Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 8(8): 19413–19426.
DOI: 10.3934/math.2023990
Received: 07 March 2023
Revised: 23 May 2023
Accepted: 28 May 2023
Published: 08 June 2023

Research article

Spatial patterns for a predator-prey system with Beddington-DeAngelis functional response and fractional cross-diffusion

Pan Xue* **and Cuiping Ren**

School of General Education, Xi'an Eurasia University, Xi'an, Shaanxi 710065, China

* **Correspondence:** Email: xuepan871215@163.com; Tel: +86 15771739695.

Abstract: In this paper, we investigate a predator-prey system with fractional type cross-diffusion incorporating the Beddington-DeAngelis functional response subjected to the homogeneous Neumann boundary condition. First, by using the maximum principle and the Harnack inequality, we establish a priori estimate for the positive stationary solution. Second, we study the non-existence of non-constant positive solutions mainly by employing the energy integral method and the Poincaré inequality. Finally, we discuss the existence of non-constant positive steady states for suitable large self-diffusion d_2 or cross-diffusion d_4 by using the Leray-Schauder degree theory, and the results reveal that the diffusion d_2 and the fractional type cross-diffusion d_4 can create spatial patterns.

Keywords: fractional type cross-diffusion; predator-prey system; Beddington-DeAngelis functional response; stationary solution; Leray-Schauder degree
Mathematics Subject Classification: 35B32, 35J65, 92D25

1. Introduction

In this paper, we investigate the following predator-prey system with cross-diffusion incorporating the Beddington-DeAngelis functional response subjected to the homogeneous Neumann boundary condition

$$\begin{cases} u_t - d_1\Delta[(1+d_3v)u] = u(1-u-\frac{v}{1+au+bv}), & (x,t)\in\Omega\times(0,\infty),\\ v_t - d_2\Delta[(1+\frac{d_4}{1+u})v] = v(-d+\frac{cu}{1+au+bv}), & (x,t)\in\Omega\times(0,\infty),\\ u(x,0)=u_0(x)\geq 0, v(x,0)=v_0(x)\geq 0, & x\in\Omega,\\ \partial_\nu u=\partial_\nu v=0, & (x,t)\in\partial\Omega\times(0,\infty), \end{cases} \tag{1.1}$$

where $\Omega\subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$, $N\geq 1$ is an integer, $\Delta=\sum_{i=1}^N$ is the Laplace operator in R^N, ν represents the outward unit normal vector on the boundary $\partial\Omega$ with $\partial_\nu=\frac{\partial}{\partial\nu}$, and the homogeneous Neumann boundary condition means that the individuals do not cross the habitat boundary, u and v represent the densities of prey and predator, respectively. The parameters a, b, c and d are all positive constants. The interaction between the prey and the predator of system (1.1) is the most usually used the Beddington-DeAngelis functional response

$$p(u, v) = \frac{u}{1 + au + bv},$$

which was introduced by Beddington and DeAngelis, where, the parameters $a, b, c > 0$ are the saturation constant for an alternative prey, the predator interference and the consumption rate, respectively. The term au in the denominator describing mutual interference among the preys while the term bv describing that among the predators. It is well known that the Beddington-DeAngelis functional response has desirable qualitative features of ratio-dependent form but takes care of their controversial behaviors at low densities. Compared with Holling-II functional response, Beddington and DeAngelis response, which considered both the mutual interference among the predator and the handing time of each prey, is more reasonable. One can refer to [1–4] for more details on the background of this functional response.

In view of the inhomogeneous distribution of the predator and prey in different spatial locations with a fixed domain Ω at any given time and the natural tendency of each species to diffuse, we take into account the predator-prey system (1.1), with self- and cross-diffusions. The role of diffusion into the modelling has been extensively studied. Generally speaking, the diffusion process usually gives rise to a stabilizing effect so that generates a constant equibrium state, namely, the spatial pattern of morphogen or chemical concentration. System (1.1) implies that, in addition to the dispersive force, the diffusion also depends population pressure from other species. The flux of diffusion to the predators of the system is

$$-\nabla(1 + \frac{d_4}{1 + u})v = -(1 + \frac{d_4}{1 + u})\nabla v + \frac{d_4 v}{(1 + u)^2}\nabla u.$$

The part $\frac{d_4 v}{(1+u)^2}\nabla u$ of the diffusion flux is directed toward the increasing densities of the prey, which implies the preys respond to attack of team for the movement of predators. The part $-(1 + \frac{d_4}{1+u})\nabla v$ of the diffusion flux is directed toward the decreasing densities of the predators, which indicates that the predators move towards the preys. The interplay between these diffusion terms and the population dynamics given by (1.1) can lead to complex spatial patterns in the predator-prey system. For example, it may be possible for regions with high predator density to drive local extinction of the prey, leading to further reduction in predator density and eventual recolonization of the area by prey. This kind of spatial dynamics is often observed in natural ecological systems, and can have important implications for conservation efforts and management of ecosystems. For more details about the biological significance, one can see [5–7,14,15] for references, in which the predator-prey system with self- and cross-diffusions were considered.

In [8], by employing the Fixed point index theory, the authors studied the existence of the non-constant steady state of a predator-prey with the Beddington-DeAngelis functional response, in which the cross-diffusions are linear. Paper [9] considers the stationary problem of the Holling-Tanner prey-predator model with fractional type cross-diffusion terms, and result reveals that the large cross-diffusion can create spatial patterns. Papers [10,11] mainly consider the existence of the non-constant positive solutions by making use of the Leray-Schauder degree theory, Furthermore, the authors also discussed the Turing instability of a Gause-type predator-prey system with self-and cross-diffusions mainly by considering the influence of the diffusion terms.

Although lots of researchers have investigated the predator-prey system with nonlinear diffusions(see[12–15] for reference), there are still many open problem on the spatial patterns cased by the linear and nonlinear cross-diffusions. To our knowledge, there are few works focused on system (1.1), which included the Beddington-DeAngelis functional response and a fractional cross-diffusion for the predator. Paper [16] considers a strongly coupled partial differential equation model with a non-monotonic functional response—a Holling type-IV function in a bounded domain with no flux boundary condition. The authors proved a number of existence and non-existence results concerning non-constant steady states of the underlying system. The main purpose of this paper is to research into the effect of the self- and cross-diffusions on the non-constant positive solutions of system (1.1), namely, we investigate the existence and non-existence of the non-constant positive solutions to the following elliptic system

$$\begin{cases} -d_1\Delta[(1+d_3v)u] = u(1-u-\frac{v}{1+au+bv}), & x\in\Omega,\\ -d_2\Delta[(1+\frac{d_4}{1+u})v] = v(-d+\frac{cu}{1+au+bv}), & x\in\Omega,\\ \partial_\nu u = \partial_\nu v = 0, & x\in\partial\Omega. \end{cases} \tag{1.2}$$

In system (1.2), the cross-diffusions implies that the movement of the species at any spatial location is influenced by the gradient of the concentration of the interacting species at that location. By taking these facts into account, the system can capture much more richer phenomena, and this deserves our careful study and discussion.

The organization of this article is as follows. In Section 2, we give a priori estimate for the positive stationary solution by using the maximum principle and the Harnack inequality. In Section 3, we study the non-existence of non-constant positive solutions mainly by employing the energy integral method and the Poincaré inequality. Moreover, we also discuss the existence of non-constant positive steady states for suitable self- and cross-diffusion coefficients by employing the Leray-Schauder degree theory. The results reveal that the diffusion d_2 and the fractional type cross-diffusion d_4 can create spatial patterns.

2. The priori estimate of non-constant positive solutions

We know that there exist three non-negative constant solutions $(0,0)$,$(1,0)$ and (u^*,v^*) for system (1.1), where

$$u^* = \frac{1}{2bc}[ad+bc-c+\sqrt{(ad+bc-c)^2+4bcd}],\quad v^* = \frac{c}{d}(1-u^*)u^*,$$

and $u^*<1$ provided by the condition $c>d(1+a)$. Therefore, it is necessary to assume that $c>d(1+a)$ holds throughout this paper so as to (u^*,v^*) is the unique positive constant solution of system (1.2). For convenience, we denote $\Theta=\Theta(a,b,c,d)$ in the sequel.

In this section, in order to obtain a priori estimates of the positive solution of system (1.2), we first present the following lemmas, named the Maximum principle and Harnack inequality[17,18], respectively.

Lemma 2.1. *Let $\phi(u,x)\in C(\Omega\times R^1)$. If $u\in C^2(\Omega)\cap C^1(\overline{\Omega})$ satisfies $\Delta u(x)+\phi(u,x)\ge 0$ in Ω, $\partial_\nu u=0$ on $\partial\Omega$ and achieves its maximum at a point $x_0\in\overline{\Omega}$, then $-\Delta u(x_0)\ge 0$.*

Lemma 2.2. *Assume that $c(x)\in C(\overline{\Omega})$ and let $\omega(x)\in C^2(\Omega)\cap C^1(\overline{\Omega})$ be a positive solution to*

$$\Delta\omega(x)+c(x)\omega(x)=0,\quad x\in\Omega,\quad \partial_\nu\omega=0,\quad x\in\partial\Omega.$$

Then there exists a positive constant $C^* = C^*(\|c\|_\infty, \Omega)$, such that $\max_{\overline{\Omega}} \omega \le C^* \min_{\overline{\Omega}} \omega$.

Theorem 2.3. *Let C_1, D_1, D_2, D_3 be given positive constants. Then there exists a positive constant $C = C(\Theta, D_1, D_2, D_3, \Omega)$ which is independent of $d_i (i = 1, 2, 3, 4)$, such that every positive solution (u, v) of system (1.2), satisfies $C^{-1} < u(x), v(x) < C$ for $d_1 \ge D_1$, $d_2 \ge D_2$, $d_3 \le D_3$, $C_1 > \frac{1}{a+c}$ and $b < 1$.*

Proof. Let $\phi_1 = d_1[(1 + d_3 v)u], \phi_2 = d_2[(1 + \frac{d_4}{1+u})v]$. Assume that there exists a point $x_0 \in \overline{\Omega}$ such that $\phi_1(x_0) = \max_{\overline{\Omega}} \phi_1$. According to Lemma 2.1, we obtain $u(x_0) < 1, v(x_0) \le \frac{1+a}{1-b}$, then

$$\max_{\overline{\Omega}} u \le \frac{1}{d_1} \max_{\overline{\Omega}} \phi_1 \le \frac{1}{d_1}(1 + d_3 v(x_0))u(x_0) \le \frac{1}{D_1}(1 + D_3 \frac{1+a}{1-b}) \triangleq C_1,$$

where $b < 1$, D_1 and D_3 are given positive constants with $d_1 \ge D_1$, $d_3 \le D_3$. Let $x_1 \in \overline{\Omega}$ be a point such that $\phi_2(x_1) = \max_{\overline{\Omega}} \phi_2$. Then, by the maximum principle, we have

$$v(x_1) \le \frac{(a + c)C_1 - 1}{b},$$

where $C_1 > \frac{1}{a+c}$. Therefore, we can obtain

$$\max_{\overline{\Omega}} v \le \frac{1}{d_2} \frac{\max_{\overline{\Omega}} \phi_2}{\min_{\overline{\Omega}} (1 + \frac{d_4}{1+u(x)})} = \frac{1 + d_4}{1 + \frac{d_4}{1+C_1}} v(x_1) \le (1 + C_1) \frac{(a + c)C_1 - 1}{b} \triangleq C_2.$$

as $\frac{1+d_4}{1+\frac{d_4}{1+C_1}}$ is strictly increasing with respect to d_4 and satisfies $\lim_{d_4 \to +\infty} \frac{1+d_4}{1+\frac{d_4}{1+C_1}} = 1 + C_1$. Thus, we obtain the upper bounds of the solution (u, v).

Hereinafter, we show that (u, v) has a lower bound. For convenience, we set

$$c_1(x) = \frac{1}{d_1(1 + d_3 v)}(1 - u - \frac{v}{1 + au + bv}), \;\; c_2(x) = \frac{1}{d_2(1 + \frac{d_4}{1+u})}(-d + \frac{cu}{1 + au + bv}).$$

Then, system (1.2) can be written as

$$\begin{cases} -\Delta\phi_1(x) = c_1(x)\phi_1(x), & x \in \Omega, \\ -\Delta\phi_2(x) = c_2(x)\phi_2(x), & x \in \Omega, \\ \partial_\nu \phi_1 = \partial_\nu \phi_2 = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Since $\|c_1(x)\|_\infty < C_3(\Theta, D_1, D_2, D_3, \Omega)$, according to Lemma 2.2, we know that there exists a positive constant $C_4 = C_4(\Theta, D_1, D_2, D_3, \Omega)$ and $C_5 = C_5(\Theta, D_1, D_2, D_3, \Omega)$ such that

$$\max_{\overline{\Omega}} \phi_1 \le C_4 \min_{\overline{\Omega}} \phi_1,$$

and

$$\frac{\max_{\overline{\Omega}} u}{\min_{\overline{\Omega}} u} \le \frac{\max_{\overline{\Omega}} \phi_1}{\min_{\overline{\Omega}} \phi_1} \cdot \frac{1 + d_3 \max_{\overline{\Omega}} v}{1 + d_3 \min_{\overline{\Omega}} v} \le C_4(1 + d_3 \max_{\overline{\Omega}} v) \le C_4(1 + D_3 C_2) \triangleq C_5. \tag{2.2}$$

Similarly, as $\|c_2(x)\|_\infty < \frac{1}{D_2}(d + \frac{c}{a}) \triangleq C_6$, Lemma 2.2 holds for ϕ_2, that is,

$$\max_{\overline{\Omega}} \phi_2 \le C_7 \min_{\overline{\Omega}} \phi_2,$$

for some positive constant C_7. Therefore, we have

$$\frac{\max_{\overline{\Omega}} v}{\min_{\overline{\Omega}} v} \leq \frac{\max_{\overline{\Omega}} \phi_2}{\min_{\overline{\Omega}} \phi_2} \cdot \frac{\max_{\overline{\Omega}}(1+\frac{d_4}{1+u})}{\min_{\overline{\Omega}}(1+\frac{d_4}{1+u})} \leq C_7 \frac{1+d_4}{1+\frac{d_4}{1+C_1}} \leq C_7(1+C_1) \triangleq C_8, \tag{2.3}$$

for some positive constant C_8. Thus, if the positive solution (u, v) does not have positive lower bound, then there exists a sequence $\{(d_{1i}, d_{2i}, d_{3i}, d_{4i})\}$ satisfying $d_{1i} \geq D_1, d_{2i} \geq D_3$ and $d_{3i} \leq D_3$ such that the corresponding solutions of system (1.2) satisfy $\min_{\overline{\Omega}} u_i \to 0$, or $\min_{\overline{\Omega}} v_i \to 0$ as $i \to \infty$. Due to the Harnack inequality, we have $\max_{\overline{\Omega}} u_i \to 0$, or $\max_{\overline{\Omega}} v_i \to 0$ as $i \to \infty$.

By integrating the second equation of system (1.2) over Ω, we obtain $\int_\Omega v(-d+\frac{cu}{1+au+bv})dx = 0$. Thus, there must exists a point $x_1 \in \overline{\Omega}$ such that $\frac{cu(x_1)}{1+au(x_1)+bv(x_1)} = d$, which implies $u(x_1) \geq \frac{d}{c}$. Thus, combined with (2.2), we have

$$\min_{\overline{\Omega}} u \geq \frac{\max_{\overline{\Omega}} u}{C_5} = \frac{d}{cC_5}.$$

Similarly, by integrating the first equation of system (1.2) over Ω, we obtain

$$\int_\Omega u(1-u-\frac{v}{1+au+bv})dx = 0.$$

Thus, there must exists a point $x_2 \in \Omega$ such that

$$1-u(x_2)-\frac{v(x_2)}{1+au(x_2)+bv(x_2)} = 0.$$

Furthermore, we get

$$\max_{\overline{\Omega}} v \geq v(x_2) \geq \frac{v(x_2)}{1+au(x_2)+bv(x_2)} = 1-u(x_2) > 0.$$

According to the inequality (2.3), if $\min_{\overline{\Omega}} v \to 0$, we have $\max_{\overline{\Omega}} v \to 0$. Hence, we get a contradiction. This shows v has a positive lower bound. This completes the proof.

Remark 2.4. Theorem 2.3 shows that if d_1 is not too large or not too small, d_2 is not too small and d_3 is not too large, then the solutions of system (1.2) are bounded, that is to say, there exists a ball $\boldsymbol{B}(C)$ such that all the positive solution (u, v) of (1.2) satisfying $(u, v) \in \boldsymbol{B}(C)$. We can also conclude that the bound of the solution is not constrained by the cross diffusion d_4.

3. Non-existence and existence of the non-constant positive stationary solutions

The purpose of this section is to study the non-existence and existence of non-constant positive stationary solutions of system (1.2) by taking the self- and cross-diffusions as parameters. The main method used to prove the existence of non-constant positive solutions is the Leray-Schauder degree. The results show that the the self-diffusion d_2 and the fractional type cross-diffusion d_4 can create spatial patterns while d_1 and d_3 failed.

3.1. Non-existence of the non-constant positive stationary solutions

This subsection is devoted to investigating the non-existence of non-constant positive solutions of system (1.2). We mainly use the energy integral method and the well-known Poincaré inequality. Let μ_1 be the smallest positive eigenvalue of the operator $-\Delta$ subject to the homogeneous Neumann boundary condition. For convenience, we set

$$\tilde{u} = \frac{1}{|\Omega|}\int_\Omega u dx \qquad \text{and} \qquad \tilde{v} = \frac{1}{|\Omega|}\int_\Omega v dx.$$

Through some calculations and analysis, we obtain the following result.

Theorem 3.1. *Let d_1, d_2, d_4 be fixed positive constants and $d_3 = 0$. If there exists a positive constant $\tilde{C} = \tilde{C}(\Theta, D_1, \Omega)$, such that $d_1 > \max\{D_1, \tilde{C}(1 + d_2^2 d_4^2)\}$, $d_2 > D_2$ and $d_4 < D_4$, then system (1.2) has no non-constant positive solution.*

Proof. Assume that (u, v) is a positive solution of system (1.2). We multiply $\frac{u-\tilde{u}}{u}$ and $\frac{v-\tilde{v}}{v}$ to the equations of system (1.2), respectively, and integrate the equation by parts in Ω. Then, we have

$$\begin{aligned}
&d_1\int_\Omega \frac{\tilde{u}}{u^2}(1 + d_3 v)|\nabla u|^2 dx + d_2\int_\Omega \frac{\tilde{v}}{v^2}(1 + \frac{d_4}{1+u})|\nabla v|^2 dx - d_2 d_4\int_\Omega \frac{\tilde{v}}{v(1+u)^2}\nabla u\nabla v dx\\
= &\int_\Omega\Big[-1 + \frac{a\tilde{v}}{(1 + a\tilde{u} + b\tilde{v})(1 + au + bv)}\Big](u - \tilde{u})^2 dx\\
&-\int_\Omega\Big[\frac{bc\tilde{u}}{(1 + a\tilde{u} + b\tilde{v})(1 + au + bv)}\Big](v - \tilde{v})^2 dx\\
&+\int_\Omega\Big[\frac{b\tilde{v} - (1 + a\tilde{u} + b\tilde{v})}{(1 + a\tilde{u} + b\tilde{v})(1 + au + bv)} + \frac{c(1 + a\tilde{u} + b\tilde{v}) - ac\tilde{u}}{(1 + a\tilde{u} + b\tilde{v})(1 + au + bv)}\Big](u - \tilde{u})(v - \tilde{v}) dx.
\end{aligned}$$

According to Theorem 2.3, for d_1 with a fixed small D_1, there exists a large enough positive constant $C_1 = C_1(\Theta, D_1, D_2, D_4, \Omega)$, such that

$$C_1^{-1} < u(x), v(x), \tilde{u}, \tilde{v} < C_1.$$

Therefore, we have

$$\begin{aligned}
&\frac{d_1}{C_1^3}\int_\Omega |\nabla u|^2 dx + \frac{d_2}{C_1^3}\int_\Omega |\nabla v|^2 dx\\
\le &\int_\Omega aC_1(u - \tilde{u})^2 dx - \int_\Omega \frac{bc}{(1 + aC_1 + bC_1)^2 C_1}(v - \tilde{v})^2 dx\\
&+\int_\Omega (bcC_1 + c)|u - \tilde{u}||v - \tilde{v}| dx + d_2 d_4 C_1^2\int_\Omega |\nabla u||\nabla v| dx.
\end{aligned}$$

It follows the Young's inequality[19–21],

$$\begin{aligned}
&\frac{d_1}{C_1^3}\int_\Omega |\nabla u|^2 dx + \frac{d_2}{C_1^3}\int_\Omega |\nabla v|^2 dx\\
\le &\int_\Omega\Big[aC_1 + \frac{(bcC_1 + c)^2}{4K}\Big](u - \tilde{u})^2 dx + \int_\Omega\Big[K - \frac{bc}{(1 + aC_1 + bC_1)^2 C_1}\Big](v - \tilde{v})^2 dx
\end{aligned}$$

$$+\frac{d_2^2 d_4^2 C_1^4}{4K}\int_\Omega |\nabla u|^2 dx + K\int_\Omega |\nabla v|^2 dx,$$

where K is a arbitrary small positive constant. For convenience, we take $d_3 = 0$ and $K = \min\{\frac{d_2}{C_1^3}, \frac{bc}{(1+aC_1+bC_1)^2 C_1}\}$. By employing the Cauchy inequality and Poincaré inequality[22,23], we have

$$\begin{aligned}
&\frac{d_1}{C_1^3}\int_\Omega |\nabla u|^2 dx \le \int_\Omega \Big[aC_1 + \frac{(bcC_1+c)^2}{4K}\Big](u-\tilde{u})^2 dx + \frac{d_2^2 d_4^2 C_1^4}{4K}\int_\Omega |\nabla u|^2 dx \\
\le\ & C_2(\Theta, D_1, D_2, D_4, \Omega)\int_\Omega (u-\tilde{u})^2 dx + d_2^2 d_4^2 C_3(\Theta, D_1, D_2, D_4, \Omega)\int_\Omega |\nabla u|^2 dx \\
\le\ & \Big(\frac{C_2(\Theta, D_1, D_2, D_4, \Omega)}{\mu_1} + d_2^2 d_4^2 C_3(\Theta, D_1, D_2, D_4, \Omega)\Big)\int_\Omega |\nabla u|^2 dx \\
\le\ & C_4(\Theta, D_1, D_2, D_4, \Omega)(1 + d_2^2 d_4^2)\int_\Omega |\nabla u|^2 dx.
\end{aligned}$$

Therefore, we can assert that $u \equiv \tilde{u}, v \equiv \tilde{v}$ if $d_1 > \max\{D_1, \tilde{C}(1 + d_2^2 d_4^2)\}$, $d_2 > D_2$ and $d_4 < D_4$.

3.2. Existence of non-constant positive solutions

In this subsection, we mainly consider the existence of non-constant positive solution of system (1.2) by taking the self- and cross- diffusion coefficients as parameters. Particularly, combing with Theorems 2.3 and 3.1, we consider the cases that the self-diffusion d_2 or the cross-diffusion d_4 is large enough. The key method used in this subsection to prove the existence of non-constant positive solutions is the well-known Leray-Schauder degree theory [24–26], which has been extensively used in many different papers. In order to establish the existence of stationary patterns of system (1.2), for convenience, we first introduce some notations and definitions. We define

$$\boldsymbol{\omega} = (u, v)^T, \quad \boldsymbol{G}(\boldsymbol{\omega}) = \Big(u\Big(1 - u - \frac{v}{1+au+bv}\Big), v\Big(-d + \frac{cu}{1+au+bv}\Big)\Big)^T,$$

$$\boldsymbol{\Phi}(\boldsymbol{\omega}) = (\phi_1(\boldsymbol{\omega}), \phi_2(\boldsymbol{\omega}))^T = \Big(d_1(1+d_3 v)u, d_2(1 + \frac{d_4}{1+u})v\Big)^T, \quad \Lambda = (d_1, d_2, d_3, d_4)$$

and set

$$\boldsymbol{X} = \{\boldsymbol{\omega} = (u, v)^T \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^2 | \partial_\nu u = \partial_\nu v = 0 \text{ on } \partial\Omega\},$$

$$\boldsymbol{X}^+ = \{\boldsymbol{\omega} \in \boldsymbol{X} | u > 0, v > 0 \ on\ \overline{\Omega}\}.$$

$$\boldsymbol{B}(C) = \{(u, v)^T \in \boldsymbol{X} | C^{-1} < u, v < C\},$$

where C is a positive constant provided by Theorem 2.3. Let $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$ be the eigenvalues of the operator $-\Delta$ and $\{\mu_i, \psi_i\}_{i=0}^\infty$ be a complete set of eigenpairs for the operator $-\Delta$ in Ω under homogeneous Neumann boundary condition. Moreover, we can decompose $\boldsymbol{X} = \oplus_{i=0}^\infty \boldsymbol{X}_i$ and $\boldsymbol{X}_i = \oplus_{j=1}^{\dim E(\mu_i)} \boldsymbol{X}_{ij}$, where $\boldsymbol{X}_i$ is the eigenspace corresponding to the eigenvalue μ_i.

Therefore, system (1.2) can be rewritten as

$$\begin{cases} -\Delta\boldsymbol{\Phi}(\boldsymbol{\omega}) = \boldsymbol{G}(\boldsymbol{\omega}), & x \in \Omega, \\ \partial_\nu \boldsymbol{\omega} = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

It is clear that system (3.1), as well as system (1.2), has a constant positive equilibrium point, denoted by $\boldsymbol{\omega}^* = (u^*,\ v^*)^T$, where $u^*,\ v^*$ are given in Section 2.

By direct computation, we obtain $\frac{\partial \boldsymbol{\Phi}(u,v)}{\partial(u,v)} = \begin{bmatrix} d_1 + d_1 d_3 v & d_1 d_3 u \\ \frac{-d_2 d_4 v}{(1+u)^2} & d_2 + \frac{d_2 d_4}{1+u} \end{bmatrix}$ and it is easy to show that $\det\left[\frac{\partial \boldsymbol{\Phi}(u,v)}{\partial(u,v)}\right] > 0$ for all non-negative solutions $(u,v)^T$. Therefore, we know that $\boldsymbol{\Phi}_\omega^{-1}$ exists and $\det\left[\frac{\partial \boldsymbol{\Phi}(u,v)}{\partial(u,v)}\right]^{-1}$ is positive. Then, system (1.2) can also be rewritten as

$$\boldsymbol{F}(\Lambda;\boldsymbol{\omega}) \triangleq \boldsymbol{\omega} - (\boldsymbol{I} - \Delta)^{-1}\{\boldsymbol{\Phi}_\omega^{-1}(\omega)[\boldsymbol{G}(\omega) + \nabla\omega\boldsymbol{\Phi}_{\omega\omega}(\omega)\nabla\omega] + \boldsymbol{\omega}\} = 0, \tag{3.2}$$

where $\boldsymbol{I}$ is the identity operator and $(\boldsymbol{I} - \Delta)^{-1}$ is the inverse of the operator $\boldsymbol{I} - \Delta$ on $\boldsymbol{X}$ with the homogeneous Neumann boundary condition. Since $\boldsymbol{F}(\Lambda;\cdot)$ is a compact perturbation of the identity operator, the Leray-Schauder degree $\deg(\boldsymbol{F}(\Lambda;\cdot),\boldsymbol{0},\boldsymbol{B})$ is well-defined if $\boldsymbol{F}(\Lambda;\boldsymbol{\omega}) \neq 0$ for all $\boldsymbol{\omega} \in \partial\boldsymbol{B}$. Furthermore, we notice that the linearizition of the operator $\boldsymbol{F}(\Lambda;\boldsymbol{\omega})$ at the equilibrium point $\boldsymbol{\omega}^*$ is

$$D_\omega\boldsymbol{F}(\Lambda;\boldsymbol{\omega}^*) = \boldsymbol{I} - (\boldsymbol{I} - \Delta)^{-1}[\boldsymbol{\Phi}_\omega^{-1}(\omega^*)\boldsymbol{G}_\omega(\omega^*) + \boldsymbol{I}],$$

and $\boldsymbol{X}_i$ is invariant under $D_\omega\boldsymbol{F}(\Lambda;\omega^*)$ for every integer $i \geq 0$. What's more, one can check that λ is an eigenvalue of the operator $D_\omega\boldsymbol{F}(\Lambda;\omega^*)$ on $\boldsymbol{X}_i$ if and only if λ is an eigenvalue of the matrix

$$\boldsymbol{I} - \frac{1}{1+\mu_i}[\boldsymbol{\Phi}_\omega{}^{-1}(\omega^*)\boldsymbol{G}_\omega(\omega^*) + \boldsymbol{I}] = \frac{1}{1+\mu_i}[\mu_i\boldsymbol{I} - \boldsymbol{\Phi}_\omega{}^{-1}(\omega^*)\boldsymbol{G}_\omega(\omega^*)].$$

Denote

$$H(\Lambda,\omega^*;\mu_i) \triangleq \det[\mu_i\boldsymbol{I} - \boldsymbol{\Phi}_\omega{}^{-1}(\omega^*)\boldsymbol{G}_\omega(\omega^*)] = \det\{[\boldsymbol{\Phi}_\omega(\omega^*)]^{-1}\}\det[\mu\boldsymbol{\Phi}_\omega(\omega^*) - \boldsymbol{G}_\omega(\omega^*)].$$

As the sign of the

$$\det\{\boldsymbol{I} - \frac{1}{1+\mu_i}[\boldsymbol{\Phi}_\omega{}^{-1}(\omega^*)\boldsymbol{G}_\omega(\omega^*) + \boldsymbol{I}]\}$$

is determined by the number of negative eigenvalue of the matrix

$$\boldsymbol{I} - \frac{1}{1+\mu_i}[\boldsymbol{\Phi}_\omega{}^{-1}(\omega^*)\boldsymbol{G}_\omega(\omega^*) + \boldsymbol{I}],$$

then both $H(\Lambda,\boldsymbol{\omega}^*;\mu_i)$ and $\det\{\boldsymbol{I} - \frac{1}{1+\mu_i}[\boldsymbol{\Phi}_\omega{}^{-1}(\omega^*)\boldsymbol{G}_\omega(\omega^*) + \boldsymbol{I}]\}$ have the same sign. Hence, if $H(\Lambda,\boldsymbol{\omega}^*;\mu_i) \neq 0$, the number of eigenvalues with negative real parts of $\boldsymbol{D}_\omega\boldsymbol{F}(\Lambda;\boldsymbol{\omega}^*)$ on $\boldsymbol{X_i}$ is odd if and only if $H(\Lambda,\boldsymbol{\omega}^*;\lambda_i) < 0$.

If $H(\Lambda,\boldsymbol{\omega}^*;\mu_i) \neq 0$ for all integer $i \geq 0$, then 0 is not an eigenvalue of the operator $\boldsymbol{D}_\omega\boldsymbol{F}(\Lambda;\boldsymbol{\omega}^*)$. This indicates that $\boldsymbol{D}_\omega\boldsymbol{F}(\Lambda;\omega^*)$ is a homeomorphism operator from the space $\boldsymbol{X}$ to $\boldsymbol{X}$. Then the implicit function theorem shows that the equilibrium point $\boldsymbol{\omega} = \boldsymbol{\omega}^*$ is an isolated solution of equation $\boldsymbol{F}(\Lambda;\omega) = 0$. In summary, according to Leray-Schauder degree theory, we present the following results (One can refer to [27,28]).

Lemma 3.3. *Assume that the matrix $\mu_i\boldsymbol{I} - \boldsymbol{\Phi}_\omega{}^{-1}(\omega^*)\boldsymbol{G}_\omega(\omega^*)$ is non-singular for each $i > 0$. Then*

$$\text{index}(\boldsymbol{F}(\Lambda;\cdot),\omega^*) = (-1)^\tau, \quad \tau = \sum_{i\geq 0, H(D,\omega^*;\mu_i)<0} \dim \boldsymbol{E}(\mu_i).$$

In order to calculate index$(\boldsymbol{F}(\Lambda;\cdot),\boldsymbol{\omega}^*)$, we will consider the sign of $H(\Lambda,\boldsymbol{\omega}^*;\mu_i)$ in detail. Notice that $\det\{\boldsymbol{\Phi}_\omega(\boldsymbol{\omega}^*)^{-1}\}>0$, so we only need to consider $\det[\mu\boldsymbol{\Phi}_\omega(\boldsymbol{\omega}^*)-\boldsymbol{G}_\omega(\boldsymbol{\omega}^*)]$. Review that

$$\begin{cases} u^* = \frac{1}{2bc}[ad+bc-c+\sqrt{(ad+bc-c)^2+4bcd}],\\ v^* = \frac{c}{d}(1-u^*)u^* \end{cases}$$

and

$$1-u^*-\frac{v^*}{1+au^*+bv^*}=0,\quad -d+\frac{cu^*}{1+au^*+bv^*}=0.$$

Then, by direct calculation, we obtain

$$\boldsymbol{\Phi}_\omega(\boldsymbol{\omega}^*)=\begin{bmatrix} d_1(1+d_3v^*) & d_1d_3u^* \\ \frac{-d_2d_4v^*}{(1+u^*)^2} & d_2+\frac{d_2d_4}{1+u^*}\end{bmatrix},$$

$$\begin{aligned}\boldsymbol{G}_\omega(\boldsymbol{\omega}^*)&=\begin{bmatrix} 1-2u^*-\frac{v^*(1+bv^*)}{(1+au^*+bv^*)^2} & -\frac{u^*(1+au^*)}{(1+au^*+bv^*)^2} \\ \frac{cv^*(1+bv^*)}{(1+au^*+bv^*)^2} & -\frac{bcu^*v^*}{(1+au^*+bv^*)^2}\end{bmatrix}\\ &=\begin{bmatrix} 1-2u^*-(1-u^*)^2(b+\frac{1}{v^*}) & -\frac{d^2}{c^2}(a+\frac{1}{u^*}) \\ c(1-u^*)^2(b+\frac{1}{v^*}) & bd(u^*-1)\end{bmatrix}\end{aligned}$$

and

$$\det[\mu\boldsymbol{\Phi}_\omega(\boldsymbol{\omega}^*)-\boldsymbol{G}_\omega(\boldsymbol{\omega}^*)]=A\mu^2+B\mu+C\triangleq\psi(\mu),\tag{3.3}$$

where

$$\begin{aligned} A&=d_1(1+d_3v^*)(d_2+\frac{d_2d_4}{1+u^*})+d_1d_2d_3d_4\frac{u^*v^*}{(1+u^*)^2}>0,\\ B&=(1-u^*)\Big[d_1d_3du^*(1+bv^*)+\frac{d_2d_4c(1-u^*)(1+bv^*)}{(1+u^*)^2}\\ &\quad +d_1bd(1+d_3v^*)-\Big(d_2+\frac{d_2d_4}{1+u^*}\Big)\frac{c(1+bv^*)}{1+au^*+bv^*}\Big],\\ C&=\frac{d^2}{c^2}(a+\frac{1}{u^*})c(1-u^*)^2(b+\frac{1}{v^*})-\Big[1-2u^*-(1-u^*)^2\Big(b+\frac{1}{v^*}\Big)\Big]bd(1-u^*).\end{aligned}$$

Denote $\mu_1(\Lambda)$ and $\mu_2(\Lambda)$ be the two roots of the equation $A\mu^2+B\mu+C=0$ with $\mathrm{Re}\mu_1(\Lambda)\le\mathrm{Re}\mu_2(\Lambda)$. We notice that $C>0$ guaranteed by $2u^*+(1-u^*)^2\Big(b+\frac{1}{v^*}\Big)>1$, and we obtain

$$\mu_1(\Lambda)\mu_2(\Lambda)=\frac{C}{A}>0.$$

Moreover, we have

$$\lim_{d_2\to+\infty}\frac{\psi(\mu)}{d_2}=A_1\mu^2+B_1\mu,\quad \lim_{d_4\to+\infty}\frac{\psi(\mu)}{d_4}=A_2\mu^2+B_2\mu,$$

where

$$A_1=\lim_{d_2\to+\infty}\frac{A}{d_2}=d_1(1+d_3v^*)(1+\frac{d_4}{1+u^*})+d_1d_3d_4\frac{u^*v^*}{(1+u^*)^2}>0,$$

$$B_1 = \lim_{d_2\to+\infty} \frac{B}{d_2} = (1-u^*)\Big[\frac{d_4 c(1-u^*)(1+bv^*)}{(1+u^*)^2} - \Big(1+\frac{d_4}{1+u^*}\Big)\frac{c(1+bv^*)}{1+au^*+bv^*}\Big],$$
$$A_2 = \lim_{d_4\to+\infty} \frac{A}{d_4} = d_1 d_2(1+d_3 v^*)\frac{1}{1+u^*} + d_1 d_2 d_3 \frac{u^* v^*}{(1+u^*)^2} > 0,$$
$$B_2 = \lim_{d_4\to+\infty} \frac{B}{d_4} = B = (1-u^*)\Big[\frac{d_2 c(1-u^*)(1+bv^*)}{(1+u^*)^2} - \frac{d_2}{1+u^*}\frac{c(1+bv^*)}{1+au^*+bv^*}\Big].$$

We notice that A_1, A_2 are always positive while B_1, B_2 may change the sign. In conclusion, we can obtain the following lemmas.

Lemma 3.4. *Assume that $d_i(i = 1,3,4)$ are all fixed and $B_1 < 0$. Then there exists a positive constant $\hat{d}_2$, such that for $d_2 \geq \hat{d}_2$, the two roots $\tilde{\mu}_1(d_2)$ and $\tilde{\mu}_2(d_2)$ of the equation $\psi(\mu) = 0$ are all real and satisfy*

$$\lim_{d_2\to+\infty} \tilde{\mu}_1(d_2) = 0, \quad \lim_{d_2\to+\infty} \tilde{\mu}_2(d_2) = -\frac{B_1}{A_1},$$

where

$$\begin{cases} \psi(\mu, d_2) < 0, & if\ \mu \in (\tilde{\mu}_1(d_2), \tilde{\mu}_2(d_2)), \\ \psi(\mu, d_2) > 0, & if\ \mu \in (-\infty, \tilde{\mu}_1(d_2)) \cup (\tilde{\mu}_2(d_2), +\infty). \end{cases} \tag{3.4}$$

Lemma 3.5. *Assume that $d_i(i = 1,2,3)$ are all fixed and $B_2 < 0$. Then there exists a positive constant $\hat{d}_4$, such that for $d_4 \geq \hat{d}_4$, the two roots $\tilde{\mu}_1(d_4)$ and $\tilde{\mu}_2(d_4)$ of the equation $\psi(\mu) = 0$ are all real and satisfy*

$$\lim_{d_4\to+\infty} \tilde{\mu}_1(d_4) = 0, \quad \lim_{d_4\to+\infty} \tilde{\mu}_2(d_4) = -\frac{B_2}{A_2},$$

where

$$\begin{cases} \psi(\mu, d_4) < 0, & if\ \mu \in (\tilde{\mu}_1(d_4), \tilde{\mu}_2(d_4)), \\ \psi(\mu, d_4) > 0, & if\ \mu \in (-\infty, \tilde{\mu}_1(d_4)) \cup (\tilde{\mu}_2(d_4), +\infty). \end{cases} \tag{3.5}$$

In the following, by using Leray-Schauder degree theory [29,30], we investigate the existence of non-constant positive solutions to system (1.2) with respect to the diffusion coefficients $d_i, i = 1,2,3,4$. We mainly consider the cases that d_2 or d_4 is large enough and in view of Lemmas 3.4 and 3.5, we can obtain the following theorems. We only prove Theorems 3.6 and 3.7 can be finished similarly.

Theorem 3.6. *Assume that $d_i(i = 1,2,3)$ are all fixed and $B_1 < 0$. For $\tilde{\mu}$ be given by the limit of Lemma 3.5, if $\tilde{\mu} \in (\mu_n, \mu_{n+1})$ for some integer $n \geq 1$ and the sum $\tau_n = \sum_{i=1}^{n} \dim \boldsymbol{E}(\mu_i)$ is odd, then there exists a positive constant $\hat{d}_4$ such that system (1.2) has at least one non-constant positive solution when $d_4 \geq \hat{d}_4$.*

Proof. According to Lemma 3.4, there exists a positive constant $\hat{d}_4$, such that (3.4) holds and $0 = \mu_0 < \tilde{\mu}_1 < \tilde{\mu}_2$, $\tilde{\mu}_2 \in (\mu_n, \mu_{n+1})$ when $d_4 \geq \hat{d}_4$.

We shall prove the result by contradiction. Suppose on the contrary that the result is not true for some $d_4 = \tilde{d}_4 \geq \hat{d}_4$, that is, system (1.2) does not have any positive non-constant positive solution and index$(\boldsymbol{F}(\Lambda;\cdot), \omega^*) = 1$ when $d_4 \geq \hat{d}_4$. If we take $\Lambda^* = (\hat{d}_1, d_2, d_3, \hat{d}_4)$, then,

$$\deg(\boldsymbol{F}(\Lambda^*;\cdot), 0, \boldsymbol{B}) = 1, \tag{3.6}$$

where, $\hat{d}_1$ is a moderately large constant provided by Theorem 2.3.

For $t \in [0,1]$, we define a homotopy as

$$\begin{cases} -\Delta[(td_1 + (1-t)\hat{d}_1 + td_1d_3v)u] = u(1 - u - \frac{v}{1+au+bv}), & x \in \Omega, \\ -\Delta[(td_2 + ((1-t)\hat{d}_4 + td_4)\frac{d_2}{1+u})v] = v(-d + \frac{cu}{1+au+bv}), & x \in \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega. \end{cases} \tag{3.7}$$

Thus, $\boldsymbol{\omega}$ is a non-constant positive solution of system (3.7) if and only if $\boldsymbol{\omega}$ is a non-constant positive solution of the following problem

$$\tilde{\boldsymbol{F}}(t,\Lambda;\boldsymbol{\omega}) \triangleq \boldsymbol{\omega} - (\boldsymbol{I}-\Delta)^{-1}\{\tilde{\boldsymbol{\Phi}}_\omega\}^{-1}(\omega,t)[\boldsymbol{G}(\boldsymbol{\omega}) + \nabla\omega\tilde{\boldsymbol{\Phi}}_{\omega\omega}(\omega,t)\nabla\omega] + \omega\} = 0, \tag{3.8}$$

on $\boldsymbol{X}$, where $\tilde{\boldsymbol{\Phi}} = \Big((td_1 + (1-t)\hat{d}_1 + td_1d_3v)u, (td_2 + ((1-t)\hat{d}_4 + td_4)\frac{d_2}{1+u})v\Big)$. It is clear that

$$\boldsymbol{\Phi}(\boldsymbol{\omega}) = \tilde{\boldsymbol{\Phi}}(\boldsymbol{\omega},1), \quad \boldsymbol{F}(\Lambda;\boldsymbol{\omega}) = \tilde{\boldsymbol{F}}(1,\Lambda;\boldsymbol{\omega}), \quad \boldsymbol{F}(\Lambda^*;\boldsymbol{\omega}) = \tilde{\boldsymbol{F}}(0,\Lambda;\boldsymbol{\omega}).$$

Theorem 3.1 shows that $\tilde{\boldsymbol{F}}(0,\Lambda;\boldsymbol{\omega}) = 0$ only has the constant positive solution $\boldsymbol{\omega}^*$ in $\boldsymbol{X}$. Through some calculation, we obtain

$$D_\omega\tilde{\boldsymbol{F}}(t,\boldsymbol{\omega}^*) = \boldsymbol{I} - (\boldsymbol{I}-\Delta)^{-1}[\boldsymbol{\Phi}_\omega{}^{-1}(t,\boldsymbol{\omega}^*)\boldsymbol{G}_\omega(\boldsymbol{\omega}^*) + \boldsymbol{I}].$$

Furthermore, in view of Lemma 3.5, for $t = 1$, we have

$$\begin{cases} H(\Lambda,\omega^*;\mu_i) < 0, & when\ 1 \le i \le n, \\ H(\Lambda,\omega^*;\mu_i) > 0, & when\ i > n. \end{cases}$$

Then, 0 is not an eigenvalue of the matrix $\mu_i\boldsymbol{I} - [\boldsymbol{\Phi}_\omega]^{-1}\boldsymbol{G}_\omega(\boldsymbol{\omega}^*)$, and $\tau_n = \sum_{i=1}^n \dim \boldsymbol{E}(\mu_i)$ is odd. Moreover, in view of Lemma 3.3, we know that

$$\text{index}(\tilde{\boldsymbol{F}}(1,\Lambda;\cdot),\boldsymbol{\omega}^*) = (-1)^\tau = -1.$$

According to Theorem 2.3, we know that all the positive solutions (u,v) of the system (1.2) are in $\boldsymbol{B}(C)$ for large enough constant C. Therefore, system (3.6) has no solution on $\partial\boldsymbol{B}$ for any $t \in [0,1]$, and $\deg(\tilde{\boldsymbol{F}}(t,\Lambda;\cdot),0,\boldsymbol{B})$ is well defined. According to the homotopy invariance of the Leray-Schauder degree, we obtain

$$\deg(\tilde{\boldsymbol{F}}(0,\Lambda;\cdot),0,\boldsymbol{B}) = \deg(\tilde{\boldsymbol{F}}(1,\Lambda;\cdot),0,\boldsymbol{B}). \tag{3.9}$$

Therefore, we have

$$\deg(\tilde{\boldsymbol{F}}(0,\Lambda;\cdot),0,\boldsymbol{B}) = \deg(\boldsymbol{F}(\Lambda^*;\cdot),0,\boldsymbol{B}) = \text{index}(\boldsymbol{F}(\Lambda;\cdot),\boldsymbol{\omega}^*) = 1, \tag{3.10}$$

$$\deg(\tilde{\boldsymbol{F}}(1,\Lambda;\cdot),0,\boldsymbol{B}) = \deg(\boldsymbol{F}(\Lambda;\cdot),0,\boldsymbol{B}) = \text{index}(\tilde{\boldsymbol{F}}(1,\Lambda;\cdot),\boldsymbol{\omega}^*) = -1. \tag{3.11}$$

From (3.9)–(3.11), we obtain a contradiction. Hence, system (1.2) has at least one non-constant positive solution and the proof is finished.

Theorem 3.7. *Assume that $d_i (i = 1,3,4)$ are all fixed and $B_2 < 0$. For $\tilde{\mu}'$ be given by the limit of Lemma 3.4, if $\tilde{\mu}' \in (\mu_n, \mu_{n+1})$ for some integer $n \ge 1$ and the sum $\tau_n = \sum_{i=1}^n \dim \boldsymbol{E}(\mu_i)$ is odd, then there exists a positive constant $\hat{d}_2$ such that system (1.2) has at least one non-constant positive solution when $d_2 \ge \hat{d}_2$.*

4. Conclusions

This paper investigates the existence of positive stationary solutions of a predator-prey system with Beddington-DeAngelis functional response and fractional cross-diffusion d_4 subjected to the homogeneous Neumann boundary condition. The priori estimate result shows that if d_1 is not too large or not too small, d_2 is not too small and d_3 is not too large, then the solutions of system (1.2) are bounded and we can assert that the bound of the solution is not constrained by the cross diffusion d_4. Moreover, from the proof the non-existence of non-constant positive solution, we obtain the sufficient condition for the non-existence, that is, $d_2 d_4$ is small enough. Finally, we discuss the existence of non-constant positive solution and the results indicate that the system admits a non-constant positive solution provided by the self-diffusion d_2 or the cross-diffusion d_4 is large enough, which means the diffusion d_2 and the fractional type cross-diffusion d_4 can create spatial patterns.

Furthermore, this study contributes to the growing field of fractional diffusion models in ecology. Fractional diffusion is a generalization of classical diffusion that allows for non-local interactions and has been shown to better capture the long-range effects observed in many ecological systems. The results of this paper demonstrate the potential for fractional cross-diffusion to play an important role in determining the existence and properties of positive solutions in predator-prey systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We would like to give our sincere thanks to the anonymous referees and the handing editors for their valuable comments and helpful suggestions on this manuscript.

This work is supported in part by the Basic Research Program of Natural Science of Shaanxi Province (No. 2022JM-055), and also by the Science Research Program of Xi'an Eurasia University (No. 2022GCPY01).

Conflict of interest

No conflict of interest exits in the submission of this manuscript, and manuscript is approved by all authors for publication. I would like to declare on behalf of my co-authors that the work described was original research that has not been published previously, and not under consideration for publication elsewhere, in whole or in part. All the authors listed have approved the manuscript that is enclosed.

References

1. W. Chen, M. Wang, Qualitative analysis of predator-prey models with Beddington-DeAngelis functional response and diffusion, *Math. Comput. Model.,* **42** (2005), 31–44. http://dx.doi.org/10.1016/j.na.2006.08.022

2. R. S. Cantrell, C. Cosner, On the dynamics of predator-prey models with the Beddington-DeAngelis functional response, *J. Math. Anal. Appl.,* **257** (2001), 206–222. https://doi.org/10.1006/jmaa.2000.7343
3. D. T. Dimitrov, H. V. Kojouharov, Complete mathematical analysis of predator-prey models with linear prey growth and Beddington-DeAngelis, *Appl. Math. Comput.,* **162** (2005), 523–538. https://doi.org/10.1016/j.amc.2003.12.106
4. D. Luo, Q. Wang, Global bifurcation and pattern formation for a reaction-diffusion predator-prey model with prey-taxis and double Beddington-DeAngelis functional responses, *Nonlinear Anal.-Real,* **67** (2022), 103638. https://doi.org/10.1016/j.nonrwa.2022.103638
5. Y. Lou, W. M. Ni, Diffusion, self-diffusion and cross-diffusion, *J. Differ. Equations,* **131** (1996), 79–131. https://doi.org/10.1006/jdeq.1996.0157
6. Y. Lou, W. M. Ni, Diffusion vs cross-diffusion: An elliptic approach, *J. Differ. Equations,* **154** (1999), 157–190. https://doi.org/10.1006/jdeq.1998.3559
7. F. Yi, Turing instability of the periodic solutions for reaction-diffusion systems with cross-diffusion and the patch model with cross-diffusion-like coupling, **281** (2021), 379–410. https://doi.org/10.1016/j.jde.2021.02.006
8. Q. Meng, L. Yang, Steady state in a cross-diffusion predator-prey model with the Beddington-DeAngelis functional response, *Nonlinear Anal.-Real,* **45** (2019), 401–413. https://doi.org/10.1016/j.nonrwa.2018.07.012
9. J. F. Zhang, Spatial patterns of a fractional type cross-diffusion Holling-Tanner model, *Comput. Math. Appl.,* **76** (2018), 957–965. https://doi.org/10.1016/j.camwa.2018.05.033
10. P. Xue, Y. Jia, C. Ren, X. Li, Non-constant positive solutions of a general Gause-type predator-prey system with self- and cross- diffusions, *Math. Model. Nat. Pheno.,* **16** (2021), 1–25. https://doi.org/10.1051/mmnp/2021017
11. Y. Jia, P. Xue, Effects of the self- and cross-diffusion on positive steady states for a generalized predator-prey system, *Nonlinear Anal.-Real,* **32** (2016), 229–241. https://doi.org/10.1016/j.nonrwa.2016.04.012
12. C. Li, Existence of positive solution for a cross-diffusion predator-prey system with Holling type-II functional response, *Chaos Soliton. Fract.,* **99** (2017), 226–232. https://doi.org/10.1016/j.chaos.2017.04.001
13. S. Li, Y. Yamada, Effect of cross-diffusion prey-preydator model with a protection zone II, *J. Math. Anal. Appl.,* **461** (2018), 971–992. https://doi.org/10.1016/j.jmaa.2017.12.029
14. A. Okubo, *Diffusion and ecological problems: Mathematical models*, Springer-Verlag, Berlin, 1980.
15. J. Cao, H. Sun, P. Hao, P. Wang, Bifurcation and turing instability for a predator-prey model with nonlinear reaction cross-diffusion, *Appl. Math. Model.,* **89** (2021), 1663–1677. https://doi.org/10.1016/j.apm.2020.08.030
16. X. Chen, Y. Qi, M. Wang, A strongly coupled predator-prey system with non-monotonic functional response, *Nonlinear Anal.,* **67** (2007), 1966–1979. https://doi.org/10.1016/j.na.2006.08.022
17. Y. Lou, W. M. Ni, S. Yotsutani, On a limiting system in the Lotka-Volterra competition with cross-diffusion, *Discrete Cont. Dyn. Syst.,* **10** (2004), 435–458.

18. J. H. P. Dawes, M. O. Souza, A derivation of Holling's type I, II and III functional responses in predator-prey systems, *J. Theor. Biol.,* **327** (2013), 11–22. https://doi.org/10.1016/j.jtbi.2013.02.017
19. D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equation of second order*, Springer-Verlag, Berlin, 2001.
20. X. Zeng, Non-constant positive steady states of a prey-predator system with cross-diffusions, *J. Math. Anal. Appl.*, **332** (2007), 989–1009. https://doi.org/10.1006/jmaa.2000.7343
21. R. K. Ghaziani, W. Govaerts, C. Sonck, Resonance and bifurcation in a discrete-time predator-prey system with Holling functional response, *Nonlinear Anal.-Real,* **13** (2012), 1451–1465. https://doi.org/10.1016/j.nonrwa.2011.11.009
22. R. Peng, J. P. Shi, Non-existence of non-constant positive steady states of two Holling-type II predator-prey systems: Strong interaction case, *J. Differ. Equations,* **247** (2009), 866–886. https://doi.org/10.1016/j.jde.2009.03.008
23. Y. Jia, A sufficient and necessary condition for the existence of positive solutions for a prey-predator system with Ivlev-type functional response, *Appl. Math. Lett.,* **24** (2011), 1084–1088. https://doi.org/10.1016/j.aml.2011.01.027
24. J. Sugie, Two-parameter bifurcation in a predator-prey system of Ivlev type, *J. Math. Anal. Appl.,* **217** (1998), 349–371. https://doi.org/10.1006/jmaa.1997.5700
25. Y. Jia, J. Wu, H. K. Xu, Positive solutions of a Lotka-Volterra competition model with cross-diffusion, *Comput. Math. Appl.,* **68** (2014), 1220–1228. https://doi.org/10.1016/j.camwa.2014.08.016
26. W. Ko, K. Ryu, A qualitative study on general Gause-type predator-prey models with constant diffusion rates, *J. Math. Anal. Appl.,* **344** (2008), 217–230. https://doi.org/10.1016/j.jmaa.2008.03.006
27. K. Oeda, Effect of cross-diffusion on the stationary problem of a prey-predator model with a protection zone, *J. Differ. Equations,* **250** (2011), 3988–4009. https://doi.org/10.1016/j.jde.2011.01.026
28. Y. Peng, T. Zhang, Turing instability and pattern induced by cross-diffusion in a predator-prey system with Allee effect, *Appl. Math. Comput.,* **275** (2016), 1–12. https://doi.org/10.1016/j.amc.2015.11.067
29. L. Nirenberg, *Topics in nonlinear functional analysis*, American Mathematical Socity, Providence, Rhode Island, 2001.
30. H. Wang, P. Liu, Pattern dynamics of a predator-prey system with cross-diffusion, Allee effect and generalized Holling IV functional response, *Chaos Soliton. Fract.,* **171** (2023), 113456. https://doi.org/10.1016/j.chaos.2023.113456

AIMS Press