathrm{d}y \\
&\leq C\sum_{n=2}^{\infty}\frac{(\log n)^2}{nb_n^\tau}\sum_{i=1}^{n}\int_{b_n^\tau}^{\infty}\mathbb{V}\left(|a_{ni}X|^\tau > y\right)\mathrm{d}y < \infty.
\end{aligned} \tag{4.12}$$

For K_{213}, for $0 < \alpha \leq 1$, by similar proof of J_{12} of Theorem 3.1, we see that

$$\begin{aligned}
K_{213} &\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2t^{2/\tau}}\left(\sum_{i=1}^{n}\mathbb{E}[|Y'_{ni}|]\right)^2\mathrm{d}t \\
&\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2t^{2/\tau}}\left(\sum_{i=1}^{n}\mathbb{E}[|a_{ni}X|I\{|a_{ni}X| \leq b_nt^{1/\tau}\} + b_nt^{1/\tau}I\{|a_{ni}X| > b_nt^{1/\tau}\}]\right)^2\mathrm{d}t \\
&\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2t^{2/\tau}}\left(\sum_{i=1}^{n}C_{\mathbb{V}}\left\{|a_{ni}X|I\{|a_{ni}X| \leq b_nt^{1/\tau}\} + b_nt^{1/\tau}I\{|a_{ni}X| > b_nt^{1/\tau}\}\right\}\right)^2\mathrm{d}t \\
&\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2t^{2/\tau}}\left(\sum_{j=1}^{\infty}\sum_{i\in I_{nj}}\int_0^{b_nt^{1/\tau}}\mathbb{V}\left\{|X| > xn^{-1/\alpha}j^{1/\alpha}\right\}\mathrm{d}x\right)^2\mathrm{d}t \\
&\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2t^{2/\tau}}\left(\sum_{j=1}^{\infty}\#I_{nj}n^{1/\alpha}j^{-1/\alpha}\int_0^{(\log n)^{3/\gamma}j^{1/\alpha}t^{1/\tau}}\mathbb{V}\left\{|X| > x\right\}\mathrm{d}x\right)^2\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2t^{2/\tau}}\left(\sum_{j=1}^{\infty}\#I_{nj}n^{1/\alpha}j^{-1/\alpha}\sum_{k=0}^{j-1}\int_{(\log n)^{3/\gamma}k^{1/\alpha}t^{1/\tau}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}t^{1/\tau}}\mathbb{V}\{|X|>x\}\,\mathrm{d}x\right)^2\mathrm{d}t\\
&\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2t^{2/\tau}}\left(\sum_{k=0}^{\infty}\int_{(\log n)^{3/\gamma}k^{1/\alpha}t^{1/\tau}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}t^{1/\tau}}\mathbb{V}\{|X|>x\}\,\mathrm{d}x\sum_{j=k+1}^{\infty}\#I_{nj}n^{1/\alpha}j^{-1/\alpha}\right)^2\mathrm{d}t\\
&\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{(\log n)^{6/\gamma}t^{2/\tau}}\left(\sum_{k=0}^{\infty}\int_{(\log n)^{3/\gamma}k^{1/\alpha}t^{1/\tau}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}t^{1/\tau}}\mathbb{V}\{|X|>x\}\,\mathrm{d}x(k+1)^{1-1/\alpha}\right)^2\mathrm{d}t\\
&\leq K_{2130}+C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{(\log n)^{6/\gamma}t^{2/\tau}}\left(\sum_{k=1}^{\infty}\int_{(\log n)^{3/\gamma}k^{1/\alpha}t^{1/\tau}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}t^{1/\tau}}\mathbb{V}\{|X|>x\}\,\mathrm{d}x(k+1)^{1-1/\alpha}\right)^2\mathrm{d}t\\
&\leq K_{2130}+\begin{cases}C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{(\log n)^{6\alpha/\gamma}t^{2\alpha/\tau}}\left(\sum_{k=1}^{\infty}\int_{(\log n)^{3/\gamma}k^{1/\alpha}t^{1/\tau}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}t^{1/\tau}}x^{\alpha-1}\mathbb{V}\{|X|>x\}\,\mathrm{d}x\right)^2\mathrm{d}t, & \text{for } \alpha>\gamma,\\ C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{(\log n)^{6}t^{2\gamma/\tau}}\left(\sum_{k=1}^{\infty}\int_{(\log n)^{3/\gamma}k^{1/\alpha}t^{1/\tau}}^{(\log n)^{3/\gamma}(k+1)^{1/\alpha}t^{1/\tau}}\mathbb{V}\{|X|>x\}\,\mathrm{d}xk^{1-\alpha/\gamma}\right)^2\mathrm{d}t, & \text{for } \alpha\leq\gamma,\end{cases}\\
&\leq K_{2130}+\begin{cases}C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{(\log n)^{6\alpha/\gamma}t^{2\alpha/\tau}}(C_{\mathbb{V}}\{|X|^{\alpha}\})^2\,\mathrm{d}t<\infty, & \text{for } \alpha>\gamma,\\ C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{(\log n)^{6}t^{2\gamma/\tau}}(C_{\mathbb{V}}\{|X|^{\gamma}\})^2\,\mathrm{d}t<\infty, & \text{for } \alpha\leq\gamma,\end{cases}
\end{aligned}\tag{4.13}
$$

where

$$
\begin{aligned}
K_{2130}&=C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{(\log n)^{6/\gamma}t^{2/\tau}}\left(\int_0^{(\log n)^{3/\gamma}t^{1/\tau}}\mathbb{V}\{|X|>x\}\,\mathrm{d}x\right)^2\mathrm{d}t\\
&\leq C\int_2^{\infty}\frac{1}{y}\int_1^{\infty}\frac{(\log y)^2}{(\log y)^{6/\gamma}t^{2/\tau}}\mathrm{d}t\mathrm{d}y\int_0^{(\log y)^{3/\gamma}t^{1/\tau}}\mathbb{V}\{|X|>z\}\,\mathrm{d}z\int_0^{z}\mathbb{V}\{|X|>x\}\mathrm{d}x\\
&\leq C\int_2^{\infty}\frac{1}{y}\int_1^{\infty}\frac{(\log y)^2}{(\log y)^{6/\gamma}t^{2/\tau}}\mathrm{d}t\mathrm{d}y\int_0^{(\log y)^{3/\gamma}t^{1/\tau}}z\mathbb{V}\{|X|>z\}\,\mathrm{d}z\\
&\leq C\int_0^{\infty}z\mathbb{V}\{|X|>z\}\,\mathrm{d}z\int_1^{\infty}\mathrm{d}t\int_{\max\{2,e^{(z/t^{1/\tau})^{\gamma/3}}\}}^{\infty}\frac{(\log y)^2}{y(\log y)^{6/\gamma}t^{2/\tau}}\mathrm{d}y\\
&\leq C\int_0^{\infty}z\mathbb{V}\{|X|>z\}\,\mathrm{d}z\int_1^{\infty}\mathrm{d}t\int_{\max\{2,e^{z^{\gamma/3}}\}}^{\infty}\frac{(\log y)^2}{y(\log y)^{6/\gamma}t^{2/\tau}}\mathrm{d}y\\
&\leq C\int_0^{\infty}z^{\gamma-1}\mathbb{V}\{|X|>z\}\,\mathrm{d}z\int_1^{\infty}t^{-2/\tau}\mathrm{d}t\leq CC_{\mathbb{V}}\{|X|^{\gamma}\}<\infty.
\end{aligned}
$$

When $1<\alpha<2$, by $\mathbb{E}(a_{ni}X_{ni})=0$ and the similar proof of J_{12} for $1<\alpha<2$ in Theorem 3.1, we have

$$
\begin{aligned}
K_{213}&\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2t^{2/\tau}}\left(\sum_{i=1}^{n}|\mathbb{E}[Y'_{ni}]-\mathbb{E}[a_{ni}X_{ni}]|\right)^2\mathrm{d}t\\
&\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2t^{2/\tau}}\left(\sum_{i=1}^{n}\mathbb{E}[|Y'_{ni}-a_{ni}X_{ni}|]\right)^2\mathrm{d}t\\
&\leq C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2t^{2/\tau}}\left(\sum_{i=1}^{n}\mathbb{E}[|a_{ni}X_{ni}-b_nt^{1/\tau}|I\{|a_{ni}X_{ni}|>b_nt^{1/\tau}\}]\right)^2\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2t^{2/\tau}}\left(\sum_{i=1}^{n}\mathbb{E}[|a_{ni}X-b_nt^{1/\tau}|I\{|a_{ni}X|>b_nt^{1/\tau}\}]\right)^2\mathrm{d}t\\
&\le C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2t^{2/\tau}}\left(\sum_{i=1}^{n}C_{\mathbb{V}}\left\{|a_{ni}X-b_nt^{1/\tau}|I\{|a_{ni}X|>b_nt^{1/\tau}\}\right\}\right)^2\mathrm{d}t\\
&\le C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{(\log n)^2}{b_n^2t^{2/\tau}}\left(\sum_{i=1}^{n}C_{\mathbb{V}}\left\{|a_{ni}X|I\{|a_{ni}X|>b_nt^{1/\tau}\}\right\}\right)^2\mathrm{d}t\\
&\le C\sum_{n=2}^{\infty}\frac{1}{n}\frac{(\log n)^2}{b_n^2}\left(\sum_{i=1}^{n}C_{\mathbb{V}}\{|a_{ni}X|I\{|a_{ni}X|>b_n\}\}\right)^2<\infty. \qquad (4.14)
\end{aligned}
$$

Combining (4.10)–(4.14) results in (4.8). By the similar proof of J_2 of Theorem 3.1, for $0<\mu<1$, we have

$$
\begin{aligned}
J_2 &\le C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}\mathbb{V}\left\{|a_{ni}X_{ni}|>b_nt^{1/\tau}\right\}\mathrm{d}t\\
&\le C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\mathrm{d}t\sum_{j=1}^{\infty}\frac{\#I_{nj}}{j+1}(j+1)\mathbb{V}\left\{|X|>\mu j^{1/\alpha}(\log n)^{3/\gamma}t^{1/\tau}\right\}\\
&\le C\sum_{n=2}^{\infty}\frac{1}{n}\int_1^{\infty}\mathrm{d}t\max_{y\ge1}y\cdot\mathbb{V}\left\{|X|>\mu y^{1/\alpha}(\log n)^{3/\gamma}t^{1/\tau}\right\} \qquad (4.15)\\
&\le \begin{cases} C\sum_{n=2}^{\infty}\frac{1}{n(\log n)^{3\alpha/\gamma}}\int_1^{\infty}t^{-\alpha/\tau}\mathrm{d}t\max_{y\ge\mu^{\alpha}}y(\log n)^{3\alpha/\gamma}t^{\alpha/\tau}\cdot\mathbb{V}\left\{|X|^{\alpha}>y(\log n)^{3\alpha/\gamma}t^{\alpha/\tau}\right\}, & \text{for } \alpha>\gamma,\\ C\sum_{n=2}^{\infty}\frac{1}{n(\log n)^{3}}\int_1^{\infty}t^{-\gamma/\tau}\mathrm{d}t\max_{y\ge\mu^{\alpha}}y^{\gamma/\alpha}(\log n)^{3}t^{\gamma/\tau}\cdot\mathbb{V}\left\{|X|^{\gamma}>y^{\gamma/\alpha}(\log n)^{3}t^{\gamma/\tau}\right\}\cdot y^{1-\gamma/\alpha}, & \text{for } \alpha\le\gamma,\end{cases}\\
&\le \begin{cases} C\sum_{n=2}^{\infty}\frac{1}{n(\log n)^{3\alpha/\gamma}}\int_1^{\infty}t^{-\alpha/\tau}\mathrm{d}t<\infty, & \text{for } \alpha>\gamma,\\ C\sum_{n=2}^{\infty}\frac{1}{n(\log n)^{3}}\int_1^{\infty}t^{-\gamma/\tau}\mathrm{d}t<\infty, & \text{for } \alpha\le\gamma.\end{cases}
\end{aligned}
$$

(4.15) together with (4.6)–(4.9) completes the proof of Theorem 3.2. □

Proof of Theorem 3.3. The proof here is similar to that of Corollary 3.1 of Xu and Kong [6] with Theorem 3.1 here in place of Theorem 3.1 of Xu and Kong [6] (also cf. the proof of Theorem 2.11 of Yan [22]), hence the proof here is omitted. This completes the proof. □

5. Conclusions

We have obtained new results about complete convergence and complete moment convergence for maximal weighted sums of extended negatively dependent random variables under sub-linear expectations. Results obtained in our article generalize those for extended negatively dependent random variables in probability space, and Theorems 3.1–3.3 complement the results of Xu et al. [5], Xu and Kong [6] in some sense. In addition, Lemma 2.3 is the Rosenthal-type inequality for extended negatively dependent random variables, which is another main innovation point of this article.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This study was supported by Science and Technology Research Project of Jiangxi Provincial Department of Education of China (No. GJJ2201041), Doctoral Scientific Research Starting Foundation of Jingdezhen Ceramic University (No. 102/01003002031), Academic Achievement Re-cultivation Project of Jingdezhen Ceramic University (Grant No. 215/20506277).

Conflict of interest

The author states no conflict of interest in this article.

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